A Reliable Iterative Error Tracking Method for Approximate Symbolic Pole/Zero Analysis

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Abstract – This paper presents a new error tracking approach for symbolic pole/zero analysis by approximation of generalized eigenvalue problems (GEP) [1]. Efficient and reliable error control is achieved by tracking eigenvalue shifts due to device parameter eliminations numerically using an iterative GEP solver in combination with the modal assurance criterion. The applicability of the enhanced algorithm to industrial circuit design problems will be demonstrated by the symbolic analysis of a CMOS folded-cascode operational amplifier.

1 Introduction

Symbolic calculation of poles and zeros plays an important role in design-oriented analysis of analog circuits, in particular in the analysis of feedback amplifiers. As the numerator and denominator polynomials of transfer functions are generally irreducible and have degrees much larger than two, it is rarely possible to find mathematically exact expressions for the poles and zeros of an electronic circuit. Consequently, approximations must be introduced to extract at least the technically relevant eigenvalues in analytical form.

In [1] the principles of a matrix-based simplification before-generation (SBG) method for direct approximation of a linear symbolic system of circuit equations

\[ T(p, s)x = b \]

with respect to a selected eigenvalue \( \lambda \) were introduced. The idea behind this method is to first decompose \( T \) into a matrix pencil \( (A(p), B(p)) \) and estimate the influence of each parameter \( p_i \in (A, B) \) on the solution \( \lambda \) of the corresponding GEP \( (1a)/(1b) \) by calculating the eigenvalue sensitivities \( \partial \lambda / \partial p_i \) [2].

\[
\begin{align*}
(A - \lambda B)u &= 0 \quad (1a) \\
v^H(A - \lambda B) &= 0 \quad (1b)
\end{align*}
\]

Where \( v^H \) denotes the Hermitian conjugate of the left eigenvector \( v \).

Then, discarding all parameters with negligible influence on \( \lambda \) from the pencil results in a simplified GEP whose determinant yields a reduced-order approximation of the characteristic polynomial of the original problem. Provided that the pole or zero of interest is located sufficiently far apart from other eigenvalues, the degree of the approximated polynomial becomes low enough so that its roots can be calculated symbolically.

In this paper two enhancements of the symbolic pole/zero analysis method presented in [1] will be discussed. First, the problem of potentially false eigenvalue pairing will be addressed, and a more reliable error control approach will be proposed. Secondly, the computational efficiency of the error tracking process will be increased by employing a fast iterative GEP solver instead of the QZ algorithm to calculate approximation errors.

2 The Eigenvalue Pairing Problem

A central task of the SBG algorithm is to determine whether eliminating a particular parameter \( p_i \) is admissible for a given error bound \( \epsilon_{\text{max}} \). Assume that \( (\lambda_1, u_1) \) is an eigenpair of \( (1a)/(1b) \), where \( \lambda_1 \) denotes the eigenvalue we wish to extract, and that \( (\lambda_2^*, u_2^*) \) is an eigenpair of the perturbed GEP

\[
(A^* - \lambda^* B^*)u^* = 0 \quad (2)
\]

obtained by setting a parameter \( p_i \in (A, B) \) to zero. If (2) is a valid approximation of the original GEP with respect to \( \lambda_1 \), and if \( (\lambda_1^*, u_1^*) \) and \( (\lambda_2, u_2) \) are corresponding eigenpairs, we have

\[
\lambda_2^* = \lambda_1 \Leftrightarrow \text{error}(\lambda_1, \lambda_1^*) < \epsilon_{\text{max}} . \quad (3)
\]

However, condition (3) alone is not sufficient to ensure that \( \lambda_2^* \) is indeed identical to the element \( \lambda_1^* \) of the spectrum of \( (A^*, B^*) \) which corresponds to \( \lambda_1 \). In fact, \( \lambda_2^* \) may just be some other eigenvalue of the perturbed pencil that has incidentally moved to a position close to \( \lambda_1 \) while \( \lambda_1^* \) has been shifted far away from its original position \( \lambda_1 \). This situation is illustrated in Fig. 1, where
In both cases a small number of iterations should be
fails to converge, then the eigenvalue shift is too large.
is supplied as initial guess. Conversely, if the method
the design-point value of the original eigenpair
eigenpair of the perturbed GEP very rapidly if
iterative method should converge to the corresponding
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3 Iterative Eigenpair Tracking

When approximating a system of circuit equations with
respect to a single pole or zero, only one eigenpair
(λ, u) of the corresponding GEP needs to be tracked
numerically. Approximation errors can thus be calculat-
ed more efficiently with an iterative GEP solver than
with the QZ algorithm, which always computes the en-
tire spectrum of the GEP. Moreover, if the error induced
by eliminating a parameter from (A, B) is small, an it-
erative method should converge to the corresponding
eigenpair (λ*, u*) of the perturbed GEP very rapidly if
the design-point value (λ₀, u₀) of the original eigenpair
is supplied as initial guess. Conversely, if the method
fails to converge, then the eigenvalue shift is too large.
In both cases a small number of iterations should be
enough to either correct the eigenpair or recognize that
a perturbation is inadmissible.

A suitable method for this purpose is the Jacobi or-
thogonal correction method (JOCM) [4, 5]. Given an
initial guess (θ₀, y₀) for the sought eigenpair (λ*, u*),
the JOCM yields iterates \( y_{i+1} = y_i + z_{i+1} \) and
\( ϑ_{i+1} = (w^H \lambda^* y_{i+1}) / (w^H B^* y_{i+1}) \) by solving the fol-
lowing linear matrix equation repeatedly for the correc-
tion vector \( z_{i+1} \) until the norm of the residual vector
\( r := (A^* - ϑ_i B^*) y_i \) is small enough.

\[
\begin{bmatrix}
A^* - ϑ_i B^*
\end{bmatrix} \begin{bmatrix}
w \\
y
\end{bmatrix} = \begin{bmatrix}
0 \\
n\end{bmatrix}
\]

Where w, w̃, and ̃y denote appropriately chosen pro-
jection vectors which ensure convergence of the it-
erates, and ε is a scalar variable whose value is irrelevant
in this context.

As opposed to the QZ algorithm, the JOCM automatic-
atically computes the eigenvector associated with the
sought eigenvalue; therefore, an eigenvector correla-
tion test can be integrated into the error tracking process with
almost no extra implementation effort and computa-
tional cost. By checking if

\[
\text{MAC}(u, y_{i+1}) > m_{\text{min}}
\]

after each iteration, the iterates (θ_i, y_i) can be prevent-
ed from converging to some other eigenpair of the per-
turbed pencil than the one corresponding to the original
solution (λ, u).

An enhanced version of the symbolic pole/zero ex-
traction algorithm described in [1] is outlined in Fig. 2;
for a complete discussion see reference [5]. Note that
the new algorithm differs from the original approach
only in the error tracking method.

input: symbolic matrix pencil (A, B) + design point
compute numerical reference solution (λ, u, v)
compute parameter ranking based on eigenvalue
sensitivities (see [1, 5])

for all parameters p_i from ranking

discard p_i from matrix pencil
compute perturbed solution (λ*, u*) by JOCM
if error(λ₁, λ*) > ε_max or MAC(u, u*) < m_min
put p_i back into matrix pencil

compute characteristic polynomial of approximated
matrix pencil, \( P(s) = \det(A - sB) \)
compute root(s) of \( P(s) \) if the polynomial degree is
less than 3

output: root(s) of \( P(s) \)

Figure 2: A symbolic pole/zero extraction algorithm with
reliable iterative error tracking.
4 Symbolic Analysis of a CMOS Folded-Cascode Operational Amplifier

The pole/zero extraction algorithm presented in this paper has been implemented in Mathematica as part of the symbolic circuit analysis toolbox Analog Insydes 2 [7, 8]. To demonstrate the practical applicability of the algorithm, the poles of the CMOS folded-cascode opamp shown in Fig. 3 will be computed symbolically.

Fig. 4 displays the frequency response of the opamp’s open-loop differential-mode voltage gain. The Bode diagram shows a peak near 10 MHz, caused by a parasitic complex pole pair located close to the imaginary axis (Fig. 5). The analysis task can now be formulated as follows: Extract an approximated symbolic formula for the parasitic pole pair in order to determine the circuit elements which have dominant influence on the peak. Then try to find suitable circuit modifications for compensating the amplifier’s frequency response [6].

In the first step of the analysis procedure, the netlist is read into Analog Insydes 2. Once the netlist and all necessary model, operating-point, and small-signal parameters have been read in, appropriate models have to be selected for the MOSFET devices. For better interpretability of the resulting symbolic expressions, the MOS devices were modeled using the SPICE Level 3 AC model (Fig. 3) instead of the BSIM3 AC model.

After model expansion, the netlist contains 321 primitive components, leading to a $29 \times 29$ system of modified nodal equations. The differential-mode voltage transfer function has 19 poles and 19 zeros (Fig. 5). In fully expanded symbolic form, it would contain more than $5 \times 10^{19}$ product terms.
To extract the pole pair at \( s = (-2.1 \pm 8.3j) \times 10^7 \) symbolically, the algorithm outlined in Fig. 2 is applied to the system of modified nodal equations, using an error bound of \( \varepsilon_{\text{max}} = 0.1 \) and a MAC acceptance threshold of \( m_{\text{min}} = 0.95 \). Simplifying the resulting approximated equations algebraically yields the \( 4 \times 4 \) system displayed in Fig. 6. Its determinant can be calculated easily, yielding a polynomial of degree 2 which can be solved explicitly for the sought pole pair. Fig. 7 shows the approximate symbolic expression thus obtained. The CPU time required for the approximation is of the order of 20 s on a 266 MHz Pentium-II PC.

The extracted formula reveals that for a given load \( C_L \), compensation capacitance \( C_{\text{CD}} \), and fixed operating points, the value of the gate-source capacitance of the device MP15 should be increased in order to decrease the value of the imaginary parts of the pole pair; this can be achieved by adding a shunt capacitor between the gate and source terminals of MP15.

Fig. 8 shows a root locus plot of the amplifier calculated from the original (unspecified) system of equations as \( C_{\text{GS,MP15}} \) is swept from 1 pF to 10 pF. The plot shows that the above conclusion drawn from the approximated symbolic pole expression is valid.

5 Conclusions

A new, reliable error tracking strategy for a matrix-based symbolic pole/zero extraction algorithm has been presented. Control of the nominal approximation error is achieved by computing eigenvalue shifts numerically using an iterative eigenpair improvement procedure based on the Jacobi orthogonal correction method. Correct pairing of exact and perturbed eigenvalues is ensured by means of the modal assurance criterion. The algorithm was implemented in Analog Insydes 2 and applied successfully to an industrial design problem.

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References


