Cramér-Rao Bounds for Direction of Arrival and Range Estimation of Near-Field Sources

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Abstract — In this paper the performance of the conditional maximum likelihood location estimator for the near-field sources is studied based on the derivation of Cramér-Rao bounds. The Cramér-Rao bound results are further analyzed for the one source case to provide insight into the dependence of estimation accuracy on signal to noise ratio and the number of antenna elements. Some insights into the achievable performance of the conditional maximum likelihood algorithm is obtained by numerical evaluation of the Cramér-Rao bounds for different test cases.

1 Introduction

In recent years, several algorithms have appeared in array signal processing literature for passive source localization. However, majority of the algorithms deals with the case in which the sources are assumed to be in the far-field of the array. That is, the sources are located relatively far from the array, and hence waves emanating from the sources can be considered as plane waves at the sensor array [1]. Consequently, each source location can be parametrized by only the azimuth (bearing). If the sources move into 'Fresnel' region of the array aperture (i.e., near-field), the far-field signal model needs to be modified to approximate the effects of spherical curvature of the wavefronts by incorporating the range information. Thus, to localize a near-field source, one needs to estimate both the azimuth and the range [1].

Recently, various near-field source localization algorithms have been proposed based on second and higher order statistics which allow eigen decomposition techniques [1], [2], [3]. In [4], conditional (deterministic) maximum likelihood (ML) formulation of the near-field localization problem is considered and Expectation/Maximization (EM) algorithm is proposed to solve resulting optimization problem.

In this paper we study the performance of the near-field conditional ML estimation algorithm proposed in [4] based on the derivation of Cramér-Rao bounds (CRBs). We also analyze the CRB for the single source to provide further insight into the dependence of estimation accuracy on signal to noise ratio and the number of antenna elements.

We use the standard narrowband observation for $d$ near-field sources impinging on an array of $M$ sensors [1]. Letting $\boldsymbol{\mu} \in \mathbb{R}^{d \times 1}$ and $\boldsymbol{\zeta} \in \mathbb{R}^{d \times 1}$ denote the vectors of near-field source location parameters to be estimated, $M$ sensor outputs $\boldsymbol{x}(t_n) = [x_{k_{\min}}(t_n), \ldots, x_{k_{\max}}(t_n)]^T$, can be written in matrix form as

$$\boldsymbol{x}(t_n) = \boldsymbol{A}(\boldsymbol{\mu}, \boldsymbol{\zeta})\boldsymbol{s}(t_n) + \boldsymbol{n}(t_n), \quad 1 \leq t_n \leq N$$

(1)

where $\boldsymbol{s} \in \mathbb{C}^{d \times 1}$ is the source signal vector, $\boldsymbol{s} = \begin{bmatrix} s^T(1) & s^T(2) & \cdots & s^T(N) \end{bmatrix} \in \mathbb{C}^{N \times d \times 1}$. $\boldsymbol{A}(\boldsymbol{\mu}, \boldsymbol{\zeta}) = [\mathbf{a}(\mu_1, \zeta_1), \ldots, \mathbf{a}(\mu_d, \zeta_d)]$ is the array steering matrix in the near-field case which is known as a function of unknown set of parameters $\{\mu, \zeta\}$ and $\mathbf{a}(\mu_i, \zeta_i)$ is the $i^{th}$ array steering vector in the following form

$$\mathbf{a}(\mu_i, \zeta_i) = \begin{bmatrix} e^{j(k_{\min} + k_{\min} \zeta_1)} \\ \vdots \\ e^{j(k_{\max} + k_{\max} \zeta_1)} \\ \vdots \\ e^{j(k_{\min} + k_{\min} \zeta_d)} \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

(2)

The following assumptions are imposed on (1):

**AS1**: The source signals are deterministic but as unknown quantities.

**AS2**: $\boldsymbol{n}(t_n)$ is modeled as a zero-mean, temporally and spatially white Gaussian process with standard deviation $\sigma$.

Based on assumptions AS1, AS2 and the array observation model, the probability density function of the received data (conditioned on $\boldsymbol{s}$, $\boldsymbol{\mu}$, $\boldsymbol{\zeta}$) is:

$$f(\boldsymbol{x} | \boldsymbol{s}, \boldsymbol{\mu}, \boldsymbol{\zeta}) = \pi^{-NM/2} (\sigma^2 \det(\mathbf{I}))^{-NM/2} \exp \left( -\frac{1}{\sigma^2} \sum_{t_n=1}^{N} [\mathbf{x}(t_n) - \boldsymbol{A}(\boldsymbol{\mu}, \boldsymbol{\zeta})\mathbf{s}(t_n)]^H \right)$$

$$\times \mathbf{I}^{-1} [\mathbf{x}(t_n) - \boldsymbol{A}(\boldsymbol{\mu}, \boldsymbol{\zeta})\mathbf{s}(t_n)]$$

(3)

and its negative log-likelihood function (after neglecting unnecessary terms) is given by

$$\mathcal{L}(\boldsymbol{x} | \boldsymbol{s}, \boldsymbol{\mu}, \boldsymbol{\zeta}) = \sum_{t_n=1}^{N} [\mathbf{x}(t_n) - \boldsymbol{A}(\boldsymbol{\mu}, \boldsymbol{\zeta})\mathbf{s}(t_n)]^H$$

$$\times [\mathbf{x}(t_n) - \boldsymbol{A}(\boldsymbol{\mu}, \boldsymbol{\zeta})\mathbf{s}(t_n)]$$

(4)
where \( \mathbf{x} = [x^T(1) \ x^T(2) \ \cdots \ x^T(N)]^T \).

2 Performance Analysis

The CRB is a lower bound on the covariance matrix of any unbiased estimator. Suppose \( \hat{\eta} \) is an unbiased estimator of a vector of deterministic unknown parameters \( \eta \) (i.e., \( E[\hat{\eta}] = \eta \)) then the estimator’s covariance matrix satisfies

\[ J^{-1}(\eta) \leq E \{ (\eta - \hat{\eta})(\eta - \hat{\eta})^T \} \]

where \( J(\eta) \) is the Fisher information matrix (FIM) defined by

\[ J(\eta) = E \left( \begin{bmatrix} \frac{\partial L(\mathbf{x}; \eta)}{\partial \eta} \ \frac{\partial L(\mathbf{x}; \eta)}{\partial \eta} \end{bmatrix}^T \right) . \]

2.1 Computation of the Derivatives

We now start constructing the FIM by calculating the derivative of (4) with respect to \( \eta = [s_1^T(1) \ s_1^T(N) \ \cdots \ s_1^T(N)]^T \) where

\[ s_n = \begin{bmatrix} s_1(n), \ldots, s_d(n) \end{bmatrix}^T \]

\[ \mathbf{s}_n = \begin{bmatrix} s_1(n), \ldots, s_d(n) \end{bmatrix}^T \]

\[ \mathbf{u} = [\mu_1, \ldots, \mu_d]^T, \ \mathbf{\zeta} = [\zeta_1, \ldots, \zeta_d]^T \]

For notational simplicity, we omit \((\mathbf{u}, \mathbf{\zeta})\) in the sequel. Taking the partial derivatives of \( L(\mathbf{x}; \eta) \) (for \( t_n = 1, \ldots, N \) and \( i = 1, \ldots, d \)), we have

\[ \frac{\partial L}{\partial s_n(t_n)} = \frac{2}{\sigma^2} \text{Re} \{ A^H \mathbf{n}(t_n) \} \]

\[ \frac{\partial L}{\partial s_n(t_n)} = \frac{2}{\sigma^2} \text{Im} \{ A^H \mathbf{n}(t_n) \} \]

\[ \frac{\partial L}{\partial s_n(t_n)} = \frac{2}{\sigma^2} \sum_{t_n=1}^N \text{Re} \left\{ s_1^*(t_n) \frac{\partial A^H}{\partial \mu_i} \mathbf{n}(t_n) \right\} \]

\[ \frac{\partial L}{\partial s_n(t_n)} = \frac{2}{\sigma^2} \sum_{t_n=1}^N \text{Re} \left\{ s_1^*(t_n) \frac{\partial A^H}{\partial \zeta_i} \mathbf{n}(t_n) \right\} \]

If we write the partial derivatives of the log-likelihood function with respect to the near-field parameters more compactly,

\[ \frac{\partial L}{\partial \tau} = \frac{2}{\sigma^2} \sum_{t_n=1}^N \text{Re} \left\{ S^H(t_n) \mathbf{D}^H \mathbf{n}(t_n) \right\} \]

then the elements of the information matrix can be obtained as

\[ E \left\{ \frac{\partial L}{\partial s_r(t_n)} \frac{\partial L}{\partial s_r(t_n)} \right\} = \frac{2}{\sigma^2} \text{Re} \{ A^H \mathbf{A} \} \delta_{n,m} \]

\[ E \left\{ \frac{\partial L}{\partial s_r(t_n)} \frac{\partial L}{\partial s_c(t_n)} \right\} = -\frac{2}{\sigma^2} \text{Im} \{ A^H \mathbf{A} \} \delta_{n,m} \]

\[ E \left\{ \frac{\partial L}{\partial s_r(t_n)} \frac{\partial L}{\partial \tau} \right\} = \frac{2}{\sigma^2} \text{Re} \{ A^H \mathbf{D} \mathbf{r} S(t_n) \} \]

\[ E \left\{ \frac{\partial L}{\partial \tau} \frac{\partial L}{\partial \tau} \right\} = \frac{2}{\sigma^2} \text{Im} \{ A^H \mathbf{D} \mathbf{r} S(t_n) \} \]

\[ E \left\{ \frac{\partial L}{\partial \tau^T} \frac{\partial L}{\partial \tau^T} \right\} = \frac{2}{\sigma^2} \sum_{t_n=1}^N \text{Re} \left\{ S^H(t_n) \mathbf{D}^H \mathbf{D} \mathbf{r} S(t_n) \right\} \]

2.2 Evaluation of the FIM Matrix

We need the following assumption and results to further proceed, (see e.g., [6]):

\[ E[\mathbf{n}(t_n) \mathbf{n}^H(t_m)] = \sigma^2 \mathbf{I} \]

\[ E[\mathbf{n}(t_n) \mathbf{n}^H(t_m)] = 0 \]

\[ E[\mathbf{n}^H(t_n) \mathbf{n}^H(t_m)] = 0 . \]

Then the FIM can be written in partitioned form as

\[ J(\eta) = \begin{bmatrix} \mathbf{H} & 0 \\ 0 & \mathbf{A} \end{bmatrix} \]

where

\[ \mathbf{A} = \begin{bmatrix} A_r^T(1) & A_r^T(N) & A_c^T(1) & \cdots & A_c^T(N) \end{bmatrix} \]

\[ J(\tau) = \begin{bmatrix} \mathbf{H} & 0 \\ 0 & \mathbf{A}^T \end{bmatrix} \]

\[ \Lambda = \begin{bmatrix} \mathbf{A}_r(T) & \mathbf{A}_r(N) & \cdots & \mathbf{A}_c(T) & \mathbf{A}_c(N) \end{bmatrix} \]
and

$$H = \begin{bmatrix} A_r & -A_c \\ A_c & A_r \end{bmatrix}.$$  \hfill (19)

If we employ a standard result on the inverse of the partitioned matrix, we obtain

$$\text{CRB}^{-1}(\tau) = J(\tau) - A^T \begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix}^{-1} A.$$  \hfill (20)

Finally, we arrive at

$$\text{CRB}^{-1}(\tau) = \frac{2}{\sigma^2} \sum_{t_n=1}^{N} \text{Re} \left\{ S^H(t_n) D^H \right\} \times \left[ I - A(A^H A)^{-1} A^H \right] D \tau S(t_n) \} \right.$$  \hfill (21)

A more explicit individual CRB expressions for the near-field parameters $\mu$ and $\zeta$ can be obtained by using a result on the partitioned matrix and its inverse. We then have

$$\text{CRB}^{-1}(\mu) = \alpha - \beta \gamma^{-1} \beta^T$$

$$\text{CRB}^{-1}(\zeta) = \gamma - \beta^T \alpha^{-1} \beta$$

where

$$\text{CRB}^{-1}(\tau) = \begin{bmatrix} \alpha & \beta \\ \beta^T & \gamma \end{bmatrix}.$$  \hfill (23)

### 2.4 CRB for the Special Case: One Source

As a special case, we consider a near-field scenario in which a narrowband signal from a single source ($d = 1$) located at $\{\theta, r\}$ impinges upon a $M$ element centro-symmetric antenna array. In this case ($d = 1$ and $N = 1$), localization involves the estimation of $\{\theta, r\}$ given the observation $x(1)$. The $\text{CRB}(\tau)$ for this particular case can be derived using the general result in (21) as

$$\text{CRB}(\tau) = \frac{\sigma^2}{|S|^2} \begin{bmatrix} 6 & \frac{0}{M(M^2 - 1)} \\ \frac{0}{M(M^2 - 1)} & \frac{90}{M(M^2 - 1)(M^2 - 4)} \end{bmatrix}.$$  \hfill (24)

Since the $\text{CRB}(\tau)$ is a diagonal matrix, the $\text{CRB}$ for the estimate of $\mu$ is decoupled from the $\text{CRB}$ of $\zeta$. In other words

$$\text{CRB}(\mu) = \frac{6\sigma^2}{|S|^2} \frac{1}{M(M^2 - 1)},$$

$$\text{CRB}(\zeta) = \frac{90\sigma^2}{|S|^2} \frac{1}{M(M^2 - 1)(M^2 - 4)}.$$  \hfill (25)

Intuitively, the decoupling exists since in the single source case, no co-channel interference with other sources occurs. Also note that $\text{CRB}(\mu)$ and $\text{CRB}(\zeta)$ are the functions of the antenna array element $M$ and $\text{SNR}$ and are independent of the array steering matrix $A$. In other words, the $\text{CRB}$ of the near-field parameters for this special case does not depend on the parametric model used for the array steering matrix.

### 3 Simulations

Some numerical simulations are presented in the sequel. We consider the following two different cases:

**Case 1:** We consider a uniform linear array consisting of $M = 7$ antennas with element spacing $\frac{\lambda}{2}$. We let two equal power sources impinge on the linear array. Source 1 is located at $-5^0$ and $3\lambda$ whereas source 2 is located at $3^0$ and $1.4\lambda$. The estimation of DOA and range parameters is done using $N = 1000$ snapshots and $K = 500$ independent runs. We compute and plot CRBs of the estimated DOA and range parameters. We also tested the EM based conditional ML method [4] for different signal to noise ratios ($\text{SNR} = 0 - 20dB$). In addition to the performance obtained with conditional ML method, we have also plotted the results achieved with the Unitary ESPRIT method proposed in [2]. In each trial, the $\text{RMSE}$ of the estimations for $\{\theta_1, r_1\}$ and $\{\theta_2, r_2\}$ were recorded and only the result for the Source 1 is presented due to lack of space in the Figure 1 and Figure 2 respectively.

**Case 2:** In this case we consider a near-field scenario in which a narrowband signal from a single source located at $\{\theta, r\} = \{20^0, 3\lambda\}$ impinges upon centro-symmetric antenna arrays with inter-element spacing $\frac{\lambda}{M}$. Two different antenna array setup were used ($M = 7$ and $M = 9$). The number of the snapshots ($N$) was set to 100. The CRBs are computed from (25) for these two cases. The CRB results only for DOA is presented in the Figure 3 together with the conditional ML results.

In all cases we use the following $\text{RMSE}$s:

$$\text{RMSE}_{\theta_i} = \sqrt{\frac{1}{K} \sum_{k=1}^{K} (\hat{\theta}_{ik} - \theta_{ik})},$$

$$\text{RMSE}_{r_i} = \sqrt{\frac{1}{K} \sum_{k=1}^{K} (\hat{r}_{ik} - r_{ik})}, \quad i = 1, \cdots, d.$$  \hfill (26)

Based on the simulation results we made the following observations: -For high $\text{SNRs}$ the $\text{RMSE}$s obtained from the simulations becomes almost identical to CRBs provided in (21). -In case 2, we observe that if we increase the number of the antenna elements the proposed estimator performs better. This result can also be obtained from (25).
In case 1, the estimates obtained from the EM based conditional ML method are very close to the CRBs. However it is not surprising that the ML method yields better performance compared with the Unitary ESPRIT since ML methods provide optimal solution to the resulting estimation problems. Therefore the estimates from the Unitary ESPRIT method can be used to initialize the conditional ML algorithm.

4 Conclusions

In this paper, we derived CRBs for direction of arrival and range estimation of near-field sources. We presented Monte Carlo simulations to verify the theoretically predicted estimator’s performance. Moreover, we also presented CRB results together with conditional ML and Unitary ESPRIT results for two different test scenarios.

References