On the Loss of Complete Stability of Neural Networks Caused by Hopf Bifurcations

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Abstract — The paper establishes an easily testable analytical criterion which is necessary and sufficient for the existence of supercritical Hopf bifurcations arbitrarily close to nominal neural networks with a symmetric neuron interconnection matrix. The importance of the result is discussed in relation with the fundamental issue of robustness of complete stability of symmetric neural networks with respect to errors in the electronic implementation.

1 Introduction

Most neural network applications to signal processing problems require that the dynamics be completely stable, i.e., each trajectory of the differential equations converges towards some equilibrium point (a stationary state) [1, 2, 3, 4, 5].

The most basic results on complete stability have been established for neural networks where the neuron interconnection matrix is symmetric [2, 3, 5]. Conversely, it is known that largely nonsymmetric neural networks may exhibit complex behaviors, including limit cycles and even chaotic attractors [1, 6, 7].

Clearly, the symmetry of the interconnecting structure is not a generic property. Indeed, in any hardware (e.g., VLSI) implementation, it is impossible to realize perfectly symmetric neuron interconnections, so that robustness of complete stability with respect to small perturbations is an extremely important issue [8, 9].

In a recent paper [10], a class of third-order Cellular Neural Networks (CNNs) [5] with competitive (i.e., inhibitory) interconnections between distinct neurons has been introduced. Such neural networks exhibit large-size persistent oscillations and rather complex bifurcations, even if their interconnection matrix is arbitrarily close to some symmetric one. Therefore, symmetry of the interconnection matrix is not sufficient for robustness of complete stability.

In this paper, robustness of complete stability of general neural networks with an additive interconnecting structure, of arbitrarily large size, is considered. These networks include the Hopfield model [2], the CNNs [5], and other classes of neural architectures of wide interest [1, 3]. To this end, Hopf bifurcations (HBs) that cause the loss of complete stability for general additive neural networks are analyzed. The main goal is to discover whether such bifurcations may be found even arbitrarily close to the interconnection symmetry condition, and to analytically characterize the possible occurrence of such critical situations.

Notation. \(\mathbb{R}^n\): real n-space; \(\mathbb{R}^+\): \(\{x \in \mathbb{R} : x > 0\}\); \(x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n\): column vector; \(\|x\|_2 = \left[\sum_{i=1}^{n} x_i^2\right]^{1/2}\) : Euclidean norm of \(x\); \(A = [A_{ij}] \in \mathbb{R}^{n \times n}\) : square matrix; \(A^t\) : transpose of \(A\); \(A^{-1}\) : inverse of \(A\); \(\ker A\) : kernel of \(A\); \(\|A\|\) : a matrix norm; \(E_n \in \mathbb{R}^{n \times n}\) : identity matrix; \(D = \text{diag}(d_1, \ldots, d_n) \in \mathbb{R}^{n \times n}\) : diagonal matrix with \(D_{ii} = d_i, i = 1, \ldots, n\); \(f \in C^k(\mathbb{R})\) : \(k\)-time continuously differentiable function on \(\mathbb{R}\).

2 Neural Network Model

The neural networks considered in this paper satisfy the system of differential equations

\[\dot{x} = -Dx + A_0G(x),\]  

(N)

where \(x \in \mathbb{R}^n\) is the vector of the neuron state variables, \(A_0 \in \mathbb{R}^{n \times n}\) describes the neuron interconnections, and \(D = \text{diag}(d_1, \ldots, d_n) \in \mathbb{R}^{n \times n}\), \(d_i > 0\), \(i = 1, \ldots, n\), models the neuron self-inhibitions.
Function $G(x) = (g_1(x_1), \ldots, g_n(x_n))^t : \mathbb{R}^n \rightarrow \mathbb{R}^n$ has components $g_i$ modeling the nonlinear input-output activations of the neurons, which are assumed to satisfy the following hypotheses.

**Assumption 1** For $i = 1, \ldots, n$, $g_i(x_i) \in C^k(\mathbb{R})$, $k \geq 5$, $g_i(0) = 0$, $g_i$ is bounded on $\mathbb{R}$, and $g_i'(x_i) > 0$ for $x_i \in \mathbb{R}$. Moreover, $g_i'(0) = \max_{x_i \in \mathbb{R}} g_i'(x_i)$, and for $x_i \in (-\delta, \delta)$, where $\delta$ is some positive constant, it results $x_ig''(x_i) \leq 0$, $g'''(x_i) < 0$.

For future reference we define the matrix

$$K = \text{diag}(g_1'(0), \ldots, g_n'(0)).$$  \hspace{1cm} (1)

Assumption 1 states that each $g_i$ is bounded and strictly increasing on $\mathbb{R}$, and it ensures the technical smoothness conditions on $g_i$ needed to apply the standard results on Hopf-bifurcation theory. Assumption 1 is satisfied by all sigmoidal functions used in practice, see, e.g., [2, 4]. Finally, observe that under the stated hypotheses, the origin is always an equilibrium point of (N).

In (N) we supposed that the input vector be zero. This is not a limitation, since a more general model with non-zero inputs can be transformed into the form (N) with a suitable change of variables [4].

In the studied problem, the following hypothesis on the nominal model (N) is fundamental.

**Assumption 2** The interconnection matrix $A_0$ is symmetric, i.e., $A_0^t = A_0$.

In addition to the nominal model (N), we consider the perturbation model defined as

$$\dot{x} = -Dx + AG(x),$$  \hspace{1cm} (P)

where $A = A_0 + \Delta A$,$\Delta A \in \mathbb{R}^{n \times n}$ in general nonsymmetric. Note that (P) is obtained from (N) by modifying the interconnection matrix, only.

Some definitions characterizing specific dynamical behaviors of models (N) and (P) are recalled next.

**Definition 1** The neural network (N) (or (P)) is said to be Completely Stable (CS) if each trajectory converges towards an equilibrium point, as $t \rightarrow \infty$.

**Definition 2** The neural network (N) (or (P)) is said to be Almost Completely Stable (ACS), if there is at most a set of zero measure of initial conditions originating non-convergent trajectories.

Though slightly weaker than CS, the property of ACS is actually as useful as CS for the applications, since such non-convergent trajectories cannot be observed in physical experiments. Obviously, the ACS property is lost if there exists an open set of initial conditions for which the corresponding trajectories do not converge to an equilibrium point.

It is well known that the symmetry of $A_0$ ensures that the nominal model (N) is ACS [1, 2]. The fundamental problem addressed in this paper is to discover cases where the perturbation model (P) is not ACS for (arbitrarily) small $\|\Delta A\|$, i.e., as close as one pleases to the nominal symmetric case.

More specifically, we study Hopf bifurcations (HBs) at the equilibrium point at the origin which cause the loss of ACS for (P). In this respect, it is important to note that if the HB is subcritical [11], (P) may still remain ACS. In the supercritical case only, the presence of a stable limit cycle ensures that (P) is no longer ACS for small bifurcation parameters. Thus, we need to evaluate the sign of the stability coefficient $\beta_2$, which determines whether or not a supercritical HB occurs [11].

**3 Main Results**

In this section we discover and analytically characterize situations where there are supercritical HBs ($\beta_2 < 0$) of the origin of (P), which are arbitrarily close to the nominal symmetric case (N) (i.e., $\|\Delta A\| \rightarrow 0$). The proofs of the results presented are omitted and can be found in [14].

We assume for the nominal model (N) that the Jacobian matrix at the origin, $J_0(0) = -D + A_0K$ (see (1)), satisfies the following basic degeneracy condition.
Assumption 3 Suppose that Assumptions 1-2 hold. We assume that $J_0(0) = -D + A_0K$ has a double zero eigenvalue, and $n - 2$ real negative eigenvalues $\lambda_1, \ldots, \lambda_{n-2}$. In particular, we have

$$\text{rank } J_0(0) = \text{rank } (-D + A_0K) = n - 2.$$  \hfill (2)

Under Assumption 3, we want to construct a special class of perturbations $\Delta A$ originating supercritical HBs at the origin of (P). To this end, let us introduce the symmetric matrix

$$M_0 = -D + K\hat{x}A_0K\hat{x},$$

where $K\hat{x} = \text{diag}(\sqrt{g_1'(0)}, \ldots, \sqrt{g_n'(0)})$. Note that $M_0$ is similar to $J_0(0) = -D + A_0K$, since $-D + K\hat{x}A_0K\hat{x} = K\hat{x}(-D + A_0K)K^{-\frac{1}{2}}$. Hence, the two matrices have the same eigenvalues and, in particular, rank $M_0 = n - 2$. Moreover, let us denote with

$$O_n = \{u, v \in \mathbb{R}^n : \|u\|_2 = \|v\|_2 = 1 \text{ and } u^tv = 0\},$$

the set of the pairs of orthonormal vectors in $\mathbb{R}^n$.

The sought class of perturbations has the form

$$\Delta A(\mu) = \mu K^{-1} + \omega K^{-\frac{1}{2}}(uv^t - v^tu)K^{-\frac{1}{2}},$$  \hfill (3)

where $\omega \in \mathbb{R}^+, \mu \in \mathbb{R}$, and $u, v \in O_n$ are vectors that span ker $M_0$.

Therefore, we investigate HBs of the equilibrium point at the origin for the following perturbed model

$$\dot{x} = F(x, \mu) = -Dx + [A_0 + \Delta A(\mu)]g(x) \quad \text{(P1)}.$$  

Note that $\mu$ is the only variable parameter (Hopf-bifurcation parameter), while $\omega$ and the components of $u, v$ are fixed parameters. The role of $\omega$, $u, v$ is different. We will see in Theorem 1 that by choosing $\omega$ small enough, it is possible to construct HBs of (P1) at $\mu = 0$, for arbitrarily small $\|\Delta A\|$.

The first basic result for (P1) is as follows.

Property 1 Suppose that Assumptions 1-2 hold. Then, Assumption 3 guarantees the existence of Hopf-generated periodic solutions, which emerge from the origin of (P1) at $\mu = 0$, for each fixed parameter $\omega \in \mathbb{R}^+$.

The above property follows from the fact that under the stated assumptions, the neural network (P1) satisfies the hypotheses of Theorem 7.2.3 in [11].

The next result concerns the quantitative local aspects of the HBs, i.e., the analytical evaluation of $\beta_2$, and the stability of the periodic solutions.

Theorem 1 Suppose that Assumptions 1-3 hold. Then, for each fixed parameter $\omega \in \mathbb{R}^+$, in relation to the HB at the origin of (P1) at $\mu = 0$, we have

$$\beta_2 = \frac{1}{8} \sum_{i=1}^{n} d_i \frac{g_i'''(0)}{[g_i'(0)]^2} (u_i^2 + v_i^2) < 0. \quad (4)$$

Theorem 1 states that the perturbations $\Delta A(\mu)$ defined in (3) indeed give rise to supercritical HBs, hence they originate stable limit cycles emerging from the origin of (P1), for small parameters $\mu > 0$.

The crucial point to note is that the expression for $\beta_2$ obtained in (4) is independent of $\omega$. Moreover, from (3) it is seen that by choosing $\omega$ small enough, it is possible to make $\|\Delta A(\mu)\|$ arbitrarily small, for small $\mu$. Therefore, there are supercritical HBs for (P1) as close as one pleases to the symmetric case (N).

The next result addresses some global aspects of the HBs of (P1).

Theorem 2 Suppose that Assumptions 1-3 hold. Then: \(a\) for $\mu \leq 0$, the origin is a Globally Asymptotically Stable (GAS) equilibrium point of (P1); \(b\) for $0 < \mu < \mu_1$, where $\mu_1$ is sufficiently small, almost all trajectories of (P1) converge towards the stable Hopf-generated periodic solution, as $t \to \infty$.

Theorem 2 actually states that at $\mu = 0$ there is a transfer of global stability from the equilibrium point at the origin of (P1), to the Hopf-generated periodic solutions. Part a) of the theorem is proved by using the method devised in [13], which is suitable to analyze GAS for nonsymmetric neuron interconnection matrices.

Some concluding comments are in order.

Remark 1 We have shown in Theorem 1 that Assumption 3 is sufficient to ensure the existence of supercritical HBs arbitrarily close to symmetry for
Let us address the necessity of such a condition. It is not difficult to show that Theorem 1 is still valid when (2) in Assumption 3 is replaced by the more general condition
\[
\text{rank} (−D + A_0K) \leq n - 2. \tag{5}
\]
Now, take into account the following observation.

**Property 2** Suppose that Assumptions 1-2 hold. If \( \text{rank} (−D + A_0K) \geq n - 1 \), then there exists \( δ > 0 \) such that when \( \| \Delta A \| < δ \), the Jacobian at the origin of the vector field defining (P) has no purely imaginary eigenvalues.

On the basis of Theorem 7.2.3 in [11], Property 2 implies that (5) is necessary to have HBs bifurcations at the origin of (P), for small \( \| \Delta A \| \). Summing up, under the hypothesis that \( A_0 \) is symmetric, the following holds: the condition \( −D + A_0K \) has non-positive eigenvalues and \( \text{rank} (−D + A_0K) \leq n - 2 \), is necessary and sufficient for the presence of supercritical HBs at the origin of (P), arbitrarily close to the symmetric case (N).

**Remark 2** The analytic expression of \( β_2 \) obtained in (4) is also useful to quantitatively evaluate the sensitivity of the amplitude of the emerging limit cycles on \( μ \) [11]. This is important in view of addressing the effects of tolerances in the practical implementation of the neural networks [14].

**References**


