A Rigorous Approach to Electro-Thermal Network Modeling

Lorenzo Codecasa*, Dario D'Amore* and Paolo Maffezzoni*

Abstract — **In this paper the notion of thermal network associated to an electric circuit is univocally defined. The definition of the temperatures of the elements of the electric circuit and of the power densities they dissipate are deduced. The relation among these variables is shown to be naturally modeled through a distributed thermal network preserving the relevant physical properties of the thermal problem. The linear thermal problem is considered in details and the particular structure of the distributed linear thermal network is deduced.**

1 Introduction

In a growing number of power electronics applications, the behavior of an electric circuit is affected by its self-heating. The ability to analyze the coupled electrical and thermal problems is consequently becoming a key point for a reliable and accurate design.

The commonly adopted approach is to couple the electric network modeling the electric problem to a suitable thermal network modeling the thermal problem. In this way the coupled electro-thermal analysis can be carried out by means of a circuit simulator. A critical point is the derivation of the thermal network from the thermal problem. In fact, in the thermal problem, the dissipated power density and the temperature are functions of spatial coordinates while, in the thermal network one power and one temperature, which do not depend on spatial coordinates, are introduced for each electric element.

Electro-thermal networks have been extensively employed in literature for electro-thermal analysis. However, no special attention has been given to the question of defining thermal networks.

In this paper, extending [1], we analyze the problem of modeling the electro-thermal problem through an electro-thermal network. The result of this analysis leads to a univocal definition of the temperatures of the elements of the electric circuit and of the power densities they dissipate. The relationship between the temperatures and the powers of the electric elements is determined by the thermal problem and is naturally modeled through a univocally defined distributed thermal network which preserves the main properties of the thermal problem.

In section 2 the dissipated power densities and temperature rises of the elements of the electric circuit are defined. In section 3 the thermal problem is formulated and the main properties of its solutions are outlined. In section 4 the thermal network is derived and its fundamental properties are deduced. In section 5 the linear thermal problem is considered. In section 6 the linear thermal network is derived and its fundamental properties are deduced.

2 Thermal variables *definition*

Let us consider the distributed electro-thermal problem in an electric circuit extending in a space region ω. We suppose that the dissipating elements of the electric circuit are n linear resistors, extending in n non-overlapping sub-regions ω_i.

2.1 Electro-thermal *problem*

In the i-th resistor the electrical phenomenon is described by the electric field $\mathbf{e}(\mathbf{r},t)$ and the current density $\mathbf{j}(\mathbf{r},t)$, and the thermal phenomenon is described by the temperature rise $\mathrm{T}(\mathbf{r},t)$ with respect to the ambient temperature, functions of position vector $\mathbf{r}$ and time t. The electrical phenomenon is governed by the following equations.

1. *Irrotationality* of the electric field $\mathbf{e}(\mathbf{r},t)$
2. *Solenoidality* of the current density $\mathbf{j}(\mathbf{r},t)$
3. *Ohm*'s law
$$\mathbf{j}(\mathbf{r},t) = \sigma(\mathrm{T}(\mathbf{r},t))\mathbf{e}(\mathbf{r},t)$$
in which the electric conductivity σ is function of the temperature rise $\mathrm{T}(\mathbf{r},t)$.
4. *Boundary conditions.* Current flows into the i-th resistor through two equipotential surfaces.

The thermal phenomenon is governed by the following equations

1. *Heat conduction equation* in which the dissipated power density $\mathrm{G}(\mathbf{r},t)$ is given by *Joule*'s law
$$\mathrm{G}(\mathbf{r},t) = \mathbf{j}(\mathbf{r},t) \cdot \mathbf{e}(\mathbf{r},t).$$
2. *Boundary conditions.*

---

*Politecnico di Milano, Dipartimento di Elettronica e Informazione, Milan, Italy.

2.2 Electro-thermal *network*

When the electro-thermal problem is approximated by an electro-thermal *network* one temperature rise $T_i(t)$ is introduced for the i-th resistor and Ohm's law is approximated by

$$\boldsymbol{j}(\mathbf{r},t) = \sigma(T_i(t))\boldsymbol{e}(\mathbf{r},t). \quad (1)$$

The electrical phenomenon is then approximated by electric field $\boldsymbol{e}(\mathbf{r},t)$ and current density $\boldsymbol{j}(\mathbf{r},t)$ and the thermal phenomenon is approximated by temperature rise $T(\mathbf{r},t)$. It can be easily proved that

Definition 1 *The power density dissipated by the element of the electric circuit in the ω_i region is the product of the dissipated power $P_i(t)$ and of a non-negative function $g_i(\mathbf{r})$ with*

$$\int_{\omega_i} g_i(\mathbf{r})\, d\mathbf{r} = 1.$$

Therefore

$$\boxed{G(\mathbf{r},t) = \mathbf{g}^{\mathbf{T}}(\mathbf{r})\mathbf{P}(t)} \quad (2)$$

in which $\mathbf{g}(\mathbf{r})$ is a vector whose i-th element is $g_i(\mathbf{r})$ and $\mathbf{P}(t)$ is a vector whose i-th element is $P_i(t)$. If the electro-thermal network approximates the electro-thermal problem then in each ω_i

$$\sigma(\mathrm{T}(\mathbf{r},t)) \approx \sigma(T_i(t)) \quad (3)$$

and, as can be proved,

Definition 2 *The temperature rise $T_i(t)$ of the element of the electric circuit in the ω_i region is the mean of the temperature rise $T(\mathbf{r},t)$ weighted by the function $g_i(\mathbf{r})$.*

Therefore

$$\boxed{\mathbf{T}(t) = \int_{\omega} \mathbf{g}(\mathbf{r})T(\mathbf{r},t)\, d\mathbf{r}.} \quad (4)$$

in which $\mathbf{T}(t)$ is a vector whose i-th element is $T_i(t)$.

Hereafter definitions (2) and (4) are assumed for the power densities and the temperature rises of the elements of electric circuits.

3 The thermal *problem*

3.1 Formulation

The relation between the power density and the temperature rise distribution in ω is determined by the following equations

1. *Heat conduction equation*

$$\nabla \cdot (-k(\mathbf{r},T)\nabla T(\mathbf{r},t)) + \\ + c(\mathbf{r},T)\frac{\partial T}{\partial t}(\mathbf{r},t) = G(\mathbf{r},t). \quad (5)$$

   in which $k(\mathbf{r},T)$ is the thermal conductivity, $c(\mathbf{r},T)$ is the volumetric heat capacity and $G(\mathbf{r},t)$ is the power density.

2. *Boundary conditions*
   The conditions, on the boundary σ, of outward normal unit vector $\boldsymbol{\nu}$, are

$$-k(\mathbf{r},T)\frac{\partial T}{\partial \nu}(\mathbf{r},t) = h(\mathbf{r},T)T(\mathbf{r},t), \quad (6)$$

   in which $h(\mathbf{r},T)$ is a *non-negative* heat transfer coefficient.

3. *Initial condition*
   It is assumed, for the sake of simplicity

$$T(\mathbf{r},0) = 0. \quad (7)$$

3.2 Main *properties*

The main properties satisfied by the solution of the thermal problem are here outlined.

1. If the dissipated power density $G(\mathbf{r},t)$ is *non-negative* the temperature rise $T(\mathbf{r},t)$ is *non-negative*,

$$T(\mathbf{r},t) \geq 0 \textit{ if } G(\mathbf{r},t) \geq 0.$$

2. It can be proved that

$$W(t) = \int_0^t d\tau \int_{\omega} G(\mathbf{r},\tau)T(\mathbf{r},\tau)\, d\mathbf{r}$$

   is such that

$$W(t) \geq 0, \quad (8)$$
$$W(t) = 0 \textit{ iff } G(\mathbf{r},t) = 0 \textit{ and } T(\mathbf{r},t) = 0. \quad (9)$$

4 The thermal *network*

4.1 Formulation

The relation between the dissipated powers and the temperature rises is univocally determined by the heat conduction equation and can be modeled by a network. Precisely, the dissipated power $P_i(t)$ and the temperature rise $T_i(t)$ are the *current* and the *voltage* of the i-th port of an n-port. This n-port *extends* the notion of thermal resistance and can be assumed as the *distributed thermal network* associated to the electric circuit.

4.2 Main *properties*

The thermal network satisfies the same properties of the solution of the thermal problem.

1. The thermal network is *positive.* That is, if the dissipated powers are non-negative then the temperatures are non-negative

   $$\mathbf{T}(t) \geq \mathbf{0} \textit{ if } \mathbf{P}(t) \geq \mathbf{0}.$$

2. The thermal network is *strictly passive.* In fact from equations (2) and (4) we have

   $$W(t) = \int_0^t \mathbf{T}^{\mathbf{T}}(\tau)\mathbf{P}(\tau)\,d\tau. \qquad (10)$$

   Therefore, from equations (8) and (10), the thermal network is *passive.* Now let us suppose that

   $$W(t) = 0.$$

   Then, from equation (9), we have

   $$\mathbf{T}(t) = \mathbf{0} \textit{ and } \mathbf{P}(t) = \mathbf{0}$$

   and the thermal network is *strictly* passive.

5 The *linear* thermal *problem*

We consider the case in which the thermal conductivity $k(\mathbf{r})$, the volumetric heat capacity $c(\mathbf{r})$ and the heat transfer coefficient $h(\mathbf{r})$ *do not* depend on temperature. Then the thermal problem is linear.

5.1 Main properties

The properties stated in section 3 for the general nonlinear case still hold. In addition, for the linear case, the following properties hold.

1. *Reciprocity* property. Let $T_1(\mathbf{r},t)$, $T_2(\mathbf{r},t)$ be the temperature rises relative to the dissipated power density $G_1(\mathbf{r},t)$, $G_2(\mathbf{r},t)$ and let $\mathsf{T}_1(\mathbf{r},s)$, $\mathsf{T}_2(\mathbf{r},s)$ and $\mathsf{G}_1(\mathbf{r},s)$, $\mathsf{G}_2(\mathbf{r},s)$ be their Laplace transforms. It can be proved that

   $$\int_\omega \mathsf{G}_1(\mathbf{r},s)\mathsf{T}_2(\mathbf{r},s)\,d\mathbf{r} = \int_\omega \mathsf{G}_2(\mathbf{r},s)\mathsf{T}_1(\mathbf{r},s)\,d\mathbf{r}.$$

2. *Expansion* of the solution of the linear thermal problem [2]. If the ω region is *bounded* the solution of the thermal problem (5), (6), (7) can be expressed by the expansion

   $$T(\mathbf{r},t) = \sum_{j\,1}^{\infty} a_j(t) z_j(\mathbf{r})$$

   $$a_j(t) = e^{-\lambda_j t} * \int_\omega z_j(\mathbf{r}) G(\mathbf{r},t)\,d\mathbf{r},$$

   in which $z_j(\mathbf{r})$ are the eigenfunctions and λ_j are the eigenvalues of the eigenvalue problem associated to the thermal problem

   $$\nabla \cdot (-k(\mathbf{r})\nabla z_j(\mathbf{r})) = \lambda_j c(\mathbf{r}) z_j(\mathbf{r})$$

   with boundary conditions

   $$-k(\mathbf{r})\frac{\partial z_j}{\partial \nu}(\mathbf{r}) = h(\mathbf{r}) z_j(\mathbf{r}).$$

6 The *linear* thermal *network*

The definitions of dissipated power vector $\mathbf{P}(t)$ and temperature rise vector $\mathbf{T}(t)$ given in section 3 allow us to determine a linear thermal network.

6.1 Main properties

The properties stated in section 4 for the general nonlinear case still hold. In addition, for the linear case, the following properties hold.

1. *Reciprocity* property. Let $\mathbf{T}_1(t)$, $\mathbf{T}_2(t)$ be the temperature rises relative to the dissipated powers $\mathbf{P}_1(t)$, $\mathbf{P}_2(t)$ and let $\boldsymbol{\mathsf{T}}_1(s)$, $\boldsymbol{\mathsf{T}}_2(s)$ and $\boldsymbol{\mathsf{P}}_1(s)$, $\boldsymbol{\mathsf{P}}_2(s)$ be their Laplace transforms. Then

   $$\boldsymbol{\mathsf{T}}_2^{\mathbf{T}}(s)\boldsymbol{\mathsf{P}}_1(s) = \int_\omega \mathsf{G}_1(\mathbf{r},s)\mathsf{T}_2(\mathbf{r},s)\,d\mathbf{r} =$$

   $$= \int_\omega \mathsf{G}_2(\mathbf{r},s)\mathsf{T}_1(\mathbf{r},s)\,d\mathbf{r} = \boldsymbol{\mathsf{T}}_1^{\mathbf{T}}(s)\boldsymbol{\mathsf{P}}_2(s)$$

   and the thermal network is *reciprocal.*

2. *Expansion* of the solution of the linear thermal network. It results

   $$\boldsymbol{\mathsf{T}}(s) = \boldsymbol{\mathsf{Z}}(s)\boldsymbol{\mathsf{P}}(s)$$

   in which

   $$\boldsymbol{\mathsf{Z}}(s) = \sum_{j\,1}^{\infty} \frac{\boldsymbol{\Gamma}_j \boldsymbol{\Gamma}_j^{\mathbf{T}}}{s + \lambda_j},$$

   $$\boldsymbol{\Gamma}_j = \int_\omega z_j(\mathbf{r})\mathbf{g}(\mathbf{r})\,d\mathbf{r}.$$

   $\boldsymbol{\mathsf{Z}}(s)$ is the *thermal impedance* matrix of the linear thermal network.

7 Example

We consider a slab ω of length L and area A, of thermal conductivity k and volumetric heat capacity c in which power is uniformly dissipated. On the lower face of the boundary σ the temperature rise is set to zero. On the rest of the boundary σ the thermal flux is set to zero. The solution of this thermal problem can be derived in closed form as a series solution. Solving the eigenvalue problem by separation of variables we determine eigenvalues

$$\lambda_j = \pi^2 \left(j - \frac{1}{2}\right)^2 \frac{k}{L^2 c}.$$

The thermal impedance of the linear thermal network is then

$$\mathsf{Z}(s) = \sum_{1}^{\infty}{}_{j} \frac{R_j}{1 + s/\lambda_j}.$$

If we assume as temperature rise of the slab the *mean* temperature rise then

$$R_j = \frac{2}{\pi^4 \left(j - \frac{1}{2}\right)^4} \frac{L}{kA},$$

and, as expected, we get passive thermal network $\mathcal{T}_1$. Otherwise if we assume as temperature rise of the slab the *maximum* temperature rise then

$$R_j = (-1)^{j-1} \frac{2}{\pi^3 \left(j - \frac{1}{2}\right)^3} \frac{L}{kA},$$

and, as can be proved, we get a *not* passive thermal network $\mathcal{T}_2$. This can lead to numerical instabilities which do not correspond to real physical instabilities. As an example, in the circuit $\mathcal{N}$ of figure 1, power is mainly dissipated by mosfet M_2. Mosfets M_1 and M_2 have practically the same temperature since they are close in space. Modeling mosfets as in [3], the electro-thermal small-signal network $\mathcal{M}$ of figure 2 can be derived in which

$$C = 12pF \quad R = 300k\Omega \quad m = 30mA^2/K$$

Using the thermal impedence $\mathsf{Z}(s)$ of the non passive thermal network $\mathcal{T}_2$ with

$$L/kA = 200K/W \quad LAc = 12nJ/K$$

the small-signal network $\mathcal{M}$ becomes unstable, having two conjugate complex natural frequencies with positive real parts. As a consequence in the electro-thermal simulation of $\mathcal{N}$ a spurious numerical oscillation arises [3]. This oscillation is prevented using passive thermal networks.

8 Conclusions

A rigorous approach to the modeling of the electro-thermal problem through an electro-thermal network has been presented. The approach is based on the precise definition of the temperatures which appear in the constitutive relation of the elements of the electric circuit and leads to a univocally defined distributed thermal network that preserves the relevant physical properties of the thermal problem. For the linear case the distributed thermal network assumes a particular structure in which the thermal impedence matrix of the thermal network is identified.

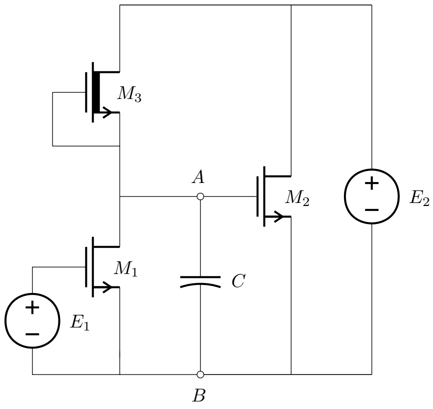

Figure 1: Circuit example $\mathcal{N}$.

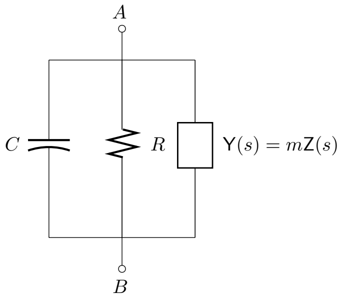

Figure 2: Electro-thermal small signal network $\mathcal{M}$.

References

[1] L. Codecasa, M. Santomauro, "A new approach to model self-heating in electric circuits through thermal networks", *ECCTD*, pp. 563–566, Stresa, 1999.

[2] Y. C. Gerstenmaier, G. Wachutka, "Time Dependent Temperature Fields Calculated Using Eigenfunctions and Eigenvalues of the Heat Conduction Equation", *THERMINIC*, pp. 55–61, Budapest, 2000.

[3] L. Codecasa, D. D'Amore, P. Maffezzoni, "Electro-thermal resonance in MOSFET devices", *Electronics Letters*, vol. 37, no. 1, pp. 57–58, 2001.