

Anticausal Inverses for Digital Filter Banks

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Abstract — Time-reversed i.e. noncausal inversion of digital filters is important for the maximally decimated filter banks based on a polyphase framework. Vaidyanathan & Chen studied the subject extensively in several of their papers. This paper re-examines the case where the filters are to be implemented in the canonical direct form II structure and introduces a new technique for the anticausal inversion of maximum-phase FIR filters.

1 Introduction

In a maximally decimated filter bank based on a polyphase implementation the perfect reconstruction property of the filter bank depends on the polyphase inverses. In most cases anticausal inverses are needed, especially if the polyphase filters are allpass filters. The theory and practical use of anticausal inverses can be found in the works of Vaidyanathan & Chen [1, 2, 3] and Creusere & Mitra [4, 5].

This paper is organized as follows. Section 2 explains the need for anticausal inverses in maximally decimated filter banks. Section 3 gives the initial state conditions for the anticausal filter when both the filter and its time-reversed inverse are implemented in a canonical direct form II structure. Also in this section a new embedded filter states technique is developed for the time-reversed inversion of maximum-phase FIR filters. Finally Section 4 presents the conclusion.

2 Maximally Decimated Filter Banks

The filter banks discussed here are based on the modulated versions of a prototype filter, $H(z)$ in such a way that the entire frequency range is covered by the modulated filters, see Figure 1, where $\mathbf{T}$ (e.g. DFT matrix) is the modulation matrix and $E_k(z)$ are the polyphase filters of $H(z)$

$$H(z) = \sum_{k=0}^{M-1} E_k(z) z^{-k}. \quad (1)$$

2.1 Perfect Reconstruction

The filter bank shown in Figure 1 will have perfect reconstruction, i.e. $\hat{x}(n) = x(n)$ if the synthesis polyphase filters are the inverses of the analysis polyphase filters, i.e.

$$R_k(z) = \frac{1}{E_k(z)} \quad (2)$$

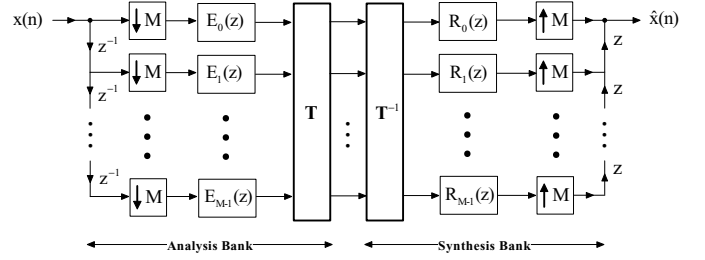


Figure 1: Polyphase implementation of a uniform maximally decimated filter bank

where $0 \leq k \leq M-1$. The branch subscript k will be dropped to improve readability when we are referring to the polyphase filters.

In general $E(z)$ contains maximum-phase components which will cause causal $R(z)$ to be unstable. Instability of $R(z)$ can be overcome if the unstable components of $R(z)$ are implemented in a noncausal manner [5]. To achieve this, first $E(z)$ needs to be separated into its maximum- and minimum-phase parts.

$$E(z) = G(z)B(z) \quad (3)$$

where $G(z)$ is the maximum-phase part of $E(z)$ and $B(z)$ is the minimum-phase part of $E(z)$. To satisfy perfect reconstruction, $R(z)$ can be implemented as a cascade of two inverse filters

$$R(z) = \frac{1}{B(z)} \frac{1}{G(z^{-1})} \quad (4)$$

where the signal needs to be time-reversed before and after being processed by the filter $\frac{1}{G(z^{-1})}$, see Figure 2.

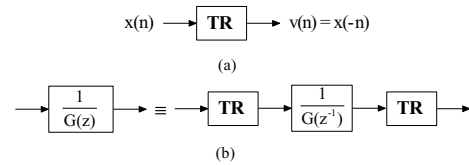


Figure 2: (a) Ideal time-reversed operator; (b) implementation of a noncausal filtering structure using time-reversal operators.

3 Implementing Anticausal Inverses

Some structures for anticausal inverse were studied previously by Vaidyanathan and Chen [3]. Here we

will consider the case that $G(z)$ and its noncausal inverse are both implemented in the direct form II structure (e.g. *filter* function in MATLAB).

3.1 Selecting the Initial States for Anticausal Inverses

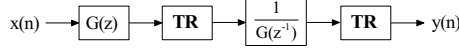


Figure 3: Signal $x(n)$ is processed first by $G(z)$ and then with its noncausal inverse $1/G(z^{-1})$.

Consider the transfer function of $G(z)$ is given as

$$G(z) = \frac{\sum_{n=0}^N p_n z^{-n}}{1 + \sum_{n=1}^D q_n z^{-n}} \quad (5)$$

where $N \geq D$. The transfer function of its noncausal inverse will be

$$\frac{1}{G(z^{-1})} = \frac{z^{D-N} \sum_{n=0}^{D-1} q_{D-n} z^{-n} + z^{-N}}{\sum_{n=0}^N p_{N-n} z^{-n}}. \quad (6)$$

Figure 3 shows a diagram for the causal-anticausal inverse pair. In order to recover the original signal at the output, i.e. $y(n) = x(n)$, the initial states of the noncausal inverse should be selected as the reverse ordered final states of $G(z)$ with a scale factor of $1/p_N$. More elaborately

$$w_i(0) = \frac{s_{N-i+1}(L-1)}{p_N}, \quad 1 \leq i \leq N \quad (7)$$

where $s_i(n)$ are the states of $G(z)$, $w_i(n)$ are the states of $1/G(z^{-1})$ and L is the length of the input signal $x(n)$. The proof is presented in Appendix A. This result is an addition to the previous work of Vaidyanathan and Chen [3].

3.2 Reconstruction Method When $G(z)$ is FIR

If $G(z)$ is an FIR filter

$$G(z) = \sum_{n=0}^N p_n z^{-n} \quad (8)$$

then its anticausal inverse will be

$$\frac{1}{G(z^{-1})} = \frac{z^{-N}}{\sum_{n=0}^N p_{N-n} z^{-n}}. \quad (9)$$

Let us modify anticausal inverse by multiplying with z^N in order to remove the pure delay in the numerator. The transfer function for the modified anticausal inverse will be

$$\frac{z^N}{G(z^{-1})} = \frac{1}{\sum_{n=0}^N p_{N-n} z^{-n}}. \quad (10)$$

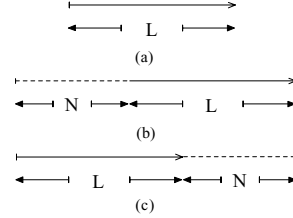


Figure 4: (a) Original signal, L -samples; (b) Final filter states of $G(z)$ (N -samples) appended after filtering; (c) First N -samples are removed in order to revert back the length of the signal to L -samples.

Exact reconstruction will be achieved if the following procedure is implemented.

- Processing with $G(z)$:
 - Process the L -sample signal with the filter $G(z)$
 - Append the signal with the final states of $G(z)$, see Figure 4b
 - Revert the length of the signal back to L -samples by removing the first N -samples, see Figure 4c.
- Recovery with the modified anticausal inverse:
 - Time-reverse the signal
 - Process the signal with the modified anticausal inverse $\frac{z^N}{G(z^{-1})}$
 - Time-reverse the signal.

The proof is presented in Appendix B.

This new method eliminates the need for sending the final filter states explicitly by embedding them into the output of the filter without increasing the signal length. The method is useful as the numerator of a polyphase filter can be splitted into its maximum- and minimum-phase parts. Figure 5 shows the comparison between the new method and the circular convolution method [6] for a 5-scale lowpass-iterated QMF filter bank, based on second order allpass filters [7], applied to 512×512 Barbara¹ image with EZW [8] coding. As the graph

¹<http://links.uwaterloo.ca/bragzone.base.html>

indicates the new method achieves better PSNR results than the traditional circular convolution approach.

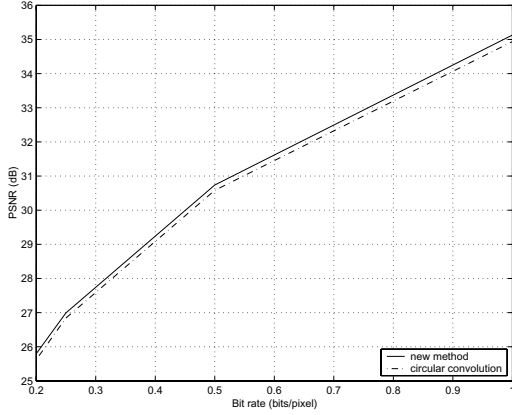


Figure 5: Comparative evaluation of the new embedded filter states technique versus the well-known circular convolution technique on 512×512 Barbara image in terms of the reconstruction PSNR values versus bits/pixel rates.

4 Conclusion

In this paper we presented the required initial conditions for the anticausal inverses when the filters are implemented in the direct form II structure. Initial filter states for the synthesis filters need to be transferred explicitly between the analysis and the synthesis bank [9], and the adverse effect of this extra information on the transmission bit rate is generally removed by the coding algorithms [5]. A new method is introduced for the anticausal inversion of the maximum-phase FIR filters, eliminating the need for sending the filter state information explicitly. Use of circular convolution also eliminates the need for the transmission of filter states and the example in Figure 5 shows that our new technique performs better than the circular convolution technique in terms of the reconstruction PSNR values at the same bit rates.

A Initial conditions for $\frac{1}{G(z^{-1})}$

If the filter $G(z)$ given in (5) is implemented in the direct form II framework, the state space equations will be

$$s_i(n+1) = s_{i+1}(n) + p_i x(n) - q_i y(n) \quad (11)$$

$$y(n) = s_1(n) + p_0 x(n) \quad (12)$$

where $1 < i \leq N$, $s_{N+1}(n) = 0$, $q_i = 0$ for $i > D$ and $x(n)$, $y(n)$ and $s(n)$ are the input, output and

states of the filter respectively. Similarly the state space equations for $1/G(z^{-1})$ will be

$$w_i(n+1) = w_{i+1}(n) + \frac{q_{N-i}}{p_N} u(n) - \frac{p_{N-i}}{p_N} v(n) \quad (13)$$

$$v(n) = w_1(n) + \frac{q_N}{p_N} u(n) \quad (14)$$

where $1 < i \leq N$, $w_{N+1}(n) = 0$, $q_i = 0$ for $i > D$ and $u(n)$, $v(n)$ and $w(n)$ are the input, output and states of the filter respectively.

The question is what are the initial conditions $w_i(0)$ for $1/G(z^{-1})$ in order to satisfy $v(n) = x(L-n-1)$ where $u(n) = y(L-n-1)$. To provide the proof, let us start by writing equation for the next states of $G(z)$ in terms of the current input and output, from (11) and (12),

$$s_i(n+1) = \sum_{m=0}^{N-i} [p_{N-m} x(n-N+i+m) - q_{N-m} y(n-N+i+m)] \quad (15)$$

and writing the equations for the current states of $1/G(z^{-1})$ in terms of the current input and output from (13) and (14)

$$w_i(n) = \frac{1}{p_N} \sum_{m=0}^{i-1} [p_{N-m} v(n+i-1-m) - q_{N-m} u(n+i-1-m)] \quad (16)$$

Using the facts that $u(n)$ is $u(n) = y(L-n-1)$ and $v(n)$ should be $v(n) = x(L-n-1)$, we can rewrite (16) as follows

$$w_i(n) = \frac{1}{p_N} \sum_{m=0}^{i-1} [p_{N-m} x(L-n-i+m) - q_{N-m} y(L-n-i+m)] \quad (17)$$

To illustrate the relationship between $w_i(n)$ and $s_i(n)$, let us modify (15) using the substitutions $i = N-k+1$ and $n+1 = L-t$:

$$s_{N-k+1}(L-t) = \sum_{m=0}^{k-1} [p_{N-m} x(L-t-k+m) - q_{N-m} y(L-t-k+m)] \quad (18)$$

Now from (17) and (18) it is apparent that

$$w_i(n) = \frac{s_{N-i+1}(L-n)}{p_N} \quad (19)$$

and this completes the proof of the statement in (7).

B Proof of the embedded filter states method

The state space equations for the FIR filter $G(z)$ given in (8) are

$$s_i(n+1) = s_{i+1}(n) + p_i x(n) \quad (20)$$

$$y(n) = s_1(n) + p_0 x(n) \quad (21)$$

where $1 < i \leq N$, $s_{N+1}(n) = 0$, and $x(n)$, $y(n)$ and $s_i(n)$ are the input, output and states of the filter respectively. Similarly the state space equations for $z^N/G(z^{-1})$ will be

$$w_i(n+1) = w_{i+1}(n) - \frac{p_{N-i}}{p_N} v(n) \quad (22)$$

$$v(n) = w_1(n) + \frac{1}{p_N} u(n) \quad (23)$$

where $1 < i \leq N$, $w_{N+1}(n) = 0$, and $u(n)$, $v(n)$ and $w_i(n)$ are the input, output and states of the filter respectively.

Here we will show that if $u(n)$ equals

$$u(n) = \begin{cases} s_{N-n}(L) & \text{if } 0 \leq n < N \\ y(L-1-n+N) & \text{if } N \leq n < L \end{cases} \quad (24)$$

then $v(n)$ satisfies $v(n) = x(L-1-n)$ while $w_i(0) = 0$. Let us start by expressing $u(n)$ given in (24) in terms of $x(n)$, using (20), (21) and (15):

$$u(n) = \begin{cases} \sum_{m=0}^n p(N-m)x(L-1-n+m) & \text{if } 0 \leq n < N \\ \sum_{m=0}^N p(N-m)x(L-1-n+m) & \text{if } N \leq n < L \end{cases} \quad (25)$$

Now let us express $w_1(n)$ in terms of $v(n)$, using (23) and (22), where $w_i(0) = 0$

$$w_1(n) = \begin{cases} -\frac{1}{p_N} \sum_{m=1}^n p(N-m)v(n-m) & \text{if } 0 \leq n < N \\ -\frac{1}{p_N} \sum_{m=1}^N p(N-m)v(n-m) & \text{if } N \leq n < L \end{cases} \quad (26)$$

Let us use the method of induction and replace $v(n)$ with $v(n) = x(L-1-n)$:

$$w_1(n) = \begin{cases} -\frac{1}{p_N} \sum_{m=1}^n p(N-m)x(L-1-n+m) & \text{if } 0 \leq n < N \\ -\frac{1}{p_N} \sum_{m=1}^N p(N-m)x(L-1-n+m) & \text{if } N \leq n < L \end{cases} \quad (27)$$

If we evaluate (23) using (25) and (27) then we will obtain our initial assumption: $v(n) = x(L-1-n)$. Hence this completes the proof.

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