A Novel Technique for the Estimation of Stability in Feedforward and Multiple-Feedback Oversampled Σ–Δ A/D Converter Configurations

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Abstract — This paper presents a novel statistical approach for the determination of the maximum DC input signal amplitude for stable operation in feedforward and multiple-feedback family of Σ–Δ A/D converters. The hitherto statistical techniques are based on the restrictive assumption that the quantizer input signal has a Gaussian distribution. However, in feedforward and multiple-feedback A/D converters, the quantizer input signal is a weighted sum of signals with Gaussian-like distributions, leading to an overall quantizer input signal with non-Gaussian distribution. In the proposed statistical approach, the Gram-Charlier series is employed to model the actual (non-Gaussian distributed) quantizer input signal. An application example is given to demonstrate the resulting improvements achieved in estimating the maximum DC input signal amplitude.

1 Introduction

Recently, feedforward and multiple-feedback family of Σ–Δ A/D converters have found widespread use in signal processing applications [1]. Fig. 1a shows a generic Σ–Δ A/D converter, where \( U(z) \) represents the z-transformed input signal, where \( Y(z) \) represents the z-transformed output signal, and where \( E(z) \) represents the z-transformed quantizer input signal. In the practical design of feedforward and multiple-feedback Σ–Δ A/D converter configurations, the constituent quantizer is frequently replaced by a uniformly distributed additive white noise source and a variable gain element \( k \) as depicted in Fig. 1b, where \( Q(z) \) represents the z-transformed quantization noise. Then, the A/D converter is characterized by the signal and noise transfer functions

\[
\text{STF}(z) = \frac{Y(z)}{U(z)} \equiv S(z)/D(z) \quad \text{and} \quad \text{NTF}(z) = \frac{Y(z)}{Q(z)} \equiv N(z)/D(z),
\]

where \( D(z) \) represents the denominator polynomial common to both \( \text{STF}(z) \) and \( \text{NTF}(z) \), and where \( S(z) \) and \( N(z) \) represent the numerator polynomials of \( \text{STF}(z) \) and \( \text{NTF}(z) \), respectively.

In [2], it was shown that by determining the noise transfer function \( \text{NTF}(z) \) for the feedforward and multiple-feedback Σ–Δ A/D converters, one can determine the corresponding signal transfer function \( \text{STF}(z) \) and subsequently proceed to realize the A/D converter configuration proper. Unfortunately, the maximum DC input signal amplitude for stable A/D converter operation is unknown prior to extensive computer simulations. To address this problem, [3] and [4] developed a statistical technique to estimate the maximum DC input signal amplitude for stable Σ–Δ A/D converter operation. This technique is based on several assumptions, the most restricting of which is that the quantizer input signal has a Gaussian distribution. However, in feedforward and multiple-feedback Σ–Δ A/D converter configurations, the quantizer input signal is a weighted sum of signals with Gaussian-like distributions \(^1\), leading to an overall quantizer input signal with non-Gaussian distribution [5]. It was further demonstrated in [5] that the existing statistical technique can fail to accurately predict the maximum DC input signal amplitude, particularly when multiple regions of instability are present.

This paper is concerned with the development of a novel statistical approach for the determination of the maximum DC input signal for stable A/D operation in feedforward and multiple-feedback Σ–Δ A/D converters.

2 The Hitherto Statistical Technique

This section is concerned with a derivation of the hitherto statistical technique in order to highlight its main fundamental assumptions and its shortcomings, particularly with regard to the assumption that the quantizer input signal has a Gaussian distribution.

\(^1\)originating from integrator output signals, feedforward signals, and feedback signals.
In the design of feedforward and multiple-feedback $\Sigma-\Delta$ \$/D converters, having determined the noise transfer function $NTF(z)$ (starting from a set of high-level system design specifications), a modified noise transfer function can be defined as

$$\overline{NTF}(z) = \frac{N(z)}{kD(z) - kN(z) + N(z)}.$$  \hspace{1cm} (1)

Then, one can define the noise power gain (NPG) as [4]

$$NPG = \frac{2}{\pi} \int_{-\pi}^{\pi} [\overline{NTF}(e^{jwT})]^2 dw.$$  \hspace{1cm} (2)

By using this equation, one can obtain a graphical representation of $NPG$ versus $k$, leading to the empirical result that for higher order $\Sigma-\Delta$ \$/D converters, $NPG$ is a convex function of $k$ with a minimum occurring, say, at $NPG_{\text{min}}$. In the statistical technique, a relationship is derived between $NPG_{\text{min}}$ and the maximum DC input signal amplitude $m_{\text{max}}$, based on the following three assumptions, a) the input signal $u(n)$ is a DC signal of amplitude $m_u$, b) the quantizer input signal $e(n)$ is the sum of a zero mean Gaussian distributed component $G_e$ and a DC bias $m_e$, and c) the quantization noise $Q(z)$ is a white noise with zero mean.

Based on the above assumptions, the variance of the output signal $y(n)$ can be calculated in two ways, namely,

$$\sigma_y^2 = E\{y(n)^2\} - E^2\{y(n)\} = 1 - m_u^2 \hspace{1cm} (3)$$

or

$$\sigma_y^2 = \sigma_N^2 NPG \hspace{1cm} (4)$$

where $E\{\cdot\}$ is the expectation operator. By combining Eqns. 3 and 4, one obtains the relationship

$$NPG = (1 - m_u^2) / \sigma_N^2. \hspace{1cm} (5)$$

Next, it is desired to determine $\sigma_N^2$ in terms of the DC input signal amplitude $m_u$. To this end, from Fig. 1, the output signal $y(n)$ can be expressed as

$$y(n) = ke(n) + q(n). \hspace{1cm} (6)$$

But, since $q(n)$ has zero mean and since the mean of $y(n)$ is $m_y$, $y(n)$ can be written in the form

$$y(n) = m_u + k[e(n) - m_e] + q(n). \hspace{1cm} (7)$$

Moreover, in [3] it was argued that since the quantizer input signal $e(n)$ is an integration of the white noise $q(n)$, its distribution can be assumed to be Gaussian. In this way, let $G_e$ represent the zero mean Gaussian distributed component of the quantizer input signal $e(n)$ given by $G_e = e(n) - m_e$. Then, Eqn. 7 can be written as

$$y(n) = m_u + kG_e + q(n). \hspace{1cm} (8)$$

The variance of $y(n)$ may be determined from Eqn. 8 as

$$\sigma_y^2 = E\{[m_u + kG_e + q(n)]^2\} - E^2\{m_u + kG_e + q(n)\} = k^2\sigma_G^2 + \sigma_N^2. \hspace{1cm} (9)$$

By substituting $\sigma_y^2$ from Eqn. 3 into 9, one gets

$$\sigma_N^2 = 1 - m_u^2 - k^2\sigma_G^2. \hspace{1cm} (10)$$

To determine the value of $k$ in terms of the DC input amplitude $m_u$, it is convenient to ensure that the white noise $q(n)$ is uncorrelated with the Gaussian component of the quantizer input $G_e$, yielding $E\{G_e\cdot q(n)\} = 0$. The variable gain $k$ may then be determined from

$$Cov\{G_e, y(n)\} = E\{(G_e - E\{G_e\})(y(n) - m_u)\} = k\sigma_G^2. \hspace{1cm} (11)$$

By rearranging Eqn. 11 one can write

$$k = \frac{Cov\{G_e, y(n)\}}{\sigma_G^2} = \int_{-\infty}^{\infty} G_e y(n) e^{-\frac{(y-n)^2}{2\sigma^2}} \frac{e^{-\frac{m^2}{2\sigma^2}}}{\sigma_G \sqrt{2\pi}} dG_e. \hspace{1cm} (12)$$

This integral can be simplified if one realizes that $y(n)$ will be $-1$ if the input to the quantizer is negative, and $+1$ if the input to the quantizer is positive, leading to

$$k = \frac{1}{\sigma_G \sqrt{2\pi}} \int_{m_e}^{\infty} G_e 1 e^{-\frac{(y-n)^2}{2\sigma^2}} e^{-\frac{m^2}{2\sigma^2}} dG_e. \hspace{1cm} (13)$$

By carrying out the integration, one arrives at

$$k = \frac{2}{\sigma_G \sqrt{2\pi}} e^{-\frac{m_e^2}{2\sigma_G^2}}. \hspace{1cm} (14)$$

Next, the mean of the output $m_y$ can determined as [6]

$$m_y = Prob(G_e > -m_e) - Prob(G_e < -m_e). \hspace{1cm} (15)$$

But, since $G_e$ has a Gaussian distribution, $m_y$ may be expressed as

$$m_y = \frac{1}{2} \left[ 1 + erf(\frac{m_e}{\sigma_G \sqrt{2}}) \right] - \frac{1}{2} \left[ 1 - erf(\frac{-m_e}{\sigma_G \sqrt{2}}) \right] = erf(\frac{-m_e}{\sqrt{2}\sigma_G}). \hspace{1cm} (16)$$

By substituting $m_u$ for $m_y$ in Eqn. 16, and by solving for $m_u$ one obtains the relationship

$$\frac{m_e}{\sqrt{2}\sigma_G} = erf^{-1}(m_u). \hspace{1cm} (17)$$
By substituting Eqn. 17 into Eqn. 14, one gets

$$k = \frac{2}{\sigma_G \sqrt{2\pi}} \pi e^{-2\sigma^2} (m_c)^2. \tag{18}$$

By substituting $k$ from Eqn. 18 into Eqn. 10, the following expression is obtained for $\sigma_N^2$

$$\sigma_N^2 = 1 - m_u^2 + \frac{2}{\pi} \pi e^{-2\sigma^2} (m_u)^2. \tag{19}$$

In this way, $\sigma_N^2$ is a function of the DC input level $m_u$ only. By substituting $\sigma_N^2$ from Eqn. 19 into Eqn. 5

$$NPG = \frac{1 - m_u^2}{1 - m_u^2 + \frac{2}{\pi} \pi e^{-2\sigma^2} (m_u)^2}, \tag{20}$$

which is the same result as reported in [4]. A graphical representation of $NPG$ versus $m_u$ can be obtained as shown in Fig. 3. By using the method presented in [4], the maximum DC input signal amplitude $m_{\text{min}}$ corresponds to $NPG_{\text{min}}$.

3 The Proposed Statistical Approach

This section presents a novel statistical approach based on Gram-Charlier series to model the quantizer input signal without making any recourse to the assumption that the quantizer input signal is Gaussian distributed.

The Gram-Charlier series is defined as

$$f_N(z) = \sum_{j=0}^{N} C_j H_j(z) \frac{1}{\sqrt{2\pi}} e^{-z^2}, \quad z = \frac{x - \mu}{\sigma}, \tag{21}$$

where $N$ represents the number of terms in the series expansion (controlling the precision). Moreover, $H_n(z)$ represents the $n$-th Hermite polynomial,

$$H_0 = 1, \quad H_1 = z, \quad H_2 = z^2 - 1, \tag{22}$$

$$H_n = z H_{n-1}(z) - (n-1) H_{n-2}(z), \tag{23}$$

and $C_n$ represents the $n$-th Hermite coefficient,

$$C_n = \frac{n!}{k!} \sum_{k=0}^{n} \left( \frac{1}{2} \right)^k \frac{n! v_n - 2k}{k!(n-2k)!}. \tag{24}$$

with $v_n = E[(x - \mu)^n] / \sigma^2$ representing the $n$-th normalized central moment of $x$.

From Section 2, the distribution of the quantizer input enters into the derivation of the statistical technique in Eqs. 12 and 16. Therefore, since $C_1 \approx 0$ in practical situations, Eqn. 12 can be modified to include the Gram-Charlier series leading to a new expression for $k$:

$$k = \frac{\sqrt{2}}{\sigma_G \sqrt{2\pi}} \pi e^{-2\sigma^2} (m_c)^2. \tag{25}$$

Similarly, Eqn. 16 can be modified to the following form

$$m_y = \frac{2}{\sqrt{2\pi}} \pi e^{-2\sigma^2} \left[ \frac{m_c}{\sqrt{2\sigma^2}} + \sum_{i=2}^{N} C_i (-m_c) e^{-m_c^2 / 2\sigma^2} \right]. \tag{26}$$

As in the case of the existing statistical technique, $m_y$ can be replaced by $m_u$. However, due to the complexity of Eqn. 26, it is very difficult to solve for $m_c$ in terms of $m_u$ analytically. The easiest approach to this problem is to solve for $m_c$ through an iterative optimization technique, e.g. by using the interval halving optimization [7].

Having determined $m_c$, one can invoke its value in Eqn. 25 to obtain a corresponding value for $k$. Moreover, having determined $k$, one can invoke its value in Eqn. 10 to obtain $\sigma_N^2$. Finally, having determined $\sigma_N^2$, one can invoke its value in Eqn. 5 to obtain $NPG$. In this way, a value for $NPG$ can be obtained for a corresponding value of $m_u$, and a graphical representation of $NPG$ versus $m_u$ can be obtained.

4 Demonstration of the Accuracy of the Proposed Statistical Approach

This section is concerned with an investigation of the improvements achieved by using the proposed statistical approach over the hitherto statistical technique. The starting point of this investigation is an actual (nonlinear) Matlab simulation of a 6th order cascade-of-integrators $\Sigma$-$\Delta$ A/D converter for the determination of the relationships between $k$ and $m_u$ on the one hand, and that between $NPG$ and $m_u$ on the other, where the A/D converter has an oversampling ratio of 16 and $NPG_{\text{min}} = 1.92$.

Fig. 2 shows the relationship between $k$ and $m_u$ with particulars as indicated in the following:

- Curve A: Obtained through Matlab simulation,
- Curve B: Obtained by employing Eqn. 18 of the hitherto statistical technique, and
- Curves C and D: Obtained by employing Eqn. 25 of the proposed statistical approach for $N = 4$ and $N = 20$, respectively.

The moments of the quantizer input signal, which are required in the Gram-Charlier series, were obtained through (nonlinear) Matlab simulation of the A/D converter. As is evident from Fig. 2, the result obtained from the hitherto statistical technique differs substantially from the Matlab simulation result, whereas the result obtained from the

\[2\text{Using the optimization technique outlined in the previous section.}\]

\[3\text{Analytic techniques exist for first- and second-order }\Sigma\Delta\text{ A/D converters only.}\]
The proposed statistical approach more closely approximates the Matlab result. It is also evident that a more accurate result may be obtained in the proposed statistical approach by increasing the number of terms $N$ in the Gram-Charlier series approximation. Choosing $15 < N < 30$ is usually sufficient for an accurate approximation of the quantizer input signal distribution. Fig. 3 shows the relationship between $NPG$ and $m_u$ with particulars as indicated in the following:

Curves A, B: Obtained through Matlab simulation, Curve C: Obtained by employing Eqn. 20 of the hitherto statistical technique, and Curve D: Obtained by employing Eqn. 25 of the proposed approach using Eqns. 10 and 5 as described in the previous section for $N = 4$ and $N = 20$, respectively.

From Fig. 3, the difference between the result obtained through the hitherto statistical technique and the result obtained using Matlab simulation leads to a mean squared error of 0.4070. The most important error is at the instability point $m_u = 0.59$, where the Matlab result falls below $NPG_{min}$, whereas the result obtained through the hitherto statistical technique maintains its value. In this way, for $N = 20$ an improvement of 80% is achieved for the estimation of the noise power gain, and, subsequently, the accuracy of the estimation of the maximum DC input signal amplitude. Furthermore, the proposed statistical approach successfully predicts the instability point found at $m_u = 0.59$. Investigations into other feedforward and multiple-feedback A/D converters have resulted in similar improvements.

5 Conclusion

This paper has presented a novel statistical approach for the estimation of the maximum DC input signal amplitude for stable operation in feedforward and multiple-feedback Σ-∆ A/D converters. This statistical approach employs Gram-Charlier series to model the distribution of the quantizer input signal. A typical application example has been given demonstrating that the proposed statistical approach leads to an 80% increase in the accuracy of estimating the noise power gain as compared to the hitherto statistical technique. This in turn gave rise to an improved accuracy in the estimation of the maximum DC input signal amplitude for stable A/D converter operation.

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References


