Coefficient Sensitivity of Polynomial-Predictive FIR Differentiators: Analysis

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Abstract — **In this paper, coefficient sensitivity of polynomial-predictive FIR differentiators (PPFD) is investigated for implementations in fixed point environments, and advantageous guidelines for creating robust designs are given. By joint selection of an appropriate filter length and implementation structure, sufficient prediction and differentiation properties can be maintained even in short word-length implementations.**

I. INTRODUCTION

By their nature, digital devices handle numbers using a finite number of bits per digit [1]. On the other hand, digital filters are typically designed using general purpose computers. When the target application has the same computation precision as the filter design environment, there are usually no implementation problems if the filter itself was appropriately designed. Many times this is not the case, however, but the filters are operating within inexpensive (fixed-point) processors, or in embedded applications using highly optimized application specific integrated circuit (ASIC) designs. In these cases, there might be a great difference between the calculation precisions of the filter design environment and the final operation platform. This obviously results in filter quality degradation and possibly even in totally unintended kind of filtering operation. In this paper, this important matter is addressed for polynomial-predictive FIR differentiators [2], and practical guidelines for short word length robust implementations are given. In [3], a genetic algorithm based approach to quantized-coefficient predictive FIR differentiator design is given by the authors.

In many engineering disciplines, accurate control of processes is absolutely necessary. In turn many of the real world physical process parameters exhibit more or less smooth transitions. Noisy measurements of these parameters are then used for process control after a delay. Our examples of closed loop control include motion control of an elevator car [2,4], and mobile phone power control [5]. In the latter, the inherent closed loop control delays make it a lucrative environment to apply polynomial-predictive techniques since the received power fluctuations can in many cases be modeled as Rayleigh distributed signals which in turn can be accurately modeled as piece-wise low-degree polynomials. Accurate control of an elevator car can effectively utilize, not only predicted position, but also predicted velocity and acceleration information. This information can be made available to the controller by predictive differentiation. Here again, the position and velocity of the elevator car can be accurately modeled as piece-wise low-degree polynomials.

In Section II, polynomial predictive FIR differentiators are shortly reviewed. Coefficient quantization is discussed in Section III where also methods for robust implementation are proposed. Section IV concludes the paper.

II. PREDICTIVE FIR DIFFERENTIATORS

General predictive filtering theory has been well-established [1]. Here we concentrate on polynomial-predictive FIR differentiators. Heinonen-Neuvo (H-N) polynomial predictors [6] assume a low-degree polynomial input signal contaminated by Gaussian noise. They provide for exact prediction while minimizing the noise gain

$$NG = \sum_{k=-\infty}^{\infty} \left|h(k)\right|^2 \tag{1}$$

where $h(k)$ are the filter coefficients. Predictive behavior can be identified from a group delay plot as a negative group delay region near the zero frequency; Fig. 1 b) illustrates this for the PPFD of length $N = 11$. In [2] polynomial differentiators are derived similarly to H-N predictors, except that the filter output is defined to be a predicted derivative of the input [2]:

$$\sum_{k=0}^{N-1} h(k)x(n-k) = \dot{x}(n+m) \tag{2}$$

where $x(n)$ are filter input samples, m is a prediction step, N is filter length, and the dot denotes time derivative. From (2) a set of linear equations on filter coefficients can be derived. They can be solved using the method of Lagrange multipliers [2] to yield closed form expressions for the filter coefficients. For the second degree one-step-ahead PPFDs, the filter coefficients are given by [2]

$$h(k) = \frac{6\left[(30N-30)k^2 + (-32N^2+38)k + 6N^3 - 11N^2 - 9N + 14\right]}{(N-2)(N-1)N(N+1)(n+2)},$$
$$k = 0, 1, \ldots, N-1 . \tag{3}$$

In [4], a feedback extension to FIR differentiators is given to provide considerable noise attenuation while maintaining the prediction and differentiation properties. Short word-length robust predictive FIR differentiators given in this paper are good basis filters for the feedback extension of [4].

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III. Coefficient Quantization

Differentiation and especially prediction properties of the PPFDs are highly affected by the available computation precision. In this Section, it is illustrated that the conventional methods for creating word length robust implementations are very well applicable also to predictive FIR differentiators. We consider three cases; naturally robust filters, lattice implementations, and filter sectioning. For fixed-point presentation of filter coefficients, two's complement presentation is used with magnitude truncation. Location of the binary point is set so that maximum accuracy is achieved given the range of filter coefficient values. PPFDs [2] for second degree polynomial input signals are considered. Differentiators for first and third order polynomials are to be treated analogously in the filter design and implementation process.

In Fig. 1 a), a typical example of exact and degraded frequency responses is shown for a one-step-ahead predictive FIR differentiator of length $N = 11$ with the quantized coefficient word length of 8 bits, along with the ideal differentiator frequency response. The corresponding group delays are shown in Fig. 1 b).

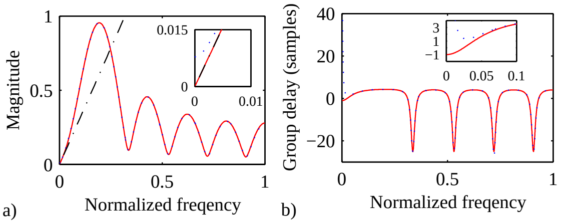

Fig. 1. a) Frequency responses of the one-step-ahead PPFD of length $N = 11$ with quantized coefficient word length of 8 bits (dotted) and with exact coefficients (solid), along with the frequency response of the ideal differentiator (dash-dot). b) Group delay of the same filter with quantized (dotted) and exact (solid) coefficients.

As also seen in the example in Fig. 1, the differentiation property is generally more robust to the coefficient quantization than the prediction property which can be lost already with the coefficient word length of 16 bits. Second degree polynomial differentiation property is set by zero magnitude response at zero frequency along with a ramp-shaped response within a desired differentiation band. The prediction property can be seen as the negative unity group delay at the zero frequency, Fig. 1 b).

A. Born to Be Robust

There exists a small set of predictive FIR differentiators that are word length robust by nature, i.e., their exact coefficients lie within a close vicinity of some quantized values. These differentiators can be found by an exhaustive search which is well feasible since a possible set of filters to take into account in the search could include filters for polynomials of degrees 1, 2, 3, and lengths $N = 3, \ldots, 100$, for example. Three is the minimum number of coefficients that are required in the second degree case.

Magnitude response F, group delay G, and noise gain NG errors are defined, respectively, as

$$e_F = \left|F(0) - F_{quant}(0)\right| \tag{3}$$

$$e_G = \left|G(\varepsilon) - G_{quant}(\varepsilon)\right| \tag{4}$$

$$e_{NG} = NG - NG_{quant} \tag{5}$$

where $F(0)$ denotes the value of the magnitude response at zero frequency, $G(\varepsilon)$ is the group delay at normalized frequency ε, here $\varepsilon = 0.0001$, and the noise gain NG is given by (1). Quantities with the subscript *quant*, refer to the filters with quantized coefficients. Calculating group delay values at normalized frequency $\varepsilon > 0$ (4), is due to the numerical problems near zero frequency. In Figs. 2 and 3, the normalized errors of the magnitude response, group delay, and noise gain, (3), (4), and (5), respectively, for the second degree PPFDs of lengths $N = 3, \ldots, 100$ with coefficients quantized to 8 and 16 bits, respectively, are shown normalized by the largest absolute errors appearing on each curve.

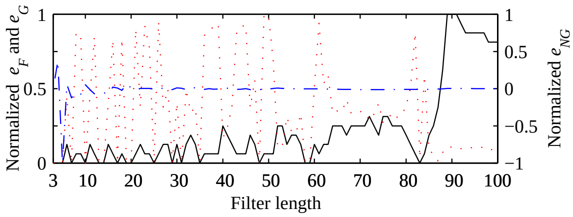

Fig. 2. Normalized magnitude response (solid), group delay (dotted) and noise gain (dash-dot) errors for the second degree PPFDs of lengths $N = 3, \ldots, 100$ with coefficients quantized to eight bits.

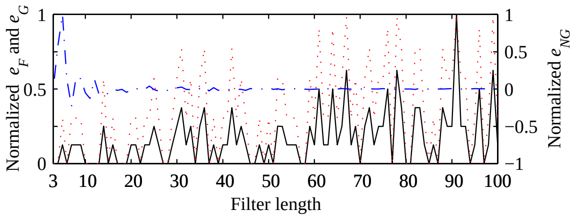

Fig. 3. Normalized magnitude response (solid), group delay (dotted) and noise gain (dash-dot) errors for the second degree PPFDs of lengths $N = 3, \ldots, 100$ with coefficients quantized to 16 bits.

Ten best filters according to each error type are listed in Tables 1 and 2 for the coefficient quantizations to 8 and 16 bits, respectively. Filters appearing in both of the lists for magnitude response and group delay errors are shaded similarly; they are the best choices for short word length fixed point applications. For the reference, the corresponding errors for the filter length $N = 11$, Fig. 1, are also listed. In all the Tables 1 through 4, also the errors for the filter of length $N = 40$ are shown for error magnitude comparisons. The filter of length $N = 40$, represents an "arbitrarily" chosen filter that is not one of the best nor one of the worst filters if the coefficients are quantized. Magnitudes of the errors for the filters of lengths $N = 11$ and $N = 40$ clearly demonstrate that attention has to be paid to the coefficient word-length effects.

In Tables 1 and 2, the errors are shown unnormalized. The quantized coefficients of the second degree polynomial predictive FIR differentiator of length $N = 3$ are still exact, the group delay error of the filter appearing in all the Tables 1 through 4 is due to calculation of the error at ε (4). Its frequency response and group delay are shown in Fig. 4. If these responses are adequate for your design, this is the filter of your choice. Besides, $N = 3$ is a good basis for the feedback extension of [4].

The original filters [2] were designed to minimize noise gain (1) while providing for exact prediction and differentiation at zero frequency. Therefore, a negative noise gain error in Tables 1 and 2, and also in Tables 3 and 4, implies that some amount of prediction and/or differentiation properties has necessarily been lost since the noise gain less than that of the optimized filters has been obtained.

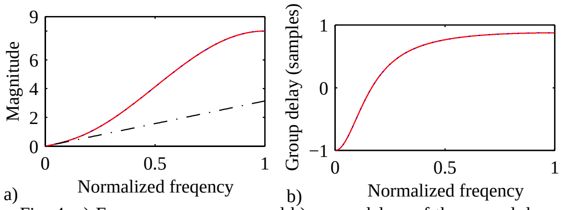

Fig. 4. a) Frequency responses and b) group delays of the second degree one-step-ahead PPFD of length $N = 3$ with exact coefficients (solid) and quantized to 8 bits (dotted). In a) also the ideal differentiator frequency response is shown (dash-dot).

B. Robust Lattices

Lattice structures are known to be often more robust to the finite word-length effects than the direct-form implementations, though this does not always hold [1]. In the case of predictive FIR differentiators, we can find several more robust filters by using lattice structures than by direct implementation. Still, the effects have to be verified filter by filter since using the lattice structure does not give any guarantees that the resulting filter would actually be robust to coefficient quantization.

In Figs. 5 and 6, the magnitude response, group delay, and noise gain errors, (3), (4), and (5), respectively, are shown for the second degree polynomial predictive FIR differentiators of lengths $N = 3, \ldots, 100$ with coefficients quantized to 8 and 16 bits, respectively. The errors are normalized with the largest absolute errors appearing on each curve. From Figs. 5 and 6, it is seen that with lattice structures, robustness with respect to magnitude response error does not imply robustness against the group delay error. With direct-form implementations, Figs. 2 and 3, simultaneous robustness against both errors was evident. In Fig. 7, the same filter as in Fig. 1 is shown for the case of coefficient quantization to 8 bits. In Fig. 7, the frequency response at zero frequency is very close to zero but the one-step-ahead prediction at zero frequency has been lost.

Ten best filters according to each error are listed in Tables 3 and 4 for the coefficient quantizations to 8 and 16 bits, respectively. For the reference, the corresponding errors for the filter length $N = 11$, c.f. Fig. 7, for the coefficient precision of 8 bits, are also listed. In Tables 3 and 4, the errors are shown unnormalized. Comparing Tables 1 and 3, and 2 and 4, respectively, it is seen that although lattice structures yield filters more robust against magnitude response errors, the same is generally not true for the group delay errors, i.e., at least in this case, lattice structures are not very effective in retaining phase characteristics in coefficient quantization.

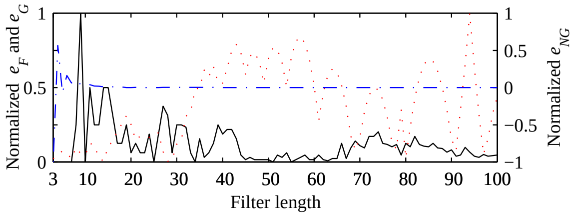

Fig. 5. Normalized magnitude response (solid), group delay (dotted) and noise gain (dash-dot) errors of the second degree lattice PPFDs of lengths $N = 3, \ldots, 100$ with coefficients quantized to eight bits.

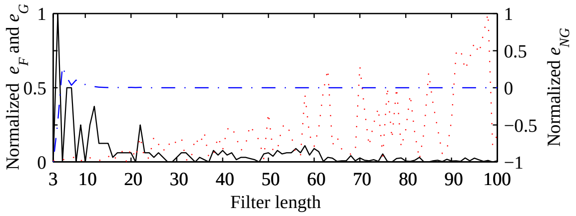

Fig. 6. Normalized magnitude response (solid), group delay (dotted) and noise gain (dash-dot) errors of the second degree lattice PPFDs of lengths $N = 3, \ldots, 100$ with coefficients quantized to 16 bits.

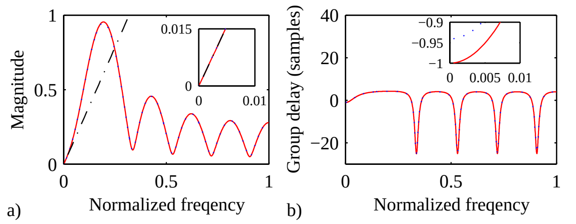

Fig. 7. a) Frequency response of the lattice second degree one-step-ahead PPFD of length $N = 11$ with coefficient word length of 8 bits (dotted) along with frequency responses of the corresponding unquantized (solid) and the optimum (dash-dot) differentiators. b) Group delay of the same filter with quantized (dotted) and exact (solid) coefficients.

C. Robust Sectioning

It is generally known that forming FIRs of first and second order sections could also result in more robust predictive differentiators [1]. In our case, the sectioning of the filters is given by the nature of the filters; they are predictive bandlimited differentiators. The differentiation property is determined by the zero at real unity. This forms the first section $D(z)$ of our cascade filter. Next, prediction in the second degree case is determined by the zero on the real axis at p

between zero and one. This forms the second section $P(z)$ in our cascade. Finally, the rest of the zeros of the FIR are used for overall spectral shaping $F(z)$ which here means minimizing the noise gain (1). The frequency response of the sectioned filters is thus given by

$$H(z) = D(z)P(z)F(z) = (1 - z^{-1})(1 - pz^{-1})F(z) \qquad (6)$$

where $0 < p < 1$. By nature, this sectioning decouples the dependencies which otherwise exist between all the zeros and all the quantized coefficients. Without sectioning, quantization of all the coefficients affects the locations of all the zeros, which results in a highly complex system to optimize, should one desire to set zeros exactly to real unity and p, and at the same time obtain desired spectral shaping properties.

IV. Conclusions

In this paper, we have shown that although using short word lengths implies degraded filtering performance, it is possible to select the length and implementation method of polynomial-predictive FIR differentiators so that the final production filter with quantized coefficients is sufficiently good for many applications. This can be done by traditional means; by lattice implementations or by filter sectioning. Also, some naturally word length robust predictive FIR differentiators are identified. It has been clearly shown that attention has to be paid to coefficient quantization effects of polynomial-predictive FIR differentiators, but with the correct selection of filter length and implementation form, filters with excellent prediction and differentiation properties can be found even with coefficients quantized to 8 bits.

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TABLE I

Ten most robust second degree PPFDs between lengths $N = 3, \ldots, 100$, based on their frequency response, group delay, and noise gain errors, e_F, e_G, and e_{NG}, respectively. The coefficient word length is set to 8 bits. Also shown are the corresponding errors for the filter length $N = 11$, Fig. 1, with coefficients quantized to eight bits. Filters appearing in both the lists for e_F and e_G are shaded similarly. Errors for the filter length $N = 40$ are shown for error magnitude comparisons regarding Fig. 2.

N	3	4	14	58	48	83	59	29	35	17	11	40
e_F	0	0	0	0	$8.67\cdot10^{-19}$	$8.67\cdot10^{-19}$	$2.60\cdot10^{-18}$	$3.47\cdot10^{-18}$	$5.20\cdot10^{-18}$	$6.94\cdot10^{-18}$	$7.81\cdot10^{-3}$	$1.56\cdot10^{-2}$
N	3	25	7	10	5	4	48	13	14	20	11	40
e_G	$2.22\cdot10^{-8}$	$1.95\cdot10^{-3}$	$2.71\cdot10^{-2}$	$3.31\cdot10^{-2}$	$3.85\cdot10^{-2}$	$3.97\cdot10^{-2}$	$5.09\cdot10^{-2}$	$6.47\cdot10^{-2}$	$1.37\cdot10^{-1}$	$1.39\cdot10^{-1}$	$1.34\cdot10^{2}$	50.00
N	3	97	100	99	96	40	35	98	45	95	11	40
e_{NG}	0	$-9.44\cdot10^{-7}$	$2.65\cdot10^{-6}$	$-3.38\cdot10^{-6}$	$-7.77\cdot10^{-6}$	$-8.45\cdot10^{-6}$	$-9.66\cdot10^{-6}$	$-9.66\cdot10^{-6}$	$1.16\cdot10^{-5}$	$-1.49\cdot10^{-5}$	$2.31\cdot10^{-4}$	$-8.45\cdot10^{-6}$

TABLE II

Ten most robust second degree PPFDs between lengths $N = 3, \ldots, 100$, based on their frequency response, group delay, and noise gain errors, e_F, e_G,, and e_{NG}, respectively. The coefficient word length is set to 16 bits. Also shown are the corresponding errors for the filter length $N = 11$ with coefficients quantized to 16 bits. Filters appearing in both the lists for e_F and e_G are shaded similarly. Errors for the filter length $N = 40$ are shown for error magnitude comparisons regarding Fig. 3.

N	3	4	6	12	15	39	58	97	70	85	11	40
e_F	0	0	0	0	0	0	0	$2.17\cdot10^{-19}$	$4.34\cdot10^{-19}$	$4.34\cdot10^{-19}$	$5.55\cdot10^{-17}$	$1.53\cdot10^{-5}$
N	3	12	13	4	6	17	11	15	19	10	11	40
e_G	$2.22\cdot10^{-8}$	$3.82\cdot10^{-5}$	$1.22\cdot10^{-4}$	$1.53\cdot10^{-4}$	$1.60\cdot10^{-4}$	$2.14\cdot10^{-4}$	$2.90\cdot10^{-4}$	$4.12\cdot10^{-4}$	$4.96\cdot10^{-4}$	$5.04\cdot10^{-4}$	$2.90\cdot10^{-4}$	$1.49\cdot10^{3}$
N	3	88	71	74	94	92	81	56	85	80	11	40
e_{NG}	0	$2.57\cdot10^{-9}$	$2.65\cdot10^{-9}$	$1.95\cdot10^{-8}$	$2.08\cdot10^{-8}$	$2.80\cdot10^{-8}$	$3.03\cdot10^{-8}$	$-3.23\cdot10^{-8}$	$3.39\cdot10^{-8}$	$3.42\cdot10^{-8}$	$5.38\cdot10^{-6}$	$1.21\cdot10^{-7}$

TABLE III

Ten most robust lattice second degree PPFDs between lengths $N = 3, \ldots, 100$, based on their frequency response, group delay, and noise gain errors, e_F, e_G,, and e_{NG}, respectively. The lattice parameter word length is set to eight bits. Also shown are the corresponding errors for the filter length $N = 11$ with coefficients quantized to eight bits which is illustrated in Fig. 7. Filters appearing in both the lists for e_F and e_G are shaded similarly. Errors for the filter length $N = 40$ are shown for error magnitude comparisons regarding Fig. 5.

N	3	4	5	6	7	10	25	34	51	55	11	40
e_F	0	0	0	0	0	0	0	0	0	0	$2.77\cdot10^{-17}$	$1.04\cdot10^{-17}$
N	28	14	29	3	7	80	13	10	4	97	11	40
e_G	$2.67\cdot10^{-3}$	$2.96\cdot10^{-3}$	$6.30\cdot10^{-3}$	$9.80\cdot10^{-3}$	$1.01\cdot10^{-2}$	$1.27\cdot10^{-2}$	$1.27\cdot10^{-2}$	$2.19\cdot10^{-2}$	$2.41\cdot10^{-2}$	$2.48\cdot10^{-2}$	$5.93\cdot10^{-2}$	$2.51\cdot10^{-1}$
N	19	10	99	98	85	86	97	84	87	88	11	40
e_{NG}	$3.92\cdot10^{-7}$	$-2.15\cdot10^{-6}$	$-2.39\cdot10^{-6}$	$-2.73\cdot10^{-6}$	$-3.17\cdot10^{-6}$	$-3.19\cdot10^{-6}$	$-3.21\cdot10^{-6}$	$-3.29\cdot10^{-6}$	$-3.34\cdot10^{-6}$	$-3.58\cdot10^{-6}$	$-1.70\cdot10^{-3}$	$-1.70\cdot10^{-3}$

TABLE IV

Ten most robust lattice second degree PPFDs between lengths $N = 3, \ldots, 100$, based on their frequency response, group delay, and noise gain errors, e_F, e_G, and e_{NG}, respectively. The lattice parameter word length is set to 16 bits. Also shown are the corresponding errors for the filter length $N = 11$ with coefficients quantized to 16 bits. Filters appearing in both the lists for e_F and e_G are shaded similarly. Errors for the filter length $N = 40$ are shown for error magnitude comparisons regarding Fig. 6.

N	3	5	8	10	21	28	29	34	37	48	11	40
e_F	0	0	0	0	0	0	0	0	0	0	$2.78\cdot10^{-17}$	$8.67\cdot10^{-18}$
N	19	9	16	33	3	13	6	12	32	5	11	40
e_G	$2.03\cdot10^{-5}$	$2.57\cdot10^{-5}$	$2.69\cdot10^{-5}$	$3.73\cdot10^{-5}$	$3.81\cdot10^{-5}$	$4.09\cdot10^{-5}$	$6.89\cdot10^{-5}$	$7.29\cdot10^{-5}$	$7.80\cdot10^{-5}$	$8.35\cdot10^{-5}$	$1.39\cdot10^{-4}$	$3.93\cdot10^{-4}$
N	62	63	61	58	30	64	70	100	65	50	11	40
e_{NG}	$3.94\cdot10^{-9}$	$6.05\cdot10^{-9}$	$1.06\cdot10^{-8}$	$1.06\cdot10^{-8}$	$1.53\cdot10^{-8}$	$1.64\cdot10^{-8}$	$1.76\cdot10^{-8}$	$1.90\cdot10^{-8}$	$1.93\cdot10^{-8}$	$1.99\cdot10^{-8}$	$-3.77\cdot10^{-6}$	$-3.77\cdot10^{-6}$