On the Spectrum of Signals Obtained by Driving FM Modulators with Chaotic Sequences

Sergio Callegari* Riccardo Rovatti* Gianluca Setti†

***Abstract* — This paper is a first attempt at characterizing the spectrum of signals obtained by feeding chaotic sequences into FM modulators. First of all, it is shown that previous analytic results about FM modulators driven by random PAM signals do not generally hold when chaotic PAM signals are used. Secondly, two novel theorems are proposed: the first one states that the *random-FM* spectrum formulas can in fact remain valid for the chaotic case as a limit condition; the second establishes some constraints sufficient for achieving symmetric *chaotic-FM* spectra.**

1 Introduction

Driving an FM modulator with random sequences — or more precisely with random Pulse Amplitude Modulated (PAM) signals — as in figure 1, is a flexible way to generate constant envelope spread spectrum signals [1]. The latter are indispensible in many novel telecommunications schemes [1] and useful for the synthesis of temporization signals suitable for digital equipment, DC-DC converters, power actuators, and any kind of *switching* device designed for environments where strict electromagnetic compatibility requirements exist [2].

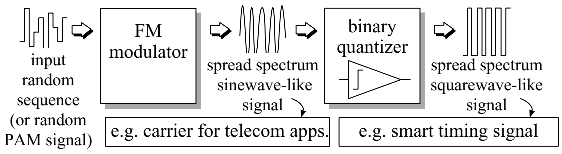

Figure 1: Applications of random/chaotic FM modulation.

For all these applications, a spectral characterization of the signal at the output of the FM block is a pre-condition. A general result in this sense was recently presented in [3], where a formula was derived, linking the FM power density spectrum to the probability density function (PDF) of the random modulating sequence. In the first instance, this result appeared sufficient to enable the design of practical spread spectrum sources following the architecture proposed in figure 1.

However, the required truly random sequences are inconvenient to synthesize in hardware, especially if they need to obey a prescribed PDF and/or to be updated at high data rates. Consequently, the exploitation of sequences generated by chaotic sources [3, 4] is currently being actively experimented. As well known, dynamical systems based on models such as

$$x_{n+1} = M(x_n) \qquad M : [-1, 1] \to [-1, 1] \tag{1}$$

can produce chaotic, *noise-like* sequences if M is appropriately chosen [5]. Furthermore, techniques exist for the synthesis of M in order to have them obey a given PDF [6].

*DEIS, Università di Bologna, Viale Risorgimento 2, 40136 - Bologna, Italia. E-mail: scallegari@deis.unibo.it, or rrovatti@deis.unibo.it.

†DI, Università di Ferrara, Via Saragat 1, 44100 - Ferrara, Italia. E-mail: gsetti@ing.unife.it.

Unfortunately, in the architecture of figure 1, the adoption of chaotic PAM signals at the input of the FM modulator is not seamless. For instance, figure 2 compares the spectra obtained by cascading an FM modulator to a random and two different chaotic sources, all characterized by the same PDF. The spectra are all notably different and the use of chaos does not always permit a correct approximation of the power distribution which is achieved in the random case.

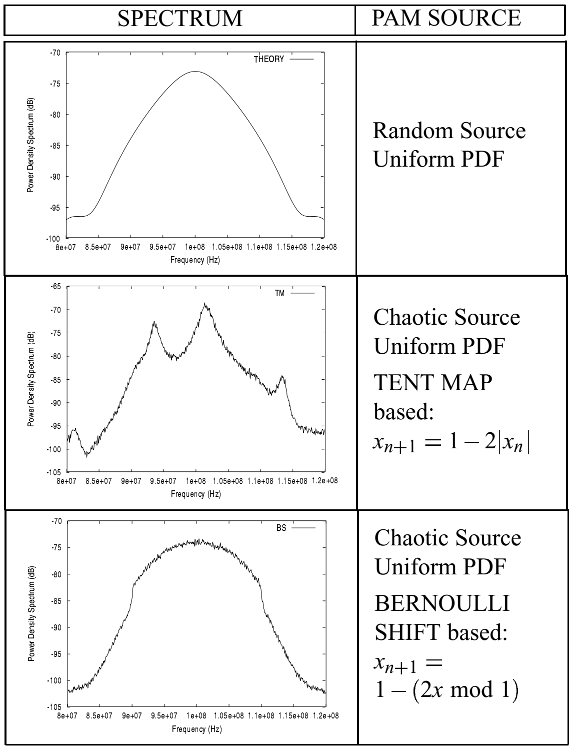

Figure 2: Spectra comparison. Carrier frequency $f_0 = 1\,\mathrm{MHz}$; Frequency deviation $\Delta f = 8.5\,\mathrm{MHz}$; Modulating signal data rate $1/T = 20\,\mathrm{MHz}$ (modulation index $m = 0.425$).

To the best of our knowledge, the peaks in the center graph have not been completely explained yet. In this paper we illustrate two novel theorems which can be employed to achieve a first characterization of the phenomena. In many cases their use can enable the spectral analysis of *chaotic-FM* signals and the synthesis of systems conforming to particular spectral requirements.

2 Spectral characterization

2.1 Preliminary concepts

In the following we shall assume that the FM signal under exam is:

$$s(t) = \cos\left[2\pi\left(f_0 t + \Delta f \int_{-\infty}^{t} \xi(\tau)\, d\tau\right)\right] \qquad (2)$$

where f_0 indicates the carrier frequency, Δf the frequency deviation, and $\xi(t)$ the input PAM signal

$$\xi(t) = \sum_{k=-\infty}^{\infty} x_k g(t-kT) \qquad (3)$$

given that $g(t)$ is a function equal to 1 for $t \in]0,T]$ and to zero elsewhere. In other terms, $\xi(t)$ is a zero-order-hold version of a sequence x_k updated at $t = kT$ and whose values are generated following the recursive rule (1). We assume that the map M used in (1) is *exact* [5]. Recall that M defined in $[-1,1]$ is said to be exact if $\lim_{k\to\infty} \mu(M^k(\mathcal{Y})) = \mu([-1,1])$ for all $\mathcal{Y} \subseteq [-1,1]$ s.t. $\mu(\mathcal{Y}) > 0$, when M^k is the k-th iteration of the map and μ denotes a measure function. The meaning of these easy-to-satisfy assumptions is to guarantee that (almost) every modulating sequence x_k is characterized by the same probability density function $\overline{\rho}$, characteristic of the map M and determinable as the unique invariant of its Perron-Frobenius operator (PFO) $\mathbf{P}$ [5]. Furthermore, it is guaranteed that the autocorrelation function of almost every sequence x_k remains bounded by an exponentially vanishing envelope whose decay rate is set by the modulus of the largest eigenvalue of $\mathbf{P}$ less than 1 [7]. This quantity will be further on referred as the system rate of mixing r_{mix}.

To proceed, it is convenient to strip the carrier information from $s(t)$, by considering the low-pass equivalent $\tilde{s}(t)$, defined so that $s(t) = \text{Re}[\tilde{s}(t)e^{\mathrm{i}2\pi f_0 t}]$. Setting $\varphi_k = 2\pi\Delta f T \sum_{j=-\infty}^{k-1} x_j$, one gets

$$\tilde{s}(t) = \sum_{k=-\infty}^{\infty} e^{\mathrm{i}[2\pi\Delta f x_k(t-kT)+\varphi_k]} g(t-kT) \qquad (4)$$

Since $s(t)$ is cyclostationary [8, 3], the autocorrelation function must be obtained by averaging [8, chapters 4.4.1–2]:

$$\phi_{\tilde{s}\tilde{s}}(\tau) = \frac{1}{2T}\int_0^T \mathbf{E}_{x_0}\left[\tilde{s}^*(t)\tilde{s}(t+\tau)\right] \qquad (5)$$

where the superscript asterisk denotes complex conjugation. The low pass power density spectrum is obviously given by

$$\Phi_{\tilde{s}\tilde{s}}(f) = \int_{-\infty}^{\infty} \phi_{\tilde{s}\tilde{s}}(\tau) e^{-\mathrm{i}2\pi f\tau} d\tau \qquad (6)$$

from which the spectrum of the original signal can be obtained as

$$\Phi_{ss}(f) = \frac{1}{2}\left[\Phi_{\tilde{s}\tilde{s}}(f-f_0) + \Phi_{\tilde{s}\tilde{s}}(-f-f_0)\right] \qquad (7)$$

Wanting to consider only positive frequencies, in practical cases (7) can be approximated by $\Phi_{\tilde{s}\tilde{s}}(f-f_0)$.

It is hence necessary to compute $\phi_{\tilde{s}\tilde{s}}$. Starting from its definition (5), the substitution of (4) into the integrand gives

$$\begin{aligned}&\mathbf{E}_{x_0}[\tilde{s}^*(t)\tilde{s}(t+\tau)] = \\ &\quad \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \mathbf{E}_{x_0}\left[e^{-\mathrm{i}\{2\pi\Delta f[x_k(t-kT)-x_l(t+\tau-lT)]+\varphi_k-\varphi_l\}}\right]\cdot \\ &\quad g(t-kT)g(t+\tau-lT)\end{aligned} \qquad (8)$$

By splitting the two sums into the cases $k = l$, $k > l$ and $k < l$

$$\begin{aligned}&\mathbf{E}_{x_0}[\tilde{s}^*(t)\tilde{s}(t+\tau)] = \\ &\quad \sum_{k=-\infty}^{\infty} \mathbf{E}_{x_0}\left[e^{\mathrm{i}2\pi\Delta f x_k\tau}\right] g(t-kT)g(t+\tau-kT) + \\ &\quad \sum_{k=-\infty}^{\infty}\sum_{n=1}^{\infty} \mathbf{E}_{x_0}\Big[e^{-\mathrm{i}2\pi\Delta f[x_k(t-kT-T)-x_{k+n}(t+\tau-(k+n)T)]}\cdot \\ &\quad\quad e^{\mathrm{i}2\pi\Delta f T\sum_{j=k+1}^{k+n-1}x_j}\Big] g(t-kT)g(t+\tau-(k+n)T) + \\ &\quad \sum_{k=-\infty}^{\infty}\sum_{n=1}^{\infty} \mathbf{E}_{x_0}\Big[e^{-\mathrm{i}2\pi\Delta f[x_{k+n}(t-(k+n)T)-x_k(t+\tau-kT-T)]}\cdot \\ &\quad\quad e^{-\mathrm{i}2\pi\Delta f T\sum_{j=k+1}^{k+n-1}x_j}\Big] g(t-(k+n)T)g(t+\tau-kT)\end{aligned} \qquad (9)$$

Exploiting the stationarity of the modulating sequence, $\phi_{\tilde{s}\tilde{s}}$ can thus be expressed as

$$\begin{aligned}&\phi_{\tilde{s}\tilde{s}}(\tau) = \frac{1}{2T}\mathbf{E}_{x_0}\left[e^{\mathrm{i}2\pi\Delta f x\tau}\right]\int_0^T g(t)g(t+\tau)dt + \\ &\quad \frac{1}{2T}\int_0^T \sum_{n=1}^{\infty} \mathbf{E}_{x_0}\Big[e^{-\mathrm{i}2\pi\Delta f[x_0(t-T)-x_n(t+\tau-nT)]}\cdot \\ &\quad\quad e^{\mathrm{i}2\pi\Delta f T\sum_{j=1}^{n-1}x_j}\Big] g(t)g(t+\tau-nT)dt + \\ &\quad \frac{1}{2T}\int_0^T \sum_{n=1}^{\infty} \mathbf{E}_{x_0}\Big[e^{-\mathrm{i}2\pi\Delta f[x_n(t-\tau-nT)-x_0(t-T)]}\cdot \\ &\quad\quad e^{-\mathrm{i}2\pi\Delta f T\sum_{j=1}^{n-1}x_j}\Big] g(t)g(t-\tau-nT)dt\end{aligned} \qquad (10)$$

where only one of the two latter sums can be non-null, depending on the sign of τ. Furthermore, $\phi_{\tilde{s}\tilde{s}}(\tau) = \phi^*_{\tilde{s}\tilde{s}}(-\tau)$, as it is always the case with autocorrelation functions, meaning that it is possible to consider only positive values for τ without loosing any information. By setting $w = e^{\mathrm{i}2\pi\,\Delta f T}$, (6) can thus be recast as

$$\Phi_{\tilde{s}\tilde{s}}(f) = 2\,\text{Re}\left(\int_0^{\infty}\phi_{\tilde{s}\tilde{s}}(\tau)w^{-\frac{f\tau}{\Delta f T}}d\tau\right) \qquad (11)$$

where $\phi_{\tilde{s}\tilde{s}}(\tau)$ is simplified into

$$\begin{aligned}&\phi_{\tilde{s}\tilde{s}}(\tau) = \frac{1}{2T}\mathbf{E}_{x_0}\left[w^{x_0\frac{\tau}{T}}\right](T-\tau)g(\tau) + \\ &\quad \frac{1}{2T}\int_0^T \sum_{n=1}^{\infty} g(t)g(t+\tau-nT)E_n(t,\tau)\,dt\end{aligned} \qquad (12)$$

thanks to the definition of

$$E_n(t,\tau) = \mathbf{E}_{x_0}\left[w^{x_0\left(1-\frac{t}{T}\right)+\sum_{j=1}^{n-1}x_j+x_n\left(\frac{t+\tau}{T}-n\right)}\right] \qquad (13)$$

2.2 Theorems

Theorem 1 (Convergence to random FM for $r_{\text{mix}} \to 0$).

$$\begin{aligned}&\lim_{r_{mix}\to 0}\Phi_{\tilde{s}\tilde{s}}(f) = \\ &\quad \mathbf{E}_{x_0}[K_1(x_0,f)] + \text{Re}\left\{\frac{\mathbf{E}^2_{x_0}[K_2(x_0,f)]}{1-\mathbf{E}_{x_0}[K_3(x_0,f)]}\right\}\end{aligned} \qquad (14)$$

where

$$K_1(x,f) = \frac{1}{2} T \operatorname{sinc}^2(\pi T(f-\Delta f x))$$
$$K_2(x,f) = \mathrm{i}\frac{e^{-\mathrm{i}2\pi T(f-\Delta f x)}-1}{2\pi\sqrt{T}(f-\Delta f x)} \tag{15}$$
$$K_3(x,f) = e^{-\mathrm{i}2\pi T(f-\Delta f x)}$$

Remarks: Equations (14) and (15) are the same formula given in [3] for the spectrum of FM signals obtained by the modulation of a *random* sequences. Hence this theorem can be informally restated by saying that for the generation of constant envelope spread spectrum carrier via the FM of a noisy PAM signal, a chaotic source can be seamlessly substituted for a random one if its rate of mixing is sufficiently good. In this condition the spectrum shape can be determined analytically and it depends exclusively on the FM modulation index $m = \Delta f T$ and on the probability density function $\overline{\rho}$ of the modulating signal. This theorem has a strong operational value as it opens the way to an iterative design process in which chaotic sources characterized by progressively better r_{mix} can be sequentially tested, until a sufficiently good behavior is obtained.

Proof: Define

$$G_{n,i}(x) = \begin{cases} w^{x\left(1-\frac{t}{T}\right)} & i=0 \\ w^x & 0<i<n \\ w^{x\left(\frac{t+\tau}{T}-n\right)} & i=n \end{cases} \tag{16}$$

$$F_{n,k}(x) = \begin{cases} F_{n,k-1}(x)G_{n,k-1}(M^{k-1}(x)) & 1\le k\le n \\ 1 & k=0 \end{cases} \tag{17}$$

With this, the term E_n in (12) can be recast as

$$E_n(t,\tau) = \int_{-1}^{1} F_{n,n}(x)G_{n,n}(M^n(x))\overline{\rho}(x)\,dx \tag{18}$$

Now, consider a generic correlation term built as

$$L_{n,k} = \int_{-1}^{1} F_{n,k}(x)G_{n,k}(M^k(x))\overline{\rho}(x)\,dx \tag{19}$$

From the very definition of r_{mix} it follows that [7] $\exists C\in\mathbb{R}^+$ so that for any t and τ the following inequality holds:

$$\left|L_{n,k} - \int_{-1}^{1} F_{n,k}(x)\,\overline{\rho}(x)\,dx \int_{-1}^{1} G_{n,k}(x)\overline{\rho}(x)\right| \le C\left\|F_{n,k}\overline{\rho}\right\|_{\text{BV}} (r_{\text{mix}})^k\,dx \tag{20}$$

Now, define

$$a_{n,k} = L_{n,k} - \prod_{i=0}^{k}\int_{-1}^{1} G_{n,i}(x)\overline{\rho}(x)\,dx$$
$$b_{n,k} = \int_{-1}^{1} F_{n,k}(x)\,\overline{\rho}(x)\,dx \int_{-1}^{1} G_{n,k}(x)\overline{\rho}(x)\,dx - \prod_{i=0}^{k}\int_{-1}^{1} G_{n,i}(x)\overline{\rho}(x)\,dx \tag{21}$$

so that equation (20) can be rewritten as

$$|a_{n,k}-b_{n,k}| \le C\left\|F_{n,k}\overline{\rho}\right\|_{\text{BV}} (r_{\text{mix}})^k \tag{22}$$

Exploiting the inequality $|a-b| \ge \big||a|-|b|\big|$, this becomes

$$|a_{n,k}| \le C\left\|F_{n,k}\overline{\rho}\right\|_{\text{BV}} (r_{\text{mix}})^k + |b_{n,k}| \tag{23}$$

By means of the recursive definition (17), the term $|b_{n,k}|$ can be expressed as

$$|b_{n,k}| = \left|L_{n,k-1}\int_{-1}^{1} G_{n,k}(x)\overline{\rho}(x)\,dx - \prod_{i=0}^{k}\int_{-1}^{1} G_{n,i}(x)\overline{\rho}(x)\,dx\right| = \left|\int_{-1}^{1} G_{n,k}(x)\overline{\rho}(x)\right| |a_{n,k-1}| \le |a_{n,k-1}| \tag{24}$$

Hence, we have the recursive inequality

$$|a_{n,k}| \le C\left\|F_{n,k}\overline{\rho}\right\|_{\text{BV}} (r_{\text{mix}})^k + |a_{n,k-1}| \tag{25}$$

Since $|a_{n,0}|$ is obviously zero, for the case $k=n$ one can write

$$|a_{n,n}| \le C\sum_{i=1}^{n}\left\|F_{n,i}\overline{\rho}\right\|_{\text{BV}} (r_{\text{mix}})^i \tag{26}$$

which vanishes for $r_{\text{mix}}\to 0$. Since the product in the definition of $a_{n,k}$ (21) is a product of expectations, this means that

$$\lim_{r_{\text{mix}}\to 0} E_n(t,\tau) = \mathbf{E}_{x_0}\left[w^{x_0\left(1-\frac{t}{T}\right)}\right]\left(\mathbf{E}_{x_0}\left[w^{x_0}\right]\right)^{n-1}\mathbf{E}_{x_0}\left[w^{x_0\left(\frac{t\tau}{T}-n\right)}\right] \tag{27}$$

In order to exploit this factorization we must consider the summands in (12). First of all, intuitively accept that the integral and the sum can swap, even if space constraints prevent us from formally proving it here. The general summand is thus

$$\int_0^T g(t)g(t+\tau-nT)E_n(t,\tau)\,dt \xrightarrow[r_{\text{mix}}\to 0]{} H^{n-1}(T)\int_0^T g(t)g(t+\tau-nT)H(T-t)H(t+\tau-nT)dt = \int_{-\infty}^{\infty} g(t)H(t)g(\tau-(n-1)T-t)H(\tau-(n-1)T-t)dt = (gH*gH)(\tau-(n-1)T) \tag{28}$$

where $H(\xi) = \mathbf{E}_{x_0}\left[w^{x_0\xi/T}\right]$ and $*$ stands for convolution. The last equality is obtained by substituting $t+\tau-nT$ for t and exploiting the vanishing of g when its argument lays outside $[0,T]$. Note that for any given n the limit holds uniformly in τ. Now consider the expression of the spectrum $\Phi_{\tilde{s}\tilde{s}}(f)$ when (12) is substituted into (11). Since it is not obvious whether it is legitimate to compute $\lim_{r_{\text{mix}}\to 0}\Phi_{\tilde{s}\tilde{s}}(f)$ by moving the limit inside the infinite sum, we shall split the spectrum into the following terms:

$$\Phi^{(1)}_{\tilde{s}\tilde{s}}(f) = \operatorname{Re}\left(\frac{1}{T}\int_0^T H(\tau)(T-\tau)e^{-\mathrm{i}2\pi f\tau}\,d\tau\right)$$
$$\Phi^{(2)}_{\tilde{s}\tilde{s}}(\tilde{n},f) = \operatorname{Re}\left(\frac{1}{T}\sum_{1}^{\tilde{n}}\int_0^\infty\int_0^T g(t)g(t+\tau-nT)\cdot E_n(t,\tau)e^{-\mathrm{i}2\pi f\tau}\,dt\,d\tau\right) \tag{29}$$
$$\Phi^{(3)}_{\tilde{s}\tilde{s}}(\tilde{n},f) = \operatorname{Re}\left(\frac{1}{T}\sum_{\tilde{n}}^{\infty}\int_0^\infty\int_0^T g(t)g(t+\tau-nT)\cdot E_n(t,\tau)e^{-\mathrm{i}2\pi f\tau}\,dt\,d\tau\right)$$

so that for any finite $\tilde{n}$, $\Phi_{\tilde{s}\tilde{s}}(f) = \Phi^{(1)}_{\tilde{s}\tilde{s}}(f) + \Phi^{(2)}_{\tilde{s}\tilde{s}}(\tilde{n}, f) + \Phi^{(3)}_{\tilde{s}\tilde{s}}(\tilde{n}, f)$. The first term can always be rewritten as

$$\Phi^{(1)}_{\tilde{s}\tilde{s}}(f) = \int_{-1}^{1} \mathrm{Re}\left(\frac{1}{T}\int_0^T e^{\mathrm{i}2\pi(\Delta f x_0 - f)\tau}(T-\tau)\,d\tau\right)\cdot \overline{\rho}(x)\,dx = \mathbf{E}_{x_0}\left[\frac{\sin^2[\pi T(f-\Delta f x)]}{2\pi^2 T(f-\Delta f x)^2}\right] \quad (30)$$

The second term, for any given $\tilde{n}$ and for $r_{\mathrm{mix}} \to \infty$ is

$$\overline{\Phi^{(2)}_{\tilde{s}\tilde{s}}}(\tilde{n}, f) = \lim_{r_{\mathrm{mix}}\to 0} \Phi^{(2)}_{\tilde{s}\tilde{s}}(\tilde{n}, f) = \mathrm{Re}\left(\frac{1}{T}\cdot \left(\sum_{n=1}^{\tilde{n}} H^{n-1}(T)e^{-\mathrm{i}2\pi f T(n-1)}\right)\left(\int_0^T H(\tau)e^{-\mathrm{i}2\pi f\tau}d\tau\right)^2\right) \quad (31)$$

Finally, the third term vanishes for $\tilde{n} \to \infty$ for any f. This is due to the fact that $\Phi^{(3)}_{\tilde{s}\tilde{s}}(0, f)$ is nothing but the spectrum itself, which must obviously converge for any f. Therefore, for any given f, $\Phi_{\tilde{s}\tilde{s}}(f) = \Phi^{(1)}_{\tilde{s}\tilde{s}}(f) + \lim_{\tilde{n}\to\infty} \overline{\Phi^{(2)}_{\tilde{s}\tilde{s}}}(\tilde{n}, f)$. The latter limit can be recast considering that

$$\frac{1}{T}\left(\int_0^T H(\tau)e^{-\mathrm{i}2\pi f\tau}d\tau\right)^2 = \left(\frac{1}{\sqrt{T}}\int_{-1}^{1}\int_0^T e^{\mathrm{i}2\pi(\Delta f x - f)\tau}d\tau\,\overline{\rho}(x)\,dx\right)^2 = \mathbf{E}_{x_0}\left[\mathrm{i}\frac{e^{-\mathrm{i}2\pi T(f-\Delta f x_0)} - 1}{2\pi\sqrt{T}(f-\Delta f x_0)}\right] \quad (32)$$

and that

$$\sum_{n=1}^{\infty} H^{n-1}(T)e^{-\mathrm{i}2\pi f T(n-1)} = \frac{1}{1-H(T)e^{-\mathrm{i}2\pi f T}} = \left(1-\mathbf{E}_{x_0}\left[e^{-\mathrm{i}2\pi T(f-\Delta f x_0)}\right]\right)^{-1} \quad (33)$$

Note that (33) is valid thanks to the fact that $|H(T)| < 1$. Now compare $K1$, $K2$ and $K3$ in the theorem statement with equations (30), (32), and (33) to see that the thesis has been proven. □

Theorem 2 (Spectrum symmetry).

$$M(-x) = -M(x) \Rightarrow \Phi_{\tilde{s}\tilde{s}}(-f) = \Phi_{\tilde{s}\tilde{s}}(f) \quad (34)$$

Remarks: If the chaotic map is odd symmetric, then the base band spectrum is even symmetric and thus also the band pass spectrum is symmetric with regard to the axis $f = f_0$. This theorem can find applications in all the cases where a precise specification of the spectrum shape is not important, but a balanced power distribution is. For instance consider the center and bottom spectra in figure 2: both are different from the random-FM spectrum (top). Nonetheless, the bottom one is much more regular than the other, thanks to the symmetry deriving from an odd symmetric map.

Proof: First of all, note that $M(-x) = -M(x) \Rightarrow \overline{\rho}(-x) = \overline{\rho}(x)$, i.e., if the map is odd symmetric, then the invariant density is even symmetric [6]. Also, consider 2 sequences $x_k^{(A)} = M^k(x_0^{(A)})$ and $x_k^{(B)} = M^k(x_0^{(B)})$. If M is odd symmetric $x_0^{(B)} = -x_0^{(A)} \Rightarrow \forall k \in \mathbb{N}$, $x_k^{(B)} = -x_k^{(A)}$, i.e., a change in the sign of the initial value causes only a change in the sign of all the following values.

Now, consider the definition of $E_n(t,\tau)$ in equation (13) and note that it can be recast as

$$E_n(t,\tau) = \int_0^1 w^{-x\left(1-\frac{t}{T}\right)+\sum_{j=1}^{n-1}M^j(-x)+M^n(-x)\left(\frac{t+\tau}{T}-n\right)}\rho(-x)\,dx + \int_0^1 w^{x\left(1-\frac{t}{T}\right)+\sum_{j=1}^{n-1}M^j(x)+M^n(x)\left(\frac{t+\tau}{T}-n\right)}\rho(x)\,dx \quad (35)$$

and thus, if the map is odd symmetric, $E_N(t,\tau)$ is the sum of 2 complex conjugated quantities and is real. If $E_N(t,\tau)$ is real, also the autocorrelation function (12) is so. Since the Fourier transform of real functions is even, we have the thesis. □

3 Conclusions

Building upon results describing the spectrum of signals obtained by feeding random sequences to FM modulators, we have proposed mathematical tools which help the spectral characterization of chaotic-FM signals. The proposed theorems permit a complete derivation of chaotic-FM spectra in many limit cases (large r_{mix}, slow modulations) and can be interpreted as a generalization of the previously known random-FM methods. Further investigation is still necessary to broaden the range of conditions for which formal results can be proposed.

4 References

[1] M. P. Kennedy, G. Kolumbán, and Z. Jákó, *Chaotic Modulation Schemes*, ch. 6. In [9], 2000.

[2] R. Rovatti, G. Setti, and S. Graffi, "Chaos based FM of clock signals for EMI reduction", in *Proceedings of ECCTD'99*, vol. 1, (Stresa), Sept. 1999.

[3] G. Mazzini, R. Rovatti, and G. Setti, "Spectrum of randomly frequency modulated signals", in *Proceedings of NDES2000*, (Catania), May 2000.

[4] S. Callegari, R. Rovatti, and G. Setti, "Interleaving techniques for high-throughput chaotic noise generation in CMOS", in *Proceedings of MIXDES2000*, (Gdynia), 2000.

[5] A. Lasota and M. C. Mackey, *Chaos, Fractals and Noise. Stochastic Aspects of Dynamics*. Springer-Verlag, 2nd ed., 1995.

[6] W. Schwarz, M. Götz, K. Kelber, A. Abel, T. Falk, and F. Dachselt, *Statistical Analysis and Design of Chaotic Systems*, ch. 9. In [9], 2000.

[7] R. Rovatti, G. Setti, and G. Mazzini, "Chaotic complex spreading sequences for asynchronous DS-CDMA. II. Some theoretical performance bounds", *IEEE Transactions on Circuits and Systems, Part I*, vol. 45, pp. 496–506, Apr. 1998.

[8] J. G. Proakis, *Digital Communications*. McGraw Hill, 4rd ed., 2000.

[9] M. P. Kennedy, R. Rovatti, and G. Setti, eds., *Chaotic Electronics in Telecommunications*. Boca Raton: CRC International Press, 2000.