Abstract — The behavior of nonlinear analog circuits can be described mathematically by a nonlinear differential algebraic equation system (DAE). Since the behavior of nonlinear analog circuits can be described mathematically by a nonlinear differential algebraic equation system (DAE). Since this equation system is getting very large, even for small circuits, symbolic analysis without approximation techniques is not practicable. Thus, an algorithm for automated simplification of nonlinear DAE systems has been developed and extended in the past years. In this paper we describe the extension of the simplification algorithm towards the control of the index: Since the index might change during simplification, it is important to undo responsible steps in order to retain the index of the original system. We describe different methods for computing the tractability index and discuss their practicability for controlling the index during simplification. The application of the index observer strategy is then shown on an example.

1 Introduction

The behavior of a nonlinear analog circuit can be described mathematically by a nonlinear differential algebraic equation system (or short DAE) $F = \{f, g\}$, where

\[
\begin{align*}
    f(x(t), \dot{x}(t), y(t), u(t); p) &= 0 \quad \text{for all } t \in I, \\
    g(x(t), y(t), u(t); p) &= 0 \quad \text{for all } t \in I.
\end{align*}
\]

Here, $u : \mathbb{IR} \rightarrow \mathbb{IR}^r$ denotes the inputs, $x = (v, i) : \mathbb{IR} \rightarrow \mathbb{IR}^k$ denotes the vector of internal voltages and currents, $y : \mathbb{IR} \rightarrow \mathbb{IR}^s$ denotes the outputs, and $I \subset \mathbb{IR}$ denotes an interval. (Note that the presented methods can be applied to charge-oriented equations as well.)

Since we are working with symbolic equations, $f$ and $g$ are parameterized by symbolic element parameters $p = \{p_1, \ldots, p_N\}$.

The symbolic DAE system $F$ is getting very large, even for small circuits. This motivated the development of an algorithm for automated and error controlled simplification of symbolic systems, which was first successfully applied to linear circuits [11]. In [1] an algorithm was presented which extends simplifications to nonlinear systems for automated behavioral model generation. Several extensions of this algorithm have been developed since then (see for example [9, 12]). There are two major goals for applying simplifications to nonlinear DAE systems: On the one hand the generation of parameterized behavioral models which allow for faster numerical simulation due to the complexity reduction. On the other hand the generation of simplified symbolic systems, where the complexity is small enough such that they can be interpreted to obtain insight into the circuit (mis)behavior.

2 The Simplification Algorithm

Starting from a hierarchical netlist description of the circuit, the symbolic DAE system can be set-up automatically using for example Analog Insydes [5]. The simplification algorithm applies several modifications to the system and ends up with a simplified symbolic description of the circuit behavior. After each simplification step a numerical calculation is performed and the result is compared to a reference calculation. If the deviation between both solutions exceeds a user given error bound, the simplification step is undone. This assures that the behavior of the simplified system lies within a user given error bound compared to the original circuit behavior. For a detailed description of the simplification algorithm we refer to [9, 10, 12]. We implemented the algorithm in the toolbox Analog Insydes, an add-on package to the computer algebra system Mathematica [13] for modeling, analysis, and design of analog circuits.

In particular with respect to behavioral model generation it is important not to increase the index during simplifications in order to ensure numerical robustness for transient simulation. The exact definition of the index and how it can be controlled during simplification is described in the following chapter.

3 The Index Observer

We will focus on quasilinear DAE systems. To simplify the notation, we will no longer distinguish input, state, and output variables and we suppress the time variable $t$ and the symbolic parameters $p$. Thus, we are dealing with equations of the form

\[
F(x, \dot{x}) = A(x) \dot{x} + h(x),
\]

where $A$ is a matrix valued function.

The index plays an important role in the theory of DAEs (for an introduction see [2]). There exist a number of different index concepts in the literature, e.g., the global, the perturbation, the differential, or the tractability index. In this paper we will focus on the tractability index [6] which is based on the linearization of a DAE and requires only weak smoothness assumptions. If we are talking of the index we are speaking of the tractability index.

Definition 1. Let $F(x, \dot{x}) = A(x) \dot{x} + h(x)$ where $A$ is a matrix valued function and $A(x)$ is singular for each $x$.
and of constant rank. Furthermore we assume that \( N = \ker A(x) \) does not depend on \( x \). Let \( B(x, x') = D_x F \) and let \( Q \) denote a projector onto \( N \). Then the DAE system \( F \) is called index-1 tractable (or transferable) if the matrix \( A(x) + B(x, x')Q \) is regular.

We will consider the index-1 case only. For a definition of higher index conditions see for example [7]. The assumptions stated on the matrix function \( A \) in Definition 1 will be assumed to be true throughout the rest of this chapter. The following theorem provides an equivalent condition for a DAE system being index-1 tractable. It will be used later on to derive a method for computing the index.

**Theorem 1.** Let \( A, B \in \mathbb{R}^{n \times n} \), let \( Q \in \mathbb{R}^{n \times n} \) be a projector onto \( \ker A \), and let \( S = \{ x \in \mathbb{R}^n | Bx \in \text{im} A \} \). Then \( A + BQ \) is regular if and only if \( S \cap \ker A = \{0\} \).

**Proof:** This can be seen using the properties of projector matrices.

DAEs originating from electrical networks theoretically can be of arbitrary index but in real world problems they are often of index one or two. Since the simplification algorithm modifies the DAE system one can not rule out the possibility of an index change during simplification which has to be avoided. Thus, the index has to be controlled during simplification and the question arises, how to compute it.

In [4] structural properties have been stated an electrical network to have fulfills in order to assure transferability. By means of C-V-loops and L-I-cuts it is possible to predict the index based on the network topology without analyzing the equation system. But this result is not suitable for our algorithm: After some simplification steps, say cancellation of terms, the resulting system of equations may not be re-interpreted as an electrical network. For example, Kirchhoff’s current law may be violated for some nodes. Therefore in our case the index has to be calculated based on the equation system.

In the following sections we describe two standard methods to compute a projector onto the kernel of a matrix [8]. In Section 3.3 we describe how to check for transferability without computing a projector.

### 3.1 Projector via singular value decomposition

Let \( A \in \mathbb{R}^{n \times n} \) be singular and let the singular value decomposition of \( A \) be given by

\[
A = U^T \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix},
\]

where \( U \in \mathbb{R}^{n \times n}, \Sigma = \text{diag}(\sigma_1, \ldots, \sigma_l) \in \mathbb{R}^{l \times l}, V_1 \in \mathbb{R}^{l \times n}, V_2 \in \mathbb{R}^{(n-l) \times n} \) and \( \sigma_i > 0 \) for all \( i \). Then

\[
Q = V_2^T V_2
\]

is a projector onto \( \ker A \). This can be seen as follows: First, we have \( Q^2 = V_2^T V_2 V_2^T V_2^T = V_2^T V_2 = Q \), since \( V \) is orthogonal. Moreover,

\[
AQ = U^T \text{diag}(\Sigma, 0) V_2^T V_2 = 0.
\]

Finally, let \( Ax = 0 \). Then \( V_1 x = 0 \) and thus

\[
x = (V_2^T V_2)^{-1} \begin{pmatrix} 0 \\ V_2^T V_2 x \end{pmatrix} = V_2^T x = Q x \in \text{im} Q.
\]

Together this shows that \( \text{im} Q = \ker A \) and this proves \( Q \) to be a projector onto \( \ker A \).

### 3.2 Projector via Gram-Schmidt orthonormalization

Let \( A \in \mathbb{R}^{n \times n} \) be singular. Using Gram-Schmidt orthonormalization we calculate a decomposition of \( A \) such that

\[
(A^T I) = (V_1 \ V_2) R,
\]

where the non-zero columns of \( V = (V_1 \ V_2) \in \mathbb{R}^{n \times 2n} \) are orthonormal and \( R \in \mathbb{R}^{2n \times 2n} \) is upper triangular. Then

\[
Q = V_2 V_2^T
\]

is a projector onto \( \ker A \). This can be seen as follows: Let \( V_2^\ast \) denote the matrix which originates from \( V_2 \) by removing zero columns. Then \( Q = V_2 V_2^T \) and

\[
Q^2 = V_2 V_2^T V_2 V_2^T V_2 V_2^T = V_2 V_2^T = Q.
\]

since the columns of \( V_2 \) are orthonormal. Moreover \( \ker A = \text{im} V_2 \) and thus \( AQ = AV_2 V_2^T = 0 \). Finally, let \( Ax = 0 \). Then \( x \in \text{im} V_2 \), i.e., \( x = V_2^\ast v^\ast \) and thus

\[
x = V_2^\ast v^\ast = V_2^\ast V_2 V_2^T V_2^\ast v^\ast = Q V_2^\ast v^\ast \in \text{im} Q.
\]

Together this shows that \( \text{im} Q = \ker A \) and this proves \( Q \) to be a projector onto \( \ker A \).

### 3.3 Index-1 condition using structural properties

Let \( F(x, x') = A(x) x' + h(x) \) be a quasilinear DAE and let \( B(x, x') = D_x F \) be the Jacobian of \( F \) with respect to \( x \). To simplify notations in the following we omit the dependency of \( A \) and \( B \) on \( x \) and \( x' \). Without loss of generality we may assume the following matrix structure for \( A \):

\[
A = \begin{pmatrix} A_1 & A_2 \\ 0 & 0 \end{pmatrix}
\]

where \( A_1 \in \mathbb{R}^{l \times l} \) is regular. Let \( B \) be partitioned accordingly:

\[
B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}
\]

\[
B_1 = \begin{pmatrix} B_1 \\ B_2 \\ B_4 \end{pmatrix}
\]

\[
B_2 = \begin{pmatrix} B_1 \\ B_3 \\ B_4 \end{pmatrix}
\]

\[
B_3 = \begin{pmatrix} B_1 \\ B_3 \\ B_4 \end{pmatrix}
\]

\[
B_4 = \begin{pmatrix} B_1 \\ B_4 \end{pmatrix}
\]
and let \((x_1, x_2)^T \in S \cap \ker A\), where \(S = \{x | Bx \in \text{im} A\}\) as in Theorem 1. Then
\[
\begin{align*}
A_1 x_1 + A_2 x_2 &= 0 \quad (14) \\
B_3 x_1 + B_4 x_2 &= 0, \quad (15)
\end{align*}
\]
and we achieve the following equivalence:
\[
S \cap \ker A = \{0\} \iff \det(B_4 - B_3 A_1^T A_2) \neq 0. \quad (16)
\]

Theorem 1 states that the conditions in (16) are equivalent to the index-1 tractability of the DAE system \(F\).

### 3.4 Index control during simplification

The idea is to check the index during simplification in order to identify those systems which are transferable. If no more simplifications are possible without exceeding the error bound, the important step is then to return the latest transferable system. Thus, we have to track the index of each intermediate system during calculation. Using this approach one can assure that starting with an index-1 system the resulting system stays index-1, too. This is especially important for behavioral model generation.

Due to the size of the symbolic DAE systems it seems not to be practical to perform symbolic index calculations (for symbolic index calculation of linear systems we refer to [3]). Computing the index numerically, there are the following difficulties: For the singular value decomposition one has to decide the index of the first singular value which has to be assumed to be zero. For the Gram-Schmidt orthonormalization one has to decide, whether two vectors are linearly dependent or not. Using the structural method one has to decide, which matrix entries are zero. Finally, all methods require a numerical test on regularity of a matrix. Since reliability is more important than performance in our case, it is reasonable to choose the singular value decomposition for this test. All methods require a threshold value for the decision and the wrong threshold results in a wrong index calculation.

We have implemented all three methods in Analog Insydes. In our test cases the singular value decomposition for calculating a projector as well as for the regularity decision showed the best results. For a wide range of threshold values this method gives a reliable index value. Both the Gram-Schmidt orthonormalization and the structural method gave the correct result only in a much smaller threshold range.

### 4 Example

We demonstrate the index observer strategy by applying the nonlinear simplification algorithm to the operational amplifier shown in Figure 1. The circuit comprises eight bipolar transistors, modeled by the dynamic Ebers-Moll equations (including base capacitances). Using the modified nodal analysis this results in a nonlinear, transferable DAE system consisting of 73 equations and a total number of 365 terms. During simplification we control the output voltage at node 9 by a DC-transfer and a multiple AC analysis. The DC-transfer analysis is performed by sweeping the current source \(I_1\) from 1µA to 200µA in 10 steps. Additionally, for each value of \(I_1\) the system is linearized about the corresponding operating point and an AC analysis is performed on the linear system at 10 frequencies between 10Hz and 100MHz. Figure 3 shows a Bode diagram of the output voltage at node 9 of the system linearized about the operating point corresponding to the current source value \(I_1 = 1\mu A\).

As simplification methods we apply elimination of variables, cancellation of terms with an error bound of 0.1mV for the DC behavior and 1V (10%) for the AC behavior, and again elimination of variables. In the following, we will not go into any further details about the simplification itself and the properties and advances of the simplified system but will focus on the index changes during simplification.

After each successful simplification step the index is calculated and it is checked, whether the system is transferable or not. The singular value decomposition is used to calculate the projector and to check for regularity. Figure 2 shows the number of simplification steps (x axis) versus the index of the corresponding simplified system (y axis). It can be seen that indeed the index in-

![Figure 1: Schematics of the operational amplifier.](image1)

![Figure 2: Index change during simplification.](image2)
creases due to an applied simplification. But it can also be seen that the index might again decrease during subsequent simplifications. Thus, we propose the following strategy for the index observer: After each simplification step calculate and store the index. Perform the simplification, even if the index increases, since subsequent simplifications might decrease it again. If no more simplifications are possible, return the latest transferable system.

After termination the algorithm returns the simplified, transferable system consisting of 8 equations with a total number of 40 terms. Compared to the original system we achieved a complexity reduction of about one order of magnitude. Figure 3 shows the AC behavior of the simplified system (dashed line) and the original system (solid line), linearized at the operating point corresponding to the current source value $I_1 = 1\mu A$.

Figure 3: Bode plot of original (solid) and simplified (dashed) system.

5 Conclusions

The simplification algorithm has been implemented as part of Analog Insydes. We successfully applied the algorithm to many examples and for static systems it was sometimes even possible to solve the system for the output variables explicitly. We have implemented a first version of the index observer strategy which now ensures that, starting with an index-1 system, the simplified system is index-1, too. In Section 3 we described three methods for computing the index, where the singular value decomposition in our implementation turns out to be the best way both to compute the projector matrix and to check for regularity. By the application of the index observer to the simplification of an operational amplifier we showed that the index in fact increases during simplification but may again decrease by further simplifications. Thus, it seems practical to track the index changes during simplification and, as soon as the error bound is exceeded, to return the latest system which stayed index-1.

References


