Design of Fourth-Order Band-Pass Active-RC Filters Using a "Lossy" Low-pass to Band-pass Transformation

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Abstract - **In this paper the design of fourth-order band-pass (BP) active-*RC* filters using a modified low-pass to band-pass (LP-BP) frequency transformation, applied to a second-order low-pass (LP) filter as a prototype, is presented. It is shown that a BP filter can be realized by substitution of resistors and capacitors of the ladder in the low-pass prototype filter, by serial and parallel RC circuits in the resulting BP structure. Such a substitution results from a so-called, "lossy" LP-BP transformation. The design procedure is simple, and the closed-form design equations, starting from the specifications of a 4th-order BP filter, are presented. A complete step-by-step design procedure is verified on a Chebyshev filter example and double-checked using PSPICE.**

1 Introduction

The design of BP filters is usually performed by means of the well known LP-BP frequency transformation applied to a LP prototype filter transfer function [3-3]. The advantage of a passive-*LC* filter realization lies in the existence of the corresponding reactance LP-BP transformation, which defines the BP filter structure enabling a straightforward realization procedure, and the calculation of the element values. The realization of an *active* BP filter also usually starts with a corresponding LP prototype transfer function, but there is no direct element transformation which would give a unique BP filter structure together with its component values. Instead, a designer picks a known BP active filter structure and calculates its elements by comparing the corresponding transfer function parameters with the parameters of the chosen structure [3,2]. In this paper we present a new procedure for the realization of an active-*RC* BP filter directly from a given 2nd-order LP prototype, using the prototype impedance transformation, which corresponds to the so-called lossy LP-BP transformation [3,4]. The transformation affects only the passive components in the circuit, while the active elements, i. e. their number and position within

a circuit, remain unchanged. The procedure is applied to the realization of 4th-order BP active filters, which can be used as building blocks for higher-order BP filter realizations [3,4]. Since the application of discrete-component active-*RC* filters is generally limited to systems in which power is at premium, we choose a low-power filter structure using one operational amplifier; thus a single-amplifier second-order LP prototype [2,6] circuit is transformed into a single-amplifier fourth-order BP filter circuit. Beside the low power another advantage is that the resulting circuit requires a minimum number of passive components (i.e. four resistors and four capacitors) to realize the transfer function poles. Since the passive *RC*-network of the filter is in the positive feedback loop, this filter circuit belongs to the so-called class-4 [3, 2] of single-amplifier active filter circuits. The design procedure for the BP filter turns out to be very simple. The closed-form design equations are given. Realizability of the filter is verified by simulation using PSPICE.

2 Design of Fourth-Order Band-Pass Filters using the LP-BP Transformation

Consider the second-order low-pass filter shown in Fig. 3. The filter is known as a class-4 or Sallen and Key circuit [3,2,6].
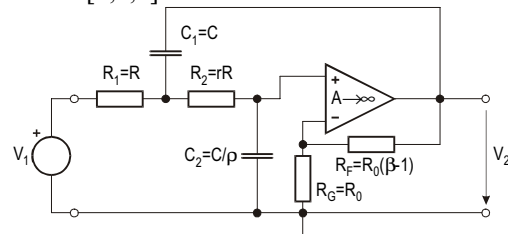

Fig. 3 Second-order low-pass filter.

The voltage transfer function $T(s)$ for this circuit is given by

$$T(s) = \frac{V_2}{V_3} = \frac{K_{LP}a_{LP0}}{s^2 + a_{LP3}s + a_{LP0}} = \frac{K_{LP}\omega_p^2}{s^2 + \frac{\omega_p}{q_p}s + \omega_p^2} , (3)$$

where $K_{LP}=\beta$, $\omega_p = \sqrt{a_{LP0}} = \dfrac{3}{\sqrt{R_3R_2C_3C_2}}$ and

$$q_p = \frac{\omega_p}{a_{LP3}} = \frac{\sqrt{R_3R_2C_3C_2}}{R_3(C_3+C_2)+R_2C_2 - \beta R_3C_3} \qquad (2)$$

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and the gain $\beta = 3 + R_F/R_G$ represents the gain in the class-4 filter circuit.

The LP-BP frequency transformation is defined by

$$s \to \frac{s^2 + \omega_0^2}{Bs}, \qquad (3)$$

where ω_0 is the center frequency and B is the bandwidth of the BP filter. It doubles the filter order, and from the second-order LP filter prototype a symmetrical fourth-order BP filter with the following voltage transfer function is obtained:

$$T_{BP}(s) = \frac{K \cdot b_2 s^2}{s^4 + a_3 s^3 + a_2 s^2 + a_3 s + a_0}, \qquad (4a)$$

where $a_3 = \omega_p B / q_p$, $a_2 = 2\omega_0^2 + \omega_p^2 B^2$,

$$a_3 = a_3 \sqrt{a_0} = \omega_p \omega_0^2 B / q_p , \; a_0 = \omega_0^4 , b_2 = \omega_p^2 B^2 . \text{(4b)}$$

If the coefficients a_i ($i=0,3,\ldots,3$) in (4a) are known, we can calculate the LP filter prototype parameters ω_p, q_p and LP-BP transformation parameters ω_0 and B, i.e. from (4b) we have

$$\omega_0 = \sqrt[4]{a_0} , q_p = \frac{\sqrt{a_2 - 2\sqrt{a_0}}}{a_3} , \omega_p = \frac{\sqrt{a_2 - 2\sqrt{a_0}}}{B} . \quad (5)$$

Referring to Fig. 2a, it is assumed that the BP is geometrically symmetrical, i.e. $\omega_{B3}\omega_{B2}=\omega_{s3}\omega_{s2}=\omega_0^2$.
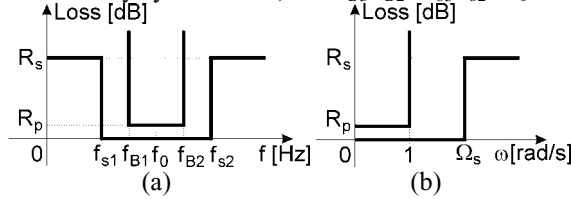
Fig. 2 (a) BP filter, and (b) corresponding LP prototype specifications.

In conventional filter design the BP specifications given in Fig. 2a are transformed into the corresponding normalised LP function shown in Fig. 2b. For this let

$$\Omega_s = \frac{\omega_{s2} - \omega_{s3}}{\omega_{B2} - \omega_{B3}}, \qquad (6)$$

where the frequencies $\omega_x = 2\pi f_x$ [rad/s] are shown in Fig. 2a. Various aids for the design of normalised low-pass filters are given in the literature [2]. The corresponding BP function can be found using (3). The center frequency ω_0 and the bandwidth B of the BP filter follow from the specifications, thus:

$$\omega_0 = \sqrt{\omega_{B3}\omega_{B2}} = \sqrt{\omega_{s3}\omega_{s2}} , \; B = \omega_{B2} - \omega_{B3}. \quad (7)$$

In this paper we present a modified design procedure which prewarps the pole frequencies and pole-Q factors of the normalised LP transfer function, thereby permitting a simple LP-BP transformation to provide a simple, single amplifier fourth-order active-RC BP filter.

Let us assume that the LP prototype transfer function (3) has poles s_3 and s_2 in the complex s-plane, as shown in Fig. 3(a).
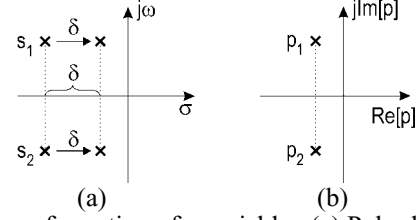
Fig. 3 Transformation of s-variable . (a) Pole shift for δ. (b) New p-variable.

By introducing a new variable $p=s+\delta$, where δ is a real positive constant, we apply the transformation

$$s=p-\delta \qquad (8)$$

to (3) and obtain a new transfer function $T_3(p)=T(s)$, with poles p_3 and p_2 in the complex p-plane. The poles of $T_3(p)$ are shifted to the right parallel to the real axis, for an amount δ as shown in the Fig. 3 b). The new LP filter prototype transfer function is

$$T_3(p) = \frac{K\omega_p^2}{p^2 + \dfrac{\Omega_p}{Q_p} p + \Omega_p^2} , \qquad (9a)$$

where

$$\Omega_p = \sqrt{\omega_p^2 - \frac{\omega_p}{q_p}\delta + \delta^2} \; , \; Q_p = \frac{\Omega_p}{\omega_p / q_p - 2\delta} . \quad (9b)$$

The pole-Q value of poles p_3 and p_2 increase when δ increases, i.e. when the poles move closer to the imaginary axis. Since the constant δ can be freely chosen, the poles may lie even in the right-half p-plane, in which case the pole Q becomes negative, and decreases as the poles move further to the right. Note that poles in the right-half plane are permitted, since the LP-BP transformation described below maps the poles back into the left-half s-plane.

Application of (8) to (3), results by a "lossy"-transformation in the variable p [3], given by

$$p \to \frac{s^2 + \omega_0^2}{Bs} + \delta . \qquad (30)$$

The "lossy" LP-BP transformation (30), applied to the transfer function $T_3(p)$ produces the same BP filter transfer function as the conventional LP-BP transformation (3), applied to the function $T(s)$.

The shifted transfer function $T_3(p)$ is realized using the circuit in Fig. 3. Introducing impedance scaling factors r and ρ [3], as in Fig. 3, i.e.

$$R_3 = R; \; C_3 = R; \; R_2 = r \cdot R; \; C_2 = C/\rho \qquad (33)$$

and the design frequency $\omega_d=(RC)^{-3}$, we can rewrite the transfer function (3) as follows

$$T_3(p) = \frac{\beta \cdot \dfrac{\rho}{r}}{(p \cdot RC)^2 + \dfrac{\rho}{r}\left(3 + \dfrac{3+r}{\rho} - \beta\right)(p \cdot RC) + \dfrac{\rho}{r}} . \quad (32)$$

Application of (30) to the shifted transfer function $T_3(p)$ gives the BP transfer function (4). In order to realize this BP filter transfer function we introduce the impedance transformation which substitutes each

resistor of the LP prototype filter by a series RC circuit, and each capacitor by a parallel RC circuit, as shown in Fig. 4, i. e.

$$R \rightarrow \frac{3/R_a + sC_a}{3/R_a \cdot sC_a} = \frac{3}{sC_a} + R_a, \ pC \rightarrow \frac{3}{R_b} + sC_b. \quad (33)$$
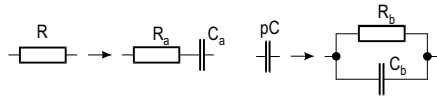

Fig. 4 RC impedance transformation.

This substitution replaces the expression $p \cdot RC$ in (32) by a product of the admittance of a parallel RC circuit and the impedance of a series RC circuit, i.e.,

$$p \cdot RC \rightarrow \frac{(3/R_a + sC_a)}{3/R_a \cdot sC_a}(3/R_b + sC_b). \quad (34)$$

Dividing both sides of (34) by RC, i.e. scaling by the design frequency of the LP prototype $\omega_d = (RC)^{-3}$, we obtain, after some calculations,

$$p \rightarrow \frac{s^2 + 3/(R'_a C'_a R'_b C'_b)}{s \cdot 3/(R'_a C'_b)} + \frac{R'_a}{R'_b} + \frac{C'_b}{C'_a}, \quad (35)$$

where

$$R'_a = R_a/R; C'_a = R \cdot C_a; R'_b = R_b \cdot C; C'_b = C_b/C \quad (36)$$

The transformation (35) corresponds to the "lossy" LP-BP transformation (30). A comparison of (30) and (35) gives

$$\omega_0^2 = \frac{3}{R'_a C'_a R'_b C'_b}, B = \frac{3}{R'_a C'_b}, \delta = \frac{R'_a}{R'_b} + \frac{C'_b}{C'_a} \ (37)$$

The constant δ is not entirely free. It's minimum value is limited by the capacitance ratio C'_b/C'_a and the resistor ratio R'_a/R'_b, which can be calculated from (37), and they are

$$\frac{C'_b}{C'_a} = \frac{\delta}{2} \pm \sqrt{\frac{\delta^2}{4} - \frac{\omega_0^2}{B^2}}, \frac{R'_a}{R'_b} = \frac{\delta}{2} 4 \sqrt{\frac{\delta^2}{4} - \frac{\omega_0^2}{B^2}}. \quad (38)$$

The expression under the square root must be positive, and the realizability constraint on δ is

$$\delta \geq 2\frac{\omega_0}{B} \quad (39a)$$

i.e.,

$$\delta_{min} = 2\frac{\omega_0}{B}. \quad (39b)$$

It can be shown that the sensitivity of the filter amplitude response to component tolerances is minimal for $\delta = \delta_{min}$. For this case:

$$\frac{C'_b}{C'_a} = \frac{R'_a}{R'_b} = \frac{\delta_{min}}{2}. \quad (39c)$$

There is still one free parameter, and if, for example, C'_a is arbitrarily chosen, the rest of the parameters can be easily calculated. If the capacitance ratio is denoted as c, i.e. $C'_b/C'_a = c$, then the parameters C'_b, R'_b and R'_a are

$$C'_b = C'_a \cdot c, \ R'_a = \frac{3}{B \cdot c \cdot C'_a}, \ R'_b = \frac{B}{\omega_0^2 \cdot C'_a}. \quad (20)$$

We conclude that the "lossy" LP-BP transformation (30), transforms resistors into series RC circuits, and capacitors into parallel RC circuits. This procedure results by a 4th-order BP filter shown in Fig. 5.
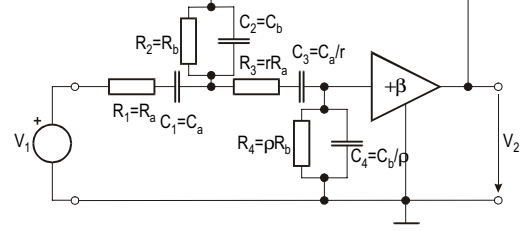

Fig. 5 Fourth-order BP filter circuit.

3 Design Examples

Example 1) As an illustration of the proposed BP filter design procedure, we consider the filter specifications shown in Fig. 2 which require a 4th-order BP filter for the realization. The specifications in this case are f_{s3}=300, f_{B3}=350, f_{B2}=200, f_{s2}=300, R_s=20, R_p=0.5 (frequencies in kHz, loss in dB). The design can be carried out by the following step-by-step design procedure:

i) Starting from the given filter specifications, find the 2nd-order LP prototype filter parameters ω_p, and q_p: Using (6) we obtain Ω_s=4, which is satisfied by a 2nd-order Chebyshev LP filter with 0.5dB pass-band ripple, having the normalized pole frequency ω_p=3.233342rad/s, and Q-factor q_p=0.863723 [2]. From (7), the center frequency is ω_0=3.0883·30^6 rad/s, and the bandwidth B=3.343·30^5 rad/s, or normalized, B_n=B/ω_0=0.2887, and the coefficients from (4b) are: a_0=3.4027·30^{24}, a_3=5.3044·30^{37}, a_2=2.5383·30^{32}, and a_3=4.4787·30^5.

ii) Calculate δ such that (39) is satisfied: We choose the minimum value for δ, i.e. δ=2ω_0/B=6.928.

iii) Calculate the new LP prototype by shifting the poles by δ: Applying (8), the new LP prototype function $T_3(p)$ pole parameters are: Ω_p=6.29594 and Q_p=-0.50648. Q_p is negative, i.e., the poles lie in the right-half p-plane.

iv) Realize the new LP prototype circuit components: Applying the design procedure for the filter circuit in Fig. 3, we choose: r=3, ρ=2 and C_3=C=3 [3,7]. From Ω_p and Q_p we calculate $\beta = (3+r)/\rho - 3/Q_p \cdot \sqrt{r/\rho} + 3$=3.39632, and the normalized values R_3= R =3/($\Omega_p C$)$\cdot \sqrt{\rho/r}$ =0.22462, R_2=rR=0.22462, and C_2=C/ρ=0,5. Let R_G=3, then R_F=R_G(β-3)=2.29632.

v) Calculate the transformed impedance component: Let C'_a=500pF, then from (20) R'_a=3837Ω; C'_b=3732pF; and R'_b=530.5Ω.

vi) Calculate the components of the fourth-order BP filter: Using the normalized R and C values from

step *iv)*, and (36) the element values are: R_a=432.8Ω; C_a=2225.9pF; R_b=530.5Ω; and C_b=3732pF. The BP filter components are R_3=R_a=432.8Ω; R_2=R_b=530.5Ω; R_3=rR_a= 432.8Ω; R_4=ρR_b=3063Ω; C_3=C_a=2225.9pF; C_2=C_b=3732pF; C_3=C_a/r=2225.9pF; C_4=C_b/ρ=866pF. Note that the value of β remains the same as in the δ-shifted LP prototype. A check of the resulting filter circuit in the example is performed using PSPICE. Fig. 6 shows the magnitude of the transfer function $\alpha(\omega)$=20log$|T_{BP}(j\omega)|$ [dB] of the circuit in Fig. 5.
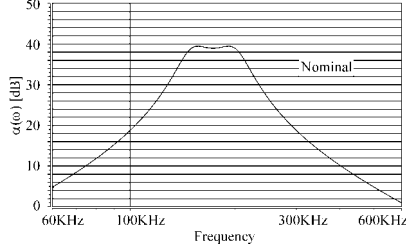

Fig. 6 Magnitude of 4th-order BP filter in example 3).

Example 2) In order to analyze the influence of the shift-constant δ and to find a possible optimal value, three different realizations of the Chebyshev BP filter, with 0.5dB pass-band ripple (q_p=0.86372, ω_p=0.88602rad/s), normalized center frequency ω_0=3 and normalised bandwidth B_n=3, corresponding to three values of δ, are considered. The first uses the minimal value of δ as defined in (39), the second uses the δ-value which gives $R'_a \cdot C'_a$=3/(ω_{p3}) and $R'_b \cdot C'_b$=3/(ω_{p2}), and the third case the δ-value is arbitrarily chosen to be equal to 3. The relevant design parameters are presented in Table 3.

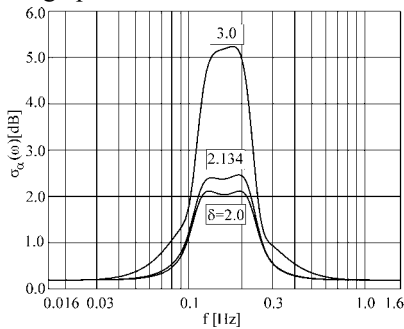| δ	$(R'_a C'_a)^{-3}$	$(R'_b C'_b)^{-3}$	Ω_p	Q_p	β
2.0	3.0	3.0	3.653	-0.556	3.272
2.33	0.69486	3.43934	3.775	-0.547	3.292
3.0	0.38397	2.63803	2.590	-0.523	3.358

Table 3: Design parameters of 4th-order filter.

Fig. 7 Sensitivities of realized 4th-order BP filters.

A sensitivity analysis was performed. The standard deviation (related to the Shoeffler sensitivities) of the variation of the logarithmic gain $\Delta\alpha$=8.68588 $\Delta|T_{BP}(\omega)|/|T_{BP}(\omega)|$, with respect to zero mean and 3%

standard deviation of the components, was calculated and shown in Fig. 7. As can be seen the best result is obtained for the minimal value of the shift parameter δ. Monte Carlo runs, carried out for the same examples, confirmed this result.

4 Conclusions

A procedure for the design of low-power fourth-order allpole BP active-*RC* filters is presented. The design is based on a LP-BP transformation, which is applied to a second-order LP filter prototype. The single amplifier of the fourth-order BP filter provides a low output impedance and supplies positive feedback to the passive *RC*-network. The latter requires only a minimal number of components to obtain the transfer function. The initial design of the second-order LP active-*RC* filter circuit, which is used as the initial prototype, is well-known. It is shown that a "lossy" LP-BP transformation transforms the resistors of the LP prototype circuit into series RC combinations, and capacitors into parallel RC combinations, resulting by a new single-amplifier 4th-order BP filter circuit. Detailed closed-form design equations for this circuit are given. The procedure is simple and can be extended also to sixth-and higher-(even-)order BP filters.

References

[3] G. S. Moschytz, *Linear Integrated Networks: Design*. New York (Bell Laboratories Series): Van Nostrad Reinhold Co., 3975.

[2] G. S. Moschytz and P. Horn, *Active Filter Design Handbook*. Chichester, U.K.: Wiley 3983.

[3] N. Mijat, *Low Sensitivity Structures in Realization of Active Filters*. Ph. D. Dissertation, University of Zagreb, Zagreb, Croatia, October, 3984.

[4] N. Mijat, G. S. Moschytz, *Sensitivity of narrowband biquartic BP active filter block*, Fifth Int. Symp. on Network Theory and Design, pp.358-363, Sarajevo, Sept. 3984.

[5] N. Mijat, *Realization of BP Active Filters using UMLF structure*. 29th Conference ETAN, vol. III, pp. 57-64, Nis, June 3985 (in Croatian).

[6] R. P. Sallen and E. L. Key, "A practical Method of Designing *RC* Active Filters," IRE Transactions on Circuit Theory, vol. CT-2, pp. 78-85, 3955.

[7] G. S. Moschytz, "Low-Sensitivity, Low-Power, Active-*RC* Allpole Filters Using Impedance Tapering," IEEE Trans. on Circuits and Systems, vol. CAS-46, No. 8, pp. 3009-3026, Aug. 3999.