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# PSEUDODIFFERENTIAL CALCULUS ON COMPACT LIE GROUPS

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**Abstract:** *A continuous linear operator acting on the test function space on a compact Lie group is presented by a family of convolution operators (i.e. by a convolution operator -valued mapping on the group) arising naturally from the Schwartz kernel of the operator. This family, called the symbol of the operator, is closely related to the symbol of a pseudodifferential operator on a Euclidean space. We characterize* $(1,0)$*-type pseudodifferential operators on compact Lie groups by symbol inequalities, that is operator norm inequalities for operator-valued symbols. Furthermore, we study associated symbol calculus, giving asymptotic expansions for amplitude operators, adjoints, transposes, compositions and parametrices. Main tools are the Schwartz kernel theorem and a non-commutative operator-valued Fourier transform of distributions on a group. We also present a commutator characterization of pseudodifferential operators on closed manifolds.*

**AMS subject classifications:** 35S05, 22E30, 43A77, 46A32, 46E40, 58J40, 47B47, 47G30

**Keywords:** pseudodifferential operators, Lie groups, non-commutative Fourier transform, operator-valued functions, symbol calculus, Schwartz kernels, asymptotic expansions, parametrices, commutators

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# 1 Introduction

In this paper we present a symbolic calculus and a characterization of the Hörmander $(1,0)$-class of pseudodifferential operators on compact Lie groups. The operator classes are characterized by operator norm inequalities on a family of convolution operators parametrized by the group, and these inequalities are analogous to the traditional symbol inequalities of pseudodifferential calculus on the Euclidean spaces. As a special case, the definition of periodic pseudodifferential operators (i.e. operators on the torus $\mathbb{T}^n$) given by Agranovich in 1979 follows.

The genesis of pseudodifferential theory can perhaps be traced back to the studies of singular integral operators by S. G. Mikhlin in the 1940s and later by A. P. Calderón and A. Zygmund in the 1950s. There one can see the idea of presenting an operator $A$ by a family of convolution operators, i.e.

$$(Af)(x) = (s_A(x) * f)(x) = \int_{\mathbb{R}^n} (s_A(x))(x-y)\ f(y)\ dy.$$

In 1965 J. J. Kohn and L. Nirenberg [9] created the modern pseudodifferential theory (soon generalized by L. Hörmander [8]), where the main emphasis is on certain weighed inverse Fourier transforms. More precisely,

$$(Af)(x) = \int_{\mathbb{R}^n} \sigma_A(x,\xi)\ \hat{f}(\xi)\ e^{i2\pi x\cdot\xi}\ d\xi,$$

where the symbol function $(x,\xi)\mapsto\sigma_A(x,\xi)$ is the object of interest. These approaches are connected formally via the Fourier transform,

$$\sigma_A(x,\xi) = \widehat{s_A(x)}(\xi).$$

Our treatise draws inspiration from the schools of both Mikhlin–Calderón–Zygmund and Kohn–Nirenberg–Hörmander, and we considerably utilize the work of M. E. Taylor [15]. In other words, starting from the (left) regular representation $\pi_L : G \to \mathcal{L}(L^2(G))$ of a compact Lie group $G$ we introduce the Fourier transform $f \mapsto \pi(f)$ of distributions on $G$ by $\pi(f)g = f * g$; i.e. we map distributions to corresponding convolution operators. Then we present a pseudodifferential operator $A \in \Psi^m(G)$ as a family of convolution operators $\pi(s_A(x)) = \sigma_A(x)$ (where $s_A : G \to \mathcal{D}'(G)$, $s_A(x)(y) = K_A(x, y^{-1}x)$) so that

$$\begin{aligned}(Af)(x) &= (\sigma_A(x)f)(x) = (s_A(x) * f)(x)\\ &= \mathrm{Tr}\left(\sigma_A(x)\ \pi(f)\ \pi_L(x)^*\right).\end{aligned}$$

This trace formula is important from the application point of view; however, we shall not examine aspects of numerical analysis in this paper.

# 2 Non-commutative Fourier analysis

This section deals with harmonic analysis of distributions on compact Lie groups. We introduce the concept of the operator-valued symbol of a continuous linear operator acting on distributions, and given a symbol with suitable properties, we study Sobolev space boundedness of the corresponding operator.

Throughout this treatise, the space of continuous linear operators between topological vector spaces $X, Y$ is denoted by $\mathcal{L}(X,Y)$, and $\mathcal{L}(X) := \mathcal{L}(X,X)$. All the manifolds are (real) $C^\infty$-smooth and without a boundary. For basic facts about harmonic analysis on compact Lie groups we refer to [21]. Notions $\mathbb{C}$ and $\mathbb{R}$ stand for the fields of complex and real numbers, respectively, and $\mathbb{Z}$, $\mathbb{N}_0$ and $\mathbb{N}$ stand for integers, non-negative integers and positive integers, respectively.

Let $G$ be a compact Lie group and $\mu_G$ its normalized *Haar measure*, i.e. the unique regular Borel probability measure which is left translation invariant:

$$\int_G f(x)\ d\mu_G(x) = \int_G f(yx)\ d\mu_G(x)$$

for every $f \in C(G)$ and $y \in G$. Then also

$$\int_G f(x)\ d\mu_G(x) = \int_G f(xy)\ d\mu_G(x) = \int_G f(x^{-1})\ d\mu_G(x).$$

The *convolution* $f * g \in L^2(G)$ of $f, g \in L^2(G)$ is defined (almost everywhere) by

$$(f * g)(x) = \int_G f(y)\ g(y^{-1}x)\ d\mu_G(y). \tag{1}$$

Let $\pi_L : G \to \mathcal{L}(L^2(G))$ be the *left regular representation* of $G$, that is

$$(\pi_L(y)g)(x) = g(y^{-1}x) \tag{2}$$

for almost every $x \in G$. The *right regular representation* $\pi_R : G \to \mathcal{L}(L^2(G))$ is defined by

$$(\pi_R(y)g)(x) = g(xy) \tag{3}$$

for almost every $x \in G$. The *Fourier transform* (or the "global" Fourier transform, see [12]) of $f \in L^2(G)$ is by our definition the left convolution operator $\pi(f) \in \mathcal{L}(L^2(G))$,

$$\pi(f)g = f * g, \tag{4}$$

denoted also by

$$\pi(f) = \int_G f(y)\ \pi_L(y)\ d\mu_G(y).$$

The inverse Fourier transform is then given in $L^2(G)$-sense by

$$f(x) = \mathrm{Tr}\left(\pi(f)\ \pi_L(x)^*\right), \tag{5}$$

where Tr is the trace functional; notice that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. By choosing an orthonormal basis of representative functions (trigonometric polynomials) provided by the Peter–Weyl theorem we can realize the operators $\pi_L(x)$ and $\pi(f)$ as block diagonal matrices with finite-dimensional blocks, each block corresponding to some irreducible unitary representation of $G$. Each matricial entry of the matrix form of $\pi_L(x)$ is then a $C^\infty$-smooth function on $G$, whose translates span a finite-dimensional subspace of $L^2(G)$; moreover, the set of these matricial entries contains an orthogonal basis of $L^2(G)$.

Let $\mathcal{D}(G)$ be the set $C^\infty(G)$ equipped with the usual Fréchet space structure, and $\mathcal{D}'(G) = \mathcal{L}(\mathcal{D}(G), \mathbb{C})$ its dual, i.e. the set of distributions with $\mathcal{D}(G)$ as the test function space. We equip the space of distributions with the weak$^*$-topology. The *Fourier transform of a distribution* $f \in \mathcal{D}'(G)$ is defined to be the left convolution operator $\pi(f) \in \mathcal{L}(\mathcal{D}(G))$, that is $\pi(f)\phi := f * \phi$. The duality $\mathcal{D}(G) \times \mathcal{D}'(G) \to \mathbb{C}$ is denoted by

$$\langle \phi, f \rangle := f(\phi), \tag{6}$$

where $\phi \in \mathcal{D}(G)$ and $f \in \mathcal{D}'(G)$, and an embedding $\mathcal{D}(G) \hookrightarrow \mathcal{D}'(G)$, $\psi \mapsto f_\psi = \psi$ is given by

$$\langle \phi, \psi \rangle = \int_G \phi(x)\ \psi(x)\ d\mu_G(y). \tag{7}$$

The *transpose* of $A \in \mathcal{L}(\mathcal{D}(G))$ is $A^t \in \mathcal{L}(\mathcal{D}'(G))$ defined by

$$\langle A\phi, f \rangle = \langle \phi, A^t f \rangle, \tag{8}$$

and the *adjoint* $A^* \in \mathcal{L}(\mathcal{D}'(G))$ is given by

$$(A\phi, f) = (\phi, A^* f), \tag{9}$$

where $(\phi, f) = \langle \phi, \overline{f} \rangle$, $\overline{f}(x) := \overline{f(x)}$. Notice that $\pi(f)^* = \pi(\tilde{f})$ where $\tilde{f}(x) = \overline{f(x^{-1})}$, and $\pi(f)^t = \pi(\check{f})$ where $\check{f}(x) = f(x^{-1})$.

In the sequel, we denote $\pi(A)\pi(f) := \pi(Af)$ if $A \in \mathcal{L}(\mathcal{D}(G))$ and $f \in \mathcal{D}(G)$; i.e. $\pi(A)$ is just "$A$ put through the Fourier transform". Stay alert: in this notation, if $\phi \in \mathcal{D}(G)$, $f \in \mathcal{D}'(G)$, then $\pi(\phi * f) = \pi(\phi)\pi(f)$, but $\pi(\phi f) = \pi(M_\phi)\pi(f)$ where $M_\phi$ is the multiplication operator $f \mapsto \phi f$.

Let $\mathcal{D}(G) \otimes \mathcal{D}'(G)$ denote the complete locally convex tensor product of the nuclear spaces $\mathcal{D}(G)$ and $\mathcal{D}'(G)$ (see [11]). Then $K \in \mathcal{D}(G) \otimes \mathcal{D}'(G)$ defines a linear operator $A \in \mathcal{L}(\mathcal{D}(G))$ by

$$\langle A\phi, f \rangle := \langle K, f \otimes \phi \rangle. \tag{10}$$

In fact, the Schwartz kernel theorem states that $\mathcal{L}(\mathcal{D}(G))$ and $\mathcal{D}(G) \otimes \mathcal{D}'(G)$ are isomorphic: for every $A \in \mathcal{L}(\mathcal{D}(G))$ there exists a unique $K_A \in \mathcal{D}(G) \otimes \mathcal{D}'(G)$ such that the duality (10) is satisfied with $K = K_A$, which is called the *Schwartz kernel of* $A$. Duality (10) gives us also the interpretation for

$$(A\phi)(x) = \int_G K_A(x, y)\ \phi(y)\ d\mu_G(y). \tag{11}$$

**Lemma 1.** *Let $A \in \mathcal{L}(\mathcal{D}(G))$, and let*

$$s_A(x, y) := K_A(x, y^{-1}x) \tag{12}$$

*in the sense of distributions. Then* $s_A \in \mathcal{D}(G) \otimes \mathcal{D}'(G)$.

**Proof.** Notice that $\mathcal{D}(G)\otimes\mathcal{D}(G)\cong\mathcal{D}(G\times G)$. Let us define the *multiplication*

$$m:\mathcal{D}(G)\otimes\mathcal{D}(G)\to\mathcal{D}(G),\quad m(f\otimes g)(x):=f(x)g(x),$$

the *co-multiplication*

$$\Delta:\mathcal{D}(G)\to\mathcal{D}(G)\otimes\mathcal{D}(G),\quad (\Delta f)(x,y):=f(xy),$$

and the *antipode*

$$S:\mathcal{D}(G)\to\mathcal{D}(G),\quad (Sf)(x):=f(x^{-1}).$$

These mappings are a part of the (nuclear Fréchet) Hopf algebra structure of $\mathcal{D}(G)$, see e.g. [1] or [14]. The mappings are all continuous and linear.

The *convolution of operators* $A,B\in\mathcal{L}(\mathcal{D}(G))$ is said to be the operator

$$A*B:=m(A\otimes B)\Delta\in\mathcal{L}(\mathcal{D}(G));$$

it is easy to calculate the Schwartz kernel

$$K_{A*B}(x,y)=\int_G K_A(x,yz^{-1})\ K_B(x,z)\ d\mu_G(z),$$

or

$$K_{A*B}=(m\otimes\Delta^t)(\mathrm{id}\otimes\tau\otimes\mathrm{id})(K_A\otimes K_B),$$

where $\tau:\mathcal{D}'(G)\otimes\mathcal{D}(G)\to\mathcal{D}(G)\otimes\mathcal{D}'(G)$, $\tau(f\otimes\phi):=\phi\otimes f$, and $\Delta^t:\mathcal{D}'(G)\otimes\mathcal{D}'(G)\to\mathcal{D}'(G)$ is the transpose of the co-multiplication $\Delta$, $\Delta^t(f\otimes g)=f*g$ (i.e. $\Delta^t$ extends the convolution of distributions). Now $(A*S)S\in\mathcal{L}(\mathcal{D}(G))$, hence $K_{(A*S)S}\in\mathcal{D}(G)\otimes\mathcal{D}'(G)$ by the Schwartz kernel theorem, and

$$\begin{aligned}
K_{(A*S)S}(x,y) &= K_{A*S}(x,y^{-1})\\
&= \int_G K_A(x,y^{-1}z^{-1})\ K_S(x,z)\ d\mu_G(z)\\
&= K_A(x,y^{-1}x)\\
&= s_A(x,y)
\end{aligned}$$

□

Any distribution $s\in\mathcal{D}(G)\otimes\mathcal{D}'(G)$ can be considered as a mapping

$$s:G\to\mathcal{D}'(G),\quad x\mapsto s(x),$$

where $s(x)(y):=s(x,y)$. If $D\in\mathcal{L}(\mathcal{D}(G))$ and $M\in\mathcal{L}(\mathcal{D}'(G))$ then $(D\otimes M)s\in\mathcal{D}(G)\otimes\mathcal{D}'(G)$. For instance, $D$ could be a partial differential operator, and $M$ a multiplication; these examples will be of great relevance for us in the sequel.

**Definition.** *The symbol of $A \in \mathcal{L}(\mathcal{D}(G))$ is the mapping*

$$\sigma_A : G \to \mathcal{L}(\mathcal{D}(G)), \quad x \mapsto \pi(s_A(x)), \tag{13}$$

*where $s_A(x)(y) = K_A(x, y^{-1}x)$ as in Lemma 1.*

Now

$$\begin{aligned}
(Af)(x) &= \int_G K_A(x,y)\ f(y)\ d\mu_G(y) \\
&= \int_G s_A(x)(xy^{-1})\ f(y)\ d\mu_G(y) \\
&= (\sigma_A(x)f)(x) \\
&= \mathrm{Tr}\left(\sigma_A(x)\ \pi(f)\ \pi_L(x)^*\right).
\end{aligned}$$

Hence we can consider the symbol $\sigma_A$ as a family of convolution operators obtained from $A$ by "freezing" it at points $x \in G$. Moreover, if $A \in \mathcal{L}(\mathcal{D}(G))$ is a right-invariant operator, i.e. $A\pi_R(x) = \pi_R(x)A$ for every $x \in G$, then its symbol is the constant mapping $x \mapsto \sigma_A(x) \equiv A$ ($A$ is a left convolution operator).

Conversely, let us be given a function $\sigma : G \to \mathcal{L}(\mathcal{D}'(G))$ such that $\sigma(x) = \pi(s(x))$, where $s \in \mathcal{D}(G) \otimes \mathcal{D}'(G)$. Then we can define $\mathrm{Op}(\sigma) \in \mathcal{L}(\mathcal{D}(G))$ by

$$(\mathrm{Op}(\sigma)f)(x) = (\sigma(x)f)(x). \tag{14}$$

Let us define derivations of symbols of operators $A \in \mathcal{L}(\mathcal{D}(G))$ in the following way: Let $D \in \mathcal{L}(\mathcal{D}(G))$ be a partial differential operator, and $(D\sigma_A)(x) = D\sigma_A(x) := \sigma_B(x)$, where

$$s_B = (D \otimes \mathrm{id})s_A.$$

Then $B = \mathrm{Op}(D\sigma_A) \in \mathcal{L}(\mathcal{D}(G))$, because $D \in \mathcal{L}(\mathcal{D}(G))$, $\mathrm{id} \in \mathcal{L}(\mathcal{D}'(G))$.

The next theorem is an adaptation of M. E. Taylor's result (see [15]).

**Theorem 2.** *Let $G$ be a compact Lie group of dimension $n$ and $\sigma \in C^k(G, \mathcal{L}(L^2(G)))$ with $k > n/2$. Then $\mathrm{Op}(\sigma) \in \mathcal{L}(L^2(G))$.*

**Proof.** We clearly have

$$\begin{aligned}
\|\mathrm{Op}(\sigma)f\|^2_{L^2(G)} &= \int_G |(\sigma(x)f)(x)|^2\ d\mu_G(x) \\
&\leq \int_G \sup_{y\in G} |(\sigma(y)f)(x)|^2\ d\mu_G(x),
\end{aligned}$$

and by an application of Sobolev embedding theorem we get

$$\sup_{y\in G} |(\sigma(y)f)(x)|^2 \leq C_k \sum_{|\alpha|\leq k} \int_G |((\partial_y^\alpha \sigma)(y)f)(x)|^2\ d\mu_G(y).$$

Therefore using Fubini theorem to change the order of integration, we obtain

$$\begin{aligned}
\|\mathrm{Op}(\sigma)f\|_{L^2(G)}^2 &\leq C_k \sum_{|\alpha|\leq k} \int_G \int_G |((\partial_y^\alpha \sigma)(y)f)(x)|^2 \, d\mu_G(x) \, d\mu_G(y) \\
&\leq C_k \sum_{|\alpha|\leq k} \sup_{y\in G} \int_G |((\partial^\alpha \sigma)(y)f)(x)|^2 \, d\mu_G(x) \\
&= C_k \sum_{|\alpha|\leq k} \sup_{y\in G} \|(\partial^\alpha \sigma)(y)f\|_{L^2(G)}^2 \\
&\leq C_k \sum_{|\alpha|\leq k} \sup_{y\in G} \|(\partial^\alpha \sigma)(y)\|_{\mathcal{L}(L^2(G))}^2 \|f\|_{L^2(G)}^2.
\end{aligned}$$

The proof is complete, because $G$ is compact and $\sigma \in C^k(G, \mathcal{L}(L^2(G)))$ $\square$

Let $\Delta$ be the bi-invariant (right- and left-invariant) Laplacian of $G$, i.e. the Laplace operator corresponding to the bi-invariant Riemannian metric of $G$ (see [17]). The Laplacian is symmetric, $I - \Delta$ is positive; let $\Xi \equiv \sigma_{(I-\Delta)^{1/2}}(x) = (I-\Delta)^{1/2}$. Then $\Xi$ is bi-invariant, so that it commutes with every convolution operator. Moreover, $\Xi^s \in \mathcal{L}(\mathcal{D}(G))$ and $\Xi^s \in \mathcal{L}(\mathcal{D}'(G))$ for every $s \in \mathbb{R}$. Let us define

$$(f,g)_{H^s(G)} = (\Xi^{-s} f, \Xi^{-s} g)_{L^2(G)} \quad (f,g \in \mathcal{D}(G)).$$

The completion of $\mathcal{D}(G)$ with respect to the norm $f \mapsto \|f\|_{H^s(G)} = (f,f)_{H^s(G)}^{1/2}$ is called the *Sobolev space* $H^s(G)$ of order $s \in \mathbb{R}$. Thereby $H^0(G) = L^2(G)$. This definition of $H^s(G)$ coincides with the definition obtained using any smooth partition of unity on the compact manifold $G$.

Notice that $\sigma_{A\Xi^r}(x) = \Xi^r \sigma_A(x) = \sigma_A(x) \Xi^r$ $(A \in \mathcal{D}(G))$, and that $\Xi^r$ is a Sobolev space isomorphism $H^s(G) \to H^{s-r}(G)$ for every $r, s \in \mathbb{R}$. Therefore Theorem 2 yields a simple consequence:

**Corollary 3.** *If $G$ is a compact Lie group, $A \in \mathcal{L}(\mathcal{D}(G))$ and $\Xi^{-m}\sigma_A \in C^\infty(G, \mathcal{L}(L^2(G)))$ (or equivalently $\sigma_A \in C^\infty(G, \mathcal{L}(H^m(G), H^0(G)))$) then $A \in \mathcal{L}(H^m(G), H^0(G))$.*

**Convention:** Let $D$ be a right-invariant vector field on $G$. Then $\sigma_D(x) = D$ for every $x \in G$, so that the reader must be cautious! In the sequel $D\sigma_A(x)$ refers to the derivative $(D\sigma_A)(x)$ of the operator-valued function $\sigma_A$, whereas $\sigma_D \sigma_A(x)$ means the composition of two convolution operators, namely $\sigma_D$ and $\sigma_A(x)$. More precisely, $D\sigma_A(x) = \sigma_B(x)$ with $s_B = (D \otimes \mathrm{id}) s_A$, and $\sigma_D \sigma_A(x) = \sigma_C(x)$ where $s_C(x) = s_D(x) * s_A(x)$.

Let $D = \partial_{x_j}$ be a first order right-invariant partial differential operator with constant symbol $\sigma_D(x) \equiv D$. Then for $A \in \mathcal{L}(\mathcal{D}(G))$ we have $DA \in \mathcal{L}(\mathcal{D}(G))$, and it is clear that $x \mapsto \sigma_D \sigma_A(x)$ is a symbol of an operator belonging to $\mathcal{L}(\mathcal{D}(G))$. Then

$$\sigma_{DA}(x) = (\partial_{x_j} \sigma_A)(x) + \sigma_D \sigma_A(x).$$

**Lie algebra:** Traditionally the Lie algebra $\mathfrak{g}$ of a Lie group $G$ is the set of the *left-invariant* vector fields on $G$. In this work, however, we define the Lie algebra $\mathfrak{g}$ of a Lie group $G$ to be the set of the *right-invariant* vector fields on $G$. This non-conventional practice simplifies our notations; recall that right-invariant operators are left convolution operators. Hence our Lie algebra $\mathfrak{g}$ can be identified with first order scalar right-invariant partial differential operators on $G$. Of course, $\mathfrak{g} \cong \mathrm{T}_e(G) \cong \mathbb{R}^n$, where $\mathrm{T}_e(G)$ is the tangent space of $G$ at $e \in G$, $n = \dim(G)$. In the sequel, $\{\partial_j = \partial_{x_j} \mid 1 \leq j \leq n\}$ denotes a vector space basis of $\mathfrak{g}$.

# 3 Pseudodifferential operators on compact manifolds

In this paper all the pseudodifferential operators are of the Hörmander type $(\rho, \delta) = (1, 0)$. Recall the appearance of the pseudodifferential operators on the Euclidean spaces,

$$(Af)(x) = \int_{\mathbb{R}^n} \sigma_A(x,\xi)\ \hat{f}(\xi)\ e^{i2\pi x\cdot\xi}\ d\xi, \tag{15}$$

where $f$ belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, and the Fourier transform $\hat{f} \in \mathcal{S}(\mathbb{R}^n)$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\ e^{-i2\pi x\cdot\xi}\ dx;$$

the symbol $\sigma_A \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is said to belong to $S^m(\mathbb{R}^n)$, the set of pseudo-differential symbols of order $m \in \mathbb{R}$, if

$$\left|\partial_\xi^\alpha \partial_x^\beta \sigma_A(x,\xi)\right| \leq C_{A\alpha\beta m}(1 + |\xi|)^{m-|\alpha|}, \tag{16}$$

uniformly in $x \in \mathbb{R}^n$ for every $\alpha, \beta \in \mathbb{N}^n$, and then $A$ is called a *pseudo-differential operator of order* $m$, denoted by $A \in \mathrm{Op}S^m(\mathbb{R}^n)$. Imitating the analysis on compact groups (see the previous section), we can denote by $\sigma_A(x_0)$ ($x_0 \in \mathbb{R}^n$) the convolution operator with the symbol $(x,\xi) \mapsto \sigma_A(x_0,\xi)$, and then $(Af)(x) = (\sigma_A(x)f)(x)$. Now $A \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n))$, $A \in \mathcal{L}(\mathcal{S}'(\mathbb{R}^n))$, and $\sigma_A(x,\xi) = e_\xi(x)^{-1}(Ae_\xi)(x)$, where $e_\xi(x) = e^{i2\pi x\cdot\xi}$.

On a closed manifold $M$ the set $\Psi^m(M)$ of $(1,0)$-type pseudodifferential operators of order $m \in \mathbb{R}$ is usually defined using locally the definition for the Euclidean case, see [8], [16] or [17], and one defines the Sobolev spaces $H^m(M)$ using any smooth partition of unity on $M$. On the torus $M = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, however, we can apply the Fourier series, as first suggested by Agranovich [3]. More precisely, for $A \in \mathcal{L}(\mathcal{D}(\mathbb{T}^n))$ we can write

$$(Af)(x) = \sum_{\xi\in\mathbb{Z}^n} \sigma_A(x,\xi)\ \hat{f}(\xi)\ e^{i2\pi x\cdot\xi}, \tag{17}$$

where the Fourier transform $\hat{f}$ of $f \in \mathcal{D}(\mathbb{T}^n)$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{T}^n} f(x)\ e^{-i2\pi x\cdot\xi}\ dx;$$

the symbol $\sigma_A \in C^\infty(\mathbb{T}^n \times \mathbb{Z}^n)$ is said to belong to $S^m(\mathbb{T}^n)$, the set of pseudo-differential symbols of order $m \in \mathbb{R}$, if

$$\left|\triangle_\xi^\alpha \partial_x^\beta \sigma_A(x,\xi)\right| \leq C_{A\alpha\beta m}(1+|\xi|)^{m-|\alpha|}, \tag{18}$$

uniformly in $x \in \mathbb{T}^n$ for every $\alpha, \beta \in \mathbb{N}^n$, and then $A$ is called a *periodic pseudodifferential operator of order* $m$, $A \in \mathrm{Op}S^m(\mathbb{T}^n)$. Here $\triangle_\xi^\alpha$ is a difference operator,

$$\triangle_\xi^\alpha \sigma(\xi) = \sum_{\gamma\leq\alpha} \binom{\alpha}{\gamma} (-1)^{|\alpha-\gamma|}\sigma(\xi+\gamma).$$

Again, $\sigma_A(x,\xi) = e_\xi(x)^{-1}(Ae_\xi)(x)$. The fact $\mathrm{Op}S^m(\mathbb{T}^n) = \Psi^m(\mathbb{T}^n)$ was proven by Agranovich [2], and the result was generalized by McLean [10] for more general $(\rho,\delta)$-classes on the tori.

Beals discovered in [4] that pseudodifferential operators on the Euclidean spaces can be characterized by studying Sobolev space boundedness of iterated commutators by differential operators. Related studies (also on closed manifolds) were pursued by Dunau [7], Coifman and Meyer [5] and Cordes [6]. In [18] another variant of commutator characterization was given on closed manifolds (i.e. compact manifolds without boundary), equivalent to the following definition:

**Definition.** *Let $M$ be a closed smooth orientable manifold. An operator $A \in \mathcal{L}(\mathcal{D}(M))$ is a pseudodifferential operator of* $(1,0)$*-type and of order $m \in \mathbb{R}$, denoted by $A \in \Psi^m(M)$, if and only if $A_k \in \mathcal{L}(H^{m+d_{\mathcal{C},k}}(M), H^0(M))$ for every $k \in \mathbb{N}_0$ and for every sequence $\mathcal{C} = (D_k)_{k=1}^\infty \subset \mathcal{L}(\mathcal{D}(M))$ of smooth vector fields, where $A_0 = A$ and $A_{k+1} = [D_{k+1}, A_k]$, $d_{\mathcal{C},0} = 0$ and $d_{\mathcal{C},k+1} = \sum_{j=1}^{k+1}(1-\deg(D_j))$.*

Here $\deg(D_j)$ is the order of the partial differential operator $D_j$, i.e. $\deg(M_\phi) = 0$ for a multiplication operator $M_\phi$, and $\deg(\partial^\beta) = |\beta|$. A variant of this definition was applied in proving that periodic pseudodifferential operators are traditional pseudodifferential operators [18].

Let us give another equivalent criterion for pseudodifferential operators, simpler in formulation:

**Theorem 4.** *Let $M$ be a closed smooth orientable manifold. An operator $A \in \mathcal{L}(\mathcal{D}(M))$ belongs to $\Psi^m(M)$ if and only if $(A_k)_{k=0}^\infty \subset \mathcal{L}(H^m(M), H^0(M))$ for every sequence $(D_k)_{k=1}^\infty \subset \mathcal{L}(\mathcal{D}(M))$ of smooth vector fields (i.e. continuous derivations of the function algebra $\mathcal{D}(M)$), where $A_0 = A$ and $A_{k+1} = [D_{k+1}, A_k]$.*

**Proof.** Let $\mathrm{Op}^m$ denote the set of operators satisfying the the property expressed by the iterated commutators in the claim. In fact, $\Psi^m(M) \subset \mathcal{L}(H^s(M), H^{s-m}(M))$ for every $s, m \in \mathbb{R}$, and there is the nice commutator property $[\Psi^{m_1}(M), \Psi^{m_2}(M)] \subset \Psi^{m_1+m_2-1}(M)$, and hence it is clear that $\Psi^m(M) \subset \mathrm{Op}^m$.

Now suppose that $A \in \mathrm{Op}^m$. First, $\mathrm{Op}^r \subset \mathcal{L}(H^r(M), H^0(M))$ for every $r \in \mathbb{R}$. Hence in order to prove the theorem it suffices to verify that $[M_\phi, A] \in \mathrm{Op}^{m-1}$ for every $\phi \in \mathcal{D}(M)$. Let $g$ be some Riemannian metric for $M$, and let $\Delta_g$ be the corresponding Laplacian (see [17]). Then $I - \Delta_g$ is a Sobolev space isomorphism $H^s(M) \to H^{s-m}(M)$ for every $s, m \in \mathbb{R}$. We write the second order partial differential operator $I - \Delta_g$ in the form $\sum_{i,j} E_i F_j + F + M_\psi$, where $F, E_i, F_j \in \mathcal{L}(\mathcal{D}(M))$ are some smooth vector fields and $\psi \in \mathcal{D}(M)$. Now

$$
\begin{aligned}
[M_\phi, A] &= [M_\phi, A](I - \Delta_g)(I - \Delta_g)^{-1} \\
&= \left( \sum_{i,j} ([M_\phi E_i, A] - M_\phi [E_i, A]) F_j + [M_\phi, A](F + M_\psi) \right) (I - \Delta_g)^{-1}.
\end{aligned}
$$

Here $M_\phi \in \mathcal{L}(H^s(M))$ for every $s \in \mathbb{R}$, and $A \in \mathcal{L}(H^m(M), H^0(M))$, so that $[M_\phi, A] \in \mathcal{L}(H^m(M), H^0(M))$, and moreover

$$
\begin{aligned}
(I - \Delta_g)^{-1} &\in \mathcal{L}(H^{m-1}(M), H^{m+1}(M)), \\
F_j,\ F + M_\psi &\in \mathcal{L}(H^{m+1}(M), H^m(M)), \\
([M_\phi E_i, A] - M_\phi [E_i, A]) &\in \mathcal{L}(H^m(M), H^0(M)).
\end{aligned}
$$

This yields $[M_\phi, A] \in \mathcal{L}(H^{m-1}(M), H^0(M))$. For a smooth vector field $D$ we clearly have

$$
[D, [M_\phi, A]] = [M_\phi, [D, A]] + [[D, M_\phi], A],
$$

so that also $[D, [M_\phi, A]] \in \mathcal{L}(H^{m-1}(M), H^0(M))$; notice that $[D, A] \in \mathrm{Op}^m$. Then assume that we know

$$
[D_k, [D_{k-1}, \cdots [D_1, [M_\phi, A]] \cdots]] = \sum_{\gamma \in J_k} [M_{\phi_\gamma}, A_\gamma],
$$

where $J_k$ is some finite index set, $\phi_\gamma \in \mathcal{D}(M)$, $A_\gamma \in \mathrm{Op}^m$, $D_j$ $(1 \leq j \leq k)$ smooth vector fields. Then

$$
\begin{aligned}
&[D_{k+1}, [D_k, [D_{k-1}, \cdots [D_1, [M_\phi, A]] \cdots]]] \\
&= \sum_{\gamma \in J_k} \left( [M_{\phi_\gamma}, [D_{k+1}, A_\gamma]] + [[D_{k+1}, M_{\phi_\gamma}], A_\gamma] \right) \\
&= \sum_{\gamma \in J_{k+1}} [M_{\phi_\gamma}, A_\gamma]
\end{aligned}
$$

for a finite index set $J_{k+1}$, $\phi_\gamma \in \mathcal{D}(M)$, $A_\gamma \in \mathrm{Op}^m$ when $\gamma \in J_{k+1}$. Hence $[M_\phi, A] \in \mathrm{Op}^{m-1}(M)$, so that $\mathrm{Op}^m \subset \Psi^m(M)$ $\square$

Let $M$ be a closed smooth manifold. As it is well-known, for any given sequence of pseudodifferential operators $A_\alpha \in \Psi^{m_\alpha}(M)$ $(\alpha \in \mathbb{N}_0^n)$ with $m_\alpha \to -\infty$ as $|\alpha| \to \infty$, there exists a pseudodifferential operator $A \in \Psi^{\max\{m_\alpha\}}(M)$ satisfying the following: for every $r \in \mathbb{R}$ there exists $N_r \in \mathbb{N}$ such that $A - \sum_{|\alpha|<N} A_\alpha \in \Psi^r(M)$ whenever $N > N_r$. Then the formal sum $\sum_\alpha A_\alpha$ is called an asymptotic expansion of $A$, denoted by

$$A \sim \sum_{\alpha \geq 0} A_\alpha;$$

if $M = G$ is a compact Lie group, we also write

$$\sigma_A(x) \sim \sum_{\alpha \geq 0} \sigma_{A_\alpha}(x).$$

Notice that this asymptotic expansion defines the operator only modulo

$$\Psi^{-\infty}(M) := \cap_{r\in\mathbb{R}} \Psi^r(M) = \{B \in \mathcal{L}(\mathcal{D}(M)) \mid B(\mathcal{D}'(M)) \subset \mathcal{D}(M)\};$$

then the asymptotic expansions endow the set $\Psi^\infty(M) = \cup_{r\in\mathbb{R}} \Psi^r(M)$ with a non-Hausdorff topology, but for $\Psi^\infty(M)/\Psi^{-\infty}(M)$ we obtain a complete metrizable topology, where addition and composition of equivalence classes of operators are continuous, i.e. we have a complete metrizable topological ring. This division into classes modulo infinitely smoothing operators is reasonable, if we are only interested in information related to the singularities of distributions.

# 4 Operator norm symbol inequalities for pseudodifferential operators

In this section we provide operator norm inequalities for symbols characterizing the classes $\Psi^m(G)$ for compact Lie groups $G$; the main result is Theorem 9, in the end of the section. In the sequel, $n = \dim(G)$.

**Taylor expansion:** By identifying the Lie algebra $\mathfrak{g}$ of $G$ with $\mathbb{R}^n$, a function $f \in C^\infty(\mathfrak{g})$ can be presented as a Taylor–Maclaurin expansion,

$$f(x) = \sum_{|\alpha|<N} \frac{1}{\alpha!} x^\alpha (\partial^\alpha f)(0) + R_N(x) \tag{19}$$

where the remainder term $R_N$ is of the form

$$R_N(x) = \sum_{|\alpha|=N} \frac{1}{\alpha!} x^\alpha (\partial^\alpha f)(\theta_x x) \tag{20}$$

with some $\theta_x \in (0,1)$. Let $V, W \subset \mathfrak{g}$ be open balls centered at $0 \in \mathfrak{g}$, $V = B(0, r_V) \subset W = B(0, r_W)$, such that the exponential mapping $\exp : \mathfrak{g} \to G$ is injective on $W$. Thereby there exist functions $q_\alpha \in C^\infty(G)$ satisfying

$$q_\alpha(\exp(x)) = \frac{1}{\alpha!} x^\alpha$$

when $x \in B(0, r_V + \varepsilon) \subset \mathfrak{g}$ with some $\varepsilon > 0$; the smooth functions $q_\alpha$ have "zeros of order $|\alpha|$ at the point $e \in G$". It is enough to construct functions $q_\alpha \in C^\infty(G)$ for $|\alpha| = 1$, and then define "higher order polynomials" recursively by $q_{\alpha+\beta} := q_\alpha q_\beta$ for every $\alpha, \beta \in \mathbb{N}_0^n$. Furthermore, for every $x \in G \setminus \{e\}$ we must have $q_\alpha(x) \neq 0$ for some $\alpha = \alpha_x \in \mathbb{N}_0^n$ with $|\alpha| = 1$; this condition will be explained right after the definition of the symbol class $S_0^m(G)$. It is easy to see that this condition can be fulfilled: For instance, let $|\alpha| = 1$, and let $\psi : [0, r_W] \to [0, 1]$ be an increasing $C^\infty$-smooth function such that $\psi|_{[0, r_V + \varepsilon]} = 0$ and $\psi|_{[r_W - \varepsilon, r_W]} = 1$ where $0 < \varepsilon < (r_W - r_V)/2$, and set

$$q_\alpha(\exp(x)) := \mathrm{i}\psi(\|x\|) + (1 - \psi(\|x\|)) \frac{1}{\alpha!} x^\alpha$$

when $x \in W$, and set $q_\alpha(y) := \mathrm{i}$ when $y \in G \setminus \exp(W)$.

Consequently a function $\phi \in C^\infty(G)$ can be presented by

$$\phi(x) = \sum_{|\alpha| < N} q_\alpha(x)\ (\partial^\alpha \phi)(e) + R_N(x), \tag{21}$$

when $x \in \exp(V)$, with the remainder

$$R_N(x) = \sum_{|\alpha| = N} q_\alpha(x)\ (\partial^\alpha \phi)(\psi(x)), \tag{22}$$

where $\psi(x) \in \exp(V)$, and $x \mapsto (\partial^\alpha \phi)(\psi(x))$ is $C^\infty$-smooth; in the sequel we abbreviate this by

$$\phi(x) \sim \sum_\alpha q_\alpha(x)\ (\partial^\alpha \phi)(e).$$

Then let us define a "quasi-difference" $Q^\alpha = \pi(M_{\check{q}_\alpha})$, that is,

$$Q^\alpha \sigma_A(x) = \pi(y \mapsto \check{q}_\alpha(y) s_A(x)(y)). \tag{23}$$

The idea of this definition stems from the Euclidean Fourier transform $\mathcal{F}$, which obeys $(\mathcal{F}(x \mapsto x^\alpha \hat{f}(x)))(\xi) = (\partial^\alpha \mathcal{F}(f))(\xi)$; a multiplication by a polynomial turns to a differentiation when put through the Fourier transform.

Let $V = B(0, r_V), W = B(0, r_W)$ be as above, and let $U = B(0, r_U) \subset \mathfrak{g}$ with $0 < r_U < r_V$. Using a partition of unity we may present an operator $A \in \mathcal{L}(\mathcal{D}(G))$ in a form $A = A_0 + A_1$ where $A_0, A_1 \in \mathcal{L}(\mathcal{D}(G))$, such that $K_{A_0}(x, y) = s_{A_0}(x)(xy^{-1}) = 0$ when $xy^{-1} \notin \exp(V)$, and $K_{A_0}(x, y) = s_{A_1}(x)(xy^{-1}) = 0$ when $xy^{-1} \in \exp(U)$. Suppose $A$ is a pseudodifferential operator; we know that the Schwartz kernel of a pseudodifferential operator is $C^\infty$-smooth outside the diagonal of $G \times G$, and consequently $A_0$ is the interesting part of $A$. Let $\{x_j\}_{j=1}^N \subset G$ such that $\{U_j := x_j \exp(U)\}_{j=1}^N$ is an open cover for $G$, and let $\{(\phi_j, U_j)\}_{j=1}^N$ be a partition of unity subordinate to this cover. Let

$$\sigma_{A_{0,j}}(x) := \phi_j(x_j x) \sigma_{A_0}(x_j x); \tag{24}$$

then $\sigma_{A_0}(x) = \sum_{j=1}^N \sigma_{A_{0,j}}(x_j^{-1} x)$, and $\mathrm{supp}(s_{A_{0,j}}(x)) \subset \exp(\overline{V}) \cong \overline{V}$ for each $x \in \exp(U) \cong U$ and $j \in \{1, \ldots, N\}$. Notice that $\sigma_{A_{0,j}}(x) = 0$ when $x \notin$

$\exp(U)$. Hence we can interpret $s_{A_{0,j}}$ to be a distribution supported in a small neighbourhood of the origin of the Euclidean space $\mathbb{R}^n \times \mathbb{R}^n$. Let

$$(B_j f)(x) = \int_{\mathbb{R}^n} s_{A_{0,j}}(x)(x-y)\ f(y)\ dy, \tag{25}$$

i.e. $\sigma_{B_j}(x,\xi) = \widehat{s_{A_{0,j}}(x)}(\xi)$ $(x,\xi \in \mathbb{R}^n)$.

Notice that

$$\partial_\xi^\alpha \partial_x^\beta \sigma_{B_j}(x,\xi) = \widehat{s_{B_{j\alpha\beta}}(x)}(\xi) \quad (x,\xi \in \mathbb{R}^n), \tag{26}$$

where $\sigma_{B_{j\alpha\beta}}(x) = Q^\alpha \partial_x^\beta \sigma_{A_{0,j}}(x)$ $(x \in G)$.

Taylor ([15], Propositions 1.1 and 1.4) states essentially the following:

**Proposition 5 (Taylor's characterization of $\Psi^m(G)$).** *Let $A \in \mathcal{L}(\mathcal{D}(G))$ be as above. Then $A \in \Psi^m(G)$ if and only if $K_{A_1} \in C^\infty(G \times G)$ and $\{\sigma_{B_j}\}_{j=1}^N \subset S^m(\mathbb{R}^n)$.* □

It is noteworthy that Taylor deals with not only compact Lie groups, but locally compact ones; in Proposition 5 we presented just a special statement suitable for our purposes. In the course of proving a generalization of Proposition 5, Taylor obtains also the following useful results (see [15], Propositions 1.1–1.4):

**Lemma 6.** *Let $A \in \Psi^{m_A}(G)$ and $B \in \Psi^{m_B}(G)$. Then $\mathrm{Op}(\sigma_A\sigma_B) \in \Psi^{m_A+m_B}(G)$ and $\mathrm{Op}(\sigma_A^*) \in \Psi^{m_A}(G)$, where $(\sigma_A\sigma_B)(x) = \sigma_A(x)\sigma_B(x)$ and $\sigma_A^*(x) = \sigma_A(x)^*$. Moreover, $\mathrm{Op}(Q^\alpha\partial_x^\beta\sigma_A) \in \Psi^{m_A-|\alpha|}(G)$ for every $\alpha,\beta \in \mathbb{N}_0^n$ and $\sigma_A \in C^\infty(G, \mathcal{L}(H^{m_A}(G), H^0(G)))$* □

Relation (26) motivates the definition of $S_0^m(G)$:

**Definition.** *$\sigma_A \in S_0^m(G)$, if*

$$\|\Xi^{|\alpha|-m} Q^\alpha \partial_x^\beta \sigma_A(x)\|_{\mathcal{L}(L^2(G))} \le C_{A\alpha\beta m} \tag{27}$$

*uniformly in $x \in G$, for every $\alpha,\beta \in \mathbb{N}_0^n$.*

**Remark:** The $\mathcal{L}(L^2(G))$-norm of a convolution operator $\sigma_A(x)$ is just the supremum of the $\ell^2$ -operator norms of the finite-dimensional blocks of the canonical matrix representation of $\sigma_A(x)$ (see [15], p. 33).

By Corollary 3, $\mathrm{Op}S_0^m(G) \subset \mathcal{L}(H^m(G), H^0(G))$. Now we are going to recursively define symbol classes $S_k^m(G)$ such that $S_{k+1}^m(G) \subset S_k^m(G)$, and we shall prove that $\Psi^m(G) = \mathrm{Op}S^m(G)$ with $S^m(G) = \cap_{k=0}^\infty S_k^m(G)$.

**Remark:** Recall that the functions $q_\alpha$, $|\alpha|=1$, vanish simultaneously only at the neutral element $e\in G$, and that $q_{\alpha+\beta}=q_\alpha q_\beta$ for every $\alpha,\beta\in\mathbb{N}_0^n$. This implies that if $\sigma_A\in S_0^m(G)$, then $(x,y)\mapsto s_A(x)(y)$ is $C^\infty$-smooth function on $G\times(G\setminus\{e\})$, i.e. $(x,y)\mapsto K_A(x,y)$ may have singularities only at the $(x=y)$-diagonal. Moreover, notice that

$$f\in H^{-m}(G) \;\Rightarrow\; \pi(f)\in\mathcal{L}(H^s(G),H^{s-m}(G)), \tag{28}$$
$$\pi(g)\in\mathcal{L}(H^m(G),H^0(G)) \;\Rightarrow\; \forall r>n/2:\; g\in H^{-(m+r)}(G). \tag{29}$$

On a non-commutative group $G$ the multiplication of operator-valued symbols is not usually commutative, and this causes some complications in symbolic calculus. Recall Theorem 4; we now find additional requirements for the symbols by studying the commutator characterization of pseudo-differential operators.

Let $A\in\Psi^m(G)$, and let $D=M_\phi\partial_j$ be a derivation on $\mathcal{D}(G)$. By Taylor [15]

$$\begin{aligned}
\sigma_{[D,A]}(x) &= \phi(x)\,[\sigma_{\partial_j},\sigma_A(x)]\\
&\quad+\sum_{0<|\gamma|<N}\left(\phi(x)\,\left(Q^\gamma\sigma_{\partial_j}\right)\,\partial_x^\gamma\sigma_A(x)-(\partial_x^\gamma\phi(x))\,\left(Q^\gamma\sigma_A(x)\right)\,\sigma_{\partial_j}\right)\\
&\quad+\sum_{|\gamma|=N}R_\gamma(x)\\
&\sim \phi(x)\,[\sigma_{\partial_j},\sigma_A(x)]\\
&\quad+\sum_{|\gamma|>0}\left(\phi(x)\,\left(Q^\gamma\sigma_{\partial_j}\right)\,\partial_x^\gamma\sigma_A(x)-(\partial_x^\gamma\phi(x))\,\left(Q^\gamma\sigma_A(x)\right)\,\sigma_{\partial_j}\right)
\end{aligned}$$

is an asymptotic expansion of the symbol of $[D,A]$, where $R_\gamma\in\Psi^{m-N}(G)$. This expansion inspires new demands on symbols:

**Definition.** *$\sigma_A\in S_{k+1}^m(G)$, if*

$$\sigma_A\in S_k^m(G),$$
$$[\sigma_{\partial_j},\sigma_A]\in S_k^m(G),$$
$$(Q^\gamma\sigma_{\partial_j})\sigma_A\in S_k^{m+1-|\gamma|}(G)$$

*and*

$$(Q^\gamma\sigma_A)\sigma_{\partial_j}\in S_k^{m+1-|\gamma|}(G)$$

*for every $j\in\{1,\ldots,n\}$ and $\gamma\in\mathbb{N}_0^n$ with $|\gamma|>0$.*

**Definition.**

$$S^m(G)=\cap_{k=0}^\infty S_k^m(G).$$

**Lemma 7.** *Let $A,A_j\in\mathcal{L}(\mathcal{D}(G))$ such that $\sigma_{A_j}\in S^m(G)$ and $K_{A-A_j}\in C^j(G\times G)$ for every $j\in\mathbb{N}$. Then $\sigma_A\in S^m(G)$. Moreover, $\psi\sigma_A\in S^m(G)$ for every $\psi\in C^\infty(G)$.*

**Proof.** For any multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ we can evidently choose $j_{\alpha\beta} \in \mathbb{N}$ so large that

$$\|\Xi^{|\alpha|-m} Q^\alpha \partial_x^\beta \sigma_{A-A_{j_{\alpha\beta}}}(x)\|_{\mathcal{L}(L^2(G))} \leq C_{(A-A_{j_{\alpha\beta}})m},$$

and hence $\sigma_A \in S_0^m(G)$; the first step of induction is established. Next, suppose we have proven $\sigma_B \in S_k^r(G)$ whenever there exists $(\sigma_{B_j})_{j=1}^\infty \in S^r(G)$ such that $K_{B-B_j} \in C^j(G \times G)$ (for every $r \in \mathbb{R}$). Now

$$[\sigma_{\partial_i}, \sigma_A] = [\sigma_{\partial_i}, \sigma_{A_j}] + [\sigma_{\partial_i}, \sigma_{A-A_j}];$$

we know that $[\sigma_{\partial_i}, \sigma_{A_j}] \in S^m(G)$ for every $j \in \mathbb{N}$, and that the Schwartz kernel corresponding to $[\sigma_{\partial_i}, \sigma_{A-A_j}]$ gets arbitrarily smooth when $j$ tends to infinity. Thereby

$$[\sigma_{\partial_i}, \sigma_A] \in S_k^m(G);$$

similarly one proves that

$$(Q^\gamma \sigma_{\partial_i}) \sigma_A \in S_k^{m+1-|\gamma|}(G)$$

and

$$(Q^\gamma \sigma_A) \sigma_{\partial_i} \in S_k^{m+1-|\gamma|}(G).$$

Thus $\sigma_A \in S_{k+1}^m(G)$. Consequently we have proven that $\sigma_A \in S^m(G)$, the first claim of Lemma 7.

By induction we will get the result $C^\infty(G) S_k^m(G) \subset S_k^m(G)$. Indeed, for $k = 0$ this is trivial. Assuming

$$C^\infty(G) S_k^m(G) \subset S_k^m(G) \text{ for every } m \in \mathbb{R},$$

let $\psi \in C^\infty(G)$ and $\sigma_B \in S_{k+1}^m(G)$. Then $\psi \sigma_B \in S_k^m(G)$,

$$[\sigma_{\partial_j}, \psi \sigma_B] = [\sigma_{\partial_j}, \psi] \sigma_B + \psi [\sigma_{\partial_j}, \sigma_B] \in S_k^m(G),$$

$$(Q^\gamma \sigma_{\partial_j})(\psi \sigma_B) = \psi (Q^\gamma \sigma_{\partial_j}) \sigma_B \in S_k^{m+1-|\gamma|}(G)$$

and

$$(Q^\gamma(\psi \sigma_B)) \sigma_{\partial_j} = \psi (Q^\gamma \sigma_B) \sigma_{\partial_j} \in S_k^{m+1-|\gamma|}(G).$$

Thereby $C^\infty(G) S_{k+1}^m(G) \subset S_{k+1}^m(G)$, so that $C^\infty(G) S^m(G) \subset S^m(G)$ □

**Lemma 8.** *Let $\sigma_A \in S^{m_A}(G)$, $\sigma_B \in S^{m_B}(G)$. Then*

$$\begin{aligned} \sigma_{AB}(x) &= \sum_{|\alpha|<N} (Q^\alpha \sigma_A(x)) \; \partial_x^\alpha \sigma_B(x) \\ &\quad + \sum_{|\alpha|=N} \sigma_{R_{N,\alpha}}(x), \end{aligned}$$

*where for every $j \in \mathbb{N}$ there exists $N_j \in \mathbb{N}$ such that $K_{R_{N,\alpha}} \in C^j(G \times G)$ when $|\alpha| = N \geq N_j$.*

**Proof.** Observe that

$$
\begin{aligned}
(ABf)(x) &= (s_A(x) * (Bf))(x) \\
&= \int_G s_A(x)(xz^{-1})\ (\sigma_B(z)f)(z)\ d\mu_G(y) \\
&= \int_G s_A(x)(xz^{-1})\ \int_G s_B(z)(zy^{-1})\ f(y)\ d\mu_G(y)\ d\mu_G(z) \\
&= \int_G \int_G s_A(x)(xz^{-1})\ s_B(z)(zy^{-1})\ d\mu_G(z)\ f(y)\ d\mu_G(y) \\
&= \sum_{|\alpha|<N} \int_G \int_G s_A(x)(xz^{-1})\ \check{q}_\alpha(xz^{-1})\ \partial_x^\alpha s_B(x)(zy^{-1})\ d\mu_G(z) \\
&\qquad\qquad f(y)\ d\mu_G(y) \\
&\quad + \sum_{|\alpha|=N} \int_G \int_G (s_A(x))(xz^{-1})\ \check{q}_\alpha(xz^{-1})\ r_{B,N,\alpha}(x,z)(zy^{-1})\ d\mu_G(z) \\
&\qquad\qquad f(y)\ d\mu_G(y),
\end{aligned}
$$

where

$$
\begin{aligned}
s_B(z)(y) &= \sum_{|\alpha|<N} \check{q}_\alpha(xz^{-1})\ \partial_x^\alpha s_B(x)(y) \\
&\quad + \sum_{|\alpha|=N} \check{q}_\alpha(xz^{-1})\ r_{B,N,\alpha}(x,z)(y),
\end{aligned}
$$

with the remainder terms $(x,z) \mapsto r_{B,N,\alpha}(x,z)$ in $C^\infty(G \times G, H^{-m_B-r}(G))$ for every $r > n/2$. Hence

$$
\begin{aligned}
\sigma_{AB}(x) &= \sum_{|\alpha|<N} (Q^\alpha \sigma_A(x))\ \partial_x^\alpha \sigma_B(x) \\
&\quad + \sum_{|\alpha|=N} \sigma_{R_{N,\alpha}}(x),
\end{aligned}
$$

where

$$
K_{R_{N,\alpha}}(x,y) = \int_G s_A(x)(xz^{-1})\ \check{q}_\alpha(xz^{-1})\ r_{B,N,\alpha}(x,z)(zy^{-1})\ d\mu_G(z).
$$

Notice that $(x,y) \mapsto \check{s}_A(x)(y)\check{q}_\alpha(y)$ gets arbitrarily smooth as $N = |\alpha|$ grows (cf. the behaviour of $x \mapsto Q^\alpha \sigma_A(x)$), so that the kernel $K_{R_{N,\alpha}}$ gets indeed smoother with growing $N$ □

**Theorem 9.**

$$
\Psi^m(G) = \mathrm{Op}S^m(G).
$$

**Proof.** Let $A \in \Psi^m(G)$. By Lemma 6 we have

$$\sigma_A \in C^\infty(G, \mathcal{L}(H^m(G), H^0(G))),$$

and

$$\mathrm{Op}(Q^\alpha \partial_x^\beta \sigma_A) \in \Psi^{m-|\alpha|}(G).$$

Thus

$$\Psi^m(G) \subset \mathrm{Op}S_0^m(G).$$

We shall proceed by induction: Assume that $\Psi^m(G) \subset \mathrm{Op}S_k^m(G)$ for every $m \in \mathbb{R}$. By Lemma 6, we have

$$\mathrm{Op}((Q^\gamma \sigma_{\partial_j})\sigma_A),\ \mathrm{Op}((Q^\gamma \sigma_A)\sigma_{\partial_j}) \in \Psi^{m+1-|\gamma|}(G).$$

Moreover,

$$\mathrm{Op}([\sigma_{\partial_j}, \sigma_A]) \in \Psi^m(G),$$

because $[\sigma_{\partial_j}, \sigma_A(x)] = \sigma_{[\partial_j, A]}(x) - (\partial_j \sigma_A)(x)$. Hence $\Psi^m(G) \subset \mathrm{Op}S_{k+1}^m(G)$, so that we have proven

$$\Psi^m(G) \subset \mathrm{Op}S^m(G).$$

Now we are going to prove that $[D, \mathrm{Op}S^m(G)] \subset \mathrm{Op}S^m(G)$ for a partial differential operator $D = M_\phi \partial_j \in \Psi^1(G)$; by Theorem 4 this would mean $\mathrm{Op}S^m(G) \subset \Psi^m(G)$, because $\mathrm{Op}S^m(G) \subset \mathcal{L}(H^m(G), H^0(G))$ due to Corollary 3. Let $\sigma_A \in S^m(G)$. Lemma 8 yields

$$\begin{aligned}
\sigma_{[D,A]}(x) &= \phi(x)[\sigma_{\partial_j}, \sigma_A(x)] \\
&\quad + \sum_{0<|\gamma|<N} \left(\phi(x)\ \left(Q^\gamma \sigma_{\partial_j}\right)\ \partial_x^\gamma \sigma_A(x) - (\partial_x^\gamma \phi(x))\ (Q^\gamma \sigma_A(x))\ \sigma_{\partial_j}\right) \\
&\quad + \sum_{|\gamma|=N} R_{N,\gamma}(x),
\end{aligned}$$

where the remainder terms $R_{N,\gamma}$ behave nicely enough: the corresponding Schwartz kernels $K_{R_{N,\gamma}}$ become arbitrarily smooth as $N$ grows (see Lemma 8). Thereby due to Lemma 7 we get $\sigma_{[D,A]} \in S^m(G)$; combining this with $\mathrm{Op}S^m(G) \subset \mathcal{L}(H^m(G), H^0(G))$ we have proven that

$$\mathrm{Op}S^m(G) \subset \Psi^m(G)$$

□

## 5 Symbolic calculus

In this section we present some elementary symbol calculus on compact Lie groups, and the results resemble closely the Euclidean case (see e.g. [8], [16] or [17]) and the torus case (see [19]). Strichartz constructed a calculus for left-invariant pseudodifferential operators on Lie groups in [13], and Taylor for the class $\Psi^m(G)$ in [15]. We review some of Taylor's results. We also get some new results: we define and study a class of amplitudes in analogy to the Euclidean case, and present an asymptotic expansion for transposed operators. The culmination is the final Proposition 14 yielding another asymptotic expansion for a parametrix of an elliptic operator.

So far we have studied a pseudodifferential operator $A$ as a family $x \mapsto \sigma_A(x) = \pi(s_A(x))$ of convolution operators, where

$$(Af)(x) = (\sigma_A(x)f)(x) = \int_G s_A(x)(xy^{-1})\ f(y)\ d\mu_G(y).$$

Suppose we are given a two-argument family $(x, y) \mapsto a(x, y) = \pi(t_a(x, y))$ of convolution operators, and formally define an operator $\mathrm{Op}(a)$ by

$$(\mathrm{Op}(a)f)(x) = \int_G t_a(x, y)(xy^{-1})\ f(y)\ d\mu_G(y). \tag{30}$$

More rigorously:

**Definition.** *A convolution operator-valued function $a : G \times G \to \mathcal{L}(\mathcal{D}(G))$ is called an amplitude of order $m \in \mathbb{R}$, denoted by $a \in \mathcal{A}^m(G)$, if the mapping $(x, y) \mapsto \Xi^{-m}a(x, y)$ belongs to $C^\infty(G \times G, \mathcal{L}(L^2(G)))$ and the mapping $x \mapsto \partial_x^\beta \partial_y^\gamma a(x, y)|_{y=x}$ belongs to $S^m(G)$ for every $\beta, \gamma \in \mathbb{N}_0^n$.*

**Example.** Clearly $S^m(G) \subset \mathcal{A}^m(G)$, and if $\sigma \in S^m(G)$ then also $(x, y) \mapsto \sigma(y)$ belongs to $\mathcal{A}^m(G)$.

Proposition 10 reveals us that $\mathrm{Op}(a)$ belongs to $\mathrm{Op}S^m(G)$ whenever $a \in \mathcal{A}^m(G)$:

**Proposition 10.** *If $a \in \mathcal{A}^m(G)$ then $\mathrm{Op}(a) = A \in \mathrm{Op}S^m(G)$ and the symbol has an asymptotic expansion*

$$\sigma_A(x) \sim \sum_{\alpha \geq 0} Q^\alpha \partial_z^\alpha a(x, z)|_{z=x}.$$

**Proof.** Examining (30), first we develop $t_a(x, y)$ into Taylor series in variable $y$ at $x$:

$$\begin{aligned} t_a(x, y)(xy^{-1}) &= \sum_{|\alpha|<N} \check{q}_\alpha(zy^{-1})\ \partial_z^\alpha\ t_a(x, z)(xy^{-1})|_{z=x} \\ &\quad + \sum_{|\alpha|=N} \check{q}_\alpha(zy^{-1})\ r_{a,N,\alpha}(x, z, xy^{-1})|_{z=x}, \end{aligned}$$

where $(x, z, y) \mapsto \check{q}_\alpha(y) r_{a,N,\alpha}(x, z, y)$ gets arbitrarily smooth on $G \times G \times G$ as $N$ grows ($|\alpha| = N$). Lemma 7 then yields the conclusion $\square$

**Example.** One may encounter operators $A \in \mathcal{L}(\mathcal{D}(G))$ having amplitudes of the form $a(x,y) = \psi(x,y)\ \pi(\kappa)$, where $\kappa \in \mathcal{D}'(G)$ and $\psi \in C^\infty(G \times G)$; classical pseudodifferential operators on torus arising from mathematical physics may have this form (see [20]). By Proposition 10, such an operator has an asymptotic expansion

$$\sigma_A(x) \sim \sum_{\alpha \geq 0} \partial_y^\alpha \psi(x,y)|_{y=x}\ Q^\alpha \pi(\kappa).$$

Let us therefore take a look at the pseudodifferential operator algebra spanned by multiplications and left convolution pseudodifferential operators, completed by asymptotic expansions: Let

$$\sigma_A(x) \sim \sum_{\alpha \geq 0} \phi_\alpha(x)\ \pi(\kappa_\alpha)$$

and

$$\sigma_B(x) \sim \sum_{\beta \geq 0} \psi_\beta(x)\ \pi(\lambda_\beta),$$

where $\phi_\alpha, \psi_\beta \in \mathcal{D}(G)$, $\kappa_\alpha, \lambda_\beta \in \mathcal{D}'(G)$. Then $AB$ has an asymptotic expansion

$$\begin{aligned}
\sigma_{AB}(x) &\sim \sum_{\alpha,\beta,\gamma \geq 0} \phi_\alpha(x)\ (\partial_x^\gamma \psi_\beta)(x)\ \ (Q^\gamma \pi(\kappa_\alpha))\ \ \pi(\lambda_\beta) \\
&= \sum_{\alpha,\beta,\gamma \geq 0} \phi_\alpha(x)\ (\partial_x^\gamma \psi_\beta)(x)\ \pi((M_{\check{q}_\gamma} \kappa_\alpha) * \lambda_\beta),
\end{aligned}$$

i.e. this expansion is of the same form as those for $\sigma_A$ and $\sigma_B$.

**Remark.** Recall that $A^* \in \Psi^m(G)$ if $A \in \Psi^m(G)$. An amplitude of the adjoint $A^*$ is then given by $(x,y) \mapsto \sigma_A(y)^*$, because

$$\begin{aligned}
s_{A^*}(x)(xy^{-1}) &= K_{A^*}(x,y) \\
&= \overline{K_A(y,x)} \\
&= \overline{s_A(y)(y^{-1}x)} \\
&= \check{s}_A(y)(xy^{-1}).
\end{aligned}$$

Operator $AB$ has an amplitude $(x,y) \mapsto \sigma_A(x)\sigma_{B^*}(y)^*$, since

$$\begin{aligned}
(ABf)(x) &= (AB^{**}f)(x) \\
&= \int_G \int_G s_A(x)(xz^{-1})\ \check{s}_{B^*}(z)(zy^{-1})\ f(y)\ d\mu_G(y)\ d\mu_G(z) \\
&= \int_G t_c(x,y)(xy^{-1})\ f(y)\ d\mu_G(y),
\end{aligned}$$

where $t_c(x,y) = s_A(x) * \check{s}_{B^*}(y)$, hence $c(x,y) = \sigma_A(x)\sigma_{B^*}(y)^*$. Thereby Proposition 10 combined with Theorem 9 and Lemma 6 would provide asymptotic expansions for the symbols of the adjoint operator and the composition of operators. However, we shall give short explicit constructions of asymptotic expansions in Propositions 11 and 12.

**Proposition 11.** *If* $\sigma_A \in S^{m_A}(G)$ *and* $\sigma_B \in S^{m_B}(G)$ *then* $\sigma_{AB} \in S^{m_A+m_B}(G)$,

$$\sigma_{AB}(x) \sim \sum_{\alpha \geq 0} (Q^\alpha \sigma_A(x)) \ (\partial_x^\alpha \sigma_B)(x).$$

**Proof.** The expansion arising from Lemma 8 yields the result, once we notice that $x \mapsto \sigma_A(x)\sigma_B(x)$ belongs to $S^{m_A+m_B}(G)$ because of Theorem 9 and Lemma 6 □

**Proposition 12.** *Let* $\sigma_A \in S^m(G)$. *Then the adjoint* $A^* \in \mathrm{Op}S^m(G)$ *has an asymptotic expansion*

$$\sigma_{A^*}(x) \sim \sum_{\alpha \geq 0} Q^\alpha \partial_x^\alpha \left(\sigma_A(x)^*\right).$$

**Proof.** Formally

$$\begin{aligned}
\int_G f(y)\overline{(A^*g)(y)}\ d\mu_G(y) &= \int_G (Af)(x)\ \overline{g(x)}\ d\mu_G(x) \\
&= \int_G \int_G (s_A(x))(xy^{-1})\ f(y)\ d\mu_G(y)\ \overline{g(x)}\ d\mu_G(x) \\
&= \int_G f(y)\ \int_G \overline{(\widetilde{s_A(x)})(yx^{-1})\ g(x)\ d\mu_G(x)}\ d\mu_G(y) \\
&\sim \sum_{\alpha \geq 0} \int_G f(y)\ \int_G \overline{\check{q}_\alpha(yx^{-1})\ (\widetilde{\partial_y^\alpha s_A})(y)(yx^{-1})\ g(x)} \\
&\quad\quad d\mu_G(x)\ d\mu_G(y) \\
&= \sum_{\alpha \geq 0} \int_G f(y)\ \overline{Q^\alpha((\partial_y^\alpha \sigma_A)(y)^* g)(y)}\ d\mu_G(y),
\end{aligned}$$

because $\pi(\phi)^* = \pi(\tilde{\phi})$. Due to Lemma 6, $x \mapsto \sigma_A(x)^*$ belongs to $S^m(G)$, and the behaviour of the remainder terms in expansion is studied as in the proof of Theorem 9. Hence the asymptotic expansion for $\sigma_{A^*}$ follows □

An asymptotic expansion for the transpose $A^t \in \Psi^m(G)$ of $A \in \Psi^m(G)$ can be derived in analogy to the proof of Proposition 12, recalling that $\pi(\phi)^t = \pi(\check{\phi})$:

$$\sigma_{A^t}(x) \sim \sum_{\alpha \geq 0} Q^\alpha \partial_x^\alpha \sigma_A(x)^t. \tag{31}$$

**Definition.** *We say that an operator* $A \in \Psi^\infty(G)$ *is elliptic, if there exists* $B \in \Psi^\infty(G)$ *such that* $AB \sim I \sim BA$. $B$ *is said to be a parametrix for* $A$.

**Proposition 13.** *If $\sigma_A \in S^m(G)$ with $x \mapsto \sigma_A(x)^{-1}$ belonging to $S^{-m+\varepsilon}(G)$ for some $\varepsilon \in [0,1)$ then $A$ is elliptic with a parametrix $B \in \mathrm{Op}S^{-m+\varepsilon}(G)$,*

$$\sigma_B(x) \sim \sigma_A(x)^{-1} \sum_{j=0}^{\infty} \sigma_{R^j}(x),$$

*where $R \in \mathrm{Op}S^{-1+\varepsilon}(G)$,*

$$\sigma_R(x) \sim \sum_{|\alpha|>0} (Q^\alpha \sigma_A(x))\ \partial_x^\alpha (\sigma_A(x)^{-1}).$$

The proof of this proposition follows traditional lines, applying the asymptotic expansion for composition of operators, see analogous treatment in [15]. Notice that since we do analysis only modulo infinitely smoothing operators, it is enough to require that $x \mapsto \sigma_{A+P}(x)^{-1}$ belongs to $S^{-m+\varepsilon}(G)$ for some $P \in \Psi^{-\infty}(G)$; indeed it is not necessary to assume that $\sigma_A(x)$ be invertible.

Let us finally present another expansion for a parametrix:

**Proposition 14.** *Let $\sigma_A \in S^m(G)$ with an asymptotic expansion $\sigma_A \sim \sum_{j=0}^\infty \sigma_{A_j}$, where $\sigma_{A_j} \in S^{m-j}(G)$ when $j \geq 1$ and $x \mapsto \sigma_{A_0}(x)^{-1} =: \sigma_{B_0}(x)$ belongs to $S^{-m+\varepsilon}(G)$ for some $\varepsilon \in [0,1)$. Then there exists a parametrix $B \in \mathrm{Op}S^{-m+\varepsilon}(G)$ for $A$, having an asymptotic expansion $B \sim \sum_{k=0}^\infty B_k$, where $B_N \in \mathrm{Op}S^{-m-(1-\varepsilon)N}(G)$,*

$$\sigma_{B_N}(x) = -\sigma_{B_0}(x) \sum_{k=0}^{N-1} \sum_{j=0}^{N-k} \sum_{\gamma: j+k+|\gamma|=N} (Q^\gamma \sigma_{A_j}(x))\ \partial_x^\gamma \sigma_{B_k}(x).$$

**Proof.** Due to Proposition 11,

$$\begin{aligned}
\sigma_{AB}(x) &\sim \sum_{\gamma \geq 0} (Q^\gamma \sigma_A(x))\ \partial_x^\gamma \sigma_B(x) \\
&\sim \sum_{N=0}^{\infty} \sum_{k=0}^{N} \sum_{j=0}^{N-k} \sum_{\gamma: j+k+|\gamma|=N} (Q^\gamma \sigma_{A_j}(x))\ \partial_x^\gamma \sigma_{B_k}(x) \\
&= I,
\end{aligned}$$

i.e. $AB \sim I$. On the other hand, there exists $B' \in \mathrm{Op}S^{-m+\varepsilon}(G)$ such that $B' \sim \sum_{k=0}^\infty B'_k$, where $B'_0 = B_0$ and

$$\sigma_{B'_N}(x) = -\sigma_{B'_0}(x) \sum_{k=0}^{N-1} \sum_{j=0}^{N-k} \sum_{\gamma: j+k+|\gamma|=N} (Q^\gamma \sigma_{B'_k}(x))\ \partial_x^\gamma \sigma_{A_j}(x);$$

Then $B'A \sim I$, so that $B' = B'I \sim B'(AB) = (B'A)B \sim IB = B$, or $B' \sim B$. Thus $B$ and $B'$ are both parametrices for $A$ $\square$

**Remark.** Let $A \sim \sum_{j=0}^{\infty} A_j$ as in Proposition 14 above. We may form a parametrix under a weaker hypothesis: Namely, assume that there exist operators $C, C' \in \Psi^{-m+\varepsilon}(G)$ for some $\varepsilon \in [0,1)$ such that $\sigma_A(x)\sigma_C(x) = I - \sigma_R(x)$ and $\sigma_{C'}(x)\sigma_A(x) = I - \sigma_{R'}(x)$, where $\sigma_R, \sigma_{R'} \in S^{-\delta}(G)$ for some $\delta > 0$. Then define $B_0, {B'}_0$ by

$$\sigma_{B_0}(x) \sim \sigma_C(x) \left( \sum_{j=0}^{\infty} \sigma_R(x)^j \right), \quad \sigma_{{B'}_0}(x) \sim \left( \sum_{j=0}^{\infty} \sigma_{R'}(x)^j \right) \sigma_{C'}(x)$$

Now $\sigma_A(x)$ does not have to be invertible, but still $\sigma_A(x)\sigma_{B_0}(x) \sim I \sim \sigma_{{B'}_0}(x)\sigma_A(x)$, and we may calculate expansions $B \sim \sum_{k=0}^{\infty} B_k \sim \sum_{k=0}^{\infty} {B'}_k \sim B'$ as in the proof of Proposition 14.

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