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Locating multiple inclusions from sweep data of electrical impedance tomography

Master's thesis submitted in partial fulfilment of the requirements for the degree of Master of Science in Technology in the Degree Programme in Engineering Physics and Mathematics.

Espoo, 18th November 2011

Supervisor: Professor Olavi Nevanlinna

Instructor: Academy Research Fellow Nuutti Hyvönen

Preface

First, I wish to express my gratitude to my thesis instructor Nuutti Hyvönen for his remarkable dedication and the excellent advice I received throughout the writing process. In addition, I would like to acknowledge Lauri Harhanen for providing his code for the numerical simulations and the Finnish Funding Agency for Technology and Innovation TEKES for supporting this work (contract 40370/08). For me, this thesis has been an interesting opportunity to learn mathematics and dive into the current inverse problem research.

Thanks are also due to my friends and family who always brighten my life outside work and studies. Especially I would like to thank my parents Johanna and Pekka, my sister Essi and my girlfriend Niina.

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Abstract of the Master's Thesis

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Title:	Locating multiple inclusions from sweep data of electrical impedance tomography	
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Abstract: Electrical impedance tomography (EIT) is the practice of estimating the position-dependent electrical properties of a body from current and voltage measurements on its boundary. It has numerous present and prospective applications in, among others, medical imaging, geophysics and non-destructive material testing.

This thesis studies *sweep data* of EIT, which is a recent concept associated with a special two-electrode measurement introduced in [21] by Hyvönen, Harhanen and Hakula. Based on the recent paper [16] by Hanke, where a similar analysis is carried out for a related novel EIT measurement, the *backscatter data*, a method for locating *inclusions* of different conductivities in an otherwise homogeneous disk-shaped object is devised.

The cornerstone of the analysis is a certain factorization of the difference Neumann-to-Dirichlet map, which is proven valid under somewhat weaker assumptions than in [16]. The factorization is subsequently used to construct an asymptotic small inclusion expansion and prove that sweep data can be interpreted as the boundary value of a complex analytic function.

As a new result, the method presented here has the capability of extracting information about the conductivities and sizes of the inclusions. Even though inspired by the devised properties, the algorithm is not entirely backed by theory, but the numerical results strongly indicate that it works as desired.

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Aalto-yliopisto
Perustieteiden korkeakoulu

Diplomityön tiivistelmä

Tekijä:	Otto Seiskari		
Työn nimi:	Useiden inkluusioiden paikantaminen impedanssitomografian sweep-datasta		
Työn valvoja: **Työn ohjaaja:**	Professori Olavi Nevanlinna Akatemiatutkija Nuutti Hyvönen		
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Pääaine:	Matematiikka	**Sivuaine:**	Tietotekniikka
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Tiivistelmä: Impedanssitomografiassa (EIT) kappaleen paikkariippuvia sähköisiä ominaisuuksia yritetään selvittää sen pinnalla mitattujen sähkövirtojen ja -jännitteiden perusteella. Sillä on sovelluksia esimerkiksi lääketieteellisessä kuvantamisessa, geofysiikassa sekä materiaalien testauksessa.

Tässä diplomityössä tutkitaan impedanssitomografian *sweep-dataa*, joka on Hyvösen, Harhasen ja Hakulan artikkelissa [21] esitelty uusi, tiettyyn kahden elektrodin EIT-mittaukseen liittyvä käsite. Diplomityössä esitellään menetelmä, jossa muuten homogeenisesta kiekkomaisesta kappaleesta paikannetaan sähkönjohtavuudeltaan poikkeavia *inkluusioita*. Menetelmä pohjautuu Hanken artikkeliin [16], jossa analysoidaan vastaavalla tavalla EIT:n *takaisinsirontadataa*, joka on sweep-datan kaltainen ja tähän läheisesti liittyvä uusi käsite.

Työssä teoreettinen tarkastelu nojautuu tiettyyn Neumann–Dirichlet-erotuskuvauksen faktorointiin, jonka todistetaan pätevän heikommilla oletuksilla, kuin artikkelissa [16]. Faktoroinnin avulla muodostetaan asymptoottinen pieninkluusiokehitelmä ja todistetaan, että sweep-data voidaan tulkita kompleksianalyyttisen funktion reuna-arvona.

Uutena tuloksena esitetty menetelmä kykenee laskemaan inkluusioiden johtavuuksiin ja kokoihin liittyviä tietoja. Vaikka algoritmi mukailee sweep-datan teoreettisia ominaisuuksia, eivät todistetut tulokset takaa sen toimivuutta. Numeeriset esimerkit kuitenkin viittaavat vahvasti siihen, että menetelmä toimii halutulla tavalla.

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Avainsanat: impedanssitomografia, inversio-ongelmat, layer-potentiaali, johtavuusyhtälö, Sobolev-avaruus, Laurent–Padé-approksimantti

Contents

Notation

The following notation is used in this work. It is assumed that the reader is familiar with these concepts. Consult e.g. [13] for details.

$\mathbb{N}$, $\mathbb{N}_0$	natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$.
$\mathbb{Z}$	integers
$\mathbb{R}$, $\mathbb{C}$	real and complex numbers $\mathbb{C} \supset \mathbb{R}$
$[a, b)$	half-open interval $\{x \in \mathbb{R} : a \leq x < b\}$
$\mathbb{F}^n$	n-dimensional vectors with elements in $\mathbb{F}$
$\mathbb{F}^{n\times m}$	$n \times m$ matrices with elements in $\mathbb{F}$ ($\mathbb{F}^n = \mathbb{F}^{n\times 1}$)
$\lvert\cdot\rvert$	Euclidean norm in $\mathbb{C}^n$
$e_1, \ldots, e_n$	orthonormal system of coordinate unit vectors in $\mathbb{R}^n$
$\lvert D\rvert$	$\int_D \mathrm{d}x$, (Lebesgue) measure of $D \subset \mathbb{R}^n$
$⨍_D u\,\mathrm{d}x$	$1/\lvert D\rvert \int_D u\,\mathrm{d}x$, average of u over D
$B_r(x)$	open unit ball $\{y \in \mathbb{R}^n : \lvert y - x\rvert < r\}$ in $\mathbb{R}^n$
ω_n	area (surface measure) of the unit sphere $\partial B_1(0)$ in $\mathbb{R}^n$
χ_D	indicator function: $\chi_D(x) = 1$ if $x \in D$ and 0 otherwise
$U \subset\subset D$	compactly contained set: $\overline{U} \subset \operatorname{Int} D$
$\overline{D}$, ∂D	closure and boundary of $D \subset (\mathbb{R}^n, \lvert\cdot\rvert)$
$\overline{z}$	complex conjugate of $z \in \mathbb{C}$
D^α	partial derivative of multi-order $\alpha \in \mathbb{N}_0^n$
$\nabla\times, \nabla\cdot, \nabla$	curl, divergence, gradient
Δ	Laplace operator
$\cdot$	(bilinear) dot product in $\mathbb{C}^n$
M^T	transpose of the matrix M
O	$f(x) = O(g(x))$ as $x \to z \Leftrightarrow \limsup_{x\to z} \frac{\lvert f(x)\rvert}{\lvert g(x)\rvert} < \infty$.
o	$f(x) = o(g(x))$ as $x \to z \Leftrightarrow \lim_{x\to z} \frac{f(x)}{g(x)} = 0$.
$\lceil x\rceil$	the least integer greater than or equal to $x \in \mathbb{R}$
$\lfloor x\rfloor$	the greatest integer less than or equal to x
A', A^*	topological dual and anti-dual of the vector space A
$(\cdot,\cdot)_V$	sesquilinear inner product in a Hibert space V
$\langle\cdot,\cdot\rangle_V$	dual evaluation between V' and V
$\lVert\cdot\rVert$	no subscript: operator norm (unless stated otherwise)

1 Introduction

Electrical impedance tomography

The spatial variations of electrical properties, such as conductivity and permittivity, of the medium inside an object can reveal valuable information about its contents. For instance, the conductivity of certain tumours differs clearly from that of healthy tissue. It is also possible to find underground mineral deposits or detect various structural defects in materials on the grounds of the conductivity or permittivity differences in the relevant media. The core idea of *electrical impedance tomography* (EIT) is to determine the position-dependent electrical properties of a body based on current and voltage measurements on its boundary. It has a wide range of potential applications and it has already been successively applied in, among others, medical imaging, non-destructive material testing and geophysics. EIT research was pioneered by Calderón in the 1980's and, along with other related inverse problems, continues to be an active field today. [5][7][30]

Inverse problems

In the beginning of the 20th century, Jacques Hadamard stated the definition of a *well-posed* mathematical problem: a solution of the problem must *exists*, be *unique* and *depend continuously on the data*. Problems that do not meet these three conditions are called *ill-posed* and were long considered to be outside the scope of meaningful mathematical analysis. However, it appears that a large class of relevant problems do not fulfil Hadamard's conditions. In particular, many problems are unstable in the sense that small changes in their input data can cause arbitrarily large variations in their solutions. In the case of real-life computational problems, where the data includes noise and systematic measurement errors, such a phenomenon is disastrous if not treated properly.

The keys to solving ill-posed problems are *regularization* and *a priori* knowledge about the solution. In practical terms, regularized solutions can be seen as compromises between resolution and error. These subjects are studied by *inverse mathematics* where the ill-posed questions are typically formulated as certain "backwards" versions of well-posed and often more well-understood problems. An archetypal example is EIT, in which the well-posed *forward problem* is, informally: given the conductivity of a body, determine the relationship between the currents and voltages on its boundary. The EIT *inverse problem* can be stated as: given the relationship between the currents and voltages, determine the conductivity. The latter problem is severely non-linear and ill-posed by nature.

This thesis

This work studies a special two-electrode EIT measurement, where one electrode is stationary and the other moves around a circular object (as illustrated in Figure 1(a) on page 39). This leads one to the notion of *sweep data*, which was introduced in the recent paper [21] by Hyvönen, Harhanen and Hakula. A method for locating inhomogeneities (also called *inclusions*) in an otherwise homogeneous medium from a noisy measurement is devised. The method relies on the so-called Laurent-Padé approximants and it is based on a similar method introduced by Hanke [16] for *backscatter data* of EIT [18], which is likewise a novel concept and closely related to sweep data. The term backscatter originates from inverse scattering theory (e.g. RADAR), where an analogous concept is commonly encountered.

The approach in this thesis is mostly theoretical, but numerical examples are also presented in the end. A lot of effort is put into elaborating the necessary modern mathematical machinery to a reader with undergraduate mathematical background. Section 2 provides some preliminary concepts, which represent well-known mathematics from the first half of the 20th century and before. These are included for convenience and in order to remove ambiguity from the definitions of the important concepts. The material in Sections 3 and 4 represents basic theory of elliptic partial differential equations (PDEs) in distributional Sobolev spaces, that has been well-established since the 1980's — mostly 1970's or earlier.

Section 5 presents the mathematical definitions of the central forward models and inverse problems in EIT, augmented by some physical arguments. In particular, the *conductivity equation*, an elliptic PDE, which models behaviour of static and time-harmonic electromagnetic fields in inhomogeneous media, and is thus the basis of EIT, is explained in detail. The definitions of sweep and backscatter data are also given in this section.

EIT problems with certain *a priori* assumptions on the regularity of the inclusions are studied in Section 6 using layer potential techniques. The utilized factorization and polarization tensor methods represent recent inverse mathematics from the late 1990's and 2000's (see the book by Ammari & Kang [1] and references therein). Section 6.6 further specializes to the two-dimensional problem with circular boundary and constructs a (complex) asymptotic expansion for sweep data as a new result. Much of the analysis in this section is based on the recent paper [16] by Hanke, but the asymptotic expansions are proven valid under weaker regularity assumptions (on the inclusion boundaries) than in [16], and for distributional input currents unlike in [1].

It is demonstrated in Section 7 how the theoretical properties of sweep data can be used to inspire the design of a numerical reconstruction algorithm. The

method is mostly a straightforward reformulation of the algorithm devised by Hanke for backscatter data, but a certain new modification makes it possible to appraise whether the detected inclusions are conductive or resistive compared to the background. Furthermore, a certain combined measure of size and conductivity of the inclusion, the *net conductivity effect* introduced on page 63, can be approximated for separate inclusions. Still, there is a theoretical gap between the reconstruction method and the preceding analysis. Any result that would yield appropriate theoretical justification for the method has not been devised. In addition to showing one possible direction for future research, this increases the need for numerical experiments to test the eligibility of the method. A set of numerical examples and some implementation details are presented in Section 7.3. The examples strongly suggest that the method works as desired — indeed better than could be expected on the grounds of known theory.

2 Preliminaries

This section is provided for completeness, convenience of the reader and foremost to remove ambiguity from the definitions of the important concepts (such as $\mathscr{C}^k$-smooth functions). Consult e.g. [13], [32] and [20, Vol. I] for more detailed introductions to these concepts.

2.1 Complex analysis

Definition 2.1 (Holomorphic functions)**.** *A function $f : D \to \mathbb{C}$, where D is an open subset of $\mathbb{C}$ is called* holomorphic *(or* complex analytic*) if it is complex differentiable, that is, the limit*

$$\frac{\mathrm{d}f}{\mathrm{d}z}(z) := \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

exists[1] *for all $z \in D$. [20, Vol. I §4.2]*

Complex differentiability is a very strong condition. From the existence of one complex derivative, it follows that the complex derivatives of *all* orders exist and are holomorphic [20, Vol. I §7.5]. In consequence, holomorphic functions are smooth when interpreted as vector fields (vector-valued functions) in $\mathbb{R}^2$.

For any z_0 in the domain D of a holomorphic function f, there exists a power series representation

$$f(z) = \sum_{j=0}^{\infty} c_j (z - z_0)^j \tag{2.1}$$

that converges absolutely in any open disk $B_R(z_0) = \{z : |z - z_0| < R\} \subset D$. On the other hand, any function representable in this form (in some $B_R(z_0)$) is holomorphic in $B_R(z_0)$. The power series representation is always unique and given by the Taylor coefficients

$$c_j = \frac{\mathrm{d}^j f}{\mathrm{d}z}(z_0)/j! \tag{2.2}$$

[20, Vol. I §§5.5 & 8.1].

Examples of functions that are holomorphic in their domain of definition include polynomials, rational functions, complex logarithms and the exponential function. [20, Vol. I]

[1] In other words, $\forall \epsilon > 0\ \exists \delta > 0$ s.t $|\frac{f(z+h)-f(z)}{h} - \frac{\mathrm{d}f}{\mathrm{d}z}(z)| < \epsilon$ for all h s.t. $|h| < \delta$. Notice that with the norm $|\cdot|$ of the complex plane, this usual definition of limit implies that the result must be independent of "the direction from which h approaches zero".

Definition 2.2 (Laurent series)**.** *Let $f : D \to \mathbb{C}$ be holomorphic in an annulus $A = \{z : 0 \leq R_1 < |z - z_0| < R_2 \leq \infty\}$. Then, inside A, f is given by the unique absolutely convergent* Laurent series

$$f(z) = \sum_{j=-\infty}^{\infty} c_j (z - z_0)^j \tag{2.3}$$

where

$$c_j = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{j+1}} \, \mathrm{d}z \tag{2.4}$$

is a anti-clockwise line integral over any circle $C = \partial B_r(z_0)$ such that $R_1 < r < R_2$. [20, §8.3]

The Laurent coefficient $c_{-1} =: \mathrm{Res}(f, z_0)$ is called the *residue* of f at z_0, and by (2.4) it holds that

$$\oint_C f(z) \, \mathrm{d}z = 2\pi i c_{-1}$$

for any anti-clockwise line integral over C as in Definition 2.2.

The Laurent coefficients of a function f on the origin-centred unit circle $\partial B_1(0) \subset \mathbb{C}$ are given by

$$c_j = \frac{1}{2\pi i} \oint_{\partial B_1(0)} \frac{f(z)}{z^{j+1}} \, \mathrm{d}z = \frac{1}{2\pi} \int_0^{2\pi} f(e^{is}) e^{-ijs} \, \mathrm{d}s, \tag{2.5}$$

that is, they coincide with the *Fourier coefficients*

$$\hat{g}_j := \frac{1}{2\pi} \int_0^{2\pi} g(s) \overline{e^{ijs}} \, \mathrm{d}s, \quad j \in \mathbb{Z} \tag{2.6}$$

of the function $g : [0, 2\pi) \to \mathbb{C}$, $g(s) = f(e^{is})$ — in the basis $\{x \mapsto e^{ijx}\}_{j \in \mathbb{Z}}$ and with respect to the inner product $\frac{1}{2\pi}(\cdot, \cdot)_{L^2((0,2\pi))}$, cf. (2.13) in Section 2.3 [32].

For a sample $z_1, \ldots, z_N$ of complex numbers, the *discrete Fourier coefficients* can be defined as

$$c_j = \frac{1}{N} \sum_{k=0}^{N-1} z_{k+1} e^{-\frac{2\pi i j k}{N}}, \qquad j \in \mathbb{Z}. \tag{2.7}$$

Obviously, any system $j_1, \ldots, j_N$ of modulo N incongruent integers reveals all such coefficients for $j \in \mathbb{Z}$. It is thus common to define (2.7) only for $j = 0, \ldots, N - 1$, which yields the *discrete Fourier transform* (DFT). For real data z, it holds that $c_{-j} = \overline{c}_j$.

A holomorphic function $f : D \to \mathbb{C}$ has a *root of multiplicity* m at $z_0 \in D$ if $c_j = 0$ for all $j = 0, \dots, m-1$ and $c_m \neq 0$ in (2.2). Let z_0 (not necessarily in D) be such that f is holomorphic in $B_R(z_0) \setminus \{z_0\}$ for some $R > 0$. Then z_0 is called an *isolated singularity* of f. The type of the singularity is defined by the Laurent series representation:

Definition 2.3 (Isolated singularities)**.** *Let $c_j, j \in \mathbb{Z}$ be the Laurent coefficients of f in $B_R(z_0) \setminus \{z_0\}$, $R > 0$, and denote by $m = \inf_{j \in \mathbb{Z}}\{c_j \neq 0\}$ the index of the first non-zero coefficient.*

- *If $m \geq 0$ (including $m = \infty$), then z_0 is a* removable singularity*.*
- *If m is a negative integer, then z_0 is a* pole of order $-m$*.*
- *If $m = -\infty$, then z_0 is an* essential singularity*.*

[20, Vol. I §8.4].

For polynomials and rational functions, poles and roots can be defined algebraically as follows: A polynomial p has a root of multiplicity m at $z_0 \in \mathbb{C}$ if it can be factorized as $p(z) = (z - z_0)^m \tilde{p}(z)$, where $\tilde{p}$ is a polynomial such that $\tilde{p}(z_0) \neq 0$. Let $f = p/q$ be a rational function that is irreducible, that is, the polynomials p and q have no non-trivial common factors. Then f has a root of multiplicity m at z_0 if and only if the numerator p has. It has a pole of order m at z_0 if the denominator q has a root of that multiplicity there.

The residue of a rational function at a first-order pole z_0 is given by

$$\operatorname{Res}(f, z_0) = \frac{p(z_0)}{q'(z_0)}, \tag{2.8}$$

and the residues of higher order poles can also be solved algebraically from the coefficients of p and q [20, Vol. I §9].

2.2 Continuous function spaces

Let D be an open subset of $\mathbb{R}^n$. A function $f : D \to \mathbb{C}^m$ is of class $\mathscr{E}^k(D)$ if for any multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, such that $|\alpha| := \sum_{j=1}^n \alpha_j \leq k$, the derivative $D^\alpha f := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} f$ exists and is continuous in D. If all derivatives of f up to order k are also bounded in D, then f is called *k-smooth*, denoted $f \in \mathscr{C}^k(D)$. The class of k-smooth functions is a Banach space with the norm

$$\|u\|_{\mathscr{C}^k(D)} := \sum_{|\alpha| \leq k} \sup_D |D^\alpha u| \tag{2.9}$$

[31, §1.1]. Convergence in $\mathscr{C}^k(D)$ is *uniform convergence* of the function and its derivatives up to order k in D. The restrictions of class $\mathscr{E}^k(\mathbb{R}^n)$ functions to D comprise the space $\mathscr{C}^k(\overline{D}) = \mathscr{E}^k(\overline{D})$. The elements of this space can alternatively be thought of as functions in $\mathscr{C}^k(D)$ that, along with their derivatives, extend as continuous functions to $\overline{D}$. The subspace $\mathscr{C}_0^k(D) = \mathscr{E}_0^k(D) \subset \mathscr{C}^k(D)$ consists of those k-smooth functions that have *compact support* in D, that is,

$$\overline{\operatorname{supp} u} := \overline{\{x \in D : u(x) \neq 0\}} \subset D.$$

A function of class $\mathscr{C}^\infty(D) := \cap_{k=0}^\infty \mathscr{C}^k(D)$ is simply called *smooth*, and $\mathscr{C}(D) := \mathscr{C}^0(D)$ denotes the class of continuous bounded functions. Correspondingly, functions with continuous derivatives of all orders are denoted $\mathscr{E}^\infty(D)$. It holds that $\mathscr{C}_0^\infty(D) = \mathscr{E}_0^\infty(D)$.

Remark 2.1. This notational convention is adopted from [10], [31] and [27], except that the class of smooth, possibly unbounded, functions is always denoted $\mathscr{E}^\infty$ (and not $\mathscr{C}^\infty$ as in [31]). A common alternative (as in [32]) is to denote $\mathscr{E}^k$ by $\mathscr{C}^k$ and equip it with an F-space topology.

Definition 2.4. *The class $\mathscr{C}^{k,\lambda}(D) \subset \mathscr{C}^k(D)$ (for $\lambda > 0$) is the Banach space of functions that are bounded in the norm*

$$\|u\|_{\mathscr{C}^{k,\lambda}(D)} := \|u\|_{\mathscr{C}^k(D)} + \max_{|\alpha|=k} \|D^\alpha u\|_{\mathscr{C}^{0,\lambda}},$$

where

$$\|u\|_{\mathscr{C}^{0,\lambda}} := \sup_{x,y \in D} \frac{|u(x) - u(y)|}{|x - y|^\lambda} \tag{2.10}$$

and, by convention, $\mathscr{C}^{k,0}(D) := \mathscr{C}^k(D)$.

A function that is continuous in the norm (2.10) (i.e. of class $\mathscr{C}^{0,\lambda}(D)$) is called λ-*Hölder continuous* and when $\lambda = 1$, this is called *Lipschitz continuity*. [31, §1.1] A mapping $f : D \to \tilde{D} \subset \mathbb{R}^n$ of class $\mathscr{C}^{k,\lambda}(D), k \geq 1$ is a (k,λ)-*diffeomorphism* if it also has an inverse of class $\mathscr{C}^{k,\lambda}(\tilde{D})$. In this context, the notion of diffeomorphisms can be extended to include the (non-differentiable) Lipschitz bicontinuous functions as "$(0,1)$-diffeomorphisms". [31, §2.3]

2.3 L^p spaces

Let $(D, \mathcal{F}, \mu)$ be a measure space, where μ is a Borel measure (in some topology) such that $\mu(F) < \infty$ for all compact $F \in \mathcal{F}$ (e.g. the Lebesgue measure on $D \subset \mathbb{R}^n$). A complex-valued function f is measurable, if both $\operatorname{Re} f$ and $\operatorname{Im} f$ are measurable mappings $(D, \mathcal{F}) \to (\mathbb{R}, \mathcal{B})$, where $\mathcal{B}$ is the Borel algebra on $\mathbb{R}$

[2, §2.4]. Denote by $\mathcal{L}^0(D,\mu)$ the vector space of complex-valued measurable functions on $(D,\mathcal{F})$ and set (as in [22, Ch. 2, §1])

$$L^0(D,\mu) = \mathcal{L}^0(D,\mu)/\sim,$$

where the relation $\sim$ is defined as

$$f \sim g \quad \Leftrightarrow \quad \mu(\{x \in D : f(x) \neq g(x)\}) = 0 \quad \overset{\text{def.}}{\Leftrightarrow} \quad f = g \text{ a.e.},$$

that is, equivalence *almost everywhere*. The function $\|\cdot\|_{L^p(D,\mu)} : L^0(D,\mu) \to [0,\infty]$,

$$\|f\|_{L^p(D,\mu)} = \left(\int_D |f|^p \,\mathrm{d}\mu\right)^{1/p}, \tag{2.11}$$

where $p \geq 1$, as well as

$$\|f\|_{L^\infty(D,\mu)} = \operatorname*{ess\,sup}_{(D,\mu)} |f| := \inf_{A\in\mathcal{F},\mu(A)=0} \sup_{x\in D\setminus A} |f(x)| \tag{2.12}$$

are well-defined in $L^0(D,\mu)$ (not only in $\mathcal{L}^0(D,\mu)$) because

$$f \sim g \;\Leftrightarrow\; \|f-g\|_{L^p(D)} = 0 \;\Rightarrow\; \|f\|_{L^p(D)} = \|g\|_{L^p(D)}$$

independent of $p \in [1,\infty]$.

Definition 2.5 (L^p spaces)**.** *The* Lebesgue space *of order* $p \in [1,\infty]$ *is the (quotient) space*

$$L^p(D,\mu) = \{f \in L^0(D,\mu) : \|f\|_{L^p(D,\mu)} < \infty\}$$

and together with the corresponding L^p *norm* (2.11) *or* (2.12) *it is a Banach space for any* $p \in [1,\infty]$ *[2, §2.4].*

If μ is clear from the context, the Lebesgue spaces are denoted simply as $L^p(D)$. For instance, if $D \subset \mathbb{R}^n$, then μ is assumed to be the Lebesgue measure.

A related important concept is the space of locally integrable functions [13, §6.4][2]

$$L^p_{\mathrm{loc}}(D,\mu) = \{f \in L^0(D,\mu) : f|_F \in L^p(F,\mu)\ \forall \text{ compact } F \in \mathcal{F}\}.$$

For any $1 \leq p < r \leq \infty$ it holds that

$$L^r(D,\mu) \subset L^r_{\mathrm{loc}}(D,\mu) \subset L^p_{\mathrm{loc}}(D,\mu) \subset L^0(D,\mu)$$

[2]Gariepy & Ziemer [13] consider only spaces of real-valued functions, but the cited results generalize directly to the complex case using the definition [2, §2.4], $\int f\,\mathrm{d}\mu := \int \operatorname{Re} f\,\mathrm{d}\mu + i\int \operatorname{Im} f\,\mathrm{d}\mu$, of complex-valued integrals.

and if $\mu(D) < \infty$, then $L^r(D,\mu) \subset L^p(D,\mu)$.

Particularly important is the space $L^2(D,\mu)$, which is a Hilbert space (a complete inner product space) with the inner product

$$(f,g)_{L^2(D,\mu)} := \int_D f\overline{g}\,\mathrm{d}\mu \tag{2.13}$$

[2, §3.1].

2.4 Distributions

Definition 2.6 (Distributions)**.** *Let D be an open set in $\mathbb{R}^n$ and denote by $\mathscr{D}(D) := \mathscr{C}_0^\infty(D)$ the smooth, compactly supported functions from D to $\mathbb{C}$ (called* test functions *or* fundamental functions*). The space of* distributions*, $\mathscr{D}'(D)$, is defined as the (topological) dual of $\mathscr{D}(D)$, that is, the set of continuous linear functionals from $\mathscr{D}(D)$ to $\mathbb{C}$.*

In order to properly define "continuous" in this context, it is necessary to define the topologies of $\mathscr{D}(D)$ and $\mathscr{D}'(D)$. In this work, however, only the following partial information (for the details and proofs, see e.g. Wloka [31, §1]) is required:

Lemma 2.2. *The following constructs are in $\mathscr{D}'(D)$.*

1. *For any $f \in L^1_{\mathrm{loc}}(D)$, the functional*

$$T_f : \phi \mapsto \int_D f\phi\,\mathrm{d}x.$$

2. *Dirac's delta distribution at $y \in D$:*

$$\delta_y : \phi \mapsto \phi(y).$$

3. *The* distributional derivative *of any $T \in \mathscr{D}'(D)$:*

$$D^\alpha T : \phi \mapsto \langle D^\alpha T, \phi\rangle = (-1)^{|\alpha|}\langle T, D^\alpha\phi\rangle.$$

4. *The product of a distribution and a smooth function $a \in \mathscr{C}^\infty(D)$*

$$aT : \phi \mapsto \langle aT, \phi\rangle = \langle T, a\phi\rangle.$$

In this context, a function $f \in L^1_{\text{loc}}(D)$ is often identified with the corresponding distribution $T_f \in \mathscr{D}'(D)$. If $V \subset L^1_{\text{loc}}(D)$ (for example $V = L^p(D)$) and a distribution T satisfies $T = T_f$ for some $f \in V$, then one writes $T \in V$. Especially, if for some $u, v \in L^1_{\text{loc}}(D)$ it holds that

$$D^\alpha T_u = T_v, \tag{2.14}$$

then v is called the *order-α weak derivative* of u. This is commonly written as

$$D^\alpha u = v \ \text{ in } \mathscr{D}'(D).$$

If the function u is continuously differentiable up to order α, then the condition (2.14) can be easily shown to hold for the function $D^\alpha u \in \mathscr{C}(D)$ by repeatedly applying Green's theorem (equation (4.1) on page 22) on the boundary of a sufficiently smooth domain containing the support of ϕ. This means that the weak derivative coincides with the classical derivative almost everywhere if both exist.

One also talks about supports and restrictions of distributions. The restriction $T|_U$ of a distribution $T \in \mathscr{D}'(D)$ to $U \subset D$ means simply the restriction $T|_{\mathscr{D}'(U)}$. The support of T is the complement of the largest open set for which $T = 0$. [31, §§1.4 & 1.5]

Definition 2.7 (Tempered distributions)**.** *A function $\phi \in \mathscr{E}^\infty(\mathbb{R}^n)$ is* rapidly decreasing *if*

$$p_{k,m}(\phi) := \sup_{|\alpha| \le k} \sup_x |(1 + |x|^2)^m D^\alpha \phi(x)| < \infty \tag{2.15}$$

for all k and m. The space of rapidly decreasing test functions is denoted $\mathscr{S}$, and it has an F-space structure when equipped with the seminorms (2.15)*. The topological dual*

$$\mathscr{S}' = \{S : \exists C, k, m \ s.t. \ \forall \phi \in \mathscr{S}, |S(\phi)| < C p_{k,m}(\phi)\} \subset \mathscr{D}'(\mathbb{R}^n) \tag{2.16}$$

is called the space of tempered distributions *[31, §1.9].*

Definition 2.8 (Fourier transform)**.** *The* Fourier transform*, $\hat{f} := \mathscr{F} f := \mathscr{F}(f)$ of an integrable function $f \in L^1(\mathbb{R}^n)$ is defined as*

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) \, \mathrm{d}x$$

where $\cdot$ denotes the dot product in $\mathbb{R}^n$. For a tempered distribution $S \in \mathscr{S}'$ define

$$\hat{S}(\phi) = (\mathscr{F} S)(\phi) := S(\mathscr{F}\phi).$$

The Fourier transform is a linear isomorphism from $\mathscr{S}$ to itself, and for any $\phi \in \mathscr{S}$ it holds that

$$\mathscr{F}(\frac{\partial \phi}{\partial x_j}) = i\xi_j \mathscr{F}(\phi), \qquad \mathscr{F}(x_j \phi) = i\frac{\partial}{\partial \xi_j}\mathscr{F}(\phi) \tag{2.17}$$

and

$$\phi(x) = (\mathscr{F}^{-1}\hat{\phi})(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{\phi}(\xi)\,\mathrm{d}\xi.$$

The *inverse Fourier transform* may alternatively be written as

$$\mathscr{F}^{-1}\phi = \frac{1}{(2\pi)^n}\mathscr{F}\check{\phi}, \tag{2.18}$$

where $\check{\phi}(x) := \phi(-x)$. For any $S \in \mathscr{S}'$, the Fourier transform $\hat{S}$ is likewise a tempered distribution. It then follows from the definition of distributional derivatives that the formulas (2.17) and (2.18) also hold in $\mathscr{S}'$ (because $\check{S} : \phi \mapsto S(\check{\phi})$ is in $\mathscr{S}'$ for any $S \in \mathscr{S}'$). [31, §1.9]

One can define S_f and $D^\alpha S$ in $\mathscr{S}'$ as in $\mathscr{D}'(\mathbb{R}^n)$. It follows directly from (2.16) that $D^\alpha S \in \mathscr{S}'$ for any $S \in \mathscr{S}'$, and Hölder's inequality [2, §2.4] yields that $S_f \in \mathscr{S}'$ if $f \in L^p(\mathbb{R}^n)$ for some $p \geq 1$ (cf. [31, §1.9]). This is also true for $f \in L^\infty(\mathbb{R}^n)$. Hence all weak derivatives of the (distributions corresponding to) elements $L^p(\mathbb{R}^n)$ are tempered distributions.

The Fourier transform can also be extended to a continuous mapping $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ in such a manner that the norms of f and $\hat{f}$ satisfy *Parseval's equality*

$$\|f\|_{L^2(\mathbb{R}^n)} = \frac{1}{(2\pi)^n}\|\hat{f}\|_{L^2(\mathbb{R}^n)} \tag{2.19}$$

[31, §1.9].

For any distribution $T \in \mathscr{D}'(D)$ one can define the corresponding (continuous) anti-linear functional $\overline{T}$ as $\langle \overline{T}, \phi\rangle := \langle T, \overline{\phi}\rangle$ and vice versa. All of the above distribution theory remains essentially the same if one replaces $\mathscr{D}'(D)$ with $\mathscr{D}^*(D)$, the topological anti-dual of $\mathscr{D}(D)$, which is the space of all continuous anti-linear functionals on $\mathscr{D}(D)$. For example, the existence of a weak derivative $v \in L^1_{\mathrm{loc}}(D)$ of u is equivalent to the condition $D^\alpha T_u = T_v$, where $T_u, T_v \in \mathscr{D}^*(D)$, that is, "$D^\alpha u = v$ in $\mathscr{D}^*(D)$".

3 Sobolev spaces

Sobolev spaces provide a natural framework for studying the solutions of (elliptic) partial differential equations. There are various definitions of these spaces that coincide under different assumptions. In this work, two (potentially) different classes of Sobolev spaces are required: distributional H-spaces in smooth domains and on their boundaries to represent point current sources. In addition, for computations on the Lipschitz boundaries of arbitrary inclusions, the Sobolev–Slobodeckiĭ spaces $W_2^{\pm 1/2}(\partial D)$ are needed.

Section 3.1 gives a mathematical definition for the "smoothness" of domains, and Section 3.4 describes some quotient spaces that are useful for studying problems whose solutions are unique only up to an additive constant. These subjects are not exclusive to Sobolev space theory, but appear naturally in this context in the thesis.

The results in this section are predominantly stated without proofs, which more often than not are non-trivial and laborious. The main references are the books by Lions & Magenes [27], Wloka [31] and Grisvard [15].

3.1 Regular domains and boundaries

Qualitatively, a domain boundary is regular if it locally resembles a graph of a "regular" function. This idea is formalized by the following definitions in terms of Hölder regularity (see Section 2.2).

Definition 3.1 (Class $\mathscr{C}^{k,\lambda}$ boundary)**.** *Let D be an open subset of $\mathbb{R}^n$. Its boundary ∂D is of* class $\mathscr{C}^{k,\lambda}$ *if for any $x \in \partial D$ there exists*

1. *an orthonormal coordinate system $y_1, \ldots, y_n$ for $\mathbb{R}^n$ (with origin at x) and a hypercube[3] neighbourhood $V = \{|y_j| < \alpha \text{ for all } j = 1, \ldots, n\}$ of x (for some $\alpha > 0$)*

2. *a function $\phi \in \mathscr{C}^{k,\lambda}(V')$ on $V' = \{|y_j| < \alpha \text{ for all } j = 1, \ldots, n-1\} \subset \mathbb{R}^{n-1}$ such that $|\phi(y')| \leq \alpha/2$ for all $y' \in V'$ and*

$$D \cap V = \{(y', y_n) \in V : y_n < \phi(y')\},$$
$$\partial D \cap V = \{(y', y_n) \in V : y_n = \phi(y')\}.$$

The above definition, also called the $N^{k,\lambda}$ *property* by Wloka [31, §§2.2], requires that the boundary ∂D can be assembled from patches of $\mathscr{C}^{k,\lambda}$ function graphs that are repositioned using rigid body motions. The next definition

[3]The exact shape of V in this definition is not important. An equivalent definition can be given using e.g. a cylinder or ball. [14, §6.2][31, §2.3]

(called (k, λ)-*smoothness* in [31, §§2.3]) formulates the notion of regularity in manifold terminology.

Definition 3.2 (Submanifold with boundary)**.** *Let D be an open subset of $\mathbb{R}^n$. The closure $\overline{D}$ is an* n-dimensional $\mathscr{C}^{k,\lambda}$ submanifold with boundary in $\mathbb{R}^n$ *if for each $x \in \partial D$ there exists a (k, λ)-diffeomorphism ϕ (see Section 2.2) from some neighbourhood $O \ni x$ to $Q = \phi(O) \subset \mathbb{R}^n$, such that $\phi(D \cap O) = Q^- := \{(y', y_n) \in Q : y_n < 0\}$ and $\phi(\partial D \cap O) = Q_0 := \{(y', y_n) \in Q : y_n = 0\}$. [15, §1.2.1][31, §2.3]*

For bounded domains and $k \geq 1$, the conditions of Definitions 3.1 and 3.2 are equivalent. However, in the case of *Lipschitz boundaries* (i.e. $k = 0, \lambda = 1$), Definition 3.1 induces strictly stronger conditions and is therefore necessary for defining Lipschitz domains. [15, §1.2.1]

Definition 3.3 (Regular domains)**.** *In the sequel, a domain (an open and connected set) $D \subset \mathbb{R}^n$ is called a $\mathscr{C}^{k,\lambda}$* regular domain *if it is non-empty, bounded and has a boundary of class $\mathscr{C}^{k,\lambda}$. Domains satisfying $\mathscr{C}^{0,1}$ regularity are also referred to as* Lipschitz regular*.*

Remark 3.1. Each $\mathscr{C}^\infty$ regular domain is an instance of what Lions & Magenes call a domain D, whose boundary is a *connected infinitely differentiable variety of dimension $n-1$, with D locally on one side of ∂D* [27, Ch. 1, §7.3]. For details, see [10, Ch. IV, Appendix] and [31, §2.5].

Examples of $\mathscr{C}^\infty$ regular domains include the unit sphere (in $\mathbb{R}^n$) and the images of $\mathscr{C}^\infty$ regular domains under any smooth diffeomorphic deformation of $\mathbb{R}^n$. Bounded piecewise $\mathscr{C}^1$ domains, such as connected non-empty intersections of $\mathscr{C}^1$ regular domains, with no cusps, are generally Lipschitz regular domains.

The compact boundary of a $\mathscr{C}^{k,\lambda}$ regular domain D can be covered with a finite system of charts $\{(O_j, \phi_j)\}_{j=1}^N$, which can be selected so that there exists a *partition of unity* $\{\alpha_j : \partial D \to [0,1]\}_{j=1}^N$, such that $\alpha_j \circ \phi_j^{-1}(\cdot, 0) \in \mathscr{C}_0^{k,\lambda}(Q_0)$ (that is, $\alpha_j \in \mathscr{C}^{k,\lambda}(\partial D)$ [14, §6.2]). Set

$$\theta_j(u)(y') = \begin{cases} (\alpha_j u)(\phi_j^{-1}(y', 0)) & \text{if } (y', 0) \in Q_0, \\ 0 & \text{otherwise .} \end{cases} \tag{3.1}$$

For a $\mathscr{C}^\infty$ regular domain, each θ_j is a continuous linear mapping $\mathscr{D}(\partial D) \to \mathscr{D}(\mathbb{R}^{n-1})$ and can be extended to a (continuous) mapping $\mathscr{D}'(\partial D) \to \mathscr{D}'(\mathbb{R}^{n-1})$. [10, Ch. IV, Appendix][27, Ch. 1, §7.3]

For all Lipschitz domains, there also exists a *canonical surface measure* (corresponding to an "area element" $\mathrm{d}s$) [10, Ch. IV, Appendix][15][31, §4.2]. This readily enables the construction of the spaces $L^p(\partial D)$. In Lipschitz regular

domains, it is also possible to define an *exterior unit normal vector field* $\nu : \partial D \to B_1(0) \subset \mathbb{R}^n$ almost everywhere w.r.t the surface measure on ∂D [15, §1.5.1], in other words, ν is in $L^\infty(\partial D)$. Furthermore, for class $\mathscr{C}^{k,\lambda}$ boundaries, ν is of class $\mathscr{C}^{k-1,\lambda}$ [15, §1.5.1][31, §2.4].

3.2 Sobolev spaces in domains

Definition 3.4 (Classical Sobolev spaces)**.** *Let D be an open subset of $\mathbb{R}^n$. The classical* Sobolev spaces *of order $m = 0, 1, 2, \ldots$ are defined as*

$$W_p^m(D) = \{v \in L^p(D) : D^\alpha v \in L^p(D) \text{ for all } |\alpha| \le m\},$$

where $D^\alpha v \in L^p(D)$ means that the weak derivative of v exists in $L^p(D)$.

It can be shown that $W_2^m(D)$ is a Hilbert space with the inner product

$$(f, g)_{W_2^m(D)} = \int_D \sum_{|\alpha| \le m} D^\alpha f \overline{D^\alpha g} \, \mathrm{d}x \tag{3.2}$$

[10, Ch. IV, §1]. Since $C_N(\sum_{j=1}^N |a_i|)^2 \le \sum_{j=1}^N |a_i|^2 \le (\sum_{j=1}^N |a_i|)^2$, the corresponding norm $\|v\|_{W_2^m(D)} := ((v, v)_{W_2^m(D)})^{1/2}$ is equivalent to

$$\|v\|'_{W_2^m(D)} = \sum_{|\alpha| \le m} \|D^\alpha v\|_{L^2(D)}, \tag{3.3}$$

which is often used for convenience. In particular, notice that

$$W_2^1(D) = \{u \in L^2(D) : \nabla u \in (L^2(D))^n\}$$

and it can be equipped with the (equivalent) norm

$$\|u\|_{W_1^2(D),\nabla} := \|u\|_{L^2(D)} + \||\nabla u|\|_{L^2(D)} = \|u\|_{L^2(D)} + \Big(\sum_{i=1}^n \|\tfrac{\partial u}{\partial x_i}\|^2_{L^2(D)} \Big)^{1/2}. \tag{3.4}$$

Definition 3.5 (Sobolev spaces on $\mathbb{R}^n$)**.** *In $\mathbb{R}^n$, the L^2-type Sobolev spaces can be defined as*

$$H^s(\mathbb{R}^n) = \{u \in \mathscr{S}' : (1 + |\xi|^2)^{s/2}(\mathscr{F}u) \in L^2(\mathbb{R}^n)\}$$

for any $s \in \mathbb{R}$.

It follows from Parseval's equality (2.19) and some elementary manipulations, that these spaces, equipped with the norm

$$\|u\|'_{H^s(\mathbb{R}^n)} = \|(1 + |\xi|^2)^{s/2} \mathscr{F}u\|_{L^2(\mathbb{R}^n)},$$

are isomorphic to $W_2^m(\mathbb{R}^n)$ for any (integral) $m \ge 0$ [31, §5.1].

Definition 3.6 (Fractional-order $H^s(D)$)**.** *Let D be an open subset of $\mathbb{R}^n$ and $s \geq 0$. The fractional-order H^s spaces on D can be defined as*

$$H^s(D) = \{f|_D : f \in H^s(\mathbb{R}^n)\},$$

where $H^s(\mathbb{R}^n)$ is as in Definition 3.5.

For any $s \geq 0$, the space $H^s(D)$ of Definition 3.6 is a Banach space with the norm

$$\|f\|'_{H^s(D)} = \inf_{g \in H^s(\mathbb{R}^n), g|_D = f} \|g\|'_{H^s(\mathbb{R}^n)}$$

[27, Ch. 1 §9.1][31, §5.1] and (due to the Calderón–Zygmund extension theorem) these spaces are isomorphic to the spaces $W_2^m(D)$ of Definition 3.4 if $s = m = 0, 1, 2, \ldots$ and D is a Lipschitz regular domain [31, §§2.2, 2.3 & 5.2].

Remark 3.2. There are many different fractional-order Sobolev space constructions. Lions & Magenes use certain *interpolation spaces* for most of the theorems in [27]. These Hilbert spaces always coincide with $W_2^m(D)$ for $m = 0, 1, 2, \ldots$ by definition and are are isomorphic to $H^s(D), s \geq 0$ in $\mathscr{C}^\infty$ regular domains D [27, Ch. 1, §§2 & 9]. It is also possible to define the Sobolev–Slobodeckiĭ spaces W_p^s (cf. Definitions 3.4 and 3.10 below) in domains $D \subset \mathbb{R}^n$ for all $s \geq 0$ (and $p > 1$ [15]) so that $W_2^s(\mathbb{R}^n)$ coincides with $H^s(\mathbb{R}^n)$ [31]. According to Lions & Magenes "all reasonable definitions coincide" [27, Ch. 1, §17] in $\mathscr{C}^\infty$ regular domains.

In this work, no Sobolev spaces are studied in non-Lipschitzian (or unbounded) domains and fractional-order $H^s(D)$ spaces are considered only in $\mathscr{C}^\infty$ regular domains. Hence there is no dangerous ambiguity, since all definitions of $H^s(D)$ coincide up to equivalence of norms.

Definition 3.7 ($H_0^s(D)$ and $H^{-s}(D)$)**.** *Let $s > 0$. An important closed subspace of $H^s(D)$ is*

$$H_0^s(D) = \overline{\mathscr{D}(D)}^{H^s(D)},$$

the closure of compactly supported smooth functions on D in the norm of $H^s(D)$. Using this space, one may define the negative order Sobolev spaces

$$H^{-s}(D) = (H_0^s(D))' \subset \mathscr{D}'(D)$$

as the (topological) duals of $H_0^s(D)$ (equipped with the operator norm) [27, Ch. 1, §12.1]. In addition, identify $H^0(D) = H^{-0}(D) = (H^0(D))' = L^2(D)$.

Each element of $(H_0^s(D))'$ is determined by its values on the dense subset $\mathscr{D}(D)$ of $H_0^s(D)$, that is, by a distribution. Therefore, it makes sense to write $H^{-s}(D) \subset \mathscr{D}'(D)$ while in reality each element of $(H_0^s(D))'$ is the continuous

extension of a distribution to this space (cf. [31, §17.2]). As in Section 2.4, each element of $u \in L^2(D) \supset H_0^s(D)$ is identified with the distribution $T_u \in \mathscr{D}'(D)$. This construction allows one to write

$$\mathscr{D}(D) \subset H_0^s(D) \subset L^2(D) \subset H^{-s}(D) \subset \mathscr{D}'(D). \tag{3.5}$$

The notation of Definition 3.7 is consistent with Definition 3.5 for negative s, since $\mathscr{D}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$ for all $s \in \mathbb{R}$ [27, Ch. 1, §§7.1 & 12.1] and $H^{-s}(\mathbb{R}^n) = (H^s(\mathbb{R}^n))'$ [27, Ch. 1, §1.2].

There is a slight obscurity. According to the Riesz representation theorem, the Hilbert space $H_0^s(D)$ is isomorphic to and could be identified with its own anti-dual, and the space $H_0^s(D)$ could alternatively be identified with its dual. However, this identification, i.e. $u \equiv (u, \bar{\cdot})_{H_0^1(D)}$, is different from the identification $u \equiv (u, \bar{\cdot})_{L^2(D)} = T_u$ induced by (3.5). Consequently, the elements of $H_0^s(D)$, $s > 0$ are never implicitly identified with the Riesz elements (or their conjugates) in $H^{-s}(D)$ in this context.

It is also common to identify $L^2(D)$ with its anti-dual space and consequently define $H^{-s}(D) = (H_0^s(D))^*$ (see e.g. [31, §§12.1 & 17.1][10, Ch. IV §5]). In this construction (the so-called *Gelfand triple* or *rigged Hilbert space*), the anti-dual evaluation $\langle \cdot, \bar{\cdot} \rangle_{H_0^s(D)}$ can be viewed as a continuous extension of the inner product $(\cdot, \cdot)_{L^2(D)}$ to $H^{-s}(D) \times H^s(D)$ [31, §17.1]. To avoid excessive and irrelevant complex conjugation, the opposite convention of identifying $L^2(D)$ with its dual (cf. [27, Ch. 1 §§1.1 & 12.1]) is adopted here. Using this construction, the dual evaluation is the continuous extension of the bilinear form $(\cdot, \bar{\cdot})_{L^2(D)}$ (which is not the inner product of $L^2(D)$).

Definition 3.8. *Let A and B be topological vector spaces. Then A* is contained in B by continuous injection*, which is denoted*

$$A \hookrightarrow B,$$

if there exists a continuous injective homomorphism (linear mapping) $\pi : A \to B$. This holds especially when A is a subspace of B and $\|x\|_B \leq C\|x\|_A$ for all $x \in A$.

Theorem 3.3 (Sobolev's embedding theorem)**.** *The order s of a Sobolev space is a measure of "regularity" or smoothness of its elements. Namely, if $s > \frac{n}{2} + m$, then for a $\mathscr{C}^\infty$ regular domain D (or respectively Lipschitz regular domain D, $s = 0, 1, 2, \ldots$)*

$$H^s(D) \hookrightarrow \mathscr{C}^m(\overline{D}) \tag{3.6}$$

[10, Ch. IV, §3][27, Ch. 1, §9.4][31, §6][4]*. In particular, if* $u \in H^s(D)$ *for all* $s = 0, 1, 2, \ldots$, *then* $u \in \mathscr{C}^\infty(\overline{D})$.

It also follows rather directly from the definitions that the inclusion

$$H^m(D) \hookrightarrow H^l(D)$$

is continuous for all $l \leq m \in \mathbb{R}$, which also remains true if D is replaced by $\mathbb{R}^n$ or ∂D (see below).

3.3 Boundary spaces and trace theorems

Definition 3.9 (H^s spaces on a smooth boundary)**.** *Let* D *be a* $\mathscr{C}^\infty$ *regular domain with a system of local maps* $\{(O_j, \phi_j)\}_{j=1}^N$ *. For any* $s \in \mathbb{R}$*, define*

$$H^s(\partial D) = \{u \in \mathscr{D}'(\partial D) : \theta_j(u) \in H^s(\mathbb{R}^{n-1}),\ j = 1, \ldots, N\},$$

where the maps $\{\theta_j\}_{j=1}^N$ *are as in equation* (3.1)*, for some partition of unity subordinate to the system of local maps.*

$H^s(\partial D)$ is a Hilbert space with the norm

$$\|u\|^2_{H^s(\partial D)} = \sum_{j=1}^N \|\theta_j(u)\|^2_{H^s(\mathbb{R}^{n-1})},$$

and the norms defined by different systems $\{O_j, \phi_j, \theta_j\}_{j=1}^N$ are equivalent. [27, Ch. 1, §7.3][10, Ch. IV, Appendix] It naturally holds that $H_0^s(\partial D) = H^s(\partial D)$ and $H^{-s}(\partial D) = (H^s(\partial D))'$ [27, Ch. 1, §7.3].

In terms of the above boundary spaces, one can formulate

Theorem 3.4 (Trace theorem for $H^s(D)$)**.** *Let* D *be a* $\mathscr{C}^\infty$ *regular domain and* $s > \frac{1}{2}$*. For all* $j = 0, \ldots, \mu = \lceil s - \frac{3}{2} \rceil$ *(i.e.* μ *is the greatest integer strictly less than* $s - \frac{1}{2}$*) there exists a continuous, linear* trace map

$$\gamma_j : H^s(D) \to H^{s-j-1/2}(\partial D) \tag{3.7}$$

such that $\gamma_j \phi = \frac{\partial^j \phi}{\partial \nu^j}$ *for all* $\phi \in \mathscr{D}(\mathbb{R}^n)$*. In addition,* $\mathrm{Ker}(\gamma_0, \ldots, \gamma_\mu) = H_0^s(D)$*, or in other words,*

$$H_0^s(D) = \{v \in H^s(D) : \gamma_j v = 0,\ j = 0, \ldots, \mu\}.$$

[4]Strictly speaking, this means that for $s > \frac{n}{2} + m$ there exists at least one m-smooth representative of the equivalence class $u \in H^s(D)$ — in addition to the uncountably many discontinuous representatives. The important point is that there exists a *continuous* function $\phi : H^s(D) \to \mathscr{C}^m(\overline{D})$ (where $\|u\|_{\mathscr{C}^m(\overline{D})} := \sum_{|\alpha| \leq m} \max_{x \in \overline{D}} |D^\alpha u(x)|$) that finds the m-smooth representative.

There also exists a continuous lifting *(or* right inverse*)*

$$\prod_{j=0}^{\mu} H^{s-j-1/2}(\partial D) \to H^s(D).$$

For a proof and details, see [27, Ch. 1, §§9.1 & 11.4]. The trace maps (3.7) are continuous extensions of the operators that, when applied to functions in $\mathscr{D}(\overline{D})$, yield the (interior) normal derivatives on ∂D. These extensions are well-defined and unique, because $\mathscr{D}(\overline{D})$ is dense in $H^s(D)$.

For any $\phi \in \mathscr{D}(\overline{D})$, $j \in \mathbb{N}_0$ and $s > 0$, it holds that

$$\|\gamma_j \phi\|_{H^s(\partial D)} \leq C_{D,j,s} \sum_{|\alpha| \leq j + \lceil s \rceil} \sup_{x \in \partial D} |D^\alpha \phi(x)| \tag{3.8}$$

for some $C_{D,j,s} > 0$.

Similar trace theorems also exist for less regular domains, but these are formulated (see [31] and [15]) for another classes of Sobolev spaces, one of which can be defined as follows:

Definition 3.10 (Sobolev–Slobodeckiĭ boundary spaces $W^{\pm 1/2}(\partial D)$)**.** *Let D be a Lipschitz regular domain. Define*

$$W^{1/2}(\partial D) = \{u \in L^2(\partial D) : \|u\|_{W^{1/2}(\partial D)} < \infty\}$$

where (cf. [31, §4.2])

$$\|u\|^2_{W^{1/2}(\partial D)} = \|u\|^2_{L^2(\partial D)} + \iint_{\partial D \times \partial D} \frac{|u(x) - u(y)|^2}{|x - y|^n} \, \mathrm{d}s_x \, \mathrm{d}s_y.$$

In addition, let

$$W^{-1/2}(\partial D) = (W^{1/2}(\partial D))'$$

be the dual of $W^{1/2}(\partial D)$, equipped with the operator norm.

Remark 3.5. It is common to denote the L^2 nature of these spaces with a sub- or superscript, e.g. $W^{1/2}(\partial D) = W_2^{1/2}(\partial D)$, since, as in Definition 3.4, it is possible to define $W_p^s(\partial D)$ spaces for other L^p exponents (and regularities $s \geq 0$) [15]. An alternative convention is to replace W_2 by H, but this notation is not adopted here to note the (possible) difference to the spaces of Definition 3.9.

Theorem 3.6 (Trace theorem for $H^1(D)$)**.** *Let D be a Lipschitz regular domain. Then, as in Theorem 3.4, there exists a unique, continuous and linear (extension) trace map*

$$\gamma_0 : H^1(D) \to W^{1/2}(\partial D)$$

with a continuous right inverse. [31, §8]

Since $\frac{\partial\phi}{\partial\nu} = \nu \cdot \nabla\phi|_{\partial D}$ for any $\phi \in \mathscr{D}(\mathbb{R}^n)$ and the exterior unit normal ν is in $L^\infty(\partial D)$ for all Lipschitz regular domains, it follows that the (extension) Neumann trace map

$$\gamma_1 : H^2(D) \to L^2(\partial D), \quad \gamma_1 u = \nu \cdot \gamma_0(\nabla u)$$

is well-defined and continuous.

3.4 Zero mean and definition up to constant

In the study of PDEs with Neumann boundary conditions, it is often natural to work with functions that are defined up to an additive constant factor or have a vanishing mean. These can be thought to correspond to the physical notions of an unknown potential ground level and conservation of current, respectively. Mathematically, these can be described as follows:

Definition 3.11 (Definition up to an additive constant)**.** *Let H be a subspace of $L^2(D,\mu)$ (with $\mu(D) < \infty$) that contains the constant function* 1*. Define*

$$H/\mathbb{C} := H/\mathrm{span}_{\mathbb{C}}\{1\} = \{\{u + c : c \in \mathbb{C}\} : u \in H\}. \tag{3.9}$$

Definition 3.12 (Vanishing mean)**.** *Let H be as in Definition 3.11. Functions with zero mean in H are denoted*

$$H_\diamond = \left\{ u \in H : \int_D u \,\mathrm{d}\mu = 0 \right\}.$$

Let H' be the dual (or anti-dual) of H (in some topology) and define

$$H'_\diamond = \{v \in H' : \langle v, 1\rangle_H = 0\}.$$

Examples of well-defined spaces include $\mathscr{D}(D)/\mathbb{C}$, $H^s(D)/\mathbb{C}$ and $H^l_\diamond(\partial D)$ for any $\mathscr{C}^\infty$ regular domain D and $s > 0$, $l \in \mathbb{R}$, but e.g. "$H^s(\mathbb{R}^n)/\mathbb{C}$" is not well-defined, because $|\mathbb{R}^n| = \infty$ and consequently $1 \notin H^s(\mathbb{R}^n)$.

If H is a Banach space, then, because $\mathrm{span}_{\mathbb{C}}\{1\}$ is a closed subspace of H, the quotient space $H/\mathbb{C}$ can be given the norm (see [32, Ch. I §11])

$$\|u\|_{H/\mathbb{C}} = \inf_{c\in\mathbb{C}} \|u + c\|_H. \tag{3.10}$$

For any Banach space, $H_\diamond$ is also a closed subspace of H and thus a Banach space sharing the norm of H.

Lemma 3.7. *Let X be a Banach space and M a closed subspace of X. Then there holds the isometric isomorphism*

$$(X/M)' \cong M^\perp := \{f \in X' : f(m) = 0 \text{ for all } m \in M\},$$

where X' is the topological dual (or respectively anti-dual) of X. [31, §12.2]

Proof. It is known that, X/M is a Banach space with the norm

$$\|u\|_{X/M} = \inf_{m\in M} \|x+m\|_X$$

[32, Ch. I §11]. Denote by $(X/M)^A$ and X^A the algebraic duals (resp. anti-duals), i.e., the sets of all possibly unbounded linear (resp. anti-linear) functionals, of X/M and X, respectively. Then define $\pi : (X/M)^A \to X^A$ by $(\pi(f))(x) = f(x+M)$. It readily follows that π is well-defined and linear with a well-defined inverse $(\pi^{-1}(g))(x) = g(x+M) = g(x)$ for any $g \in M^{\perp,A} = \{g \in X^A : g(m) = 0 \ \forall m \in M\}$. This means that π is an isomorphism between the vector spaces $(X/M)^A$ and $M^{\perp,A}$. Now, for any functional $f \in (X/M)^A$, the operator norm satisfies

$$\begin{aligned}\|f\|_{(X/M)'} &= \sup_{u\in X/M, u\neq 0} \frac{|f(u)|}{\|u\|_{X/M}} = \sup_{u\in X/M, u\neq 0} \frac{|f(u)|}{\inf_{m\in M}\|u+m\|_X} \\ &= \sup_{u\in X/M, u\neq 0} \sup_{m\in M} \frac{|f(u+m)|}{\|u+m\|_X} = \sup_{u+m\in X, u+m\neq 0} \frac{|(\pi(f)(u+m)|}{\|u+m\|_X} \\ &= \|\pi(f)\|_{X'},\end{aligned}$$

which means that $\pi|_{(X/M)'}$ is an isometric isomorphism between $(X/M)' = \{f \in (X/M)^A : \|f\|_{(X/M)'} < \infty\}$ and $M^\perp = \{f \in M^{\perp,A} : \|f\|_{X'} < \infty\}$. □

Corollary 3.8. *Let H be as in Definition 3.11 and a Banach space. The topological dual (resp. anti-dual) of $H/\mathbb{C}$ is isometrically isomorphic to $H'_\diamond$, where H' is the dual (resp. anti-dual) of H.*

Lemma 3.9. *Let $m \in \mathbb{N}_0$ and D be a bounded domain. Then $H^m(D)/\mathbb{C}$ is a Hilbert space that is isometrically isomorphic to $H^m_\diamond(D)$.*

Proof. Define $\pi : H^m(D) \to H^m(D)$ by

$$\pi(u) = u - \fint_D u \,\mathrm{d}x.$$

Notice that for any $C \in \mathbb{C}$,

$$\int_D \pi(u+C)\,\mathrm{d}x = 0$$

and thus π is in fact well-defined from $H^m(D)/\mathbb{C}$ to $H^m_\diamond(D)$. It is also linear and has a well-defined (canonical) inverse $\pi^{-1}(u) = u + \mathbb{C}$. It remains to show that π is norm-preserving:

$$\begin{aligned}\|u\|^2_{H^m(D)/\mathbb{C}} &= \|\pi(u)+\mathbb{C}\|^2_{H^m(D)/\mathbb{C}} = \inf_{C\in\mathbb{C}} \|\pi(u)+C\|^2_{H^m(D)} \\ &= \inf_{C\in\mathbb{C}} \{\|\pi(u)\|^2_{H^m(D)} + 2\mathrm{Re}\,(\pi(u),C)_{H^m(D)} + \|C\|^2_{H^m(D)}\} \\ &= \inf_{C\in\mathbb{C}} \{\|\pi(u)\|^2_{H^m(D)} + \|C\|^2_{H^m(D)}\} = \|\pi(u)\|^2_{H^m(D)}\end{aligned}$$

because $(\pi(u), C)_{H^m(D)} = \overline{C} \int_D \pi(u)\,\mathrm{d}x = 0$ by (3.2) and the fact that $D^\alpha C = 0$ for any multi-index $\alpha \neq 0$. Consequently $H^m(D)/\mathbb{C}$ is a Hilbert space, as its norm is identical to that of $H^m_\diamond(D)$. □

Lemma 3.10 (Trace theorem for $H(D)/\mathbb{C}$)**.** *Let $H(D)$, $H_1(\partial D)$ be such*[5] *that the (extension) trace map (as in Theorem 3.4) $\gamma_0 : H(D) \to H_1(\partial D)$ is well-defined and continuous. Then it can also be interpreted as a continuous operator*

$$\tilde{\gamma}_0 : H(D)/\mathbb{C} \to H_1(\partial D)/\mathbb{C}.$$

Furthermore, if $H_2(\partial D)$ is such that $\gamma_1 : H(D) \to H_2(\partial D)$ is well-defined and continuous, then so is

$$\tilde{\gamma}_1 : H(D)/\mathbb{C} \to H_2(\partial D).$$

Proof. By definition, γ_0 maps the constant function $c \in \mathbb{C}$ to a constant function on ∂D. For any $\tilde{u} \in H(D)/\mathbb{C}$ and $u \in \tilde{u} \subset H(D)$,

$$\begin{aligned}\|\gamma_0 u\|_{H_1(\partial D)/\mathbb{C}} &= \inf_{c\in\mathbb{C}} \|\gamma_0 u + c\|_{H_1(\partial D)} = \inf_{c\in\mathbb{C}} \|\gamma_0(u+c)\|_{H_1(\partial D)} \\ &\leq \inf_{c\in\mathbb{C}} \|\gamma_0\|\|u+c\|_{H(D)} = \|\gamma_0\|\|u\|_{H(D)/\mathbb{C}},\end{aligned}$$

that is, the operator $\tilde{\gamma}_0$ is well-defined and has a smaller operator norm than γ_0. Similarly, the operator γ_1 maps any constant function to the zero function and $\|\gamma_1 u\|_{H_2(\partial D)} \leq \|\gamma_1\| \inf_{c\in\mathbb{C}} \|u+c\|_{H(D)}$, which proves the second claim. □

[5]For example, $H(D) = H^1(D)$ and $H_1(\partial D) = W^{1/2}(\partial D)$; or $H(D) = H^s(D)$ and $H_1(\partial D) = H^{s-1/2}(\partial D)$, where $s > 1/2$.

4 Elliptic PDEs

This section presents some key concepts in elliptic PDE theory. The focus is on Laplace's equation and providing the minimum tools for formulating the necessary forward problem theory for the conductivity equation. For an accessible introduction to elliptic PDE theory, see e.g. Dautray & Lions [10].

4.1 Green's identities

Theorem 4.1 (Green's theorem)**.** *Let D be a Lipschitz regular domain and $u \in H^1(D)$. Then*

$$\int_D \frac{\partial u}{\partial x_i} \,\mathrm{d}x = \int_{\partial D} u\nu_i \,\mathrm{d}s, \tag{4.1}$$

where ν_i is the i:th component of the exterior unit normal vector field. The boundary integral is taken over the trace $\gamma_0 u \in W^{1/2}(\partial D) \subset L^2(\partial D)$ (recall Theorem 3.6).

Equation (4.1) is a certain generalization of the fundamental theorem of calculus to higher dimensions. This general form of the theorem (in $\mathbb{R}^n$) can be recovered from [15, Theorem 1.5.3.1].

Straightforward application of (4.1) to suitable functions yields

Corollary 4.2. *For any $w \in (H^1(D))^n$ (resp. $(\mathscr{C}^1(D) \cap \mathscr{C}(\overline{D}))^n$) $u \in H^1(D)$ (resp. $\mathscr{C}^1(D) \cap \mathscr{C}(\overline{D})$) and $v, h \in H^2(D)$ (resp. $\mathscr{C}^2(D) \cap \mathscr{C}^1(\overline{D})$) the following identities hold:*

$$\int_D \nabla \cdot w \,\mathrm{d}x = \int_{\partial D} w \cdot \nu \,\mathrm{d}s, \tag{4.2a}$$

$$\int_D \Delta v \,\mathrm{d}x = \int_{\partial D} \frac{\partial v}{\partial \nu} \,\mathrm{d}s, \tag{4.2b}$$

$$\int_D \nabla u \cdot \nabla v \,\mathrm{d}x = \int_{\partial D} u \frac{\partial v}{\partial \nu} \,\mathrm{d}s - \int_D u \Delta v \,\mathrm{d}x, \tag{4.2c}$$

$$\int_D (v\Delta h - h\Delta v) \,\mathrm{d}x = \int_{\partial D} \left(v \frac{\partial h}{\partial \nu} - h \frac{\partial v}{\partial \nu} \right) \mathrm{d}s, \tag{4.2d}$$

$$\int_D \nabla u \,\mathrm{d}x = \int_{\partial D} u\nu \,\mathrm{d}s. \tag{4.2e}$$

Equation (4.2a) is regularly called the divergence theorem. Equations (4.2c) and (4.2d) are known as Green's first and second identity, respectively. Equation (4.1) is occasionally referred to as *Stokes's theorem*.

Definition 4.1. *Define*

$$H^1(D,\Delta) = \{u \in H^1(D) : \Delta u \in L^2(D)\},$$

which can be equipped with the graph norm

$$\|u\|^2_{H^1(D,\Delta)} = \|u\|^2_{H^1(D)} + \|\Delta u\|^2_{L^2(D)}.$$

Clearly the above is a subspace of $H^1(D)$. In addition, the following holds:

Lemma 4.3. *The smooth functions $\mathscr{D}(\overline{D}) \cap H^1(D,\Delta)$ are dense in $H^1(D,\Delta)$. [15, Lemma 1.5.3.9]*

Lemma 4.4 (A generalized Green's identity)**.** *Let D be a Lipschitz regular domain. Then there exists a continuous extension $\gamma_1 : H^1(D,\Delta) \to W^{-1/2}(\partial D)$ (resp. $\gamma_1 : H^1(D,\Delta)/\mathbb{C} \to W^{-1/2}(\partial D)$) to the classical Neumann trace operator. In addition, the generalized Green's identity*[6]

$$\langle \gamma_1 u, \gamma_0 v\rangle_{\partial D} = \int_D v\Delta u \,\mathrm{d}x + \int_D \nabla u \cdot \nabla v \,\mathrm{d}x \tag{4.3}$$

holds for all $u \in H^1(D,\Delta)$ (resp. $u \in H^1(D,\Delta)/\mathbb{C}$) and $v \in H^1(D)$.

Proof. (cf. [10, Ch. VII §2]) Let $u \in H^1(D,\Delta)$ and *define* $\gamma_1 u$ by

$$(\gamma_1 u)(w) = \int_D (\gamma_0^{-1} w)\Delta u \,\mathrm{d}x + \int_D \nabla u \cdot \nabla(\gamma_0^{-1} w)\,\mathrm{d}x,$$

where $w \in W^{1/2}(\partial D)$ is arbitrary and γ_0^{-1} is the (linear) continuous lifting of $W^{1/2}(\partial D)$ to $H^1(D)$ (see Theorem 3.6). Clearly, $\gamma_1 u$ is linear and, due to the norm equivalence (3.4),

$$\begin{aligned}
|(\gamma_1 u)(w)| &\le \|\gamma_0^{-1} w\|_{L^2(D)} \|\Delta u\|_{L^2(D)} + \||\nabla(\gamma_0^{-1} w)|\|_{L^2(D)} \||\nabla u|\|_{L^2(D)} \\
&\le C_1 \|\gamma_0^{-1}\| \|w\|_{W^{1/2}(\partial D)} (\||\nabla u|\|_{L^2(D)} + \|\Delta u\|_{L^2(D)}) \\
&\le C_2 \|\gamma_0^{-1}\| \|u\|_{H^1(D,\Delta)} \|w\|_{W^{1/2}(\partial D)}
\end{aligned}$$

for some $C_1, C_2 > 0$ depending only on n. Therefore $\gamma_1 u \in W^{-1/2}(\partial D)$ with $\|\gamma_1 u\|_{W^{-1/2}(\partial D)} \le C_2 \|\gamma_0^{-1}\| \|u\|_{H^1(D,\Delta)}$ and consequently $\|\gamma_1\| \le C_2 \|\gamma_0^{-1}\|$.

By Lemma 4.3, for any $u \in H^1(D,\Delta)$, $v \in H^1(D)$ and $\varepsilon > 0$, there exists $u_\varepsilon \in H^2(D)$ such that

$$\begin{aligned}
&\left| \langle \gamma_1 u, \gamma_0 v\rangle - \int_D v\Delta u \,\mathrm{d}x - \int_D \nabla u \cdot \nabla v \,\mathrm{d}x \right| \\
&\le C_3 (1 + \|\gamma_0\| \|\gamma_1\|) \|v\|_{H^1(D)} \|u - u_\varepsilon\|_{H^1(D,\Delta)} \le \varepsilon
\end{aligned}$$

[6] The notation $\langle \cdot, \cdot \rangle_{\partial D}$ stands for $\langle \cdot, \cdot \rangle_{W^{1/2}(\partial D)}$ (or any other dual evaluation between spaces on ∂D). Similar shorthands are used throughout the remainder of this thesis.

because, by Green's first identity (4.2c), u_ε and v satisfy

$$\int_D v\Delta u_\varepsilon \,\mathrm{d}x + \int_D \nabla v \cdot \nabla u_\varepsilon \,\mathrm{d}x = \int_{\partial D} \frac{\partial u_\varepsilon}{\partial \nu} v \,\mathrm{d}x = \int_{\partial D} \frac{\partial u_\varepsilon}{\partial \nu} \gamma_0^{-1} v \,\mathrm{d}x$$
$$= \int_D \gamma_0^{-1} v \Delta u_\varepsilon \,\mathrm{d}x + \int_D \nabla(\gamma_0^{-1} v) \cdot \nabla u_\varepsilon \,\mathrm{d}x = \langle \gamma_1 u_\varepsilon, \gamma_0 v \rangle_{\partial D}.$$

Consequently, the generalized Green's identity (4.3) holds, and γ_1 is the continuous extension of the (Sobolev space) Neumann trace operator. The respective identities for $H^1(D,\Delta)/\mathbb{C}$ hold by Lemma 3.10 and the fact that the differential operators ∇ and Δ eliminate the additive constant. □

4.2 Laplace's equation

The partial differential equation

$$\Delta u = 0 \quad \text{in } D, \tag{4.4a}$$
$$\frac{\partial u}{\partial \nu} = f \quad \text{on } \partial D, \tag{4.4b}$$

where $D \subset \mathbb{R}^n$ is an open set, is called *Laplace's equation* with a *Neumann boundary condition* or the *Neumann problem of the Laplacian (in D)*. A twice-differentiable function $u \in \mathscr{E}^2(D)$ that satisfies (4.4a) is called *harmonic* in D. If (4.4a) holds (only) in the distributional sense, i.e., $\Delta u = 0 \in \mathscr{D}'(D)$, then u is called *weakly harmonic* in D. In fact, these notions are very close to each other since any weakly harmonic function (in $H^1(D)$) coincides almost everywhere with a function in $\mathscr{E}^\infty(D)$ [13, §11.7].

The boundary condition is understood in the sense of traces. If D is a $\mathscr{C}^\infty$ regular domain, then (4.4b) means $\gamma_1 u = f$, where γ_1 is the Neumann trace operator of Theorem 3.4. If D is a Lipschitz regular domain, then (4.4) can be studied for $f \in L^2(\partial D)$ or $f \in W^{-1/2}(\partial D)$, but (as with $f \in H^s(\partial D)$ with low enough s) more care needs to be taken to define the appropriate trace operator.

By equation (4.2b) a necessary condition for the existence of a solution $u \in \mathscr{C}^2(D) \cap \mathscr{C}^1(\overline{D})$ to (4.4) is

$$\int_{\partial D} f \,\mathrm{d}s = \int_{\partial D} \frac{\partial u}{\partial \nu} \,\mathrm{d}s = \int_D \Delta u \,\mathrm{d}x = 0. \tag{4.5}$$

Another characteristic of the problem (4.4) is that adding a constant function to any solution u also yields a solution, because the differential operators $\partial/\partial\nu$ and Δ do not "see" the constant. In other words, the constant functions are in

the null-space of these operators. This suggests searching for the solution of (4.4) in the spaces (3.9) of functions defined up to an additive constant.

The elliptic PDE theory for solving (4.4) is usually formulated for more general equations with a non-zero source term, e.g. $\Delta u = -\rho$ (Poisson's equation). In this work, the source term is not required and some unnecessary complications can be avoided by using the sourceless equation. Especially in the most generic distributional solutions spaces $H^{-s}(D)$, the proper choice of the space for ρ causes some trouble (cf. [27, Ch. 2, §6]). In addition, the sourceless forms of (4.4) and, later, of the conductivity equation allow more convenient formulation of the solvability conditions (e.g. (4.5)) in terms of Definitions 3.11 and 3.12 as demonstrated by

Theorem 4.5. *Let D be a $\mathscr{C}^\infty$ regular domain and $f \in H_\diamond^{s-3/2}(\partial D)$ for an arbitrary $s \in \mathbb{R}$. The Neumann problem* (4.4) *(understood in the distributional sense) has a unique solution $u \in H^s(D)/\mathbb{C}$ that depends linearly and continuously on f:*

$$\|u\|_{H^s(D)/\mathbb{C}} \leq C_{D,s}\|f\|_{H^{s-3/2}(\partial D)} \tag{4.6}$$

for some $C_{D,s} > 0$. In addition, there exists a linear and bounded trace operator γ_0 (that maps $\mathscr{C}(\overline{D})$ functions to their restrictions on ∂D) such that

$$\|\gamma_0 u\|_{H^{s-1/2}(\partial D)/\mathbb{C}} \leq C'_{D,s}\|f\|_{H^{s-3/2}(\partial D)} \tag{4.7}$$

for some $C'_{D,s} > 0$.

For a proof, see Lions & Magenes [27, Ch. 2, §§1.4, 2.2 & 7.3]. The above theorem follows directly from [27, Ch. 2, Remark 7.2]. Again, $\partial/\partial\nu$ in (4.4b) stands for γ_1, the continuous extension of a trace map whose restriction to smooth functions corresponds to the normal derivative, as in Theorem 3.4. In general, such an extension does *not* exist as a map from $H^s(D)/\mathbb{C}$ to $H^{s-3/2}(\partial D)$ if $s \leq \frac{3}{2}$, but it is possible to construct one in a space similar to that in Definition 4.1, as in Lemma 4.4. This also concerns γ_0, when $s \leq \frac{1}{2}$. The details and theory behind the proof (i.e. the majority of the first 200 pages of the book [27]) are beyond the scope of this thesis.

For less regular domains, there holds

Theorem 4.6. *Let D be a Lipschitz regular domain and $f \in W_\diamond^{-1/2}(\partial D)$ (respectively $f \in L^2_\diamond(\partial D)$). The Neumann problem* (4.4) *(understood in the distributional sense) has a unique solution $u \in H^1(D)/\mathbb{C}$ that depends linearly and continuously on f:*

$$\|u\|_{H^s(D)/\mathbb{C}} \leq C\|f\|_{W^{-1/2}(\partial D)} \quad \Big(\|u\|_{H^1(D)/\mathbb{C}} \leq C\|f\|_{L^2(\partial D)}\Big) \tag{4.8}$$

for some $C > 0$.

The proof for $f \in W_\diamond^{-1/2}(\partial D)$ is given in Section 4.3 below. Subsequently, the theorem for the case $f \in L^2_\diamond(\partial D)$ follows from the fact that the functional $\gamma_1 u = T_f \in W^{-1/2}(\partial D)$ can be identified with the L^2 function f and particularly $\|T_f\|_{W^{-1/2}(\partial D)} = \|f\|_{L^2(\partial D)}$. Notice that the continuous Neumann trace map γ_1 exists by Lemma 4.4 since any $H^1(D)$ solution u of (4.4) is obviously in $H^1(D, \Delta)/\mathbb{C}$.

4.3 Variational formulation

In this section, the weak formulation of the Neumann problem (4.4) is used to prove Theorem 4.6, following [10, Ch. VII §2.2], where this is proven for a similar but different problem. Different weak (or variational) and minimization formulations of PDEs are important in applications as they are the basis of the *finite element methods* (FEM) [6].

A key concept in variational theory is

Theorem 4.7 (Lax–Milgram)**.** *Let H be a Hilbert space and $a : H \times H \to \mathbb{C}$ a sesquilinear (i.e. linear in the first, anti-linear in the second argument) form that is* continuous*:*

$$|a(u, v)| \le C\|a\|_H \|u\|_H \quad \text{for all } u, v \in H \tag{4.9}$$

and (strongly) coercive*:*

$$\operatorname{Re} a(v, v) \ge c\|v\|_H^2 \quad \text{for all } v \in H \tag{4.10}$$

for some $c, C > 0$. In addition, let $L \in H^*$ be a continuous anti-linear functional on H. Then there exists a unique $u \in H$ such that

$$a(u, v) = L(v) \quad \text{for all } v \in H. \tag{4.11}$$

[10, Ch. VI §3 Theorem 7]

An immediate consequence of the Lax–Milgram theorem, particularly the (in)equalities (4.10) and (4.11), is that the solution u satisfies

$$\|u\|_H \le \frac{1}{c}\|L\|_{H^*}. \tag{4.12}$$

Definition 4.2. *Let D be a Lipschitz regular domain and $f \in W_\diamond^{-1/2}(\partial D)$. Any solution $u \in H^1(D)/\mathbb{C}$ of* (4.4) *is clearly in $H^1(D, \Delta)/\mathbb{C}$ and applying* (4.3) *shows that*

$$\int_D \nabla u \cdot \overline{\nabla v}\, \mathrm{d}x = \langle f, \gamma_0 \overline{v} \rangle_{\partial D} \quad \text{for all } v \in H^1(D)/\mathbb{C}. \tag{4.13}$$

This equation is called the weak formulation *of the Neumann problem* (4.4)*.*

The left hand side of (4.13) defines a sesquilinear form and the right hand side an anti-linear functional. The functional is continuous because $f \in W^{-1/2}(D)$, and the sesquilinear form is bounded:

$$\left| \int_D \nabla u \cdot \overline{\nabla v} \, \mathrm{d}x \right| \leq \| |\nabla u| \|_{L^2(D)} \| |\nabla v| \|_{L^2(D)} \leq \|u\|_{H^1(D)/\mathbb{C}} \|v\|_{H^1(D)/\mathbb{C}}.$$

By Lemma 3.9, it is known that $H^1(D)/\mathbb{C}$ is a Hilbert space and isometrically isomorphic to $H^1_\diamond(D)$. The coercivity of the form can be proven using

Theorem 4.8 (Poincaré's inequality in $H^1_\diamond(D)$)**.** *Let D be a bounded domain. Then there exists a constant $P_D > 0$ such that for all $u \in H^1_\diamond(D)$*

$$\|u\|_{L^2(D)} \leq P_D \| |\nabla u| \|_{L^2(D)}. \tag{4.14}$$

[10, Ch. IV §7.2]

Now for any $v \in H^1_\diamond(D)$,

$$\int_D |\nabla v|^2 \, \mathrm{d}x = \tfrac{1}{1+P_D} \| |\nabla v| \|^2_{L^2(D)} + (1 - \tfrac{1}{1+P_D}) \| |\nabla v| \|^2_{L^2(D)} \geq \tfrac{1}{1+P_D} \|v\|^2_{H^1(D)}$$

and, since for any $v \in H^1(D)/\mathbb{C}$, $\nabla v = \nabla(\pi(v))$, where $\pi(v) \in H^1_\diamond(D)$ is as in Lemma 3.9, the coercivity result also holds in $H^1(D)/\mathbb{C}$.

It has now been proven that (4.13) has a unique solution, but it is not immediately clear that the solution satisfies (4.4). Applying (4.13) to any $v \in \mathscr{D}(D)$ yields

$$\langle \Delta u, v \rangle_D = -\langle \nabla u, \nabla v \rangle_D = -(\nabla u, \overline{\nabla v})_{L^2(D)} = -\langle f, \gamma_0 v \rangle_{\partial D} = 0,$$

that is, $\Delta u = 0$ in $\mathscr{D}'(D)$ and consequently $u \in H^1(D, \Delta)/\mathbb{C}$. Hence (4.3) can be used and for any $w \in W^{1/2}(\partial D)/\mathbb{C}$ it holds that

$$\langle \gamma_1 u, w \rangle_{\partial D} = \int_D \nabla u \cdot \nabla(\gamma_0^{-1} w) \, \mathrm{d}x = \langle f, \overline{\overline{w}} \rangle_{\partial D},$$

which means that $\gamma_1 u = f$ in $W^{-1/2}(\partial D)$.

To conclude the proof of Theorem 4.6, notice that the continuity condition (4.8) of the solution is shown by the inequality (4.12).

4.4 The Neumann–Green function

The *Neumann–Green function*, N_D (of the Laplacian) for a domain D is an integral kernel that can be used to express the solution of the Neumann problem (4.4) as an integral as in e.g. equation (4.20) below. A related concept is *Green's function*, which is the corresponding entity for the Dirichlet problem (see e.g. [1][10]). Both of these functions are closely linked to the *fundamental solution* of the Laplacian.

Definition 4.3. *(Fundamental solution) The fundamental solution* Γ *of the Laplacian in the free space* $\mathbb{R}^n$ *is the function that satisfies* $\Delta\Gamma = \delta_0 \in \mathscr{D}'(\mathbb{R}^n)$, $n \geq 2$. *It is given by [1, Lemma 2.11]*

$$\Gamma(x) = \begin{cases} \frac{1}{2\pi} \log|x| & \text{if } n = 2, \\ \frac{1}{(2-n)\omega_n} |x|^{2-n} & \text{if } n \geq 3. \end{cases} \tag{4.15}$$

Definition 4.4 (Neumann–Green function)**.** *The Neumann–Green function* N_D *of the Laplacian for a Lipschitz regular domain* $D \subset \mathbb{R}^n$, $n \geq 2$ *is a real-valued function of two arguments that satisfies*

$$\Delta N_D(\cdot, y) = -\delta_y \quad \text{in } \mathscr{D}'(D), \tag{4.16a}$$

$$\frac{\partial}{\partial \nu_x} N_D(x, y) = \frac{-1}{|\partial D|} \ \text{a.e.}, \tag{4.16b}$$

$$\int_{\partial D} N_D(\cdot, y) \, \mathrm{d}s = 0 \tag{4.16c}$$

for all $y \in D$ *and* $x \in \partial D$. *[1, §2.4]*

The Neumann–Green function is related to the fundamental solution via

$$N_D(x, y) = -\Gamma(x - y) + R_D(x, y), \tag{4.17}$$

where $R_D(x, y)$ is a function that is harmonic (and thus smooth) with respect to both x and y if neither x nor y is on ∂D [1, Lemma 2.27]. It follows that the Neumann–Green function exists for any Lipschitz regular domain D and is unique because the last condition of (4.16) fixes the additive constant.

Moreover, by [1, Lemma 2.27], N_D (and consequently also R_D) is symmetric in its arguments, $N_D(x, y) = N_D(y, x)$. For all $x, y \in D, x \neq y$, the mapping $N_D(\cdot, y)$ is harmonic (and thus smooth) in the set $D \setminus \{y\}$ and it satisfies

$$\begin{aligned} N_D(x, y) &= -\Gamma(|\varepsilon|) + o(\varepsilon^{1-n}), \\ \nabla_x N_D(x, y) &= o(\varepsilon^{1-n}) \end{aligned} \tag{4.18}$$

for $x \in \partial B_\varepsilon(y)$ as $\varepsilon \to 0^+$.

From conditions (4.16) and (4.18), it follows using basic properties of integrals and Green's identities that any solution $u \in H^1(D)$ (or respectively $u \in \mathscr{C}^2(D) \cap \mathscr{C}^1(\overline{D})$) of (4.4) for $f \in L^2_\diamond(\partial D)$ (resp. $f \in \mathscr{C}(\partial D)$) must satisfy

$$u(y) = \int_{\partial D} N_D(\cdot, y) f \, \mathrm{d}s + \fint_{\partial D} u \, \mathrm{d}s. \tag{4.19}$$

On the other hand, a function u defined by (4.19) for a continuous f satisfying (4.5) is smooth in D and solves (4.4). Consequently, for any $f \in L^2_\diamond(\partial D)$ (resp.

$f \in \mathscr{C}_\diamond(\partial D)$) the solution u to (4.4) is unique up to an additive constant and given by

$$u(y) = \int_{\partial D} N_D(\cdot, y) f \, \mathrm{d}s + C, \quad C \in \mathbb{C}. \tag{4.20}$$

Example 4.9. It is easy to verify, that the Neumann–Green function of the unit disk, $D = B_1(0) \subset \mathbb{R}^2$, is [1, §2.4]

$$N_{B_1(0)}(x, y) = \begin{cases} -\frac{1}{2\pi} \left(\log|y - x| + \log\left| \frac{y}{|y|} - |y|x \right| \right) & \text{if } y \neq 0, \\ -\frac{1}{2\pi} \log|x| & \text{if } y = 0. \end{cases} \tag{4.21}$$

The next theorem proves that less regular solutions of (4.4) can also be given in terms of the Neumann–Green function.

Theorem 4.10. *Let D be a $\mathscr{C}^\infty$ regular domain and $U \subset\subset D$ an open set. For any $s \in \mathbb{R}$ and $f \in H_\diamond^{s-3/2}(\partial D)$, the solution of the Neumann problem* (4.4) *restricted to U is given by*

$$u(y) = (Bf)(y) := \langle f, N_D(\cdot, y) \rangle + C, \tag{4.22}$$

where N_D is the Neumann–Green function of D and $\langle \cdot, \cdot \rangle$ denotes the duality between $\mathscr{D}(\partial D)$ and $\mathscr{D}'(\partial D)$. The linear mapping $B : H_\diamond^{s-3/2}(\partial D) \to \mathscr{C}^\infty(U)/\mathbb{C}$ is bounded in the norm of $\mathscr{C}^k(U)/\mathbb{C}$ (recall (2.9) *and* (3.10)*) for any $k \geq 0$.*

Proof. For any $y \in U$, $N_D(\cdot, y) \in \mathscr{D}(\partial D)$ and therefore $\langle f, N_D(\cdot, y) \rangle$ is well-defined for all $f \in \mathscr{D}'_\diamond(\partial D)$. Let $d = \operatorname{dist}(U, \partial D)/2 := \inf_{y \in U, x \in \partial D} |x - y|/2$ and denote $U_d = \{x \in D : \operatorname{dist}(x, U) < d\}$. For any $h \in (-d, d)$ and coordinate unit vector e_i, $i = 1, \ldots, n$,

$$\begin{aligned} &\left| \tfrac{\langle f, N_D(\cdot, y + he_i) \rangle - \langle f, N_D(\cdot, y) \rangle}{h} - \langle f, \tfrac{\partial N_D}{\partial y_i}(\cdot, y) \rangle \right| \\ &\quad = \left| \left\langle f, \tfrac{N_D(\cdot, y + he_i) - N_D(\cdot, y)}{h} - \tfrac{\partial N_D(\cdot, y)}{\partial y_i} \right\rangle \right| \\ &\quad \leq \|f\|_{H^{s-3/2}(\partial D)} \left\| \tfrac{N_D(\cdot, y + he_i) - N_D(\cdot, y)}{h} - \tfrac{\partial N_D(\cdot, y)}{\partial y_i} \right\|_{H^{3/2-s}(\partial D)/\mathbb{C}} \\ &\quad \leq C|h| \|f\|_{H^{s-3/2}(\partial D)} \sum_{|\alpha| + |\beta| \leq \lceil s \rceil + 2} \sup_{x \in U_d} \left| D_x^\alpha D_y^\beta N_D(x, y) \right| \\ &\quad \leq C'|h| \|f\|_{H^{s-3/2}(\partial D)}, \end{aligned}$$

for some $C, C' > 0$, independent of y, due to Taylor's theorem and (3.8) (because N_D is smooth in $\partial D \times U_d$). Iterating the previous argument yields that $D^\alpha \langle f, N_D(\cdot, y) \rangle = \langle f, D^\alpha N_D(\cdot, y) \rangle$ for any multi-index α (including $\alpha = 0$). Moreover, it follows that for any $k \geq 1$,

$$\|\langle f, N_D(\cdot, y) \rangle\|_{\mathscr{C}^k(U)} \leq C_{U,k,s} \|f\|_{H^{s-3/2}(\partial D)}$$

for some $C_{U,k,s} > 0$. Consequently B, as defined in (4.22), is continuous in the norm of $\mathscr{C}^k(U)/\mathbb{C}$. It is also bounded in the norm of $H^s(U)/\mathbb{C}$, because $y \mapsto \langle f, N_D(\cdot, y)\rangle$ is clearly continuous in the norm of $H^{\max(\lceil s\rceil,0)}(U)$.

On the other hand, Theorem 4.5 states the existence of a continuous mapping

$$\tilde{B} : H_\diamond^{s-3/2}(\partial D) \to H^s(U)/\mathbb{C},\ f \mapsto u|_U$$

and, by equation (4.20), the values of $\tilde{B}$ and B agree on a dense set $\mathscr{C}_\diamond^\infty(\partial D)$. Therefore the two mappings coincide, which proves the representation (4.22). □

A consequence of the previous theorem is that the solution of (4.4) is smooth everywhere inside D, i.e. $u \in \mathscr{E}^\infty(D)/\mathbb{C}$, because for any $y \in D$, there exists a neighbourhood $U \ni y$ such that $\overline{U} \subset D$. It also follows that the representation formula (4.22) holds for any $y \in D$.

5 Electrical impedance tomography

Electrical impedance tomography is performed by applying different voltage (or current) patterns on the boundary ∂D of an object D (in $\mathbb{R}^n$ where $n = 1, 2$ or 3) and measuring the corresponding voltage (or current) patterns. It is assumed that inside D the conductivity equation, (5.3) below, holds and the goal is to recover some information on the unknown conductivity distribution σ (or admittivity γ) inside the object. Measurements and current feed are generally carried out using some number N of non-overlapping *electrodes* that partially cover the boundary (surface) of the object. There are different models for relating the current and voltage measurements at the electrodes to the electric potential u in D.

5.1 Physics of the conductivity equation

Electric and magnetic fields in various media can, to a certain extent, be modelled as the time-dependent vector fields $\mathcal{E}, \mathcal{H} : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ that satisfy the following macroscopic Maxwell's equations [28, §2.2]

$$\nabla \cdot (\epsilon(x)\mathcal{E}(x,t)) = \rho_f(x), \tag{5.1a}$$
$$\nabla \cdot (\mu(x)\mathcal{H}(x,t)) = 0, \tag{5.1b}$$
$$\nabla \times \mathcal{E}(x,t) = -\mu(x)\frac{\partial \mathcal{H}(x,t)}{\partial t}, \tag{5.1c}$$
$$\nabla \times \mathcal{H}(x,t) = \mathcal{J}_f(x,t) + \epsilon(x)\frac{\partial \mathcal{E}(x,t)}{\partial t}, \tag{5.1d}$$

where ϵ and μ are respectively the permittivity and permeability of the medium and ρ_f is the free charge density. The free current density $\mathcal{J}_f$ can be modelled by Ohm's law

$$\mathcal{J}_f(x,t) = \sigma(x)\mathcal{E}(x,t),$$

where the (possibly anisotropic) *conductivity* σ of the medium is a function whose values are symmetric positive-definite matrices.

Consider the time-harmonic fields

$$\mathcal{E}(x,t) = \operatorname{Re}(E(x)e^{i\omega t}), \qquad \mathcal{H}(x,t) = \operatorname{Im}(H(x)e^{i\omega t}),$$

where the (angular) frequency ω is fixed and E, H are complex *phasors*. Such fields could particularly be caused by a sinusoidal alternating current conducted through the medium. For these fields (5.1c) and (5.1d) yield

$$\nabla \times E(x) = -i\omega\mu(x)H(x), \tag{5.2a}$$
$$\nabla \times H(x) = (\sigma(x) + i\omega\epsilon(x))E(x). \tag{5.2b}$$

In the case of low frequencies ω, (5.2a) can be approximated by $\nabla \times E(x) = 0$ [5], which means that E can be given as the gradient of a (complex) scalar potential

$$E(x) = -\nabla u(x).$$

For any vector field H, the curl $\nabla \times H$ is source-free and thus (5.2b) yields the *conductivity equation*

$$\nabla \cdot (\gamma(x)\nabla u(x)) = 0, \tag{5.3}$$

where the complex, frequency-dependent quantity $\gamma(x) = \sigma(x) + i\omega\epsilon(x)$ is called *admittivity*.

Let D be a bounded and connected domain in which (5.3) holds. Applying the divergence theorem (4.2a) yields

$$\int_{\partial D} \nu_x \cdot \gamma(x)\nabla u(x)\, \mathrm{d}s_x = 0. \tag{5.4}$$

A straightforward calculation shows that the above equations also hold in the static case ($\omega = 0$), in which (5.3) can be simply interpreted as the statement $\nabla \cdot \mathcal{J}_f(x) = 0$, the absence of sources and sinks of electric current in D, and (5.4) becomes $\int_{\partial D} \nu_x \cdot \mathcal{J}_f(x)\, \mathrm{d}s_x = 0$, the current conservation law.

In EIT, it is physically reasonable to assume that γ is uniformly bounded (in some matrix norm $\|\cdot\|$),

$$\|\gamma(x)\| < C \quad \text{for all } x \in D, \tag{5.5}$$

and that

$$\mathrm{Re}\,(v^*\gamma v) > c|v|^2 \quad \text{for all } x \in D, v \in \mathbb{C}^n, \tag{5.6}$$

where $c, C > 0$ [5].

Example 5.1 (Two-dimensional electricity). Let $u \in \mathscr{C}^1(\overline{D}) \cap \mathscr{C}^2(D)$ solve the boundary value problem defined by (5.3) and, say (5.7) below, for some $\gamma \in \mathscr{C}^1(\overline{D}, \mathbb{C}^{2\times 2})$ and $f \in \mathscr{C}(\partial D)$ in $D \subset \mathbb{R}^2$. Then for any $z \in (0, h)$, where $h > 0$, define

$$\tilde{u}(\cdot,\cdot,z) = u, \qquad \tilde{f}(\cdot,\cdot,z) = f, \qquad \text{and} \qquad \tilde{\gamma}(\cdot,\cdot,z) = \begin{bmatrix} \gamma & \\ & 1 \end{bmatrix}.$$

Now $\tilde{u}$ solves the boundary value problem

$$\begin{aligned} \nabla \cdot (\tilde{\gamma}\nabla\tilde{u}) &= 0 \quad \text{in } \tilde{D}, \\ \nu \cdot \tilde{\gamma}\nabla\tilde{u} &= \tilde{f} \quad \text{on } \partial D \times (0,h), \\ \frac{\partial \tilde{u}}{\partial \nu} &= 0 \quad \text{on } D \times \{0,h\} \end{aligned}$$

in the cylinder $\tilde{D} = D \times (0,h) \subset \mathbb{R}^3$. Notice also that (5.4) holds for $\tilde{u}$ if and only if it holds for u in the two-dimensional case.

The above example shows how the two-dimensional versions of (5.3) and (5.4) can be used to model current flow in certain "cylindrical" problems, even though the original Maxwell's equations (5.1) do not make any sense physically or mathematically in two dimensions. Similar reasoning can be applied to justify one-dimensional conductivity equations. In the remainder of this work, equations (5.3) and (5.4) are taken as the basis for modelling (alternating) electric currents and potentials in all dimensions $n = 1, 2, 3$.

5.2 EIT forward models

In the *continuum forward model* one assumes that current through each point on the boundary is perceived, that is,

$$\nu \cdot \gamma \nabla u = f \quad \text{on } \partial D \tag{5.7}$$

for some known f, and the boundary potential $u|_{\partial D}$ is measured on all ∂D (or vice versa, potential known, current measured). The *gap model* relates the electrode measurements to the continuum model by assuming that the current density f on each electrode is a constant, equal to the measured current per contact surface area, and zero outside the electrodes. [5][7] A related concept is the *point electrode model* (PEM), in which the electrodes are modelled as point current sources and the above equation holds for some distributional f [17].

The gap model is not accurate, because it does not properly model the behaviour of current on the contact interfaces between the electrodes and the object [5][7][8][29]. In reality, the current density is not constant on the contact surface and, furthermore, a thin, highly resistive layer caused by an electrochemical phenomenon is formed at the contact. Both of these effects are taken into account by the *complete electrode model* (CEM), where the boundary conditions are

$$u + z\nu \cdot \gamma \nabla u = V_j \quad \text{on } E_j,\ j = 1, \ldots, N, \tag{5.8a}$$

$$\nu \cdot \gamma \nabla u = 0 \quad \text{on } \partial D \setminus \cup_{j=1}^{N} \overline{E}_j, \tag{5.8b}$$

$$\int_{E_j} \nu \cdot \gamma \nabla u \, \mathrm{d}s = I_j, \quad j = 1, \ldots, N. \tag{5.8c}$$

The (known) function z is called *surface impedance* or *(effective) contact impedance* and it models the formation of the resistive layer. In this model, the contact surface E_j of each electrode is assumed to be in a constant potential V_j that can also be measured, but the respective measured current I_j does not correspond to a constant current density. CEM has been shown to be capable

of predicting experimental data with great precision and is considered a very accurate model for EIT measurements. [5][7][8][17][29]

From the theoretical point of view, the continuum model and its derivatives are more appealing and are known to posses some important properties, such as unique solvability of certain inverse problems (see Section 5.3). The corresponding theory for CEM is less studied, more complicated and known to lack some important characteristics of the continuum model. See [5], [30] and [26] for more details. On the other hand, it has been shown in [17] that, for various types of *difference data*, the use of PEM is justified for small electrodes, because the model error between CEM and PEM decreases with electrode size in a suitable manner.

In the sequel, the studied object D is modelled as a $\mathscr{C}^\infty$ regular domain (see Definition 3.3) that has a connected complement and whose conductivity is assumed to be homogeneous near the boundary, that is,[7]

$$\operatorname{supp}(\gamma - I) =: \Omega \subset\subset D. \tag{5.9}$$

The sweep forward model is based on PEM difference data. It is defined as the difference between the measurement on an object D with unknown conductivity γ and the *background measurement* on an object of the same shape D but known homogeneous conductivity $\gamma = I$. This is explained in more detail in Section 5.4. In this model, the following (elliptic) boundary value problems are studied: the *full conductivity equation*

$$\nabla \cdot (\gamma \nabla u) = 0 \quad \text{in } \mathscr{D}'(D), \tag{5.10a}$$

$$\frac{\partial u}{\partial \nu} = f \quad \text{in } H^s_\diamond(\partial D) \tag{5.10b}$$

and the *background conductivity equation*,

$$\Delta u_{\mathbb{1}} = 0 \quad \text{in } \mathscr{D}'(D), \tag{5.11a}$$

$$\frac{\partial u_{\mathbb{1}}}{\partial \nu} = f \quad \text{in } H^s_\diamond(\partial D), \tag{5.11b}$$

that is, Laplace's equation with a Neumann boundary condition, which corresponds to the full conductivity equation in the case $\gamma = I$. The solvability condition (4.5) of the Neumann problem, indicated by the $\diamond$ symbol, corresponds to (5.4) and is thus not an essential restriction in the physical sense.

The regularity $s \in \mathbb{R}$ of the current pattern f is arbitrary. As in Section 4.2, the normal derivative $\partial/\partial\nu$ is understood in the sense of traces and the existence of an appropriate trace operator is non-trivial for $s \leq 1/2$. The solution

[7] Notice that for any homogeneous non-vanishing admittivity $\gamma = cI$ near the boundary, the conductivity equation can be transformed to this form by dividing f and γ by $c \in \mathbb{C}$.

$u_{\mathbb{1}}$ is called the *background potential* corresponding to the current pattern f. The next section studies the basic properties of these equations and the related solution maps, which are key concepts in the theory of the continuum forward model.

5.3 Neumann-to-Dirichlet maps

Definition 5.1. *The background Neumann-to-Dirichlet map*

$$\Lambda_{\mathbb{1}} : H^s_\diamond(\partial D) \to H^{s+1}(\partial D)/\mathbb{C}$$

is defined as $f \mapsto u_{\mathbb{1}}|_{\partial D}$, where $u_{\mathbb{1}}$ is the solution to (5.11).

According to Theorem 4.5, equation (5.11) has a unique solution in $u_{\mathbb{1}} \in H^{s+3/2}(D)/\mathbb{C}$ and the existence of the linear and continuous map $\Lambda_{\mathbb{1}}$ follows from (4.7).

The existence of a unique solution and suitable continuous linear trace operators for the full conductivity equation is stated by

Theorem 5.2. *Let $\gamma \in L^\infty(D)$ satisfy* (5.5), (5.6) *and* (5.9) *almost everywhere in D. For any $s \in \mathbb{R}$ and $f \in H^s_\diamond(\partial D)$ there exists a unique solution u of* (5.10) *in*

$$\mathcal{H}^s(D) := (H^{\min(1,s+3/2)}(D) \cap H^1_{\mathrm{loc}}(D))/\mathbb{C}.$$

Furthermore, the solution and its (well-defined) trace on ∂D depend linearly and continuously on the data

$$\|u\|_{H^{\min(1,s+3/2)}(D)/\mathbb{C}} \leq C\|f\|_{H^s(\partial D)},$$
$$\|u\|_{H^{s+1}(\partial D)/\mathbb{C}} \leq C'\|f\|_{H^s(\partial D)},$$

where C and C' depend on γ, D and s. Moreover, outside the inhomogeneities Ω, the solution is smooth: $u \in \mathscr{E}^\infty(D \setminus \Omega)/\mathbb{C}$.

Proof. This is proved in [18, Appendix]. Even though the proof is formulated for real isotropic conductivities, the variational argument also holds for the anisotropic complex admittivities of this theorem, because the sesquilinear form

$$a(u,v) = \int_D \gamma \nabla u \cdot \overline{\nabla v}\, \mathrm{d}x$$

can be proven to be bounded and coercive under the assumptions (5.5) and (5.6). □

Definition 5.2. *The Neumann-to-Dirichlet map $\Lambda_\gamma : H^s_\diamond(D) \to H^{s+1}(D)/\mathbb{C}$ is defined as $\Lambda_\gamma : f \mapsto u|_{\partial D}$, where $u \in \mathcal{H}^s(D)$ is the solution to* (5.10).

By Theorem 5.2, the map Λ_γ is well-defined, linear and continuous.

Remark 5.3. The norms of the operators Λ_γ and $\Lambda_{\mathbb{1}}$ do depend on s, the regularity of the boundary current space $H^s_\diamond(\partial D)$, as well as on the domain D. In the following analysis neither s or D "vary" in a manner that would cause problems, but the dependence is good to be kept in mind when interpreting the results.

The next property explains why the Neumann-to-Dirichlet map is commonly written between the space $H^{-1/2}_\diamond(\partial D)$ and its dual $H^{1/2}(\partial D)/\mathbb{C}$.

Lemma 5.4. *The Neumann-to-Dirichlet map $\Lambda_\gamma : H^{-1/2}_\diamond(\partial D) \to H^{1/2}(\partial D)/\mathbb{C}$ for a real (symmetric) conductivity $\gamma = \sigma$ is self-adjoint and positive-definite, that is, for all $f, g \in H^{-1/2}_\diamond(\partial D)$, it holds that*

$$\langle f, \Lambda_\sigma g\rangle = \langle \Lambda_\sigma f, g\rangle \tag{5.12}$$

and

$$\langle f, \Lambda_\sigma f\rangle \geq c\|f\|^2_{H^{-1/2}(\partial D)} \tag{5.13}$$

for some $c > 0$. [5]

Remark 5.5. It is also common to work with the Dirichlet-to-Neumann map $\Lambda_\gamma^{-1} : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$, which somewhat reduces the need for quotient spaces. From a certain theoretical point of view, it makes no difference which map one studies. Namely, as inverses of each other, Λ_γ^{-1} and Λ_γ contain the same information. [5] In more practical EIT research, on the contrary, it is more convenient to work with Λ_γ, since it is a smoothing operator and more easily related to the measurements. For example, in the gap model, it is physically reasonable to assert that the current through the surface outside the electrodes is zero and an EIT measurement can be interpreted as an evaluation of Λ_γ at a known value (current pattern). The same cannot be said if using Λ_γ^{-1}, as the potential outside the electrodes is unknown.

The choice of convention is commonly reflected to notation: Λ_γ denotes either Neumann-to-Dirichlet or Dirichlet-to-Neumann map, depending on which is the main object of study.

In the analysis of, for instance, PEM it is necessary to work with the difference boundary map $\Lambda_\gamma - \Lambda_{\mathbb{1}}$, which satisfies (see [18, Theorem A.3])

Lemma 5.6. *The difference Neumann-to-Dirichlet map $\Lambda_\gamma - \Lambda_{\mathbb{1}}$ is a linear and continuous mapping from $H^s_\diamond(\partial D)$ to $H^l(\partial D)/\mathbb{C}$ for any $l \in \mathbb{R}$.*

Analogously to Lemma 5.4, if the difference boundary map is written as $\Lambda_\sigma - \Lambda_{\mathbb{1}} : H^{-s}_\diamond(\partial D) \to H^s(\partial D)/\mathbb{C}$, then it holds that

$$\langle f, (\Lambda_\sigma - \Lambda_{\mathbb{1}})g\rangle = \langle (\Lambda_\sigma - \Lambda_{\mathbb{1}})f, g\rangle \tag{5.14}$$

for all $f, g \in H_\diamond^{-s}(\partial D)$ in case of real conductivities. This follows from e.g. [18, Corollary 3.2].

The general EIT inverse problem corresponding to the continuum forward model is formulated as follows [30]:

Definition 5.3 (Calderón's problem)**.** *Given the map* $\Lambda_\gamma^{-1} : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$*, determine* γ*.*

This means that one studies (the inverse of) the map

$$\gamma \longmapsto \Lambda_\gamma^{-1}, \tag{5.15}$$

for some class of admittivities γ. As Calderón proposed, one may as well study the map $Q : \gamma \longmapsto Q_\gamma$, where

$$Q_\gamma(f) = \langle f, \Lambda_\gamma^{-1} f \rangle \tag{5.16}$$

because knowing the quadratic form Q_γ is equivalent to knowing Λ_γ^{-1} [30]. If the map is injective, then the inverse problem has a unique solution. Injectivity results have been proven in various cases such as recently for real, isotropic $\sigma \in L^\infty(D)$ on the plane $D \subset \mathbb{R}^2$ [3]. The map (5.15) is non-linear and, despite the injectivity results, the general inverse problem is severely ill-posed [5][30].

5.4 Sweep and backscatter data

Consider the following two-electrode EIT measurement setup on the boundary of a cylindrical object, modelled as a disk $D = B_1(0) \subset \mathbb{R}^2$. One of the electrodes is held stationary at angle θ_0 and the other is circulated around the object, as shown in Figure 1(a). More specifically, assume that one has access to *two* objects with the same shape (modelled as D):

- Object A with unknown admittivity γ that is *a priori* assumed to satisfy the homogeneity condition (5.9);
- Object B with homogeneous admittivity $\gamma = I$.

Then the following measurement is performed:

1. For each $j = 2, \ldots, N$, place electrode 1 at angle $\theta_j = \frac{2\pi(j-1)}{N} + \theta_0$ and electrode 2 at angle θ_0 on the boundary of Object A. Conduct unit current from electrode 1 to electrode 2 and measure the signed voltage V_j^γ between the electrodes. In addition, define $V_1^\gamma = 0$.
2. Conduct (or simulate) the same measurement for Object B to get $V_j^{\mathbb{1}}$, $j = 1, \ldots, N$.

3. The difference data

$$\varsigma_j = V_j^\gamma - V_j^{\mathbb{1}}, \qquad j = 1, \ldots, N \tag{5.17}$$

defines the *sweep measurement*.

For simplicity, it is assumed that the electrodes (or the number N) are sufficiently small not to overlap at any angle. Notice that for any constant background conductivity, current through the electrodes and disk radius, a similar measurement can be interpreted to be of the above form by appropriate scaling and change of units.

The measurement setup modelled by *backscatter data* comprises a pair of small electrodes that are held at a small constant distance from each other — an approximation of a dipole electrode — and simultaneously circulated around the object. This is depicted in Figure 1(b). As in the case of sweep data, the backscatter measurement is the difference data of the unit-current-maintaining voltage measurements [18]. In both cases, the requirement for a background measurement is, from the practical point of view, an unwanted complication which is unfortunately very important for the theoretical analysis.

Let D be as above and denote by

$$\delta_\theta = \delta_{(\cos\theta, \sin\theta)} \tag{5.18}$$

the Dirac's delta distribution at boundary point $(\cos\theta, \sin\theta) \in \partial D$. It holds that $\delta_\theta \in H^{-1/2-\varepsilon}(\partial D)$ for any $\varepsilon > 0$ [21]. Also set

$$\delta'_\theta = \frac{d}{d\theta}\delta_\theta, \tag{5.19}$$

which is in $H_\diamond^{-3/2-\varepsilon}(\partial D)$ [18]. For each $f \in H^s(\partial D)$ with a sufficiently large s, it holds in the sense of Sobolev's embedding theorem (3.6) that

$$\delta_\theta(f) = f((\cos\theta, \sin\theta)), \qquad \delta'_\theta(f) = \frac{-\,\mathrm{d}f((\cos\theta, \sin\theta))}{\mathrm{d}\theta}.$$

Definition 5.4. Sweep data *is given as* $\varsigma_\gamma : [0, 2\pi) \to \mathbb{R}$,

$$\varsigma_\gamma(\theta) = \langle \delta_\theta - \delta_{\theta_0}, (\Lambda_\gamma - \Lambda_{\mathbb{1}})(\delta_\theta - \delta_{\theta_0}) \rangle \tag{5.20}$$

for some $\theta_0 \in [0, 2\pi)$. Backscatter data *is* $b_\gamma : [0, 2\pi) \to \mathbb{R}$,

$$b_\gamma(\theta) = \langle \delta'_\theta, (\Lambda_\gamma - \Lambda_{\mathbb{1}})\delta'_\theta \rangle. \tag{5.21}$$

[18][21]

Remark 5.7. In the sequel, sweep and backscatter data are always denoted ς_γ and b_γ, respectively, but, depending on the context, the domain of these functions is interpreted to be $[0, 2\pi)$, $\partial B_1(0) \subset \mathbb{R}^2$ or $\partial B_1(0) \subset \mathbb{C}$.

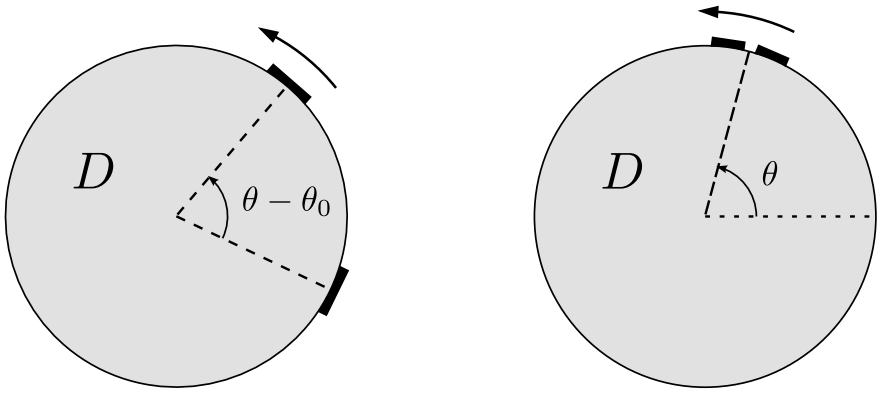
Sweep and backscatter data can be interpreted as point electrode models for the measurements defined above. That is, one can write

$$\varsigma_j \approx \varsigma_\gamma(\theta_j) + \eta_j, \tag{5.22}$$

where η_j is random variable that represents the measurement error as noise.

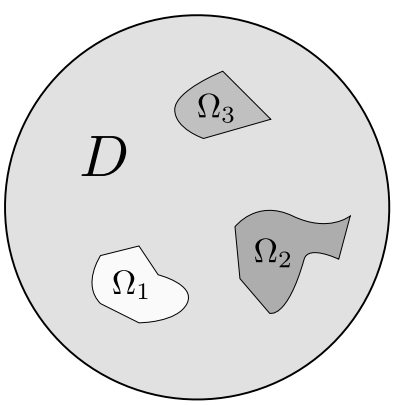
The idealistic sweep data inverse problem is: *given the sweep data* (5.20), *determine the admittivity* γ. In the sequel, the goal is to devise methods for solving the following realistic problem.

Definition 5.5 (Sweep data inverse problem)**.** *Given a sample of* N *noisy sweep data measurements* (5.17)*, find information on the admittivity* γ*.*

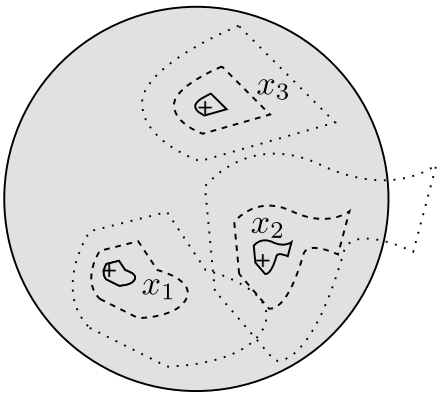


(a) sweep data (b) backscatter data

Figure 1: Sweep and backscatter measurements. The thick protruded lines depict electrodes. Unit current is conducted into D through one electrode and out through the other. The voltage required to maintain the current is measured.



(a) A domain D and $m = 3$ Lipschitz regular inclusions in $\mathbb{R}^2$. Lighter shades indicate higher conductivities.



(b) Contracting inclusions $\Omega_j(\varepsilon)$ for $\varepsilon = 1/3$ (solid line), $\varepsilon = 1$ (dashed) and $\varepsilon = 2$ (dotted). The centre points x_j are marked with crosses.

Figure 2: Regular inclusions and asymptotics

6 Regular inclusion theory

Throughout this section, it is assumed that the conductivity $\sigma : D \to (0, \infty)$ takes the following form

$$\sigma(x) = \begin{cases} \kappa_j \neq 1 & \text{if } x \in \Omega_j, \\ 1 & \text{otherwise }, \end{cases}$$

where the *inclusions*[8] $\Omega_1, \ldots, \Omega_m$ are Lipschitz regular domains with connected complements, such that

$$\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset \quad \text{for all } i \neq j \text{ and} \tag{6.1a}$$

$$\Omega := \bigcup_{j=1}^{m} \Omega_j \subset\subset D. \tag{6.1b}$$

An example is given in Figure 2(a). In some cases, also $\mathscr{C}^1$ regular inclusions are studied. As in Section 5, the object D is assumed to be a $\mathscr{C}^\infty$ regular domain with a connected complement.

[8]The term inclusion can refer to either Ω_j or the pair (Ω_j, κ_j).

6.1 Spaces on the inclusions

The following analysis is based on the behaviour of the solutions of the conductivity equation (5.10) on the boundaries of the inclusions, which requires the definition of some special function spaces. For the boundary $\partial\Omega$ of the m-component region Ω, define

$$L^2(\partial\Omega) = L^2(\partial\Omega_1) \oplus \cdots \oplus L^2(\partial\Omega_m) \tag{6.2}$$

and $W^{1/2}(\partial\Omega)$ correspondingly. The spaces $H^1(\Omega)$ and $L^2(\Omega)$ are well-defined for a disconnected region according to their original definitions, but they can also be identified with similar direct-sum structures when necessary.

Definition 6.1. *Let*

$$L^2(\partial\Omega)/\mathbb{C}^m := L^2(\partial\Omega)/\mathbb{C}^m_{\partial\Omega} = \{\{u + G : G \in \mathbb{C}^m_{\partial\Omega}\} : u \in L^2(\partial\Omega)\},$$

where

$$\mathbb{C}^m_{\partial\Omega} := \operatorname{span}_{\mathbb{C}}\{\chi_{\partial\Omega_1}, \ldots, \chi_{\partial\Omega_m}\} \subset L^2(\partial\Omega), \tag{6.3}$$

be the quotient space of L^2 functions that are defined up to an (individual) additive constant on each $\partial\Omega_j, j = 1, \ldots, m$.

In addition, define

$$L^2_*(\partial\Omega) := \{u \in L^2(\partial\Omega) : u|_{\partial\Omega_j} \in L^2_\diamond(\partial\Omega_j) \text{ for all } j = 1, \ldots, m\}$$

as the space of L^2 functions whose mean over each connected component $\partial\Omega_j, j = 1, \ldots, m$ of $\partial\Omega$ is zero.

It follows from Lemma 3.7 that $L^2(\partial\Omega)/\mathbb{C}^m$ can be identified with the dual of $L^2_*(\partial\Omega)$ because, as a finite dimensional subspace, $\mathbb{C}^m_{\partial\Omega}$ is closed in $L^2(\partial\Omega)$. Furthermore:

Lemma 6.1. *$L^2_*(\partial\Omega)$ is isometrically isomorphic to $L^2(\partial\Omega)/\mathbb{C}^m$.*

Proof. Define $\pi : L^2(\partial\Omega) \to L^2(\partial\Omega)$ by

$$\pi(f) = f - \sum_{j=1}^{m} \chi_{\partial\Omega_j} \fint_{\partial\Omega_j} f \, \mathrm{d}s. \tag{6.4}$$

It is easy to verify (cf. Lemma 3.9) that $\pi(f) \in L^2_*(\partial\Omega)$ for any $f \in L^2(\partial\Omega)$ and $\pi(f + G) = \pi(f)$ for any $G \in \mathbb{C}^m_{\partial\Omega}$. Therefore the mapping is well-defined

from $L^2(\partial\Omega)/\mathbb{C}^m$ to $L^2_*(\partial\Omega)$. In addition, it is linear and has a well-defined inverse $\pi^{-1}(f) = f + \mathbb{C}^m_{\partial\Omega}$. To see that π is norm-preserving, notice that

$$\begin{aligned}\|f\|^2_{L^2(\partial\Omega)/\mathbb{C}^m} &= \|\pi(f) + \mathbb{C}^m_{\partial\Omega}\|^2_{L^2(\partial\Omega)/\mathbb{C}^m} = \inf_{G\in\mathbb{C}^m_{\partial\Omega}} \|\pi(f) + G\|^2_{L^2(\partial\Omega)} \\ &= \inf_{G\in\mathbb{C}^m_{\partial\Omega}} \{\|\pi(f)\|^2_{L^2(\partial\Omega)} + 2\mathrm{Re}\,(\pi(f), G)_{L^2(\partial\Omega)} + \|G\|^2_{L^2(\partial\Omega)}\} \\ &= \inf_{G\in\mathbb{C}^m_{\partial\Omega}} \{\|\pi(f)\|^2_{L^2(\partial\Omega)} + \|G\|^2_{L^2(\partial\Omega)}\} = \|\pi(f)\|^2_{L^2(\partial\Omega)},\end{aligned}$$

because

$$(\pi(f), G)_{L^2(\partial\Omega)} = \sum_{j=1}^{m} |\partial\Omega_j| \cdot \overline{G}|_{\partial\Omega_j} \fint_{\partial\Omega_j} \pi(f)\,\mathrm{d}s = 0. \qquad \square$$

With the help of Definition 6.1 one can also construct the space $W^{1/2}(\partial\Omega)/\mathbb{C}^m$ and its dual $W^{-1/2}_*(\partial\Omega)$. One more special space is required:

Definition 6.2. *Let*

$$H^1(D\setminus\partial\Omega)/\mathbb{C}^{m+1} = \{\{u + G : G \in \mathbb{C}^{m+1}_{D,\Omega}\} : u \in H^1(D\setminus\partial\Omega)\},$$

where $\mathbb{C}^{m+1}_{D,\Omega} := \mathrm{span}_{\mathbb{C}}\{\chi_{D\setminus\overline{\Omega}}, \chi_{\Omega_1}, \ldots, \chi_{\Omega_m}\}$, *be the space of* $H^1(D\setminus\partial\Omega)$ *functions that are defined up to an individual additive constant on each* Ω_j *and* $D\setminus\overline{\Omega}$.

In the sequel, the notions of *interior* and *exterior* traces on the inclusion boundaries are also required. For piecewise differentiable functions w such that $w|_\Omega \in \mathscr{C}^1(\overline{\Omega})$ and $w|_{D\setminus\overline{\Omega}} \in \mathscr{C}^1(\overline{D\setminus\overline{\Omega}})$ the interior Neumann trace $\frac{\partial w}{\partial\nu}^+$ (respectively the interior Dirichlet trace w^+) and the exterior Neumann trace $\frac{\partial w}{\partial\nu}^+$ (resp. w^-) are precisely the derivatives along the exterior normal ν (resp. the limits along the normal) as the boundary is approached from the interior and exterior of Ω, respectively.

When it comes to Sobolev spaces, the interior traces in Ω are naturally defined as those of each $u|_{\Omega_j} \in H^1(\Omega_j, \Delta)/\mathbb{C}$ (resp. in $H^1(\Omega_j)/\mathbb{C}$) for $j = 1, \ldots, m$, and they are indeed well-defined by Lemma 4.4 (resp. Theorem 3.6). The well-definedness of the exterior traces can be justified as follows: Let $U \subset\subset D$ be a Lipschitz regular domain with a connected complement such that $\overline{\Omega} \subset U$. Then $u|_{U\setminus\overline{\Omega}} \in H^1(U\setminus\overline{\Omega}, \Delta)/\mathbb{C}$ (resp. $H^1(U\setminus\overline{\Omega})/\mathbb{C}$) has a well-defined trace in $W^{-1/2}(\partial(U\setminus\overline{\Omega})) = W^{-1/2}(\partial U) \oplus W^{-1/2}(\partial\Omega_1) \oplus \ldots \oplus W^{-1/2}(\partial\Omega_m)$ (resp. $W^{1/2}(\partial(U\setminus\overline{\Omega}))/\mathbb{C}^{m+1}$). The negations of the last components of this trace, which do not depend on U or nor the trace of u on ∂U, define the exterior traces on $\partial\Omega_j, j = 1, \ldots, m$.

Lemma 6.2. *Let D and Ω be as above and $u \in H^1(D/\partial\Omega)$. Then $u \in H^1(D)$ if and only if*

$$u^+ - u^- = 0 \in W^{1/2}(\partial\Omega_j) \tag{6.5}$$

for each $j = 1, \ldots, m$.

Proof. If $u \in H^1(D)$, then (6.5) holds in the dense subset $\mathscr{D}(\overline{D})$ and consequently in all $H^1(D)$ due to the continuity of the trace operator γ_0 from both sides of $\partial\Omega$.

To prove the converse, for an arbitrary $x \in \partial\Omega_j$ (for any j), take a Lipschitz regular neighbourhood $O \ni x$ such that $\overline{O} \subset D$, $\overline{O} \cap \overline{\Omega}_i = \emptyset, i \neq j$ and $O \cap \Omega_j$, $O \setminus \overline{\Omega}_j$ are simply connected and Lipschitz regular. Then for an arbitrary $\phi \in \mathscr{D}(O)$,

$$\begin{aligned}\int_O u\nabla\phi \,\mathrm{d}x &= \int_{O\setminus\overline{\Omega}_j} u\nabla\phi \,\mathrm{d}x + \int_{O\cap\Omega_j} u\nabla\phi \,\mathrm{d}x \\ &= \int_{\partial(O\setminus\overline{\Omega}_j)} u\phi\nu \,\mathrm{d}s - \int_{O\setminus\overline{\Omega}_j} \phi\nabla u \,\mathrm{d}x + \int_{\partial(O\cap\Omega_j)} u\phi\nu \,\mathrm{d}s - \int_{O\cap\Omega_j} \phi\nabla u \,\mathrm{d}x \\ &= -\int_O \phi\nabla u \,\mathrm{d}x + \int_{\partial\Omega} \phi u(\nu - \nu) \,\mathrm{d}s = -\int_O \phi\nabla u \,\mathrm{d}x,\end{aligned}$$

where (4.2e) was applied to $\nabla(u\phi)$. The integral of $\phi u\nu$ obviously vanishes on ∂O and the normal vector fields cancel out on $\partial\Omega \cap O$ as indicated. It follows that $\nabla u \in L^2(D)$ remains a valid weak derivative of $u \in L^2(D)$ in all D and the claim is proven. □

Corollary 6.3. *Let $w \in H^1(D \setminus \partial\Omega)/\mathbb{C}^{m+1}$. Then if and only if*

$$w^+ - w^- = 0 \in W^{1/2}(\partial\Omega_j)/\mathbb{C} \quad \text{for all } \ j = 1, \ldots, m, \tag{6.6}$$

there exists a unique $\tilde{w} \in H^1(D)/\mathbb{C}$ such that $w = \tilde{w} + \mathbb{C}^{m+1}_{D,\Omega}$.

6.2 Diffraction problem formulation

Theorem 6.4. *With Lipschitz regular inclusions, the conductivity equation* (5.10) *is equivalent to the* diffraction problem *(cf. [25, Ch. V, §4])*

$$\Delta u = 0 \quad \text{in } \mathscr{D}'(D \setminus \partial\Omega), \tag{6.7a}$$

$$\frac{\partial u}{\partial \nu}^+ = \kappa_j \frac{\partial u}{\partial \nu}^- \in W^{-1/2}_\diamond(\partial\Omega_j), \quad j = 1, \ldots, m, \tag{6.7b}$$

$$\frac{\partial u}{\partial \nu} = f \in H^s_\diamond(\partial D), \tag{6.7c}$$

which therefore has a unique solution in $u \in \mathcal{H}^s(D)$.

Proof. Let $u \in H^1(D)/\mathbb{C}$ satisfy $\Delta u = 0$ in $\mathscr{D}'(D \setminus \partial\Omega)$, and let O be a neighbourhood of an arbitrary $x \in \Omega_j$ as in the proof of Lemma 6.2. Then $u|_{O\cap\Omega_j} \in H^1(O \cap \Omega_j, \Delta)/\mathbb{C}$, $u|_{O\setminus\overline{\Omega}_j} \in H^1(O \setminus \overline{\Omega}_j, \Delta)/\mathbb{C}$ and thus Lemma 4.4 can be applied in these regions. Hence, for any $\phi \in \mathscr{D}(O)$,

$$\begin{aligned}
&\langle \nabla\cdot(\sigma\nabla u), -\phi\rangle_O = \int_O \sigma\nabla\phi\cdot\nabla u\,\mathrm{d}x \\
&= \int_{O\cap\Omega_j} \kappa_j\nabla\phi\cdot\nabla u\,\mathrm{d}x + \int_{O\setminus\overline{\Omega}_j} \nabla\phi\cdot\nabla u\,\mathrm{d}x \\
&= \kappa_j\left(\left\langle \tfrac{\partial u}{\partial\nu}, \phi\right\rangle_{\partial(O\cap\Omega_j)} - \int_{O\cap\Omega_j}\phi\Delta u\,\mathrm{d}x\right) + \left\langle \tfrac{\partial u}{\partial\nu}, \phi\right\rangle_{\partial(O\setminus\overline{\Omega}_j)} - \int_{O\setminus\overline{\Omega}_j}\phi\Delta u\,\mathrm{d}x \\
&= \kappa_j\left\langle \frac{\partial u}{\partial\nu}^-, \phi\right\rangle_{\partial\Omega_j} - \left\langle \frac{\partial u}{\partial\nu}^+, \phi\right\rangle_{\partial\Omega_j} = \left\langle \kappa_j\frac{\partial u}{\partial\nu}^- - \frac{\partial u}{\partial\nu}^+, \phi\right\rangle_{\partial\Omega_j}.
\end{aligned}$$

This means that, under the common condition (6.7a) on $u \in H^1_{\mathrm{loc}}(D)/\mathbb{C}$, the identity $\kappa_j\frac{\partial u}{\partial\nu}^+ = \frac{\partial u}{\partial\nu}^-$ in $(\mathscr{C}_0(O\cap\partial\Omega_j))'$ is equivalent to $\nabla\cdot(\sigma\nabla u) = 0$ in $\mathscr{D}'(O)$. It follows, by using a suitable partition of unity on D and the density of $\mathscr{C}(\partial\Omega_j)$ in $W^{1/2}(\partial\Omega_j)$ [31], that the identity (6.7b) is equivalent to (5.10a) for the piecewise constant conductivity $\gamma = \sigma$. Since the last conditions (5.10b) and (6.7c) are the same, the claim is proven. □

Lemma 6.5. *The* auxiliary diffraction problem

$$\Delta w = 0 \quad \textit{in } \mathscr{D}'(D\setminus\partial\Omega), \tag{6.8a}$$

$$w^+ - w^- = \psi_j \in W^{1/2}(\partial\Omega_j)/\mathbb{C}, \quad j = 1,\dots,m \tag{6.8b}$$

$$\frac{\partial w}{\partial\nu}^+ = \frac{\partial w}{\partial\nu}^- \in W^{-1/2}_\diamond(\partial\Omega), \tag{6.8c}$$

$$\frac{\partial w}{\partial\nu} = 0 \quad \textit{on } \partial D, \tag{6.8d}$$

where $\psi \in W^{1/2}(\partial\Omega)/\mathbb{C}^m$ has a unique solution $w \in H^1(D\setminus\partial\Omega)/\mathbb{C}^{m+1}$. Moreover, this problem is equivalent to the conductivity equation in the sense that u solves (5.10) *with the boundary value f if and only if*

$$w = (\chi_{D\setminus\overline{\Omega}} + \sum_{j=1}^m \chi_{\Omega_j}\kappa_j)u - u_{\mathbb{1}}. \tag{6.9}$$

solves (6.8) *with $\psi_j = (1-\kappa_j)u|_{\partial\Omega_j}$. Here $u_{\mathbb{1}}$ is the solution to* (5.11) *with Neumann boundary value f.*

Proof. By the arguments in the proof of Theorem 6.4 the background solution satisfies the interior boundary condition (6.8c). In addition, $u^+ - u^- = u_{\mathbb{1}}^+ - u_{\mathbb{1}}^- = 0 \in W^{1/2}(\partial\Omega)$ (by Lemma 6.2). It is thus obvious that, since u is the solution to (6.7), w as in (6.9) satisfies all the conditions (6.8). What remains unclear is whether $w \in H^1(D \setminus \partial\Omega)/\mathbb{C}^{m+1}$.

Let $\{U, V\}$ be an open cover of D such that $\overline{\Omega} \subset U \subset\subset D$, $\overline{\Omega} \cap V = \emptyset$, $V \subset D$ and U, V are $\mathscr{C}^\infty$ regular domains. Then w satisfies $w|_V \in H^{\min(1,s+3/2)}(V)/\mathbb{C}$ and solves Laplace's equation with a Neumann boundary condition in $\mathscr{C}^\infty(\partial V) \subset H^{\min(1,s+3/2)-3/2}(\partial V)$. The distribution (equivalence class) $w|_V$ must therefore coincide with the classical solution of this equation in $\mathscr{C}^\infty(\overline{V})/\mathbb{C}$ and, in particular, $w|_{D\setminus\overline{\Omega}} \in H^1(D \setminus \overline{\Omega})/\mathbb{C}$, since $u, u_{\mathbb{1}} \in H^1_{\mathrm{loc}}(D)/\mathbb{C}$; it also holds that $w|_{\Omega_j} \in H^1(\Omega_j)/\mathbb{C}$ for each $j = 1, \ldots, m$. Consequently $w \in H^1(D \setminus \partial\Omega)/\mathbb{C}^{m+1}$.

On the other hand, for any solution w of (6.8), with the given jumps $\psi_j, j = 1, \ldots .m$, set $u = (w + u_{\mathbb{1}})/(\chi_{D\setminus\overline{\Omega}} + \sum_{j=1}^m \chi_{\Omega_j}\kappa_j)$. Then u satisfies (6.6) and $u|_{U\setminus\partial\Omega} \in H^1(U \setminus \partial\Omega)/\mathbb{C}^{m+1}$, which, by Corollary 6.3, means that $u|_U$ can be identified with a unique element of $H^1(U)/\mathbb{C}$. In addition $u|_V = w|_V + u_{\mathbb{1}}|_V \in (H^{s+3/2}(V) \cap H^1_{\mathrm{loc}}(V))/\mathbb{C}$ as $w|_V$ is smooth. Therefore $u \in \mathcal{H}^s(D)$. Since u satisfies the conditions (6.7) it must be the solution to (5.10) for $\gamma = \sigma$ by Theorem 6.4.

The unique solvability of (6.8) follows from the fact that the difference $\tilde{w}$ of any two solutions satisfies (6.8) with $\psi = 0 \in W^{1/2}(\partial\Omega)/\mathbb{C}^m$ which, again by Corollary 6.3, means that $\tilde{w}$ it is identifiable with an element in $H^1(D)/\mathbb{C}$ that, after transforming (6.8c) to $\Delta\tilde{w} = 0 \in \mathscr{D}'(D)$, solves the Neumann problem (4.4) with $f = 0$. Thus $\tilde{w} = 0$ in $H^1(D \setminus \partial\Omega)/\mathbb{C}^{m+1}$. □

6.3 Layer potentials

Definition 6.3. *The* augmented single and double layer potentials *on* $\partial\Omega$ *are respectively the operators* $S_\Omega^N : L^2(\partial\Omega) \to H^1(D \setminus \partial\Omega, \Delta)$ *and* $D_\Omega^N : W^{1/2}(\partial\Omega) \to H^1(D \setminus \partial\Omega, \Delta)$ *such that*

$$(S_\Omega^N\psi)(y) = \int_{\partial\Omega} N_D(x, y)\psi(x)\,\mathrm{d}s_x$$
$$(D_\Omega^N\psi)(y) = \int_{\partial\Omega} \frac{\partial N_D}{\partial\nu_x}(x, y)\psi(x)\,\mathrm{d}s_x,$$

where N_D *is the Neumann–Green function of* D *and* ψ *is called the* density *of the layer potential.*

It is not immediately clear that ranges of the layer potentials are as stated. For the augmented double layer potential D_Ω^N this is proven by

Theorem 6.6. *Let $\psi \in W^{1/2}(\partial\Omega)$. Then $D^N_\Omega\psi$ is in $H^1(D\backslash\partial\Omega, \Delta)$ and satisfies the following (jump) conditions:*

$$\Delta D^N_\Omega\psi = 0 \qquad \text{in } D \setminus \partial\Omega, \tag{6.10a}$$

$$D^N_\Omega\psi^+ - D^N_\Omega\psi^- = \psi \qquad \text{a.e. on } \partial\Omega, \tag{6.10b}$$

$$\frac{\partial D^N_\Omega\psi^+}{\partial\nu} = \frac{\partial D^N_\Omega\psi^-}{\partial\nu} \in W^{-1/2}(\partial\Omega), \tag{6.10c}$$

$$\frac{\partial D^N_\Omega\psi}{\partial\nu} = 0 \qquad \text{a.e. on } \partial\Omega. \tag{6.10d}$$

Proof. Clearly, the augmented double layer potential can be written as a sum

$$D^N_\Omega\psi = \sum_{j=1}^{m} D^N_{\Omega_j}\psi_j$$

of augmented double layer potentials $D^N_{\Omega_j}$ over boundaries of the separate inclusions. By equation (4.17), each of these can be decomposed as

$$D^N_{\Omega_j}\psi = -D_{\Omega_j}\psi_j + R_{\Omega_j}\psi_j, \tag{6.11}$$

where

$$(R_{\Omega_j}\psi_j)(y) := \int_{\partial\Omega_j} \frac{\partial R_D}{\partial\nu_x}(x,y)\psi_j(x)\,\mathrm{d}s_x \tag{6.12}$$

and D_{Ω_j} are "standard" double layer potentials

$$(D_{\Omega_j}\psi_j)(y) = \int_{\partial\Omega_j} \frac{\partial\Gamma}{\partial\nu}(x-y)\psi_j(x)\,\mathrm{d}s_x$$

over the boundaries of the inclusions. For any Lipschitz regular domain U and $\phi \in W^{1/2}(\partial U)$ these are known to satisfy the following set of (jump) conditions [1, §§2.2 & 2.3.1]

$$\Delta D_U\phi = 0 \qquad \text{in } \mathbb{R}^n \setminus \partial U, \tag{6.13a}$$

$$D_U\phi^+ - D_U\phi^- = -\phi \qquad \text{a.e. on } \partial U, \tag{6.13b}$$

$$\frac{\partial D_U\phi^+}{\partial\nu} = \frac{\partial D_U\phi^-}{\partial\nu} \in W^{-1/2}(\partial U). \tag{6.13c}$$

Note that when $x \in \partial\Omega$, $R_D(x,y)$ is harmonic (and thus $\mathscr{E}^\infty$) for $y \in D$ and $N_D(x,y)$ for $y \notin \partial\Omega$. Therefore the order of integration and differentiation can be changed and the properties

$$\Delta R_{\Omega_j}\psi_j = 0 \qquad \text{in } D,$$

$$\frac{\partial D^N_{\Omega_j}\psi_j}{\partial\nu} = 0 \qquad \text{on } \partial D$$

follow from the harmonicity of $R_D(x,\cdot)$ in D and the property (4.16b) of N_D. In addition, any $D^N_{\Omega_j}\psi_j$ is smooth near $\partial\Omega_i$ for all $i \neq j$, and thus the claimed conditions hold.

In each inclusion Ω_j and in $D \setminus \overline{\Omega}$, the augmented double layer potential solves a Neumann problem of the Laplacian with a $W_\diamond^{-1/2}$ boundary condition (the vanishing mean results from applying (4.3)). Hence, by Theorem 4.6, the restriction of $D^N_\Omega\psi$ to each of these domains is in H^1. Consequently $D^N_\Omega\psi \in H^1(D \setminus \partial\Omega, \Delta)$, and $D^N_\Omega\psi + \mathbb{C}^{m+1}_{D,\Omega}$ solves (6.8). □

It is not hard to see from the definition of the Neumann–Green function, that D^N_Ω maps $\mathbb{C}^m_{\partial\Omega}$ to $\mathbb{C}^{m+1}_\Omega$. Thus one obtains

Corollary 6.7. *The operator $D^N_\Omega : W^{1/2}(\partial\Omega)/\mathbb{C}^m \to H^1(D \setminus \partial\Omega)/\mathbb{C}^{m+1}$ is well-defined. Furthermore, $w = D^N_\Omega\psi$ solves the auxiliary diffraction problem (6.8).*

Correspondingly, the augmented single layer potential can be written as

$$(S^N_\Omega\psi)(y) = \sum_{j=1}^m \left(-(S_{\Omega_j}\psi_j)(y) + \int_{\partial\Omega_j} R_D(x,y)\psi_j(x)\,\mathrm{d}s_x \right),$$

where the "standard" single layer potentials

$$(S_U\phi)(y) = \int_{\partial U} \Gamma(x-y)\phi(x)\,\mathrm{d}s_x$$

are known to satisfy the following conditions:

$$\begin{aligned} \Delta S_U\phi &= 0 \quad \text{in } \mathbb{R}^n \setminus \partial U, \\ S_U\phi^+ &= S_U\phi^- \quad \text{a.e. on } \partial U, \\ \frac{\partial S_U\phi^+}{\partial\nu} - \frac{\partial S_U\phi^-}{\partial\nu} &= \phi \quad \text{a.e. on } \partial U \end{aligned}$$

[1, §§2.2 & 2.3.1]. By similar arguments as in the proof of Theorem 6.6, it follows that the conditions

$$\Delta S^N_\Omega\psi = 0 \quad \text{in } D \setminus \partial\Omega, \tag{6.15a}$$

$$S^N_\Omega\psi^+ = S^N_\Omega\psi^- \quad \text{a.e. on } \partial\Omega, \tag{6.15b}$$

$$\frac{\partial S^N_\Omega\psi^+}{\partial\nu} - \frac{\partial S^N_\Omega\psi^-}{\partial\nu} = \psi \quad \text{a.e. on } \partial\Omega, \tag{6.15c}$$

$$\frac{\partial S^N_\Omega\psi}{\partial\nu} = -\frac{\int_{\partial\Omega}\psi\,\mathrm{d}s}{|\partial D|} \quad \text{a.e. on } \partial\Omega \tag{6.15d}$$

hold for the augmented single layer potentials. Obviously, the last condition implies that the Neumann trace of the augmented single layer potential vanishes on ∂D if $\psi \in L^2_\diamond(\partial D)$.

More precisely, it is known that the traces of the layer potentials satisfy

$$\frac{\partial S_U \phi^\pm}{\partial \nu} = \left(\pm \frac{1}{2} I + K_U^* \right) \phi \quad \text{a.e. on } \partial U, \tag{6.16a}$$

$$D_U \phi^\pm = \left(\mp \frac{1}{2} I + K_U \right) \phi \quad \text{a.e. on } \partial U, \tag{6.16b}$$

where I is the identity operator on $L^2(\partial U)$ and $K_U : L^2(\partial U) \to L^2(\partial U)$,

$$\begin{aligned}(K_U \phi)(y) &= \frac{1}{\omega_n} \text{p.v.} \int_{\partial U} \frac{(x-y)\cdot \nu_x}{|y-x|^n} \phi(x)\, \mathrm{d}s_x \\ &:= \frac{1}{\omega_n} \lim_{\varepsilon \to 0^+} \int_{\partial U \setminus B_\varepsilon(y)} \frac{(x-y)\cdot \nu_x}{|y-x|^n} \phi(x)\, \mathrm{d}s_x, \quad y \in \partial U\end{aligned}$$

is a bounded operator, whose adjoint is

$$(K_U^* \phi)(y) = \frac{1}{\omega_n} \text{p.v.} \int_{\partial U} \frac{(y-x)\cdot \nu_y}{|y-x|^n} \phi(x)\, \mathrm{d}s_x, \quad y \in \partial U.$$

Here "p.v." stands for *Cauchy principal value.* [1, §2.3.1] Notice that

$$\begin{aligned}(K_U \phi)(y) &= \text{p.v.} \int_{\partial U} \frac{\partial \Gamma}{\partial \nu_x}(x-y) \phi(x)\, \mathrm{d}s_x, \\ (K_U^* \phi)(y) &= \text{p.v.} \int_{\partial U} \frac{\partial \Gamma}{\partial \nu_y}(y-x) \phi(x)\, \mathrm{d}s_x.\end{aligned}$$

Definition 6.4. *Define $K_\Omega^N : L^2(\partial\Omega) \to L^2(\partial\Omega)$ (cf. [16]) via*

$$(K_\Omega^N \psi)(y) = \text{p.v.} \int_{\partial\Omega} \frac{\partial N_D}{\partial \nu_x}(x,y) \psi(x)\, \mathrm{d}s_x. \tag{6.17}$$

Applying (6.12) to (6.17) yields that

$$K_\Omega^N = -K_\Omega + R_{K,D,\Omega} \tag{6.18}$$

where $K_\Omega|_{\Omega_j} = K_{\Omega_j}$ and

$$R_{K,D,\Omega}\psi|_{\partial\Omega_j} = \left(\sum_{i \neq j} D_{\Omega_i}^N \psi_i + R_{\Omega_j} \psi_j \right) \Bigg|_{\partial\Omega_j}.$$

It is not hard to see that the adjoint of K_Ω^N is given by

$$(K_\Omega^{N^*}\psi)(y) = \text{p.v.} \int_{\partial\Omega} \frac{\partial N_D}{\partial \nu_y}(x,y)\psi(x)\,\mathrm{d}s_x$$

and it satisfies

$$K_\Omega^{N^*} = -K_\Omega^* + R_{K,D,\Omega}^*. \tag{6.19}$$

The operator K_Ω^N can be interpreted as an $m \times m$ "matrix of operators"

$$K_\Omega^N = \begin{bmatrix} K_{1,1} & \cdots & K_{1,m} \\ \vdots & \ddots & \vdots \\ K_{m,1} & \cdots & K_{m,m} \end{bmatrix}$$

acting on the vector $\psi = [\psi|_{\Omega_1}, \ldots, \psi|_{\Omega_m}]^T$ of L^2 functions. In these terms,

$$K_\Omega = \operatorname{diag}(K_{\Omega_j}, \ldots, K_{\Omega_m})$$

and $R_{K,D,\Omega}$ has the operators $\psi_j \mapsto R_{\Omega_j}\psi_j|_{\partial\Omega_j}, j = 1, \ldots, m$ on the diagonal and $\psi_j \mapsto D_{\Omega_j}^N \psi_j|_{\partial\Omega_i}, i \neq j$ as the other elements. The elements of $R_{K,D,\Omega}$ consist of integral operator with bounded kernels. Similar results hold for $K_\Omega^{N^*}$.

Combining equations (6.18), (6.19) and (6.16) with the definitions and familiar properties (from the proof of Theorem 6.6) of the relevant layer potentials yields

Lemma 6.8. *The augmented layer potentials satisfy the trace conditions*

$$\frac{\partial S_\Omega^N \psi}{\partial \nu}^{\pm} = (\mp\frac{1}{2}I + K_\Omega^{N^*})\psi, \tag{6.20a}$$

$$D_\Omega^N \psi^\pm = (\pm\frac{1}{2}I + K_\Omega^N)\psi \tag{6.20b}$$

a.e. on $\partial\Omega$.

6.4 Factorization of $\Lambda_\sigma - \Lambda_{\mathbb{1}}$

Definition 6.5. *Define* $A_\Omega : H_\diamond^s(\partial D) \to L_*^2(\partial\Omega)$ *as* $f \mapsto \frac{\partial u_{\mathbb{1}}}{\partial \nu}|_{\partial\Omega}$, *where* $u_{\mathbb{1}}$ *is the solution to* (5.11) *for the boundary current* f.

First, notice that due to Theorem 4.10

$$\nabla u_{\mathbb{1}}(x) = \langle f, \nabla_x N_D(x,\cdot)\rangle_{\partial D}\,, \quad x \in D$$

has a well-defined (continuous) restriction to $\partial\Omega$. In addition, since $\nu \in L^\infty(\partial\Omega)$ (see Section 3.1), $\frac{\partial u_{\mathbb{1}}}{\partial \nu} = \nabla u_{\mathbb{1}} \cdot \nu \in L^\infty(\partial\Omega) \subset L^2(\partial\Omega)$ is given by the "layer potential"

$$(A_\Omega f)(x) = \frac{\partial}{\partial \nu_x} \langle f, N_D(x,\cdot)\rangle_{\partial D} = \left\langle f, \frac{\partial N_D}{\partial \nu_x}(x,\cdot)\right\rangle_{\partial D}, \quad x \in \partial\Omega. \tag{6.21}$$

To see that $A_\Omega f$ is in $L^2_*(\partial\Omega)$, notice that due to the generalized Green's identity (4.3) the integral (dual evaluation with $1 \in W^{1/2}(\partial\Omega_j)$) of the flux of a harmonic function $u_{\mathbb{1}}$ over the closed surface $\partial\Omega_j$ is zero and therefore $\frac{\partial u_{\mathbb{1}}}{\partial \nu}|_{\partial\Omega_j} \in W^{-1/2}_\diamond(\Omega_j)$. It follows that, as an element of $L^2(\partial\Omega_j)$, $\frac{\partial u_{\mathbb{1}}}{\partial \nu}|_{\partial\Omega_j} \in L^2_\diamond(\partial\Omega_j)$ for any $j = 1, \ldots, m$.

Lemma 6.9. *The dual operator of A_Ω is $A'_\Omega : L^2(\partial\Omega)/\mathbb{C}^m \to H^{-s}(\partial D)/\mathbb{C}$.*

$$(A'_\Omega \psi)(y) = \int_{\partial\Omega} \frac{\partial N_D}{\partial \nu_x}(x,y)\psi(x)\,\mathrm{d}s_x + C \tag{6.22}$$

where $C \in \mathbb{C}$ is independent of $y \in \partial D$.

Proof. Define $\tilde{A} : L^2(\partial\Omega)/\mathbb{C}^m \to \mathscr{C}^\infty(\partial D)$ as the augmented double layer potential

$$(\tilde{A}\psi)(y) = \int_{\partial\Omega} \frac{\partial N_D}{\partial \nu_x}(x,y)\psi(x)\,\mathrm{d}s_x.$$

Again, $\frac{\partial N_D}{\partial \nu}(\cdot, y) \in L^\infty(\partial\Omega)$ for $y \in \partial D$ and the elements of $\mathbb{C}^m_{\partial\Omega}$ vanish under $\tilde{A}$ due to (4.3). Using similar arguments as in the proof of Theorem 4.10, one can see that for any multi-index α and some open "neighbourhood" $U \subset D$ of the boundary ∂D (i.e. $\overline{U} \supset \partial D$) such that $\overline{U} \cap \overline{\Omega} = \emptyset$ it holds that

$$\sup_{\overline{U}} |D^\alpha \tilde{A}\psi| \le |\partial\Omega|^{1/2} \sup_{x\in\partial\Omega, y\in\overline{U}} |D^\alpha_y \tfrac{\partial N_D}{\partial \nu_x}(x,y)| \|\psi\|_{L^2(\partial\Omega)/\mathbb{C}^m}$$

and hence, for any $k = 1, 2, 3, \ldots$, the expression $(\tilde{A}\psi)(y)$ defines a continuous operator from $L^2(\partial\Omega)/\mathbb{C}^m$ to $\mathscr{C}^k(\overline{U})$. Consequently, $\tilde{A}\psi \in \mathscr{C}^\infty(\partial D) \hookrightarrow H^l(\partial D)$ for any $l \in \mathbb{R}$, and therefore $\psi \mapsto \tilde{A}\psi + C$ is continuous as $L^2(\partial\Omega)/\mathbb{C}^m \to H^{-s}(\partial D)/\mathbb{C}$.

For any $f \in \mathscr{C}^\infty_\diamond(\partial D)$, $g \in L^2(\partial\Omega)/\mathbb{C}^m$ and $C \in \mathbb{C}$, it holds that

$$\begin{aligned}
\langle A_\Omega f, g\rangle_{\partial\Omega} &= \int_{\partial\Omega}\int_{\partial D} \frac{\partial N_D}{\partial \nu_x}(x,y) f(y)\,\mathrm{d}s_y g(x)\,\mathrm{d}s_x \\
&= \int_{\partial D}\int_{\partial\Omega} \frac{\partial N_D}{\partial \nu_x}(x,y) g(x)\,\mathrm{d}s_x f(y)\,\mathrm{d}s_y \\
&= \int_{\partial D} (\tilde{A}g)(y) f(y)\,\mathrm{d}s_y = \langle f, \tilde{A}g\rangle_{\partial D} = \langle f, \tilde{A}g + C\rangle_{\partial D},
\end{aligned}$$

where the change of integration order is justified, because $g \in L^1(\partial\Omega)$ and $N(x,y)f(y)$ is bounded in $\partial\Omega \times \partial D$. In consequence, the continuous functional $\tilde{A}g + C \in H^{-s}(\partial D)/\mathbb{C}$ coincides with $A'_\Omega g$ on a dense set, and thus $\tilde{A}g + C = A'_\Omega g$. Since g was arbitrary, (6.22) holds. □

Definition 6.6. *Let $v_j \in H^1(\partial\Omega_j)$ be the solution to the Neumann problem*

$$\begin{aligned} \Delta v_j &= 0 \quad \text{in } \mathscr{D}'(\Omega_j) \\ \gamma_1 v_j &= g_j \in L^2(\partial\Omega_j). \end{aligned}$$

The Neumann-to-Dirichlet map $\lambda_{\Omega_j} : L^2_\diamond(\partial\Omega_j) \to L^2(\partial\Omega_j)/\mathbb{C}$ is defined as the trace $\lambda_{\Omega_j} g_j = \gamma_0 v_j$. Combining these maps for all $j = 1, \ldots, m$ gives $\lambda_\Omega : L^2_*(\partial\Omega) \to L^2(\partial\Omega)/\mathbb{C}^m$,

$$g_1\chi_{\partial\Omega_1} + \ldots + g_m\chi_{\partial\Omega_m} \mapsto (\lambda_{\Omega_1} g_1)\chi_{\partial\Omega_1} + \ldots + (\lambda_{\Omega_m} g_m)\chi_{\partial\Omega_m}.$$

It follows from Theorems 4.6 and 3.6, that each λ_{Ω_j} is a well-defined and bounded map. Consequently, the operator λ_Ω is also well-defined and continuous.

Theorem 6.10 (Factorization of $\Lambda_\sigma - \Lambda_{\mathbb{1}}$)**.** *If each inclusion Ω_j is a $\mathscr{C}^1$ regular domain with a connected complement, then the difference Neumann-to-Dirichlet map can be factorized as*

$$\Lambda_\sigma - \Lambda_{\mathbb{1}} = A'_\Omega (E - K^N_\Omega)^{-1} \lambda_\Omega A_\Omega \tag{6.23}$$

where A_Ω is as in (6.21)*, λ_Ω is the Neumann-to-Dirichlet map of Definition 6.6 and $E : L^2(\partial\Omega) \to L^2(\partial\Omega)$ is the "diagonal" operator*

$$Ev|_{\partial\Omega_j} = \frac{1}{2}\frac{1+\kappa_j}{1-\kappa_j} v|_{\partial\Omega_j}.$$

In addition, the operator $(E - K^N_\Omega)^{-1} : L^2(\partial\Omega)/\mathbb{C}^m \to L^2(\partial\Omega)/\mathbb{C}^m$ is bounded. (cf. [16])

Proof. Let w be as in (6.9) and set $\psi = w|^+_{\partial\Omega} - w|^-_{\partial\Omega} = (1-\sigma)u|^-_{\partial\Omega} \in W^{1/2}(\partial\Omega)/\mathbb{C}^m$. Then, by Corollary 6.7 and (6.20b),

$$w|^-_{\partial\Omega} = K^N_\Omega \psi - \frac{1}{2}\psi \in L^2(\partial\Omega)/\mathbb{C}^m,$$

and on each $\partial\Omega_j$ it holds that

$$\begin{aligned} \left(\left(E - K^N_\Omega\right)\psi\right)|_{\partial\Omega_j} &= \left(\frac{1}{2}\frac{1+\kappa_j}{1-\kappa_j} - \frac{1}{2}\right)\psi_j - w|^-_{\partial\Omega_j} \\ &= \frac{\kappa_j}{1-\kappa_j}\psi_j - (\kappa_j u - u_{\mathbb{1}})|_{\partial\Omega_j} = u_{\mathbb{1}}|_{\partial\Omega_j}, \end{aligned}$$

that is,

$$(E - K_\Omega^N)\psi = u_{\mathbb{1}}|_{\partial\Omega} \in L^2(\partial\Omega)/\mathbb{C}^m. \tag{6.24}$$

On the other hand,

$$u_{\mathbb{1}}|_{\partial\Omega} = \lambda_\Omega A_\Omega f$$

in $L^2(\partial\Omega)/\mathbb{C}^m$ and by Lemma 6.5, Definitions 5.1 & 5.2, Corollary 6.7 and Theorem 6.22, it holds that

$$(\Lambda_\sigma - \Lambda_{\mathbb{1}})f = (u - u_{\mathbb{1}})|_{\partial D} = w|_{\partial D} = D_\Omega^N\psi|_{\partial D} + C = A'_\Omega\psi. \tag{6.25}$$

It remains to show that (6.24) has a unique solution ψ that depends continuously on $u_{\mathbb{1}}$, in other words, that the operator $E - K_\Omega^N$ has a bounded inverse. Each non-augmented operator $K_{\Omega_j}^*$, $j = 1, \ldots, m$ is compact if Ω_j is a $\mathscr{C}^1$ regular domain [1, §2.3.3][12, Theorem 1.2]. Consequently the whole diagonal operator K_Ω^* is compact. This also holds for K_Ω. As a sum of K_Ω^* and an integral operator with a bounded kernel, $K_\Omega^{N^*}$ (as well as K_Ω^N) is likewise a compact operator in $L^2(\partial\Omega)$ [32, Ch. X §2]. By Lemma 6.11 below, the operator $E^* - K_\Omega^{N^*} : L_*^2(\partial\Omega) \to L_*^2(\partial\Omega)$ is injective. Now it follows from Riesz–Fredholm theory [32, Ch. X, §5] that

$$E^* - K_\Omega^{N^*} = (I - K_\Omega^{N^*}(E^*)^{-1})E^*$$

has a bounded inverse, because E^* is clearly an isomorphism and 1 is not an eigenvalue of the compact operator $K_\Omega^{N^*}(E^*)^{-1}$ (see [32, Ch. VIII §1]). Riesz–Fredholm theory also implies that the adjoint

$$(I - K_\Omega^{N^*}(E^{-1})^*)^* = I - E^{-1}\tilde{K}_\Omega^N = E^{-1}(E - \tilde{K}_\Omega^N),$$

where $\tilde{K}_\Omega^N$ is the adjoint of $K_\Omega^{N^*}$ in $L_*^2(\partial\Omega)$, is invertible. It is easy to see that $\tilde{K}_\Omega^N = \pi K_\Omega^N \pi^{-1}$, where π is as in (6.4). In consequence, $E - K_\Omega^N : L^2(\partial\Omega)/\mathbb{C}^m \to L^2(\partial\Omega)/\mathbb{C}^m$ is an invertible operator. □

Lemma 6.11. *The linear operator $E^* - K_\Omega^{N^*} : L_*^2(\partial\Omega) \to L_*^2(\partial\Omega)$ is injective (cf. [1, §2.3.2]).*

Proof. First of all, the operator maps $L_*^2(\partial\Omega)$ to itself because, by Lemma 6.8 and (4.3), for any $\psi \in L^2(\partial\Omega)$ and $j = 1, \ldots, m$ it holds that

$$\int_{\partial\Omega_j} (\frac{1}{2}I + K_\Omega^{N^*})\psi \,\mathrm{d}s = \int_{\partial\Omega_j} \frac{\partial S_\Omega^N \psi}{\partial \nu}^- \mathrm{d}s = \int_{\Omega_j} \Delta S_\Omega^N \psi \,\mathrm{d}x = 0$$

and thus $\int_{\partial\Omega_j} K_\Omega^{N^*}\psi \,\mathrm{d}s = -\frac{1}{2}\int_{\partial\Omega_j} \psi \,\mathrm{d}s$.

Assume that $E^* - K_\Omega^{N*}$ is not injective, that is, there exists $\psi \neq 0$ in $L^2_*(\partial\Omega)$ such that $(E^* - K_\Omega^{N*})\psi = 0$ a.e. Define

$$A = \sum_{j=1}^{m} \kappa_j \int_{\Omega_j} |\nabla S_\Omega^N \psi|^2 \,\mathrm{d}x, \quad B = \int_{D\setminus\overline{\Omega}} |\nabla S_\Omega^N \psi|^2 \,\mathrm{d}x.$$

By the Green's formula (4.3), Lemma 6.8 and (6.15b),

$$\begin{aligned} A &= \sum_{j=1}^{m} \kappa_j \int_{\Omega_j} (\nabla S_\Omega^N \psi) \cdot \overline{(\nabla S_\Omega^N \psi)} \,\mathrm{d}x = \sum_{j=1}^{m} \kappa_j \int_{\partial\Omega_j} \frac{\partial S_\Omega^N \psi}{\partial \nu}^- \overline{S_\Omega^N \psi^-} \,\mathrm{d}s \\ &= \sum_{j=1}^{m} \kappa_j \int_{\partial\Omega_j} \left((\frac{1}{2} I + E^*)\psi \right) \cdot \overline{S_\Omega^N \psi} \,\mathrm{d}s = \sum_{j=1}^{m} \frac{\kappa_j}{1-\kappa_j} \int_{\partial\Omega_j} \psi \cdot \overline{S_\Omega^N \psi} \,\mathrm{d}s \end{aligned}$$

and

$$\begin{aligned} B &= \int_{D\setminus\overline{\Omega}} (\nabla S_\Omega^N \psi) \cdot \overline{(\nabla S_\Omega^N \psi)} \,\mathrm{d}x = \int_{\partial(D\setminus\overline{\Omega})} \frac{\partial S_\Omega^N \psi}{\partial \nu} \overline{S_\Omega^N \psi} \,\mathrm{d}s \\ &= \int_{\partial D} \frac{\partial S_\Omega^N \psi}{\partial \nu} \overline{S_\Omega^N \psi} \,\mathrm{d}s - \sum_{j=1}^{m} \int_{\partial\Omega_j} \frac{\partial S_\Omega^N \psi}{\partial \nu}^+ \overline{S_\Omega^N \psi^+} \,\mathrm{d}s \\ &= -\sum_{j=1}^{m} \int_{\partial\Omega_j} \left((-\frac{1}{2} I + E^*)\psi \right) \cdot \overline{S_\Omega^N \psi} \,\mathrm{d}s \\ &= -\sum_{j=1}^{m} \frac{\kappa_j}{1-\kappa_j} \int_{\partial\Omega_j} \psi \cdot \overline{S_\Omega^N \psi} \,\mathrm{d}s. \end{aligned}$$

That is, $A + B = 0$. Since clearly $B \geq 0$ and for each $j = 1, \ldots, m$,

$$\int_{\Omega_j} |\nabla S_\Omega^N \psi|^2 \,\mathrm{d}x \geq 0,$$

this means that

$$\int_{D\setminus\partial\Omega} |\nabla S_\Omega^N \psi|^2 \,\mathrm{d}x = 0$$

and consequently $S_\Omega^N \psi$ is constant on each connected component of $D \setminus \partial\Omega$. Thus, by (6.15c), $\psi = 0$ a.e., which is a contradiction. □

Remark 6.12. The factorization (6.23) is possibly valid for all Lipschitz regular domains. The problem is that then the operators K_Ω^N and K_Ω^{N*} are no longer compact [1, §2.3.3] and the invertibility of $E^* - K_\Omega^{N*}$ cannot be deducted from Riesz–Fredholm theory. The surjectivity of a closely related operator $sI - K_\Omega^*$ for $|s| \geq \frac{1}{2}$ is proven in [1, §2.3.3] for (connected) Lipschitz Ω.

There are also many other similar factorizations. In particular for Lipschitz regular inclusions, there holds [16]

$$\Lambda_\sigma - \Lambda_{\mathbb{1}} = B'_\Omega (E^* - K_\Omega^{N^*})^{-1} \lambda_\Omega^{-1} B_\Omega, \tag{6.26}$$

where the single layer operator $B_\Omega : H^s_\diamond(\partial D) \to L^2(\partial\Omega)/\mathbb{C}^m$ is as in Theorem 4.10 but restricted to $\partial\Omega$, and $B'_\Omega = S_\Omega^N|_{\partial D}$ is its dual.

6.5 Asymptotic expansion

Let the inclusions be given by

$$\Omega_j = \Omega_j(\varepsilon) := x_j + \varepsilon O_j, \quad j = 1, \ldots, m$$

where the "shapes" O_j are Lipschitz regular domains which have connected complements and contain the origin, and x_j are distinct points in D. Also assume that ε is sufficiently small so that the assumptions (6.1) are satisfied. This is demonstrated in Figure 2(b); for $\varepsilon = 2$, Ω_2 is not contained in D and $\Omega_1 \cap \Omega_2 \neq \emptyset$, but for $\varepsilon \leq 1$ all the requirements are met.

It holds that

$$\int_{\Omega_j(\varepsilon)} \mathrm{d}x = \varepsilon^n \int_{O_j} \mathrm{d}x \quad \text{and} \quad \int_{\partial\Omega_j(\varepsilon)} \mathrm{d}s = \varepsilon^{n-1} \int_{\partial O_j} \mathrm{d}s.$$

Lemma 6.13. *For sufficiently small $\varepsilon > 0$, the factorization* (6.23) *holds for arbitrary Lipschitz regular inclusions O_j, $j = 1, \ldots, m$ and*

$$(E - K^N_{\Omega(\varepsilon)})^{-1} = D^{-1} + R'_\varepsilon, \tag{6.27}$$

where

$$D := \operatorname{diag}(D_1, \ldots, D_m) = E + K_\Omega$$

and

$$\|R'_\varepsilon\| \leq C\varepsilon^n \tag{6.28}$$

for some $C > 0$ independent of ε.

Proof. (cf. [16]) As already discussed in Remark 6.12, the sole concern is the invertibility of the operator $E - K_\Omega^N$. By (6.19), the adjoint of this operator can be written as

$$E^* - K_\Omega^{N^*} = D^* + R^*_{K,D,\Omega}, \tag{6.29}$$

where $D^* = \operatorname{diag}(D_1^*, \ldots, D_m^*)$. It is known, that each operator

$$D_j^* = E^*|_{\partial\Omega_j} + K^*_{\Omega_j} = \frac{1}{2}\frac{1+\kappa_j}{1-\kappa_j} I + K^*_{\Omega_j}, \quad j = 1, \ldots, m$$

is invertible on $L^2_\diamond(\partial\Omega_j)$ [1, §2.3.3]. A simple change of variables shows that

$$\begin{aligned}(K^*_{\partial\Omega_j(\varepsilon)}\phi)(\varepsilon y + x_j) &= \frac{1}{\omega_n}\text{p.v.}\int_{\partial O_j}\frac{(y-x)\cdot\nu_y}{|y-x|^n}\phi(\varepsilon x + x_j)\,\mathrm{d}s_x \\ &= (K^*_{O_j}\phi(\varepsilon\cdot + x_j))(y)\end{aligned}$$

and thus the operator norm

$$\begin{aligned}&\|(D_j^*)^{-1}\|^2 \\ &= \sup_{\substack{\phi\in L^2_\diamond(\partial\Omega_j(\varepsilon))\\ \phi\neq 0}}\frac{\|\phi\|^2_{L^2(\partial\Omega_j(\varepsilon))}}{\|D_j^*\phi\|^2_{L^2(\partial\Omega_j(\varepsilon))}} \\ &= \sup_{\substack{\phi\in L^2_\diamond(\partial\Omega_j(\varepsilon))\\ \phi\neq 0}}\frac{\int_{\partial\Omega_j(\varepsilon)}|\phi|^2\,\mathrm{d}s}{\int_{\partial\Omega_j(\varepsilon)}|\frac{1}{2}\frac{1+\kappa_j}{1-\kappa_j}\phi + (K^*_{\partial\Omega_j(\varepsilon)}\phi)|^2\,\mathrm{d}s} \\ &= \sup_{\substack{\phi\in L^2_\diamond(\partial\Omega_j(\varepsilon))\\ \phi\neq 0}}\frac{\varepsilon^{n-1}\int_{\partial O_j}|\phi(\varepsilon x + x_j)|^2\,\mathrm{d}s_x}{\varepsilon^{n-1}\int_{\partial O_j}|\frac{1}{2}\frac{1+\kappa_j}{1-\kappa_j}\phi(\varepsilon x + x_j) + (K^*_{\partial\Omega_j(\varepsilon)}\phi)(\varepsilon x + x_j)|^2\,\mathrm{d}s_x} \\ &= \sup_{\substack{\varphi\in L^2_\diamond(\partial O_j)\\ \varphi\neq 0}}\frac{\int_{\partial O_j}|\varphi|^2\,\mathrm{d}s}{\int_{\partial O_j}|\frac{1}{2}\frac{1+\kappa_j}{1-\kappa_j}\varphi + (K^*_{O_j}\varphi)|^2\,\mathrm{d}s} = \left\|(\tfrac{1}{2}\tfrac{1+\kappa_j}{1-\kappa_j}I + K^*_{O_j})^{-1}\right\|^2\end{aligned}$$

is independent of ε. Therefore (see [32, Ch. VIII §6]) the same is true for each D_j and consequently $D : L^2_*(\partial\Omega) \to L^2_*(\partial\Omega)$ is invertible with $\|D^{-1}\|$ independent of ε. It follows by similar arguments as in the proof of Theorem 6.10 that the invertibility result also holds for $D : L^2(\partial\Omega)/\mathbb{C}^m \to L^2(\partial\Omega)/\mathbb{C}^m$.

On the other hand, the kernels R_{ij} of the integral operators in $R_{K,D,\Omega(\varepsilon)}$ are smooth and thus, by Taylor's theorem, for any $\psi_j \in L^2_\diamond(\partial\Omega_j(\varepsilon))$, it holds that

$$\begin{aligned}&\left|\int_{\partial\Omega_j(\varepsilon)} R_{ij}(x,y)\psi_i(x)\,\mathrm{d}s_x\right| \\ &= \left|R_{ij}(x_j,y)\int_{\partial\Omega_j(\varepsilon)}\psi_i\,\mathrm{d}s + \int_{\partial\Omega_j(\varepsilon)}(x-x_j)\cdot\nabla_x R_{ij}(x_j+\xi(x,y),y)\psi_i(x)\,\mathrm{d}s_x\right| \\ &\le C_1\varepsilon\int_{\partial\Omega_j(\varepsilon)}|\psi_i|\,\mathrm{d}s \le C_1\varepsilon^{1+\frac{n-1}{2}}\|\psi_i\|_{L^2(\partial\Omega_j(\varepsilon))} \qquad \text{for all } y\in\partial\Omega_i,\end{aligned}$$

where $|\xi(x,y)| \le |x - x_j| \le \varepsilon$ for all $x \in \partial\Omega_j$, $y \in \partial\Omega_i$. Therefore

$$\|R_{K,D,\Omega(\varepsilon)}\| \le C_1\varepsilon^n.$$

For sufficiently small $\varepsilon > 0$, it thus holds that $\|D^{-1}R_{K,D,\Omega(\varepsilon)}\| < \frac{1}{2}$, and then (6.27) and (6.28) follow from the Neumann series representation of the inverse [32, Ch. II §1]. □

Definition 6.7. *Let O be an inclusion with conductivity κ. Its* augmented Pólya-Szegö polarization tensor *is the $n \times n$ matrix*

$$M(\kappa, O) = \int_{\partial O} \nu_x \left(\frac{1}{2}\frac{1+\kappa}{1-\kappa} I - K_O \right)^{-1} x^T \, \mathrm{d}s_x, \tag{6.30}$$

where the (inverse) operator in parentheses is applied to the row vector x^T component-wise and, after left multiplication by the column vector ν_x, the resulting matrix is integrated element-wise. (see [16])

Example 6.14. Let $O = B_R(0) \subset \mathbb{R}^2$ be a two-dimensional disk of radius $R > 0$ and $\kappa \neq 1$ positive. Then for all $x, y \in \partial O$, $x \neq y$, it holds that [1, §2.2]

$$\frac{(x-y)\cdot \nu_x}{|x-y|^2} = \frac{1}{2R}$$

and consequently

$$K_O\phi = K_O^*\phi = \frac{1}{2} \fint_{\partial O} \phi \, \mathrm{d}s$$

for any $\phi \in L^2(\partial O)$. This means that $\frac{1}{2}\frac{1+\kappa}{1-\kappa} I - K_O = \frac{1}{2}\frac{1+\kappa}{1-\kappa} I : L^2(\partial O)/\mathbb{C} \to L^2(\partial O)/\mathbb{C}$ and thus

$$\left(\frac{1}{2}\frac{1+\kappa}{1-\kappa} I - K_O \right)^{-1} = 2\frac{1-\kappa}{1+\kappa} I.$$

Now the polarization tensor can be computed in closed form:

$$\begin{aligned} M(\kappa, B_R(0)) &= 2\frac{1-\kappa}{1+\kappa} \int_0^{2\pi} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} [R\cos\theta,\ R\sin\theta] R \, \mathrm{d}\theta \\ &= 2\frac{1-\kappa}{1+\kappa} R^2 \int_0^{2\pi} \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} \mathrm{d}\theta \\ &= 2\pi R^2 \frac{1-\kappa}{1+\kappa} \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}. \end{aligned} \tag{6.31}$$

Remark 6.15. In the work of Ammari & Kang, e.g. [1], there are numerous results for the "standard" Pólya-Szegö polarization tensors, whose definition is similar to (6.30) but features the operator K_O^* instead of its dual. Unfortunately, these results cannot be directly applied here, because a different polarization tensor is utilized.

Theorem 6.16. *Let $f, g \in H^s_\diamond(\partial D)$. The bilinear form $\langle g, (\Lambda_\sigma - \Lambda_{\mathbb{1}})f\rangle$ satisfies*

$$\langle g, (\Lambda_{\sigma(\varepsilon)} - \Lambda_{\mathbb{1}})f\rangle = \varepsilon^n \sum_{j=1}^{m} \nabla u_{g,\mathbb{1}}(x_j)^T M(\kappa_j, O_j) \nabla u_{f,\mathbb{1}}(x_j) + O(\varepsilon^{n+1}), \quad (6.32)$$

where $\nabla u_{f,\mathbb{1}}(x) = \langle f, \nabla_x N_D(\cdot, x)\rangle$ and $\nabla u_{g,\mathbb{1}}(x) = \langle g, \nabla_x N_D(\cdot, x)\rangle$ are the gradients of the background solutions (of equation (5.11)) for boundary currents f and g, respectively.

Proof. (cf. [16]) From Theorem 4.10 and Definition 6.5 it follows that

$$(A_\Omega g)(x) = \nu_x \cdot \nabla u_{g,\mathbb{1}}(x_j) + r_{j,\varepsilon}(x)$$

and

$$(\lambda_\Omega A_\Omega f)(x) = u_{f,\mathbb{1}}(x) = u_{f,\mathbb{1}}(x_j) + (x - x_j) \cdot \nabla u_{f,\mathbb{1}}(x_j) + r'_{j,\varepsilon}(x)$$

where for all $x \in \partial\Omega_j$, $|r_{j,\varepsilon}(x)| \le C_1 \varepsilon \|g\|_{H^s_\diamond(\partial D)}$ and $|r'_{j,\varepsilon}(x)| \le C_2 \varepsilon^2 \|f\|_{H^s(\partial D)}$ for some $C_1, C_2 > 0$ independent of $\varepsilon > 0$ and $j = 1, \dots, m$. Define $B_1 : H^s_\diamond(\partial D) \to L^2_*(\partial\Omega)$ and $B_2 : H^s_\diamond(\partial D) \to L^2(\partial\Omega)/\mathbb{C}^m$ as

$$\begin{aligned}(B_1 g|_{\partial\Omega_j})(x) &= \nu_x \cdot \nabla u_{g,\mathbb{1}}(x_j) = \nu_x \cdot \langle g, \nabla_{x_j} N_D(x_j, \cdot)\rangle \\ (B_2 f|_{\partial\Omega_j})(x) &= (x - x_j) \cdot \nabla u_{f,\mathbb{1}}(x_j) + G \\ &= (x - x_j) \cdot \langle f, \nabla_{x_j} N_D(x_j, \cdot)\rangle + G,\end{aligned}$$

where $G \in \mathbb{C}^m_{\partial\Omega}$ is independent of x. Then the operator norm estimates

$$\|A_{\Omega(\varepsilon)} - B_1\| \le C_1 \varepsilon^{1+\frac{n-1}{2}}, \qquad \|\lambda_{\Omega(\varepsilon)} A_{\Omega(\varepsilon)} - B_2\| \le C_2 \varepsilon^{2+\frac{n-1}{2}}$$

and

$$\|A_{\Omega(\varepsilon)}\| \le C_3 \varepsilon^{\frac{n-1}{2}}, \qquad \|B_1\| \le C_4 \varepsilon^{\frac{n-1}{2}}$$

are satisfied. By change of variables, it is also easy to see that

$$\|\lambda_{\Omega_j(\varepsilon)}\| = \varepsilon \|\lambda_{O_j}\|$$

and thus

$$\|\lambda_{\Omega(\varepsilon)}\| \le \varepsilon \max_{j=1,\dots,m} \|\lambda_{O_j}\|.$$

Lemma 6.13 yields that, for sufficiently small $\varepsilon > 0$,

$$\begin{aligned}\langle g, (\Lambda_\sigma - \Lambda_{\mathbb{1}})f\rangle &= \int_{\partial\Omega} (A_\Omega g) \cdot ((E - K^N_{\Omega(\varepsilon)})^{-1} \lambda_\Omega A_\Omega f)\, \mathrm{d}s \\ &= \int_{\partial\Omega} B_1 g \cdot (D^{-1} B_2 f)\, \mathrm{d}s + R(f, g, \varepsilon)\end{aligned}$$

where

$$R(f,g,\varepsilon) = \int_{\partial\Omega} \Big((A_\Omega g)\cdot(R'_\varepsilon \lambda_\Omega A_\Omega f) \\ + ((A_\Omega - B_1)g)\cdot(D^{-1}\lambda_\Omega A_\Omega f) \\ + (B_1 g)\cdot(D^{-1}(\lambda_\Omega A_\Omega - B_2)f)\Big)\,\mathrm{d}s_x.$$

By applying the Cauchy-Schwartz inequality to each term in the integral $R(f,g,\varepsilon)$, minding the relevant operator norms, it follows that

$$|R(f,g,\varepsilon)| \le C\varepsilon^{n+1}\|f\|_{H^s(\partial D)}\|g\|_{H^s(\partial D)}$$

and the leading term satisfies

$$\begin{aligned}
&\int_{\partial\Omega} B_1 g\cdot(D^{-1}B_2 f)\,\mathrm{d}s_x \\
&\quad= \sum_{j=1}^{m}\int_{\partial\Omega_j(\varepsilon)} \nabla u_{g,\mathbb{1}}(x_j)\cdot\nu_x(D_j^{-1}(y\cdot\nabla u_{f,\mathbb{1}}(x_j)))(x)\,\mathrm{d}s_x \\
&\quad= \sum_{j=1}^{m}\nabla u_{g,\mathbb{1}}(x_j)^T\int_{\partial\Omega_j(\varepsilon)}\nu_x(D_j^{-1}y^T)(x)\,\mathrm{d}s_x\nabla u_{f,\mathbb{1}}(x_j)
\end{aligned}$$

where

$$\begin{aligned}
\int_{\partial\Omega_j(\varepsilon)}\nu_x(D_j^{-1}y^T)(x)\,\mathrm{d}s_x &= \int_{\partial\Omega_j(\varepsilon)}\nu_x\left((\tfrac{1}{2}\tfrac{1+\kappa_j}{1-\kappa_j}I + K_{\Omega_j(\varepsilon)})^{-1}y^T\right)(x)\,\mathrm{d}s_x \\
&= \varepsilon^{n-1}\int_{\partial O_j}\nu_{\hat{x}}\left((\tfrac{1}{2}\tfrac{1+\kappa_j}{1-\kappa_j}I + K_{\Omega_j(\varepsilon)})^{-1}y^T\right)(\varepsilon\hat{x}+x_j)\,\mathrm{d}s_{\hat{x}} \\
&= \varepsilon^{n-1}\int_{\partial O_j}\nu_{\hat{x}}\left((\tfrac{1}{2}\tfrac{1+\kappa_j}{1-\kappa_j}I + K_{O_j})^{-1}(\varepsilon y+x_j)^T\right)(\hat{x})\,\mathrm{d}s_{\hat{x}} \\
&= \varepsilon^{n}\int_{\partial O_j}\nu_{\hat{x}}\left((\tfrac{1}{2}\tfrac{1+\kappa_j}{1-\kappa_j}I + K_{O_j})^{-1}y^T\right)(\hat{x})\,\mathrm{d}s_{\hat{x}} \\
&= \varepsilon^{n}M(\kappa_j,O_j).
\end{aligned}$$

Notice that the constant x_j vanishes on the penultimate row because D_j^{-1} maps the subspace of constant functions to itself and the integral of the exterior normal vector field ν is zero by equation (4.2e). □

6.6 Holomorphic extension

In this section, the two-dimensional problem, where D is the unit disk, is studied using complex analysis. It is assumed that the (real) conductivity σ is defined by regular homogeneous inclusions as in Section 6. In the following, the sets D, Ω and their boundaries are identified with the corresponding subsets of the complex plane when necessary.

Let $f : \partial D \to H^s_\diamond(\partial D)$ be a family of real current patterns parametrized by $y \in \partial D$ and $u_{\mathbb{1}}(\cdot, y) = u_{f(y),\mathbb{1}} : D \to \mathbb{R}$ the background potential of equation (5.11) corresponding to $f(y)$. In this section, the mapping $(x, y) \mapsto \nabla_x u_{\mathbb{1}}(x, y)$ (from $D \times \partial D \subset \mathbb{R}^2 \times \mathbb{R}^2$ to $\mathbb{R}^2$) is identified with a complex function $(\xi, \zeta) \mapsto v_f(\xi, \zeta)$, which is a mapping from $D \times \partial D \subset \mathbb{C} \times \mathbb{C}$ to $\mathbb{C}$. Also f is interpreted as a family parametrized by the (constrained) complex parameter $\zeta \in \partial D$.

Lemma 6.17. *If $f : C \to \mathbb{C}$ is the restriction $f = g|_C$ of a holomorphic function $g : U \to \mathbb{C}$ on a circle $C \subset U$, where $U \subset \mathbb{C}$ is a connected open set, then g is uniquely determined by f.*

Proof. Because U is an open set containing C, it follows that g has a Laurent series representation (2.3), which is completely determined by the values of f on C as demonstrated by (2.4). The Laurent series converges on some annulus $C \subset A \subset U$. It follows from the representation (2.1) and the properties of connected open subsets of $\mathbb{C}$ that $g|_A$ can be *analytically continued* to the larger connected open set U and that the continuation must coincide with g. See [20, Vol. II] for more details on analytic continuation. □

Let $C = \partial B_1(0) \subset \mathbb{C}$ be the unit circle centred at the origin. It also follows from the analytic continuation theory, that if the function f is real on C, then the holomorphic extension g of f exhibits the following circle inversion symmetry

$$g(z) = \overline{g(\overline{1/z})}, \quad \text{for all } z \in U \cap U^*, \tag{6.33}$$

where the following definition is utilized.

Definition 6.8. *Denote by*

$$\Omega^* := \{\zeta \in \mathbb{C} : 1/\overline{\zeta} \in \Omega\} = \{\zeta \in \mathbb{C} : \zeta\overline{\xi} - 1 = 0 \text{ for some } \xi \in \Omega\},$$

the reflection (inversion) of Ω to $\mathbb{C} \setminus \overline{D}$ with respect to the unit circle ∂D and set

$$\tilde{\Omega} := \mathbb{C} \setminus (\overline{\Omega} \cup \overline{\Omega^*}).$$

Lemma 6.18. *For any complex function $v : \partial D \to \mathbb{C}$, $\operatorname{Re} v$ and $\operatorname{Im} v$ have holomorphic extensions v_1 and v_2 to $\tilde{\Omega}$ if and only if v and $\overline{v}$ have. Denote these latter extensions by w_1 and w_2 respectively. It holds that*

$$w_1 = v_1 + iv_2 \quad \text{and} \quad w_2 = v_1 - iv_2 \tag{6.34}$$

everywhere in $\tilde{\Omega}$.

Proof. Let v_1 and v_2 be the extensions of $\operatorname{Re} v$ and $\operatorname{Im} v$. Then w_1 and w_2 are holomorphic functions with v and $\overline{v}$ as their boundary values on ∂D. By Lemma 6.17 they are thus the unique extensions of v and $\overline{v}$. The converse statement follows from an application of the same argument to $(w_1 + w_2)/2$ and $(w_1 - w_2)/(2i)$. □

Remark 6.19. The change of notation is to underline the fact that v_1 and v_2 do not generally correspond to the real and imaginary parts of v (i.e., the real and imaginary parts of w_1) away from ∂D. Neither does w_2 represent the complex conjugate of w_1 outside ∂D.

Theorem 6.20. *Assume that $\Omega_j, j = 1, \ldots, m$ are such (e.g. sufficiently small or $\mathscr{C}^1$ regular) that the factorization* (6.23) *of Theorem 6.10 holds. Let f and g be two families of current patterns, such that for each $\xi \in \partial\Omega$ the real and imaginary components $\operatorname{Re} v_f(\xi, \cdot)$, $\operatorname{Im} v_f(\xi, \cdot)$, $\operatorname{Re} v_g(\xi, \cdot)$, $\operatorname{Im} v_g(\xi, \cdot)$ have holomorphic extensions $v_{f,1}(\xi, \cdot)$, $v_{f,2}(\xi, \cdot)$, $v_{g,1}(\xi, \cdot)$, $v_{g,2}(\xi, \cdot)$ to $\tilde{\Omega}$ and $\frac{\partial v_{f,i}}{\partial \zeta}(\xi, \zeta)$ and $\frac{\partial v_{g,i}}{\partial \zeta}(\xi, \zeta)$, $i = 1, 2$ are square-integrable w.r.t $\xi \in \partial\Omega$. Then*

$$\zeta \mapsto \langle g(\zeta), (\Lambda_\sigma - \Lambda_{\mathbb{1}}) f(\zeta) \rangle \tag{6.35}$$

has a holomorphic extension to $\tilde{\Omega}$.

Proof. Define $h_f : \partial\Omega \times \tilde{\Omega} \to \mathbb{C}$ as

$$h_f(\xi, \zeta) := \nu(\xi) \cdot v_f(\xi, \zeta) = \nu_1(\xi) v_{f,1}(\xi, \zeta) + \nu_2(\xi) v_{f,2}(\xi, \zeta),$$

where ν is the exterior normal of Ω at $\xi \in \partial\Omega$ and h_g correspondingly. Since ν only depends on ξ, and the derivatives of v are square-integrable, h_f and h_g are holomorphic in ζ and the derivatives $\frac{\partial h}{\partial \zeta}(\cdot, \zeta)$ are in $L^2(\partial\Omega)$. It follows from Theorem 6.10 and Definition 6.5 that the dual evaluation (6.35) can be written as

$$\langle g(\zeta), (\Lambda_\sigma - \Lambda_{\mathbb{1}}) f(\zeta) \rangle = \int_{\partial\Omega} h_g(\xi, \zeta) ((E - K_\Omega^N)^{-1} \lambda_\Omega h_f(\cdot, \zeta))(\xi) \, \mathrm{d}s_\xi.$$

Because $(E - K_\Omega^N)^{-1}\lambda_\Omega$ can be interpreted as a bounded operator from $L^2(\partial\Omega)/\mathbb{C}^m$ to $L^2_*(\partial\Omega)$, it follows that $\zeta \mapsto (E - K_\Omega^N)^{-1}\lambda_\Omega h_f(\cdot, \zeta) \in L^2(\partial\Omega)$

is bounded and holomorphic as $U \to L^2(\partial\Omega)$ for any $U \subset\subset \tilde{\Omega}$. Since $\zeta \mapsto h_g(\cdot,\zeta) \in L^2(\partial\Omega)$ is likewise bounded and complex analytic for $\zeta \in U$, it follows that $\zeta \mapsto \langle g(\zeta), (\Lambda_\sigma - \Lambda_{\mathbb{1}})f(\zeta)\rangle$ is holomorphic in $\zeta \in \tilde{\Omega}$. This means that, as a boundary value of a complex analytic function, $\langle g(\zeta), (\Lambda_\sigma - \Lambda_{\mathbb{1}})f(\zeta)\rangle, \zeta \in \partial D$ has a unique holomorphic extension in $\tilde{\Omega}$ due to Lemma 6.17.

□

Let

$$M = M(O,\kappa) = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$$

be the polarization tensor of some inclusion. Define

$$\tilde{M} = \frac{1}{4}\begin{bmatrix} -\overline{\delta} & \alpha \\ \alpha & -\delta \end{bmatrix} \tag{6.36}$$

where

$$\begin{aligned} \alpha &:= a + b = \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right]^T M \left[\begin{smallmatrix} 1 \\ -i \end{smallmatrix}\right], \\ \delta &:= b - a - 2ic = -\left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right]^T M \left[\begin{smallmatrix} 1 \\ i \end{smallmatrix}\right]. \end{aligned} \tag{6.37}$$

The complex matrix $\tilde{M}$ is symmetric, but not (generally) Hermitian.

Theorem 6.21. *Let f and g be such that Theorem 6.20 holds and $\|f(\zeta)\|_{H^s(\partial D)}$, $\|g(\zeta)\|_{H^s(\partial D)}$ are bounded in $\zeta \in \partial D$. Denote $V(\xi,\zeta) = [v_1(\xi,\zeta), v_2(\xi,\zeta)]^T$ and $W(\xi,\zeta) = [w_1(\xi,\zeta), w_2(\xi,\zeta)]^T$. The complex form of the asymptotic expansion* (6.32) *is given by*

$$\langle g(\zeta), (\Lambda_{\sigma(\varepsilon)} - \Lambda_{\mathbb{1}})f(\zeta)\rangle = \varepsilon^2 \sum_{j=1}^{m} V_g(\xi_j,\zeta)^T M_j V_f(\xi_j,\zeta) + O(\varepsilon^3) \tag{6.38a}$$

$$= \varepsilon^2 \sum_{j=1}^{m} W_g(\xi_j,\zeta)^T \tilde{M}_j W_f(\xi_j,\zeta) + O(\varepsilon^3) \tag{6.38b}$$

as $\varepsilon \to 0^+$ and it converges uniformly for $\zeta \in \partial D$. For real-valued f and g, the expansion satisfies (6.33).

Proof. From proof of Theorem 6.16 it follows that the remainder term is $O(\varepsilon^3 \|f\|_{H^s_\delta(\partial D)} \|g\|_{H^s_\delta(\partial D)})$ and thus convergence to (6.38a) is uniform for $\zeta \in \partial D$. The equivalence of (6.38a) and (6.38b) follows from substituting (6.36), (6.37) and (6.34) to (6.38b). □

By Theorem 4.10 and (4.21), the background potential $u_{\mathbb{1}}$ corresponding to the boundary current pattern $f(x_\theta) = \delta_{x_\theta} - \delta_{x_0}$ on the unit circle ∂D is

$$u_{\mathbb{1}}(x, x_\theta) = \langle \delta_{x_\theta} - \delta_{x_0}, N_D(\cdot, x) \rangle = \frac{1}{\pi} \left(\log |x - x_0| - \log |x - x_\theta| \right) \tag{6.39}$$

for any $x \in D$ and $x_0, x_\theta \in \partial D$. The gradient of this expression with respect to x is

$$\nabla_x u_{\mathbb{1}}(x, x_\theta) = \frac{1}{\pi} \left(\frac{x - x_0}{|x - x_0|^2} - \frac{x - x_\theta}{|x - x_\theta|^2} \right). \tag{6.40}$$

Obviously, this background potential gradient, related to sweep data, corresponds to the complex function $v : D \times \partial D \to \mathbb{C}$,

$$\begin{aligned} v(\xi, \zeta) &= \frac{1}{\pi} \left(\frac{\xi - \zeta_0}{|\xi - \zeta_0|^2} - \frac{\xi - \zeta}{|\xi - \zeta|^2} \right) \\ &= \frac{1}{\pi} \left(\frac{\xi - \zeta_0}{(\xi - \zeta_0)\overline{(\xi - \zeta_0)}} - \frac{\xi - \zeta}{(\xi - \zeta)\overline{(\xi - \zeta)}} \right) \\ &= \frac{1}{\pi} \left(\frac{1}{\overline{\xi} - \overline{\zeta}_0} - \frac{1}{\overline{\xi} - \overline{\zeta}} \right) \\ &= \frac{1}{\pi} \left(\frac{1}{\overline{\xi} - \overline{\zeta}_0} - \frac{\zeta}{\zeta\overline{\xi} - 1} \right), \end{aligned} \tag{6.41}$$

because $\zeta \in \partial D$ or equivalently $\zeta\overline{\zeta} = |\zeta|^2 = 1$. On $D \times \partial D$, the complex conjugate of v is given by

$$\overline{v(\xi, \zeta)} = \frac{1}{\pi} \left(\frac{1}{\xi - \zeta_0} - \frac{1}{\xi - \zeta} \right). \tag{6.42}$$

Consequently, the functions $w_1, w_2 : D \times \tilde{\Omega} \to \mathbb{C}$

$$w_1(\xi, \zeta) = \frac{1}{\pi} \left(\frac{1}{\overline{\xi} - \overline{\zeta}_0} - \frac{\zeta}{\zeta\overline{\xi} - 1} \right), \tag{6.43a}$$

$$w_2(\xi, \zeta) = \frac{1}{\pi} \left(\frac{1}{\xi - \zeta_0} - \frac{1}{\xi - \zeta} \right) \tag{6.43b}$$

are the unique holomorphic extensions of v and $\overline{v}$ to $D \times \tilde{\Omega}$. Thus Theorem 6.20 applies and sweep data $\varsigma_\sigma(\zeta)$ has a unique holomorphic extension to $\tilde{\Omega}$ (provided that the inclusions Ω are $\mathscr{C}^1$ regular or sufficiently small). Furthermore,

it follows from Theorem 6.21, that the complex asymptotic expansion

$$\begin{aligned}\varsigma_{\sigma(\varepsilon)}(\zeta) = \frac{\varepsilon^2}{4\pi^2}\sum_{j=1}^{m}\Bigg(&-\overline{\delta}_j\left(\frac{1}{\overline{\xi}_j-\overline{\zeta}_0}-\frac{\zeta}{\zeta\overline{\xi}_j-1}\right)^2 \\ &+2\alpha_j\left(\frac{1}{\overline{\xi}_j-\overline{\zeta}_0}-\frac{\zeta}{\zeta\overline{\xi}_j-1}\right)\left(\frac{1}{\xi_j-\zeta_0}-\frac{1}{\xi_j-\zeta}\right) \\ &-\delta_j\left(\frac{1}{\xi_j-\zeta_0}-\frac{1}{\xi_j-\zeta}\right)^2\Bigg)+O(\varepsilon^3).\end{aligned} \tag{6.44}$$

holds.

Remark 6.22. Both sweep and backscatter data are holomorphic in $\tilde{\Omega}$ also for all real L^∞-conductivities σ that satisfy (5.9), (5.5) and (5.6) under only mild assumptions on the topology of Ω [21][18]. In particular, this holds for any finite number m of Lipschitz regular inclusions satisfying the conditions on page 40.

It follows from Example 6.14 that, for disk-shaped (origin-centred) inclusions $O = B_R(0)$, one has $\alpha = 4\pi R^2\frac{1-\kappa_j}{1+\kappa_j}$ and $\delta = 0$. Applying this to (6.44) yields

Corollary 6.23. *Let $\Omega'(\varepsilon) = \xi + \varepsilon O \subset \mathbb{C}$, $\xi \in D$ be a disk-shaped inclusion $O = B_{\tilde{R}}(0)$ with conductivity $\kappa \neq 1$. Then inside D, the asymptotic expansion* (6.44) *of sweep data (with Ω consisting of (Ω', κ) and possibly other inclusions) has a first-order pole at ξ with residue*

$$\operatorname{Res}(\varsigma_{\sigma(\epsilon)}, \xi) = \pi R^2\frac{1-\kappa}{1+\kappa}\frac{2}{\pi^2}\left(\frac{1}{\overline{\xi}-\overline{\zeta}_0}-\frac{\xi}{|\xi|^2-1}\right), \tag{6.45}$$

where $R = \varepsilon\tilde{R}$ is the radius of the disk Ω'.

The above corollary suggests the definition of a new quantity:

Definition 6.9. *Let $\Omega' \subset \mathbb{C}$ be a Lipschitz regular inclusion with homogeneous conductivity $\kappa \neq 1$. Its* net conductivity effect *is*

$$\rho(\Omega', \kappa) = |\Omega'|\frac{\kappa-1}{\kappa+1}. \tag{6.46}$$

Notice that there exists a lower limit

$$r_{\min}(\rho_0) := \inf\{r : |\rho(B_r, \kappa)| = \rho_0;\ \kappa, r > 0\} = \sqrt{\frac{\rho_0}{\pi}} \tag{6.47}$$

for the radius of a (disk) inclusion with absolute net conductivity effect $\rho_0 > 0$. This limit radius corresponds to an inclusion with zero or infinite conductivity, i.e. a perfect insulator or conductor.

7 Reconstruction

In this section, an algorithm for reconstructing the locations of multiple (small) inclusions in the conductivity σ of an otherwise homogeneous disk $D = B_1(0) \subset \mathbb{R}^2$ from noisy sweep data is introduced. The method is inspired by the holomorphic extension property of sweep data (see Theorem 6.20) and the complex asymptotic expansion (6.44). The algorithm is based on the reconstruction method devised by Hanke [16] for backscatter data.

The algorithm constructs a Laurent–Padé approximant, a complex rational function, which reproduces the sweep data. It is anticipated that the poles of the approximant locate the inclusions and, moreover, that the corresponding residues can be used to compute approximations of the net conductivity effects (6.46). There is no theoretical result that would appropriately justify this, but the numerical results given in Section 7.3 indicate that the method works as desired.

In order to solve an inverse problem like this from noisy data, regularization is paramount. The regularization properties of the algorithm are inherited from the discrete Fourier transform that is used first in the process.

7.1 Laurent–Padé approximants

Definition 7.1. *A* Padé approximant of type $[L/M]$ *is a rational function*

$$\frac{P^{[L/M]}(x)}{Q^{[L/M]}(x)} = \frac{\sum_{j=0}^{L} p_j x^j}{\sum_{j=0}^{M} q_j x^j}$$

that "matches" a given formal power series (Maclaurin series) up to order $L + M$*, that is,*

$$\frac{P^{[L/M]}(x)}{Q^{[L/M]}(x)} - \sum_{j=0}^{\infty} c_j x^j = O(x^{L+M+1}) \tag{7.1}$$

as $x \to 0$.[9]

Such $P^{[L/M]}$ and $Q^{[L/M]}$ exist if and only if the Hankel matrix

$$H_{[L/M]} = \begin{bmatrix} c_{L-M+1} & c_{L-M+2} & \cdots & c_L \\ c_{L-M+2} & c_{L-M+3} & \cdots & c_{L+1} \\ \vdots & & \ddots & \vdots \\ c_L & c_{L+1} & \cdots & c_{L+M-1} \end{bmatrix}, \tag{7.2}$$

[9]The order symbol $[L/M]$ is omitted from the coefficients for clarity. The coefficients of approximants of different order do not (necessarily) match. This also concerns the coefficients of the Laurent–Padé approximants in the next definition.

where $c_{-j} := 0$ for all $j = 1, 2, 3, \ldots$, is non-singular. In this case one may set $q_0 = 1$,

$$\begin{bmatrix} q_M \\ \vdots \\ q_1 \end{bmatrix} = -H_{[L/M]}^{-1} \begin{bmatrix} c_{L+1} \\ \vdots \\ c_{L+M} \end{bmatrix} \tag{7.3}$$

and

$$p_k = \sum_{j=0}^{\min(k,M)} q_j c_{k-j} \quad \text{for all } k = 0, \ldots, L. \tag{7.4}$$

If the matrix $H_{[L/M]}$ is singular, it is said that the corresponding Padé approximant does not exist. [4] The fact that the non-singularity of $H_{[L/M]}$ implies (7.1), (7.3) and (7.4) follows from elementary manipulations of series and linear algebra. The other possibility is not be considered here in any detail. If, in the numerical computations, the matrix should appear severely ill-conditioned, the algorithm is considered to have failed.

Definition 7.2. *A* Laurent–Padé approximant *of type* $[L/M]$ *of a formal Laurent series* $\sum_{j=-\infty}^{\infty} c_j z^j$ *is the (complex) function*

$$\frac{p^{[L/M]}(z)}{q^{[L/M]}(z)} = \frac{\sum_{j=-L}^{L} p_j z^j}{\sum_{j=-M}^{M} q_j z^j} \tag{7.5}$$

given by

$$\frac{p^{[L/M]}(z)}{q^{[L/M]}(z)} = \frac{\alpha^+(z)}{\beta^+(z)} + \frac{\alpha^-(z^{-1})}{\beta^-(z^{-1})}, \tag{7.6}$$

where $\alpha^{\pm}(z) = \sum_{j=0}^{\max(L,M)} \alpha_j^{\pm} z^j$, $\beta^{\pm}(z) = \sum_{j=0}^{M} \beta_j^{\pm} z^j$ *are polynomials whose coefficients satisfy, respectively,* (7.9) *and* (7.8) *below. [4, §7.4]*

Define

$$H_{[L/M],\pm} = \begin{bmatrix} c'_{\pm(L-M+1)} & c'_{\pm(L-M+2)} & \cdots & c'_{\pm L} \\ c'_{\pm(L-M+2)} & c'_{\pm(L-M+3)} & \cdots & c'_{\pm(L+1)} \\ \vdots & & \ddots & \vdots \\ c'_{\pm L} & c'_{\pm(L+1)} & \cdots & c'_{\pm(L+M-1)} \end{bmatrix}, \tag{7.7}$$

where $c'_0 = \frac{1}{2} c_0$ and $c'_j = c_j$ for all $j \neq 0$. Then

$$\begin{bmatrix} \beta_M^{\pm} \\ \vdots \\ \beta_1^{\pm} \end{bmatrix} = -H_{[L/M],\pm}^{-1} \begin{bmatrix} c_{\pm(L+1)} \\ \vdots \\ c_{\pm(L+M)} \end{bmatrix} \tag{7.8}$$

and

$$\alpha_k^{\pm} = \sum_{j=0}^{\min(k,M)} \beta_j^{\pm} c'_{k-j} \quad \text{for all } k = 0, \ldots, \max(L, M) \tag{7.9}$$

define the coefficients of the polynomials in the Laurent–Padé approximants of type $[L/M]$, provided that the matrices $H_{[L/M],+}$ and $H_{[L/M],-}$ are non-singular. It is not immediately obvious, but does follow from the definition, that the Laurent degree of the numerator p is reduced from $\max(L, M)$ to L when $L < M$ [4, §7.4].

Laurent–Padé approximants also match a formal Laurent series up to order $L+M$. Namely, a formal Laurent series can be split into two formal Maclaurin series as

$$f(z) = \sum_{j=-\infty}^{\infty} c_j z^j = f^+(z) + f^-(z^{-1}) = \sum_{j=0}^{\infty} c'_j z^j + \sum_{j=0}^{\infty} c'_{-j} (z^{-1})^j$$

where the coefficient c_0 is halved as above. The rational functions $\frac{\alpha^{\pm}(z)}{\beta^{\pm}(z)}$ in the Laurent–Padé approximant (7.6) (if exists) match $f^{\pm}$ up to order $L + M$. If $L \geq M$, then $\frac{\alpha^{\pm}(z)}{\beta^{\pm}(z)}$ are precisely the $[L/M]$ Padé approximants of $f^{\pm}$. If $L < M$ this is not the case, since $\alpha^{\pm}$ is of degree $\max(L, M) > L$, but the matching property still holds.

7.2 Sweep data reconstruction

The next corollary summarizes the previously devised key complex analytic properties of sweep data. As in Section 6, it is assumed, that the conductivity $\gamma = \sigma$ is determined by m homogeneous, simply connected[10] Lipschitz regular inclusions satisfying (6.1). Recall especially the connection (2.5) between the Fourier and Laurent coefficients, and Lemma 6.17.

Corollary 7.1. *The sweep data $\varsigma_\sigma : \partial D \to \mathbb{R}$ has a unique holomorphic extension to $\tilde{\Omega}$, that is, up to the boundary of the inclusions (and their reflections w.r.t ∂D). The extension satisfies the inversion symmetry* (6.33) *and is uniquely determined by the Fourier coefficients of $\varsigma_\sigma : [0, 2\pi) \to \mathbb{R}$.*

On the other hand, the idea of Laurent–Padé approximants is to "mimic" a Laurent series, and each Laurent–Padé approximant defines a complex rational function that is holomorphic outside the roots of its denominator. It is easy to see that, if calculated from real (non-complex) data, the Laurent–Padé

[10] A bounded set in $\mathbb{C}$ (or $\mathbb{R}^2$) is simply connected if and only if it has a connected complement.

approximants exhibit the symmetry (6.33). Moreover, it is known (see e.g. [4]) that for certain functions, many subsequences of Padé approximants converge to the functions whose power series they match. From the inverse problem point of view, it is also important that low-order Laurent–Padé approximants are computed from the low-order Fourier coefficients, which are insensitive to noise — a well-known regularizing property exploited by a plethora of numerical methods. This information suggests using low-order Laurent–Padé approximants for regularized reconstruction of the holomorphic extension of sweep data from noisy measurements.

It is also known that, for a family $\sigma(\varepsilon)$ of contracting Lipschitz regular homogeneous inclusions, the complex asymptotic expansion (6.44) holds. In addition, Corollary 6.23 states that, in this asymptotic expansion, each disk-shaped inclusion (Ω', κ) corresponds to a first-order pole, whose residue divided by a location-dependent factor is the corresponding net conductivity effect (6.46).

The following reconstruction method is based on a set of hypothesized properties of the Laurent–Padé approximants of sweep data, stemming from the characteristics of the complex small inclusion expansion. In particular, it is anticipated that the poles of the approximants reside near the inclusions (and their reflections outside the unit disk), and for each (first-order) pole $\xi \in D$ of an approximant $f = p^{[L/M]}/q^{[L/M]}$, the *effect*

$$\mu(f, \xi) = \frac{-\pi^2}{2} \frac{\operatorname{Res}(f, \xi)}{\frac{1}{\bar{\xi} - \bar{\zeta}_0} - \frac{\xi}{|\xi|^2 - 1}} \tag{7.10}$$

of the pole is viewed as an approximation to the net conductivity effect of (a part of) an inclusion near ξ. More precisely, if the poles $\xi_1, \ldots, \xi_k$ are located near an inclusion Ω' (of any shape), it is expected that

$$\rho(\Omega', \kappa) \approx \mu_{\mathrm{cl}}(f, \{\xi_1, \ldots, \xi_k\}) := \sum_{j=1}^{k} \mu(f, \xi_j). \tag{7.11}$$

It must be stressed that the suggested relationship between the inclusions and the poles in the approximant has not been proven to exist. Further considerations for closing this theoretical gap are omitted in this thesis. Instead, the usefulness of the method is assessed on the grounds of numerical experiments.

Formal extension of (7.5) yields

$$\frac{p^{[L/M]}(z)}{q^{[L/M]}(z)} = z^{M-L} \frac{\sum_{j=0}^{2L} p_{j-L} z^j}{\sum_{j=0}^{2M} q_{j-M} z^j} \tag{7.12}$$

which suggests that, when $L > M$ (respectively $L < M$), the $[L/M]$ Laurent–Padé approximant generally has a root (resp. pole) of multiplicity $z^{|L-M|}$ at $z = 0$. In case of backscatter data, the expansion corresponding to (6.38) always has a double root at the origin and therefore Laurent–Padé approximants of type $[(M-2)/M], M = 2, 3, 4, \ldots$ are used in [16]. This is not the case with sweep data and neither does the expansion (6.44) have a persistent pole at the origin. Thus the most natural choice for sweep data are the $[M/M]$ approximants.

It is assumed that the sweep data is given as $\varsigma_1, \ldots, \varsigma_N$, the noisy difference data (5.17) at N equispaced points

$$x_j = e^{\theta_j} = e^{\frac{2\pi i(j-1)}{N}}, \qquad j = 1, \ldots, N$$

on the unit circle. Approximations of the Fourier coefficients of the "underlying" sweep data ς_σ are computed as the discrete Fourier coefficients (2.7) of $\varsigma_1, \ldots, \varsigma_N$. Notice that the maximum sensible degree of an $[M/M]$ Laurent–Padé approximant for this kind of data is

$$M = \lfloor N/4 \rfloor. \tag{7.13}$$

For noisy data, the value of M is selected based on the desired amount of regularization. Namely, one can apply *Morozov's discrepancy principle*

$$\left\| \left(\frac{p^{[M/M]}(x_j)}{q^{[M/M]}(x_j)} - \varsigma_j \right)_{j=1}^{N} \right\| \approx \mathbb{E} \left\| (\varsigma(x_j) - \varsigma_j)_{j=1}^{N} \right\| \tag{7.14}$$

at the measurement points, that is, the proper choice of M is such that the residual is approximately equal to the expected value of the error in a suitable noise model (cf. [23]). Assuming the PE model (5.22) with, for simplicity, no model error, and normally distributed independent additive noise components $\eta_j \sim N(0, s^2)$, the expected error for the Euclidean norm is [11, §8.3]

$$\mathbb{E} \left| (\eta_j)_{j=1}^{N} \right| = \sqrt{s^2} \frac{\sqrt{2}\Gamma(\frac{1}{2}(N+1))}{\Gamma(\frac{1}{2}N)} \approx \sqrt{N s^2}, \tag{7.15}$$

where the exact expected value is the standard deviation $\sqrt{s^2}$ of the noise multiplied by the mean of the χ-distribution with N degrees of freedom. This is somewhat different from the approach in [16], where the error is studied in terms of the Fourier coefficients.

The final task is locating the poles and calculating the corresponding residues of the rational function (7.12) on the basis of the coefficients p_j, q_j. This can be done by first transforming (7.12) to a proper fraction by formal polynomial (long) division. Then one should find the roots of the denominator

polynomial $\tilde{q}(z)$ and finally calculate the residues as in e.g. [20, Vol. I §9]. In terms of linear algebra, finding the roots corresponds to finding the eigenvalues of the companion matrix of the polynomial.

Numerically, finding poles and residues is generally an ill-posed problem, especially in the case of multiple poles (poles of order greater than one). In fact, even constructing the (Laurent–)Padé approximants from the Laurent coefficients is a relatively unstable numerical problem [4]. These stability issues are not studied deeper here. It is merely anticipated that the regularization originating from the discrete Fourier transform suffices for obtaining reasonable low-order approximations with high probability. For example, the presence of multiple poles in numerical computations can be generally considered unlikely (cf. [16]).

7.3 Numerical examples

An ideal setting for testing the sweep data reconstruction method would be conducting physical measurements using real objects and EIT devices as described in Section 5.4. The second-best option is to simulate a realistic setup as accurately as possible using, for instance, the CEM model and finite element methods as done in [21]. Both of these can be quite laborious tasks and are not in the scope of this thesis.

There also exists somewhat simpler, layer-potential-based boundary element technique that has also been used in [21] and [19]. It can be understood in terms of the factorization (6.26), which can alternatively be written as

$$\Lambda_\sigma - \Lambda_{\mathbb{1}} = S_\Omega^N|_{\partial D}(E^* - K_\Omega^{N*})^{-1}A_\Omega.$$

To compute the value of the sweep data function ς_σ at $x_\theta \in \partial D = \partial B_1(0) \subset \mathbb{R}^2$ (for an arbitrary $x_0 \in \partial D$), one must evaluate the above operator for $\delta_{x_\theta} - \delta_{x_0}$ and then dual evaluate the resulting function with $\delta_{x_\theta} - \delta_{x_0}$ (recall Definition 5.4). First, notice that

$$\begin{aligned}\phi(y) &:= (A_\Omega(\delta_{x_\theta} - \delta_{x_0}))(y) \\ &= \nu_y \cdot \nabla_y(N_D(y, x_\theta) - N_D(y, x_0)) \\ &= -\frac{1}{\pi}\nu_y \cdot \left(\frac{y - x_\theta}{|y - x_\theta|^2} - \frac{y - x_0}{|y - x_0|^2}\right)\end{aligned}$$

can be computed exactly up to machine precision for any $y \in \partial\Omega$ such that ν_y exists (which holds a.e. on $\partial\Omega$). Then the idea is to solve a discretized version of the equation

$$(E^* - K_\Omega^{N*})\psi = \phi$$

and finally compute the value of the sweep data by numerically evaluating the integral

$$\begin{aligned}\varsigma_\sigma(x_\theta) &= \langle \delta_{x_\theta} - \delta_{x_0}, S_\Omega^N|_{\partial D}\psi\rangle = \int_{\partial\Omega} (N_D(\cdot, x_\theta) - N_D(\cdot, x_0))\psi \,\mathrm{d}s \\ &= -\frac{1}{\pi}\int_{\partial\Omega} (\log|y - x_\theta| - \log|y - x_0|)\psi(y)\,\mathrm{d}s_y.\end{aligned}$$

For details on the numerical techniques for such equations, consult e.g. [24].

The forward solver code that was used to simulate sweep data in the following numerical experiments was written by Lauri Harhanen and it utilizes the aforementioned technique. The numerical examples have been performed by first simulating sweep data at 320 equispaced boundary points (and 500 discretization points on each inclusion). The static electrode is at position $(1, 0)$, corresponding to angle $\theta_0 = 0$ in all experiments.

In the examples with *ideal data*, all simulated sweep data are used in the algorithm, and the order M of the Laurent–Padé approximant is selected by hand so that the "best" reconstructions are obtained. The purpose of these examples is to study and demonstrate upper limits for the performance of the method.

In the *noisy examples*, every 10th of the data is selected and noise — a sample of independent pseudo-random numbers $\eta_j \sim N(0, s^2)$, $j = 1, \ldots, N$ — is added to simulate a sweep measurement at $N = 32$ equispaced points corresponding to different locations of the moving electrode. Reconstructions are then computed using progressively higher orders M of the Laurent–Padé approximation until (in the noisy examples) the residual error satisfies the stopping condition

$$\left(\sum_{j=1}^{N} \left| \frac{p^{[M/M]}(x_j)}{q^{[M/M]}(x_j)} - \varsigma_j \right|^2 \right)^{\frac{1}{2}} < \frac{3}{2}\sqrt{Ns^2} \tag{7.16}$$

adapted from the discrepancy principle (7.14). The noise level is set to the, somewhat arbitrarily chosen, constant

$$\sqrt{s^2} = 5 \cdot 10^{-4},$$

that corresponds to ca. 1–2 % of the maximum value of the ideal sweep data in the experiments. It is important to notice that in real (or CEM simulated) sweep measurements, as defined in Section 5.4, the magnitude of the difference $|\varsigma_j| = |V_j^\sigma - V_j^{\mathbb{1}}|$ could be small compared to the voltages $|V_j^\sigma|$, $|V_j^{\mathbb{1}}|$. In this case a relatively small error in e.g. V_j^σ would correspond to a large error in the

sweep measurement. In particular, achieving a noise level of 2% in the sweep measurement could correspond to a noise level order of magnitude smaller in the actual voltage measurements. The quantitative relationship between the noise levels depends on the sizes of the electrodes and cannot be studied further within the PEM framework only.

Example 7.2 (Ideal data). Figure 3(a) shows a conductivity phantom σ defined by three resistive inclusions, and Figure 3(b) is a plot of the corresponding sweep data $\varsigma_\sigma(\zeta)$, $\zeta \in \partial D$. The reconstruction with $M = 13$ is illustrated in Figure 3(c). For each pole $\xi \in D$ in the Laurent–Padé approximant, the effect μ has been computed from the residue using (7.10). In this and all the following reconstruction plots, the pairs (ξ, μ) have a special visual representation which comprises a circle and a direction pointer. The circle is centred at ξ and its radius is $r_{\min}(|\mu|)$, the lower radius limit (6.47) for a net conductivity effect μ. The direction of the pointer represents the phase $\angle\mu$. On the boundary $\zeta \in \partial D$, the Laurent–Padé approximant $p^{[M/M]}(\zeta)/q^{[M/M]}(\zeta)$ matches the ideal sweep data almost up to machine precision, and is not plotted separately in Figure 3(b).

Figure 3(d) shows a certain clusterization of the poles. In this case, there is a clear one-to-one correspondence between the clusters and the inclusions. Each cluster is assigned a pair $(\xi_{\mathrm{cl}}, \mu_{\mathrm{cl}})$,

$$\mu_{\mathrm{cl}} = \sum_j \mu_j, \qquad \xi_{\mathrm{cl}} = \frac{\sum_j \xi_j |\mu_j|}{\sum_j |\mu_j|}$$

where the sums are taken over all the poles in the cluster. The pairs $(\xi_{\mathrm{cl}}, \mu_{\mathrm{cl}})$ are visualized as above. A left-pointing μ_{cl} indicates a resistive inclusion. In Table 1, the values μ_{cl} are compared to the actual real-valued net conductivity effects of the inclusions, whose locations are specified by their mass centres

$$\xi_{\mathrm{CM}} = \fint_\Omega \xi \, \mathrm{d}\xi.$$

In this example, the cluster centre points ξ_{cl} are displaced from the actual mass centres ξ_{CM} by up to some hundredths of a unit. These numbers are omitted from the table for clarity.

Example 7.3 (Ideal data). An experiment with a more complicated conductivity phantom is illustrated in Figure 4; here $M = 20$. In this case, some poles have $|\mu| < 10^{-4}$ and they are marked with small crosses, because the corresponding circle of radius $r_{\min} \approx 0.005$ would not be visible. This also reflects the fact that the effect of these poles on the overall net conductivity effects is negligible. As in Example 7.2, the net conductivity effects of the inclusions are compared to the cluster values μ_{cl} in Table 2.

Table 1: Comparison of μ_{cl} and ρ for ideal data — Example 7.2.

inclusion	location	$\rho \cdot 10^2$	$\mu_{\mathrm{cl}} \cdot 10^2$	$\frac{\lvert\rho-\mu_{\mathrm{cl}}\rvert}{\lvert\rho\rvert} \cdot 10^2$
ellipse	0.10-0.60i	-1.26	-1.33-0.05i	7.38
disk	-0.50+0.20i	-0.377	-0.37+0.01i	2.82
box	-0.20+0.30i	-0.4	-0.40-0.00i	0.20

Table 2: Comparison of μ_{cl} and ρ for ideal data — Example 7.3.

inclusion	location	$\rho \cdot 10^2$	$\mu_{\mathrm{cl}} \cdot 10^2$	$\frac{\lvert\rho-\mu_{\mathrm{cl}}\rvert}{\lvert\rho\rvert} \cdot 10^2$
polygon	0.41-0.38i	-1.17	-1.16-0.10i	8.86
box	-0.10+0.70i	1	0.97+0.04i	5.16
ellipse	0.00-0.60i	1.26	1.27+0.03i	2.54
disk	-0.50+0.20i	0.168	0.15-0.00i	10.37
disk	-0.30+0.37i	-0.513	-0.50-0.04i	7.90
disk	-0.30+0.03i	-0.513	-0.50+0.01i	2.20

The above examples indicate that ideal sweep data can be used to locate multiple inclusions and also provide some information on their shapes and sizes. In particular, it appears that the method works even though the inclusions are not particularly small. In the case of polygonal inclusions, the poles seem to be attracted to the vertices. For circular inclusions, the poles locate the centre point, and ellipses seem to attract several poles. Due to the ill-posedness of approximant construction and pole location, it is possible that the round-off errors in the computer floating point arithmetic already have a significant influence on the locations of some poles in these reconstructions.

It is not clear from Figures 3(c) and 4(c), whether or not (7.11) is a good approximation, but Tables 1 and 2 show that the relative error between the effects μ_{cl} of the clusters and the actual net conductivity effects ρ of the inclusions is, at worse, around 10% in these examples. Generally, clusterization is a rather involved problem, and the purpose of the clusterized examples is not to suggest such a procedure as a routine part of the reconstruction method, but to show that a certain interpretation allows to view the effects of the poles as parts of the net conductivity effects of the inclusions, even in the case of relatively complicated geometries. For instance, no effect μ of an individual pole explains the majority of the ρ of any elliptic inclusion in the above experiments, but the overall effects μ_{cl} of the poles in the corresponding clusters are decent approximations of the net conductivity effects of the ellipses.

The following examples further demonstrate the usefulness of the effects μ

in case of low-order noisy reconstructions.

Example 7.4 (Noisy data). Figure 5(a) shows the ideal sweep data $\varsigma_\sigma(\zeta)$, $\zeta \in \partial D$ for the conductivity phantom in Figure 3(a), a noisy measurement sample and a Laurent–Padé approximant $p^{[M/M]}(\zeta)/q^{[M/M]}(\zeta)$ for $\zeta \in \partial D$. The poles and effects corresponding to this approximant are shown in Figure 5(b). The five other pole-effect reconstructions in Figure 5 correspond to other realizations of the noisy measurement for the same sweep data. In all these reconstructions, the discrepancy principle (7.16) yielded an regularized value of $M = 2$ for the order of the approximation.

Example 7.5 (Noisy data). Figure 6 shows noisy reconstructions corresponding to the conductivity phantom of Figure 4(a). In this case the same discrepancy principle (7.16) yields a higher reconstruction order $M = 3$ to explain the data.

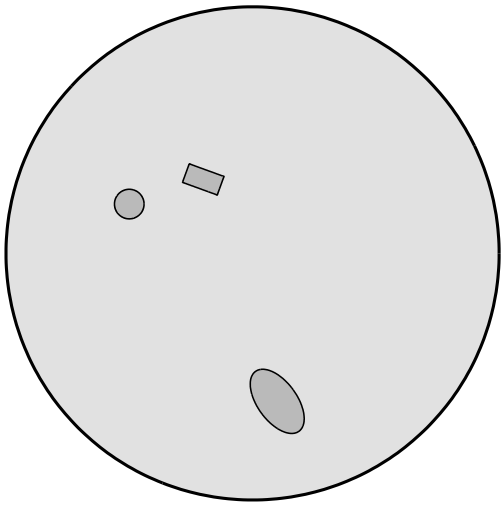
Even in the noisy case, the sweep data curve is reproduced well by the Laurent–Padé approximants, as seen in Figures 5(a) and 6(a). In Figure 5, the ellipse in the Southern half of the object is located accurately by one pole. The inclusions in the Northern half seem to be simultaneously tracked by the other pole; recall that in Figure 5, $M = 2$ and there are thus two poles inside D. In Figure 6, the outermost large inclusions are located relatively well, but the smaller inclusions deeper inside the object are not detected. In these low-order approximations, the effect μ of each pole is a decent approximation of the net conductivity effect ρ of some inclusion or inclusion group. Especially, it is clear from the reconstructions in Figure 5, that the detected inclusions are resistive. Likewise in Figure 6, the reconstructions show that there are conductive inclusions in the Northern and Southern parts of the object and resistive inclusions in the SE part.

By these examples, one might think that by a reasonable reduction of noise level, it would be possible to, for instance, locate all six inclusions in the conductivity phantom of Example 7.5. In actuality, it was not possible to locate all the inclusions from $N = 32$ measurements even if they were completely free of noise. For an increased resolution of $N = 64$, the noise level needed to be reduced by over four orders of magnitude to 10^{-8} for reasonably reliable reconstructions featuring all the inclusions. This reflects the general ill-posedness of the EIT inverse problems. From the theoretical point of view, Examples 7.4 and 7.5 show a rather successful application of regularization with the purpose of obtaining approximate solutions to an ill-posed problem.

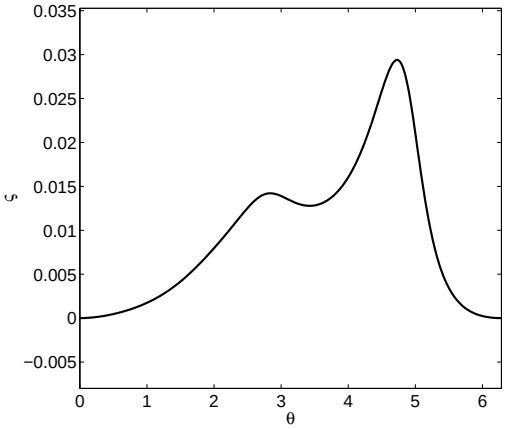
Example 7.6. Using this reconstruction algorithm, it is also possible to appraise, whether or not there are inclusions at all. In the case of a homogeneous object, the sweep data vanishes by definition. For the corresponding noisy measurement, the discrepancy principle should select the [0/0] approximant, i.e. the constant corresponding to the mean of the data, which has no poles. This

naturally indicates the absence of inclusions. On the other hand, it may occur that the sweep data caused by small enough inclusions cannot be distinguished from noisy background, which then gives a false negative result for the presence of inclusions. This is demonstrated in Figure 7. Alternatively, other statistical tests could be used to see, how likely the measurement is to represent a noisy realization of zero sweep data.

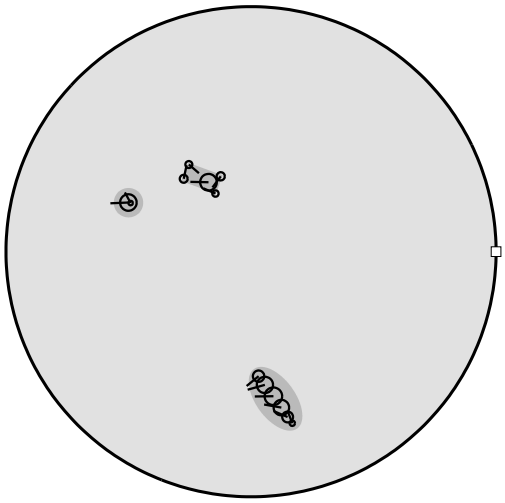
From the computational complexity point of view, the intensive parts of the method are calculating the discrete Fourier transform of the data (of N points) and an eigendecomposition of an $M \times M$ matrix in order to resolve the locations of the poles. The DFT can be computed efficiently for large datasets (e.g. $N > 10^6$ on current personal computers) using the Fast Fourier Transform (FFT) algorithms [9, §30.2]. In addition, on the grounds of the ill-posedness of the problem, it seems that only low-order reconstructions with, say, $M < 10$ are reasonable candidates for practical use with noisy data, and for matrices of this size, the required operations can be computed in milliseconds. In other words, the method is extremely fast when applied to any realistic noisy data.



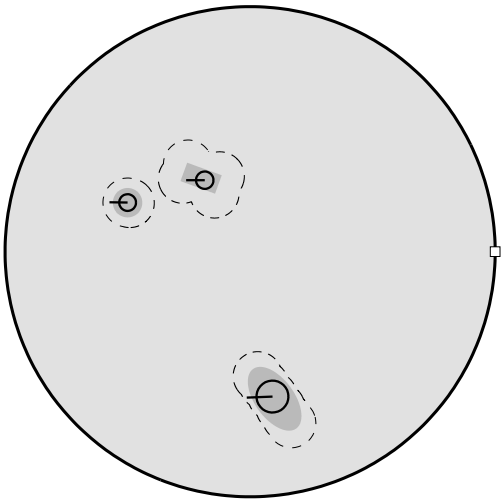
(a) Conductivity phantom, all inclusions have $\kappa = 1/2$.



(b) Sweep data.

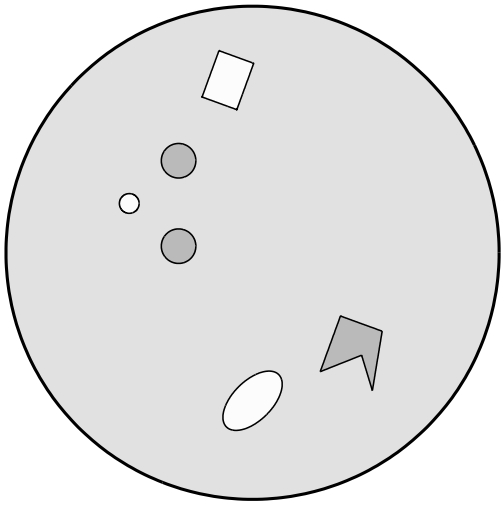


(c) Reconstruction with $M = 13$.

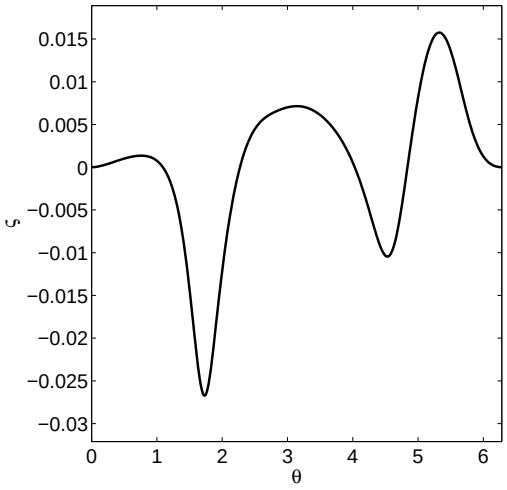


(d) Clusterized reconstruction. The dashed lines show the cluster boundaries.

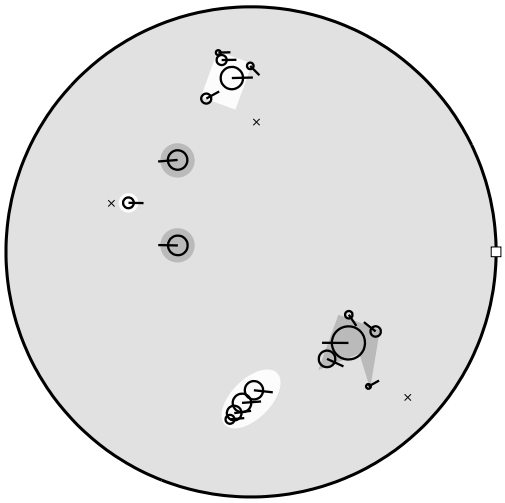
Figure 3: Reconstruction of the conductivity phantom (a) from ideal sweep data (b). The pole-effect pairs (ξ, μ) and $(\xi_{\mathrm{cl}}, \mu_{\mathrm{cl}})$ are visualized as explained in Example 7.2. The small square on the boundary marks the position of the static electrode.



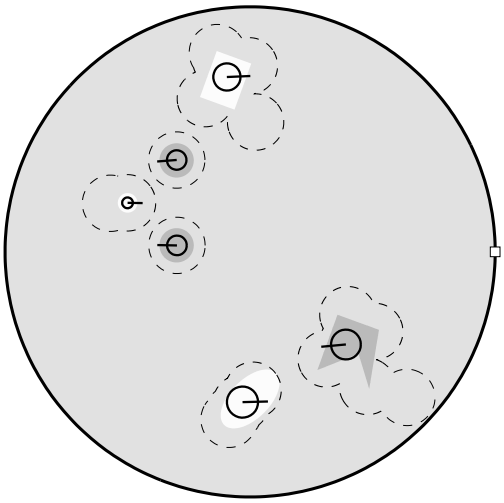
(a) Conductivity phantom. The box, the ellipse and the smallest disk have $\kappa = 2$. The conductivity of the other inclusions is $\kappa = 1/2$.



(b) Sweep data.

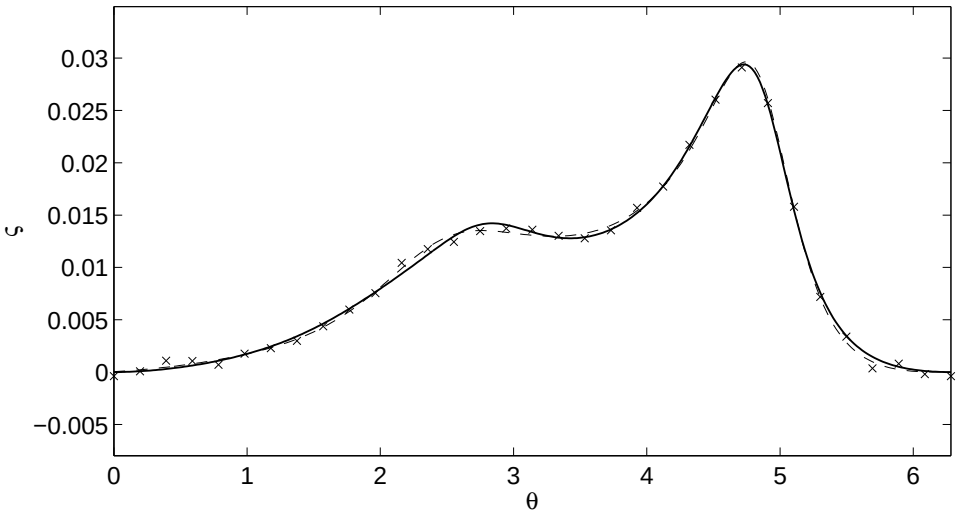


(c) Reconstruction with $M = 20$. The three poles marked by small crosses have a "subvisual" $|\mu| < 10^{-4}$.

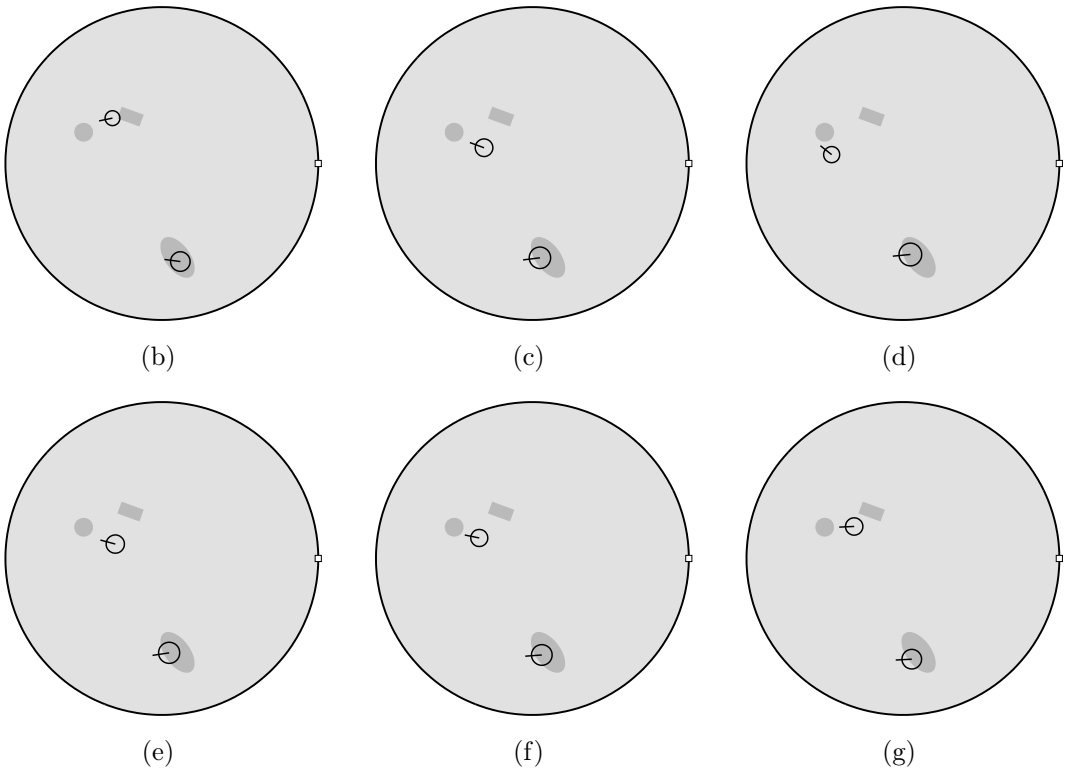


(d) Clusterized reconstruction.

Figure 4: Reconstruction of the conductivity phantom (a) from ideal sweep data (b). Poles and clusters are represented as in Figure 3.



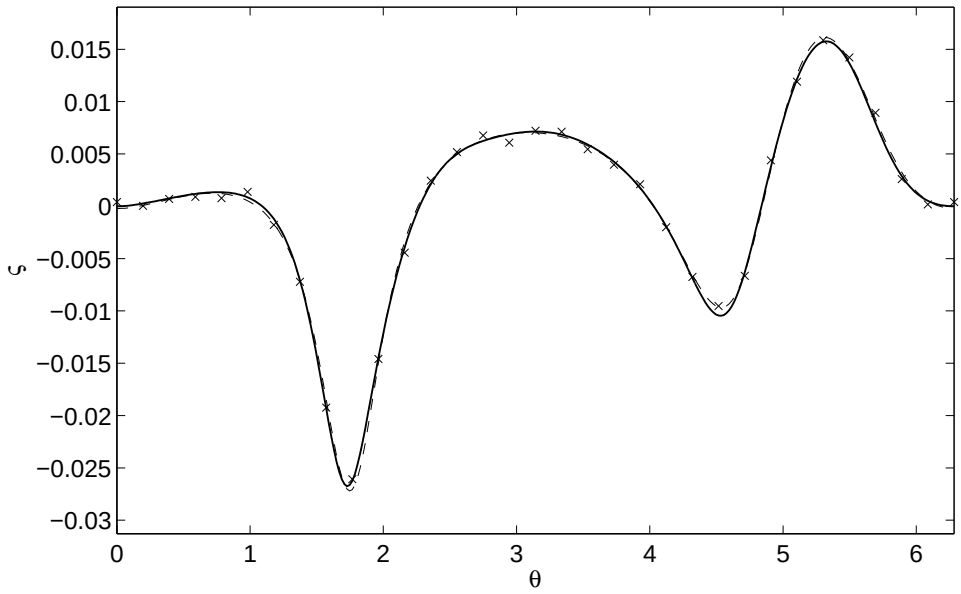
(a) Sweep data (solid line), $N = 32$ noisy samples (x) and Laurent–Padé-approximant with $M = 2$ (dashed).



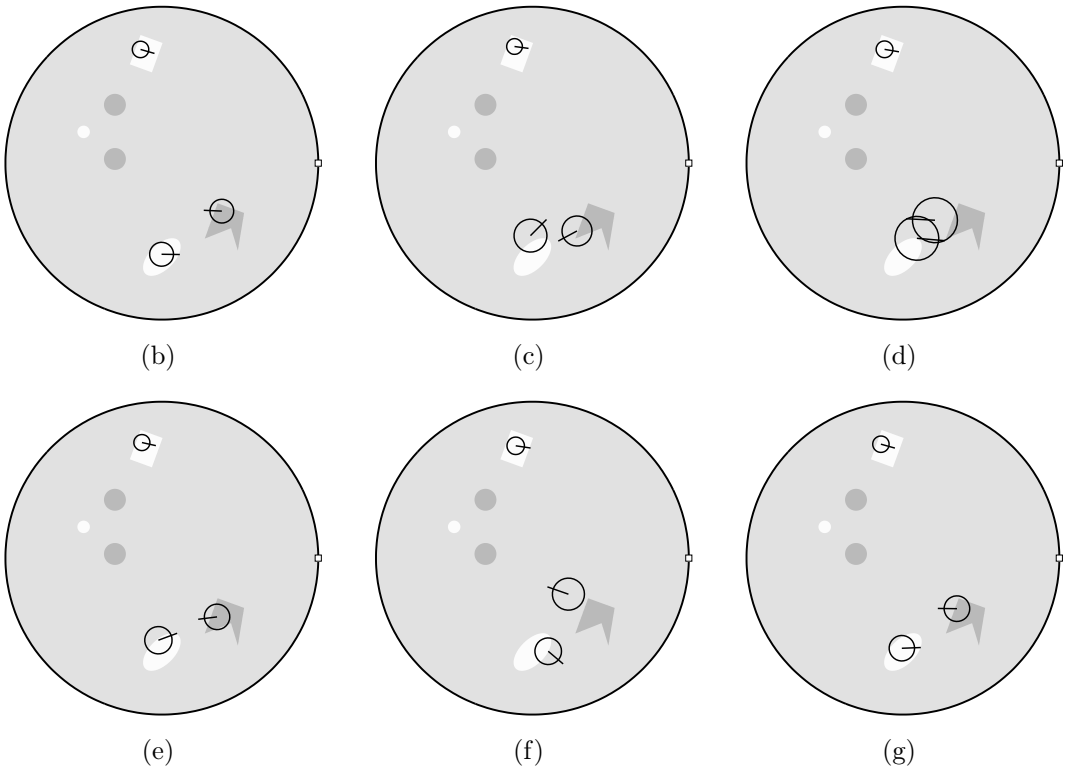
(b) (c) (d)

(e) (f) (g)

Figure 5: Reconstructions from noisy sweep data. (b) corresponds to the Laurent–Padé approximant in (a) and (c)–(g) are reconstructions from different noisy samples with the same parameters.



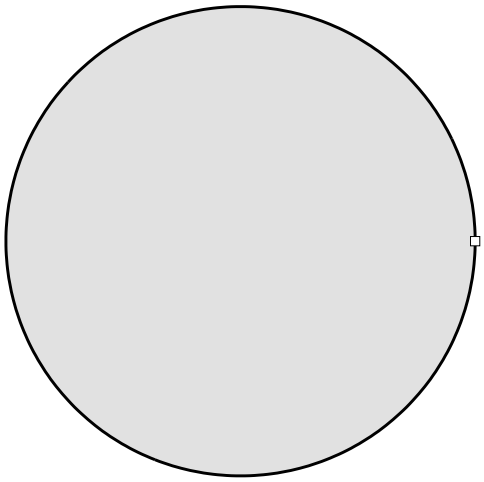
(a) Sweep data (solid line), $N = 32$ noisy samples (x) and Laurent–Padé approximant with $M = 3$ (dashed).



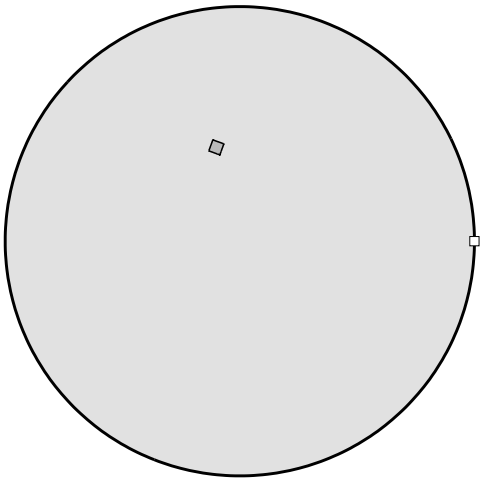
(b) (c) (d)

(e) (f) (g)

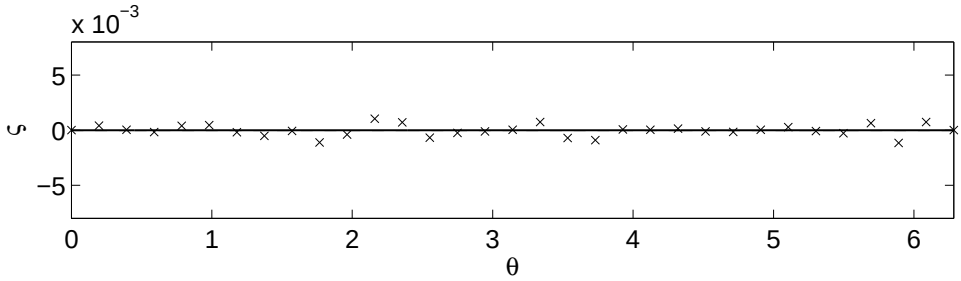
Figure 6: Reconstructions from noisy sweep data. (b) corresponds to (a) and (c)–(g) are reconstructions from different noisy samples.



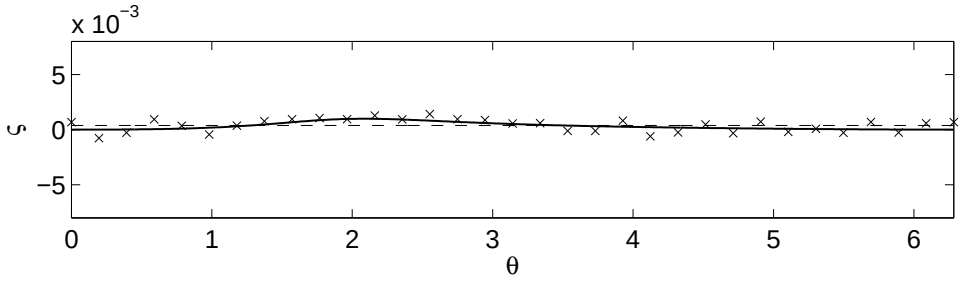
(a) Homogeneous object.



(b) A small inclusion with $\kappa = 1/2$.



(c) Sweep data (solid line), noisy samples (x) and Laurent–Padé approximant with $M = 0$ for the homogeneous object.



(d) Sweep data, samples and Laurent–Padé approximant with $M = 0$ for the object with one small inclusion.

Figure 7: The total effect of a small inclusion on noisy background. With this noise level, the reconstruction method does not detect the inclusion.

8 Concluding remarks

In Section 6, layer potential techniques were used to devise the factorization (6.23) for the difference Neumann-to-Dirichlet map $\Lambda_\sigma - \Lambda_{\mathbb{1}}$. This factorization was proven valid in the case where the real conductivity σ (i.e. the direct current case) consists of $\mathscr{C}^1$ regular or sufficiently small Lipschitz regular homogeneous inclusions, and for distributional input currents. An asymptotic expansion of the bilinear form $\langle \cdot, (\Lambda_\sigma - \Lambda_{\mathbb{1}})\cdot \rangle$ in terms of polarization tensors of the inclusions was given in Theorem 6.16. These tools were used to prove a holomorphic continuation property and to formulate a complex asymptotic expansion (6.44) for sweep data.

The introduced theoretical properties were subsequently used (in Section 7) to aid the design of a reconstruction algorithm for noisy sweep data. The algorithm is not entirely supported by theory, but is based on hypothesized properties of the utilized Laurent–Padé approximants. The performance of the method was assessed by numerical experiments, which seemed to justify the basic operational principle. The algorithm is based on a similar reconstruction method for backscatter data, presented by Hanke in [16]. Regarding the locations of the poles in the approximant, which is the basis of the inclusion-locating properties in both algorithms, the numerical results were comparable to those in [16], and withal the method devised here also showed the capability of extracting information about the conductivities and sizes of the inclusions. Namely, a quantity (6.46) dubbed the *net conductivity effect* was approximately reconstructed for individual inclusions, which is essentially a new result. As a measure combining the size and conductivity of an inclusion, the net conductivity effect could certainly be relevant in practical EIT.

For evaluating the the feasibility of the reconstruction algorithm for real measurements described in Section 5.4, further testing with more realistic CEM simulations or actual real-world measurements are required. Nevertheless, the numerical experiments conducted here suggest that the algorithm could be useful even with a modest resolution of 32 measurement points, provided that the noise level can be kept sufficiently low.

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