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An analysis of finite element locking in a parameter dependent model problem

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Summary. We consider the bilinear finite element approximation of smooth solutions to a simple parameter dependent elliptic model problem, the problem of highly anisotropic heat conduction. We show that under favorable circumstances that depend on both the finite element mesh and on the type of boundary conditions, the effect of parametric locking of the standard FEM can be reduced by a simple variational crime. In our analysis we split the error in two orthogonal components, the approximation error and the consistency error, and obtain different bounds for these separate components. Also some numerical results are shown.

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1 Introduction

Locking, or parametric error amplification is a well-known phenomenon that may arise when solving parametric elliptic problems with the finite element method. A rather challenging problem of this type is the shell problem of linear elasticity [6,4,7] where the thickness of the shell serves as the main parameter. Locking usually effectively prohibits the convergence of low-order finite element schemes when the parameter associated with the problem approaches an asymptotic value such as zero. There have been numerous attempts to reduce the effects of locking in order to obtain improved, or even optimal, convergence rates in various problems of linear elasticity. In fact, in this home field of FEM even one of the earliest finite elements, the Turner rectangle [9], may be viewed as such a formulation, see [5].

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Perhaps the most popular way to circumvent the problem of locking is to produce a variational crime of some kind, i.e. the bilinear form $\mathcal{A}(u,v)$ associated with the problem is substituted by another, usually mesh dependent form $\mathcal{A}_h(u,v)$. The aim of the modification is to generate a weaker norm in the underlying energy space $\mathcal{V}$ so as to "unlock" the problem. However, the choice of the modification is a delicate matter since introducing a variational crime entails a consistency error component that must also be treated.

In this paper we consider a simple parametric elliptic model problem, the problem of highly anisotropic heat conduction, previously studied in [1]. This problem resembles the shell problem in that, depending on the boundary conditions and on the load, the solution may fall in two different asymptotic states. In the context of heat conduction these states could be named as the *cool state* and the *hot state*, respectively. In the cool state, the main heat conduction occurs in the direction of high conductivity (as expected normally), whereas in the hot state, the conduction in the direction of low conductivity dominates. In beam theory [5] and in shell theory [7], the cool state corresponds to a deformation state where stretching dominates, while the hot state corresponds to a bending-dominated deformation.

In parametric elliptic problems like the ones mentioned, the numerical locking problem appears in the "hot" or bending-dominated asymptotic states. In problems where this is the only relevant state, like in the plate-bending problem [8], various "tricks" or variational crimes have usually been formulated so as to avoid the locking effect. The real challenge begins, however, when two (or perhaps more) asymptotic states are possible. One should then try to find a formulation that not only avoids the locking in the "hot" state but also maintains the good performance in the "cool" state. The main problem is that the more the crime helps to avoid the locking effect, the larger consistency error typically appears in the "cool" state where no locking occurs.

In general, it is far from obvious that a "dream scheme" good for all asymptotic states exists in the context of the simple low order FEM. However, the few successful examples, like the bilinear Turner rectangle for beams [5], seem to indicate that the problem at least is not hopeless. Here we consider a simple model problem which aims to model our ultimate target, the shell problem.

The plan of the paper is as follows. In Sect. 2 we present our model problem containing positive but arbitrarily small parameter ϵ. We show that as $\epsilon \to 0$, two different asymptotic solution modes are induced by different boundary conditions. In Sect. 3 we introduce the modified bilinear formulation for the model problem where the main crime is to elementwise average the heat flux in the direction of high conductivity. The error analysis of this reduced-flux scheme is carried out in Sects. 4-6. In Sect. 4 we split the error into two components, the approximation error which is treated in Sect. 5, and the consistency error treated in Sect. 6. The results of some numerical experiments are shown in Sect. 7.

We denote the kth Sobolev norm by $\|\cdot\|_k$, the corresponding seminorm by $|\cdot|_k$ and the norm over the boundary of a domain Ω by $\|\cdot\|_{k,\partial\Omega}$. The L^2-inner product is written as $\langle\cdot,\cdot\rangle$ and the induced norm as $\|\cdot\|_{L^2}$. $\mathcal{V}$ and $\mathcal{V}^0$ stand for the energy space and its subspace with proper homogeneous boundary conditions. $\mathcal{V}_h$ and $\mathcal{V}_h^0$ are the finite dimensional counterparts of these two spaces. C represents an arbitrary but finite constant, not necessarily always the same, but independent of any parameter unless noted otherwise. Finally, Q denotes a constant that depends on the exact solution u but not on the parameter ϵ.

2 The model problem

As a model problem for our analysis of locking in this paper we take the problem considered already in [1]: The anisotropic heat equation in the unit square $\Omega = (0,1)\times(0,1)$ with principal axes that are not aligned with the coordinate axis, that is

$$-\frac{\partial^2 u}{\partial \xi^2} - \epsilon^2 \frac{\partial^2 u}{\partial \eta^2} = f \text{ in } \Omega$$

with

$$\begin{cases} \xi &= \alpha x + \beta y \\ \eta &= -\beta x + \alpha y \end{cases}$$

where $\alpha^2 + \beta^2 = 1$ and $\alpha, \beta \neq 0$. From this setting we generate three different problems where the behavior of the solution is largely dictated by the boundary conditions:

A. $u = w$ on $\partial\Omega$
B. $\partial_\nu u = g$ when $x = 1$ or $y = 1$ and $u = w$ elsewhere on $\partial\Omega$
C. $\partial_\nu u = g$ on $\partial\Omega$

Here f, w and g are given functions, chosen so that the exact solution of the problem is sufficiently smooth, and ∂_ν stands for the normal component of the heat flux at the boundary, i.e.,

$$\partial_\nu u = (\alpha n_x + \beta n_y)\frac{\partial u}{\partial \xi} + \epsilon^2(-\beta n_x + \alpha n_y)\frac{\partial u}{\partial \eta},$$

where (n_x, n_y) is the outward unit normal to $\partial\Omega$. The variational formulation of these problems is: Find $u \in \mathcal{V}$ such that

$$\mathcal{A}(u,v) = \phi(v) \quad \forall v \in \mathcal{V}^0 \tag{2.1}$$

where the bilinear form (energy product) is written as

$$\mathcal{A}(u,v) = \left\langle \frac{\partial u}{\partial \xi}, \frac{\partial v}{\partial \xi} \right\rangle + \epsilon^2 \left\langle \frac{\partial u}{\partial \eta}, \frac{\partial v}{\partial \eta} \right\rangle,$$

and the linear functional as

$$\phi(v) = \langle f, v \rangle + \int_{\Gamma_N} g v d\Gamma,$$

where $\Gamma_N = \emptyset$ for Problem A, $\Gamma_N = \{(x,y) \in \partial\Omega \,|\, x = 1 \text{ or } y = 1\}$ for Problem B, and $\Gamma_N = \partial\Omega$ for Problem C. In (2.1), $\mathcal{V} = H_1(\Omega)$ and $\mathcal{V}^0 = \{v \in H_1(\Omega) \,|\, v = 0 \text{ on } \partial\Omega \setminus \Gamma_N\}$. In Problem C we assume that $\phi(1) = 0$ and impose the constraint $\langle u, 1 \rangle = 0$ on $\mathcal{V}$ so as to make the solution unique.

Due to the assumed constraint $u = w$ when $x = 0$ or $y = 0$ in Problems A,B, we may assume that heat conduction in the ξ-direction becomes dominant in these cases as $\epsilon \to 0$. This corresponds to the "cool" state where, neglecting any boundary layer effects, we may assume that

$$u \sim u^0 \text{ as } \epsilon \to 0$$

where the limiting solution u^0 satisfies

$$\left\langle \frac{\partial u^0}{\partial \xi}, \frac{\partial v}{\partial \xi} \right\rangle = \phi(v) \quad \forall v \in \mathcal{V}^0. \tag{2.2}$$

In Problem C the physical situation is rather different. In this case, due to the forced heat flux along $\partial\Omega$, the solution develops in general a "hot" component that scales like $u \sim \epsilon^{-2}$ and represents heat flux in the η-direction only. Again neglecting boundary layer effects, we may then assume that

$$u \sim \epsilon^{-2} u^0 \text{ as } \epsilon \to 0$$

where u^0 (the scaled limiting solution) satisfies

$$\frac{\partial u^0}{\partial \xi} = 0. \tag{2.3}$$

The above discussion is to motivate the following hypothesis on the solution.

Hypothesis 1 *In problems A and B assume the solution has the form*

$$u = u^0 + \epsilon^2 u^1, \tag{2.4}$$

where u^0 satisfies (2.2) with the bounds

$$\|u\|_k \le Q|||u||| \text{ for } k \le 5 \tag{2.5}$$

and

(2.6) $$\|u\|_{k,\partial\Omega} \leq Q|||u||| \text{ for } k = 1,2.$$

where $|||\cdot|||$ *is the energy norm*

$$|||u||| = \sqrt{\mathcal{A}(u,u)}.$$

In problem C assume the solution has the form

(2.7) $$u = \epsilon^{-2}u^0 + u^1$$

u^0 *satisfying (2.3) and* $u^0 \neq 0$*. Assume here further that*

(2.8) $$\|u^i\|_k \leq Q\epsilon|||u||| \text{ for } k \leq 3 \text{ and for } i = 0,1.$$

Remark 1 We note that as as rule, u^0 in (2.4) or in (2.7) is not continuously differentiable across the lines η =const. that pass through a corner of Ω at $x = 0$, $y = 0$ or at $x = 1$, $y = 1$, so in this sense the regularity assumptions (2.5), (2.6) and (2.8) are quite unrealistic. However, we choose not to discuss problems associated with the regularity of the exact solution here, but simply assume these bounds for our analysis. We could formally justify these assumptions by thinking of Ω as a fictitious subdomain of a larger domain where the physical boundary conditions are set so that $u_{|\Omega}$ is smooth.

Remark 2 In order to have $u^0 \neq 0$ in the hot state we must assume that our boundary data g and load function f are such that there is at least some $v \in \mathcal{W}$ for which $\phi(v) \neq 0$ where $\mathcal{W} = \{u \in \mathcal{V} \,|\, \frac{\partial u}{\partial \xi} = 0\}$. The scaled limiting solution u^0 in (2.7) then satisfies (2.1) with $\mathcal{W}$ replacing $\mathcal{V}$.

Remark 3 In [1] the concepts of "locking" or "robustness" were given a precise mathematical meaning in the context of a model problem of type C. Here our focus is somewhat different. We look at different solution states simultaneously, basically looking for a simple FE scheme that is "robust" with respect to the variation of both ϵ and the solution state from hot to cool.

3 Standard FE scheme vs. reduced formulation

Our main concern throughout this paper is the isoparametric bilinear element: Let τ denote the subdivision of Ω into convex disjoint quadrilaterals K that satisfy the usual shape regularity assumptions (cf. [2]). Then we set our local finite element space to be

$$\mathcal{M}_K = \{v = \hat{v} \circ \mathcal{F}_K^{-1},\ v \in \mathcal{M}_{\hat{K}}\}$$

where $\mathcal{M}_{\hat{K}}$ is the reference finite element space associated to the reference element $\hat{K} = (-1,1) \times (-1,1)$ and $\mathcal{F}_K : \hat{K} \to K$ is a bilinear map. For the reference space we take

$$\mathcal{M}_{\hat{K}} = \{\hat{v}(\hat{x},\hat{y}) = a_{00} + a_{10}\hat{x} + a_{01}\hat{y} + a_{11}\hat{x}\hat{y},\ a_{ij} \in \mathbb{R}\}$$

and for the degrees of freedom the nodal values as usual. We further denote by h_K the largest side of the element K, let $h = \max_K h_K$ be our mesh parameter, and denote by $\mathcal{V}_h$ the piecewise bilinear FE space (a subspace of $\mathcal{V}$) associated to a given mesh.

In the above notation, the standard bilinear FE approximation $u_h \in \mathcal{V}_h$ to the solution of our model problem satisfies

$$\mathcal{A}(u_h, v) = \phi(v) \quad \forall v \in \mathcal{V}_h^0 \tag{3.1}$$

together with the interpolated constraint $u_h = u$ at those nodal points of the boundary where the corresponding constraint is imposed in the exact formulation (Problems A,B). For this scheme, the standard FE error analysis together with the assumed regularity assumptions (2.5) (Problems A,B) and (2.7), (2.8) (Problem C) gives us the following error bound in the energy norm:

Theorem 1 *For the standard bilinear FE scheme (3.1) we have the error bounds*

$$\frac{|||u - u_h|||}{|||u|||} \le \begin{cases} Qh & \textit{in Problems A,B} \\ \min\{1, Q\epsilon^{-1}h\} & \textit{in Problem C} \end{cases}$$

Proof. In Problems A,B (cool state) the asserted bound follows from standard FE approximation theory based on assumption (2.5) with $k = 2$, cf. [2]. In Problem C (hot state) we have no constraints on the boundary so the projection principle gives $|||u - u_h||| \le |||u|||$. To prove the second bound we expand u_h in analogy with (2.7) as

$$u_h = \epsilon^{-2} u_h^0 + u_h^1.$$

The asserted bound then follows again by standard reasoning based on bounds (2.8) with $k = 2$. Here we also see that the dominant error contribution comes from

$$\frac{|||\epsilon^{-2}(u^0 - u_h^0)|||}{|||u|||} \sim \left\{ \epsilon^{-2} \| \frac{\partial}{\partial \xi}(u^0 - u_h^0) \|_{L^2}^2 + \| \frac{\partial}{\partial \eta}(u^0 - u_h^0)\|_{L^2}^2 \right\}^{1/2}.$$

Since $\frac{\partial u^0}{\partial \xi} = 0$, ϵ-uniform convergence is possible only under constraint $\frac{\partial u_h^0}{\partial \xi} = 0$. This, however, implies that $u_h^0 = c_1 + c_2 \eta$ for some constants c_1,

c_2, unless the mesh is carefully aligned with the ξ-axis. Thus we conclude that the asserted bound is (essentially) not improvable on a general mesh. This is also confirmed by numerical experiments, see Sect. 7. $\square$

In order to circumvent the parametric error amplification in the hot state, we now commit a variational crime choosing our new mesh dependent formulation to be: Find $u_h \in \mathcal{V}_h$ such that

$$\mathcal{A}_h(u_h, v) = \phi(v) \quad \forall v \in \mathcal{V}_h^0 \tag{3.2}$$

where

$$\begin{aligned} \mathcal{A}_h(u,v) = \left\langle R_h \frac{\partial u}{\partial \xi}, R_h \frac{\partial v}{\partial \xi} \right\rangle + \epsilon^2 \left\langle \frac{\partial u}{\partial \eta}, \frac{\partial v}{\partial \eta} \right\rangle \\ + \epsilon^2 \left\langle (I - R_h) \frac{\partial u}{\partial \xi}, \frac{\partial v}{\partial \xi} \right\rangle . \end{aligned} \tag{3.3}$$

Here R_h is a numerical flux reduction operator which we choose to be the orthogonal L^2-projection onto elementwise constant functions, i.e.

$$R_h \varphi_{|K} = \frac{1}{\text{area}(K)} \int_K \varphi \, dxdy$$

for every element K.

The basic idea in the formulation (3.2)-(3.3) is the hope that weakening the locking constraint from $\frac{\partial u^0}{\partial \xi} = 0$ to $R_h \frac{\partial u^0}{\partial \xi} = 0$ retains the approximation properties of the finite element subspace at the limit $\epsilon \to 0$ in the hot state. However, one must also keep in mind the cool state where the standard element did not suffer from locking. Thus there is an additional requirement that in the cool state the performance of the scheme should not deteriorate due to the flux reduction. The last term in (3.3) is introduced to keep the formulation at least ϵ-stable, or H_1-stable at $\epsilon = O(1)$. We finally note that the reduction operator R_h could be different, but the chosen one appears a rather natural "first choice".

In non-standard FE formulations, the regularity of the mesh may have a strong impact on the actual performance of the algorithm, cf. [5]. To study such possible effects in the present context we will distinguish between four different mesh types as follows:

1. General quadrilateral mesh.
2. Rectangular mesh.
3. Piecewise uniform rectangular mesh: Ω is divided into N subrectangles Ω_i, and each Ω_i is subdivided by a uniform rectangular mesh.
4. Uniform rectangular mesh: case 3 with $N = 1$.

The four mesh types are shown in Fig. 1. Note that on each of these meshes, the standard FE scheme suffers from locking at $\epsilon \sim 0$ in Problem C.

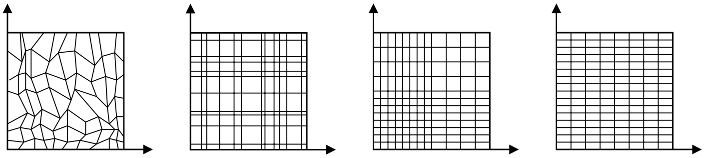

Fig. 1. Different mesh types. From left to right: General quadrilateral, rectangular, piecewise uniform and uniform mesh

4 Error analysis principles

In our error analysis of the reduced-flux scheme (3.2)-(3.3) we follow the ideas of [5] choosing our error indicator to be

$$e = \frac{|||u - u_h|||_h}{|||u|||}$$

where u and u_h are the exact and finite element solution, respectively, and $||| \cdot |||_h$ is the (semi) norm generated by the bilinear form $\mathcal{A}_h(u, v)$ on $\mathcal{V}$. Our aim is to divide the total relative error e into two parts and discuss them separately. To this end, we note that by (2.1), (3.2)

$$\mathcal{A}_h(u_h, v) = \mathcal{A}(u, v) \quad \forall v \in \mathcal{V}_h^0.$$

Let us then split u_h as $u_h = \tilde{u}_h + z_h$ where $\tilde{u}_h$ satisfies the same boundary conditions as u_h, and is defined as the best approximation in $\mathcal{V}_h$ to u with respect to the norm $||| \cdot |||_h$, so that

$$\mathcal{A}_h(\tilde{u}_h, v) = \mathcal{A}_h(u, v) \quad \forall v \in \mathcal{V}_h^0 \ ,$$

Since $z_h \in \mathcal{V}_h^0$, this implies in particular that

$$\mathcal{A}_h(u - \tilde{u}_h, z_h) = 0 \ ,$$

and therefore the orthogonal splitting of the error as

$$|||u - u_h|||_h^2 = |||u - \tilde{u}_h|||_h^2 + |||z_h|||_h^2 \ ,$$

or

$$\mathrm{e}^2 = e_A^2 + e_C^2 \ ,$$

where e_A, the approximation error, is

$$e_A = \frac{|||u - \tilde{u}_h|||_h}{|||u|||} \ , \tag{4.1}$$

and e_C, the consistency error, is

$$e_C = \frac{|||z_h|||_h}{|||u|||}.$$

We also note that since

$$\mathcal{A}_h(z_h, v) = (\mathcal{A} - \mathcal{A}_h)(u, v) \quad \forall v \in \mathcal{V}_h^0 \ ,$$

the consistency error can be written as

$$e_C = \sup_{v \in \mathcal{V}_h^0, v \neq 0} \frac{(\mathcal{A} - \mathcal{A}_h)(u, v)}{|||u||| \ |||v|||_h}. \tag{4.2}$$

In order to bound the consistency error we need some stability results for the reduced-flux scheme (3.2)-(3.3). The following lemma gives us bounds that will play a crucial role in the analysis of Sect. 6. We denote here by Γ^+ the "outflow" boundary of the primary heat-flow in the ξ-direction in Problems A, B.

Lemma 1 *In Problems A,B,C*

$$|v|_1 \leq C\epsilon^{-1}|||v|||_h \quad \forall v \in \mathcal{V}. \tag{4.3}$$

In addition, if the mesh is piecewise uniform, one has

$$\left(\|v\|_{L^2}^2 + \|v\|_{L^2(\Gamma^+)}^2\right)^{1/2} \leq C(N)|||v|||_h \quad \forall v \in \mathcal{V}_h^0 \tag{4.4}$$

in Problems A, B.

Proof. Estimate (4.3) is a direct consequence of the definition of $\mathcal{A}_h(u, v)$. To prove (4.4), assume first a uniform mesh. Then

$$R_h \frac{\partial}{\partial x} v = \frac{1}{2h_x}(v^{\mathrm{i},j} - v^{\mathrm{i}-1,j} + v^{\mathrm{i},j-1} - v^{\mathrm{i}-1,j-1}),$$

where h_x is the mesh spacing in the x-direction and the $v^{k,l}$'s denote the nodal values of v. Upon expanding $R_h \frac{\partial v}{\partial y}$ similarly we see that $R_h \frac{\partial v}{\partial \xi}$ actually defines a well-known difference approximation, the box-scheme, for solving the linear hyperbolic equation $\frac{\partial v}{\partial \xi} = f$. Assuming that f is elementwise constant, the L^2-stability of the box-scheme [3] implies (4.4):

$$\|v\|_{L^2}^2 + \|v\|_{L^2(\Gamma^+)}^2 \leq C\|f\|_{L^2}^2 = C\|R_h \frac{\partial v}{\partial \xi}\|_{L^2}^2 \leq C|||v|||_h^2.$$

The piecewise uniform case follows applying the result separately to each subdomain Ω_i. In this case $C = C(N)$. □

Remark 4 When (4.4) holds, we have

$$|v|_1 \leq C(N)h^{-1}|||v|||_h \quad \forall v \in \mathcal{V}_h^0$$

by usual inverse inequalities on shape regular meshes.

5 The approximation error

The approximation error of the reduced-flux scheme (3.2)-(3.3) was defined by (4.1). The following theorem gives bounds for this error in case of the three test problems A,B,C and the four mesh types.

Theorem 2 *In Problems A and B (the cool state) the approximation error obeys the bound*

$$e_A \leq Qh$$

on every mesh type. In Problem C (the hot state) the approximation error obeys the bound

$$e_A \leq \begin{cases} Q\epsilon^{-1}h & \textit{on a general mesh} \\ Q\epsilon^{-1}h^2 & \textit{on a rectangular mesh} \\ Q(N)h & \textit{on a piecewise uniform mesh} \end{cases}$$

Proof. The bounds on a general mesh are direct consequences of the inequalities (2.5), (2.8) (with $k = 2$) and standard approximation theory. To obtain the bound on a rectangular mesh (Problem C) we consider the standard interpolant $\hat{u}_h \in \mathcal{V}_h$ of u. We have

$$|||u - \hat{u}_h|||_h^2 = \|R_h \frac{\partial}{\partial \xi}(u - \hat{u}_h)\|_{L^2}^2 + \epsilon^2 \|\frac{\partial}{\partial \eta}(u - \hat{u}_h)\|_{L^2}^2 + \epsilon^2 \|(I - R_h)\frac{\partial}{\partial \xi}(u - \hat{u}_h)\|_{L^2}^2,$$

so by (4.1) and (2.8) the main problem is the first term on the right side. Here $\frac{\partial v}{\partial \xi} = \alpha \frac{\partial v}{\partial x} + \beta \frac{\partial v}{\partial y}$, so that

$$\|R_h \frac{\partial}{\partial \xi}(u - \hat{u}_h)\|_{L^2} \leq \|R_h \frac{\partial}{\partial x}(u - \hat{u}_h)\|_{L^2} + \|R_h \frac{\partial}{\partial y}(u - \hat{u}_h)\|_{L^2}.$$

Thus we must find a bound for terms of type

$$\|R_h \frac{\partial}{\partial x}(u - \hat{u}_h)\|_{L^2} = \|R_h \frac{\partial}{\partial x} u - R_h \frac{\partial}{\partial x}\hat{u}_h\|_{L^2} \ .$$

However, since

$$R_h \frac{\partial}{\partial x}(u - \hat{u}_h)_{|K} = 0$$

whenever u is a quadratic polynomial on K, it follows by standard reasoning that $\|R_h \frac{\partial}{\partial x}(u - \hat{u}_h)\|_{L^2(K)} \leq Ch^2 |u|_{3,K}$. Thus the asserted first bound follows using (2.7), (2.8) (with $k = 3$).

In order to prove the error bound on a piecewise uniform mesh we first consider the case $N = 1$. Here we make use of the decomposition (2.7), choosing our approximation $\hat{u}_h$ to u as

$$\hat{u}_h = \epsilon^{-2}\hat{u}_h^0 + \hat{u}_h^1,$$

where $\hat{u}_h^0$ satisfies the constraint

$$R_h \frac{\partial}{\partial \xi}\hat{u}_h^0 = 0, \tag{5.1}$$

and $\hat{u}_h^1$ is the standard interpolant of u^1. Again, (5.1) defines the box-scheme, now for solving $\frac{\partial u^0}{\partial \xi} = 0$. The box-scheme is second order accurate, so selecting inflow boundary-conditions such that $\hat{u}_h^0$ interpolates u^0 on $\partial\Omega$ we have the bound [3]

$$\|u^0 - \hat{u}_h^0\|_{L^2} \leq Ch^2\|u^0\|_3 \ .$$

From this we obtain by standard inverse inequalities the bounds

$$\|\frac{\partial}{\partial \xi}(u^0 - \hat{u}_h^0)\|_{L^2} \leq Ch\|u^0\|_3, \quad \|\frac{\partial}{\partial \eta}(u^0 - \hat{u}_h^0)\|_{L^2} \leq Ch\|u^0\|_3 \ .$$

Using these bounds together with $\frac{\partial}{\partial \xi}u^0 = R_h \frac{\partial}{\partial \xi}\hat{u}_h^0 = 0$ and standard interpolation error bounds for $u^1 - u_h^1$, we can now bound e_A as

$$\begin{aligned}
e_A^2 \leq & \frac{(\epsilon^{-2}\|R_h \frac{\partial}{\partial \xi}(u^0 - \hat{u}_h^0)\|_{L^2} + \|\frac{\partial}{\partial \xi}(u^1 - \hat{u}_h^1)\|_{L^2})^2}{|||u|||^2} \\
& + \frac{\epsilon^2\|\frac{\partial}{\partial \eta}(u - \hat{u}_h)\|_{L^2}^2 + \epsilon^2\|(I - R_h)\frac{\partial}{\partial \xi}(u - \hat{u}_h)\|_{L^2}^2}{|||u|||^2} \\
\leq & \frac{\|\frac{\partial}{\partial \xi}(u^1 - \hat{u}_h^1)\|_{L^2}^2 + \epsilon^2\|\epsilon^{-2}\frac{\partial}{\partial \eta}(u^0 - \hat{u}_h^0) + \frac{\partial}{\partial \eta}(u^1 - \hat{u}_h^1)\|_{L^2}^2}{|||u|||^2} \\
& + \frac{\epsilon^2\|\epsilon^{-2}\frac{\partial}{\partial \xi}(u^0 - \hat{u}_h^0) + \frac{\partial}{\partial \xi}(u^1 - \hat{u}_h^1)\|_{L^2}^2}{|||u|||^2} \\
\leq & \frac{C(h^2|u^1|_2^2 + \epsilon^{-2}h^2\|u^0\|_3^2 + \epsilon^2 h^2|u^1|_2^2 + \epsilon^{-2}h^2\|u^0\|_3^2 + \epsilon^2 h^2|u^1|_2^2)}{|||u|||^2} \\
\leq & Q^2h^2
\end{aligned}$$

where we needed estimates (2.8) with $k = 2, 3$. The bound for the piecewise uniform mesh ($N > 1$) is finally obtained by iterating the finite difference error bound (cf. [3])

$$\|u^0 - \hat{u}_h^0\|_{\Gamma_i^+} + \|u^0 - \hat{u}_h^0\|_{\Omega_i} \leq C(h^2\|u^0\|_{3,\Omega_i} + \|u^0 - \hat{u}_h^0\|_{\Gamma_i^-})$$

over every subdomain $\Omega_i \subset \Omega$. Here Γ_i^- and Γ_i^+ are the "inflow" and "outflow" boundaries of Ω_i. □

Remark 5 Whether the bound $e_A \leq Q\epsilon^{-1}h$ is improvable or not on a general mesh remains an open question. Our experiments indicate that the reduced-flux scheme behaves at least adequately on a general mesh.

6 The consistency error

The following theorem establishes bounds for the consistency error component (4.2).

Theorem 3 *In case of a general quadrilateral mesh the consistency error obeys the bounds*

$$e_C \leq \begin{cases} Q\epsilon^{-1}h & \text{in Problems A and B} \\ Qh & \text{in problem C.} \end{cases}$$

In case of a rectangular mesh, we have the improved bounds

$$e_C \leq \begin{cases} Q\epsilon^{-1}h^2 & \text{in Problem A} \\ Q\epsilon^{-1}h^{3/2} & \text{in Problem B} \\ Qh^2 & \text{in problem C.} \end{cases}$$

Finally, in case of a piecewise uniform mesh, the bounds are still improvable for Problems A and B as

$$e_C \leq \begin{cases} Qh^2 + Q(N)h^{3/2} & \text{in Problem A} \\ Q(N)\min\{h^{1/2}, \epsilon^{-1}h^{3/2}\} & \text{in Problem B} \end{cases}$$

with $Q(1) = 0$ in Problem A.

Proof. To have the bound $e_C \leq \delta$, we need to bound the consistency error functional

$$\begin{aligned}(\mathcal{A} - \mathcal{A}_h)(u, v) =& (1-\epsilon^2)\left\langle (I - R_h)\frac{\partial u}{\partial \xi}, \frac{\partial v}{\partial \xi}\right\rangle \\ =& (1-\epsilon^2)\left\langle \frac{\partial u}{\partial \xi}, (I - R_h)\frac{\partial v}{\partial \xi}\right\rangle\end{aligned}$$

as

$$|(\mathcal{A} - \mathcal{A}_h)(u, v)| \leq \delta |||u||| \, |||v|||_h \quad \forall v \in \mathcal{V}_h^0.$$

The asserted bounds in case of a general mesh then follow immediately from simple approximation theory, estimates (2.5), (2.8) with $k = 2$, and the stability estimate (4.3). Note that in Problem C, the leading term in (2.7) does not contribute to the consistency error.

Next, assume a rectangular mesh. In this case, since

$$\frac{\partial v}{\partial \xi} = \alpha\frac{\partial v}{\partial x} + \beta\frac{\partial v}{\partial y}$$

we need only to treat expressions of type $\left\langle \frac{\partial u}{\partial z_1}, (I - R_h)\frac{\partial v}{\partial z_2} \right\rangle$ where z_i denotes either x or y. Without loss of generality, let us consider the case $z_1 = y$ and $z_2 = x$. Noting that $\frac{\partial v}{\partial x}$ depends only on y on a rectangular mesh, we can write

$$(I - R_h)\frac{\partial v}{\partial x} = (I - \Lambda_y)\frac{\partial v}{\partial x}$$

where Λ_y is the orthogonal projection onto piecewise constant functions with respect to the y-coordinate, i.e.

$$\Lambda_y w_{|K} = \frac{1}{y_i - y_{i-1}} \int_{y_{i-1}}^{y_i} w(x, y')\, dy'$$

for every element K. Then we obtain integrating by parts

(6.1)

$$\begin{aligned}\left\langle \frac{\partial u}{\partial y}, (I - R_h)\frac{\partial v}{\partial x} \right\rangle = \left\langle \frac{\partial u}{\partial y}, (I - \Lambda_h)\frac{\partial v}{\partial x} \right\rangle &= \left\langle (I - \Lambda_y)\frac{\partial u}{\partial y}, \frac{\partial v}{\partial x} \right\rangle \\ &= \int_0^1\int_0^1 \left[(I - \Lambda_y)\frac{\partial u}{\partial y}(x,y)\right] \frac{\partial v}{\partial x}(x,y)\, dxdy \\ &= \int_0^1 \left[(I - \Lambda_y)\frac{\partial u}{\partial y}(1,y)\right] v(1,y)\, dy \\ &\quad - \int_0^1 \left[(I - \Lambda_y)\frac{\partial u}{\partial y}(0,y)\right] v(0,y)\, dy \\ &\quad - \int_0^1\int_0^1 \left[(I - \Lambda_y)\frac{\partial^2 u}{\partial x \partial y}\right] v\, dxdy,\end{aligned}$$

where we used also the fact that Λ_y and $\frac{\partial}{\partial x}$ commute.

In Problem A we have $v = 0$ on $\partial\Omega$, so the boundary terms in (6.1) vanish and we get

$$\begin{aligned}\left\langle \frac{\partial u}{\partial y}, (I - R_h)\frac{\partial v}{\partial x} \right\rangle &= -\int_0^1\int_0^1 (I - \Lambda_y)\frac{\partial^2 u}{\partial x \partial y}(I - \Lambda_y) v\, dxdy \\ &\le C h^2 |u|_4 |v|_1 \\ &\le C\epsilon^{-1} h^2 |u|_4 |||v|||_h\end{aligned}$$

applying lemma 1. Together with (2.5) ($k = 4$) this proves the assertion for Problem A.

Still considering Problem A, let us now assume a uniform mesh so that writing

$$(I - \Lambda_y)\frac{\partial^2 u}{\partial x \partial y} = h\omega(y)\frac{\partial^3 u}{\partial x \partial y^2} - \zeta(x,y)$$

where $\omega(y)$ is the piecewise linear "sawtooth" function jumping from $+1/2$ to $-1/2$ at y_i we obtain

$$\begin{aligned} -\left\langle (I - \Lambda_y)\frac{\partial^2 u}{\partial x \partial y}, v \right\rangle &= -\left\langle (I - \Lambda_y)\frac{\partial^2 u}{\partial x \partial y}, (I - \Lambda_y)v \right\rangle \\ &= -\left\langle h\omega(y)\frac{\partial^3 u}{\partial x \partial y^2}, h\omega(y)\frac{\partial v}{\partial y} \right\rangle + \\ &\quad \left\langle \zeta(x,y), h\omega(y)\frac{\partial v}{\partial y} \right\rangle \\ &= -\left\langle h^2\omega^2(y)\frac{\partial^3 u}{\partial x \partial y^2}, \frac{\partial v}{\partial y} \right\rangle + \\ &\quad \left\langle \zeta(x,y), h\omega(y)\frac{\partial v}{\partial y} \right\rangle . \end{aligned} \tag{6.2}$$

Noting that $\omega^2(y) = \frac{1}{4}\theta_2(y) + \frac{1}{12}$ where $\theta_2(y)$ is the monic piecewise Legendre polynomial of order two we get

$$\begin{aligned} \left|\left\langle (I - \Lambda_y)\frac{\partial^2 u}{\partial x \partial y}, v \right\rangle\right| &\le \left|\left\langle h^2\frac{1}{4}\theta_2(y)\frac{\partial^3 u}{\partial x \partial y^2}, \frac{\partial v}{\partial y} \right\rangle\right| \\ &\quad + \left|\left\langle \frac{1}{12}h^2\frac{\partial^3 u}{\partial x \partial y^2}, \frac{\partial v}{\partial y} \right\rangle\right| \\ &\quad + \left|\left\langle \zeta(x,y), h\omega(y)\frac{\partial v}{\partial y} \right\rangle\right| \\ &= h^2\left|\left\langle \frac{1}{4}\theta_2(y)(I - \Pi_y)\frac{\partial^3 u}{\partial x \partial y^2}, \frac{\partial v}{\partial y} \right\rangle\right| \\ &\quad + \frac{1}{12}h^2\left|\left\langle \frac{\partial^4 u}{\partial x \partial y^3}, v \right\rangle\right| \\ &\quad + \left|\left\langle \zeta(x,y), h\omega(y)\frac{\partial v}{\partial y} \right\rangle\right| \end{aligned}$$

where Π_y is the orthogonal projection to elementwise linear functions in the y-direction. Here for the last term we can apply lemma 2 ahead, writing this term as a sum over K and denoting $\psi = \frac{\partial^2 u}{\partial x \partial y}$, $\varphi = \frac{\partial v}{\partial y}$. Together with the Cauchy-Schwartz inequality and the stability estimate (4.4) this lemma gives

$$\left|\left\langle \zeta(x,y), h\omega(y)\frac{\partial v}{\partial y}\right\rangle\right| \leq Ch^3|u|_5|||v|||_h \ . \tag{6.3}$$

By standard approximation theory and by (6.3), (4.4) we have then also

$$\begin{aligned}\left|\left\langle (I-\Lambda_y)\frac{\partial^2 u}{\partial x\partial y}, v\right\rangle\right| \leq & Ch^4\|\frac{\partial^5 u}{\partial x\partial y^4}\|_{L^2}\|\frac{\partial v}{\partial y}\|_{L^2} \\ & + \frac{1}{3}h^2\|v\|_{L^2}\|\frac{\partial^4 u}{\partial x\partial y^3}\|_{L^2} + Ch^3|u|_5|||v|||_h \\ \leq & Ch^2(|u|_5+|u|_4)|||v|||_h,\end{aligned}$$

Using finally the estimate (2.5) (with $k=4,5$), it follows that the asserted improved bound for Problem A holds in case of a uniform mesh.

At this point it is tempting to ask if the procedure above could be continued to produce even higher order terms in h. However, this is not possible as can be verified from the expression $\left\langle \theta_2(y)(I-\Lambda_y)\frac{\partial^3 u}{\partial x\partial y^2}, \frac{\partial v}{\partial y}\right\rangle$ by taking $\frac{\partial^3 u}{\partial x\partial y^2} = y^2$ and choosing v such that $\frac{\partial v}{\partial y} = \frac{1}{1-h}$ when $h<x<1-h$ and $0<y<1-h$ with $v=0$ on $\partial\Omega$.

The above reasoning anyhow extends to a piecewise uniform mesh as follows. Suppose that Ω is divided into two subdomains Ω_1 and Ω_2 with mesh parameters h_1 and h_2, respectively, by a horizontal line (N=2). Denoting this line by Γ and integrating by parts in

$$\begin{aligned}\left\langle \frac{1}{3}h_i(y)^2\frac{\partial^3 u}{\partial x\partial y^2}, \frac{\partial v}{\partial y}\right\rangle = & \left\langle \frac{1}{3}h_1^2\frac{\partial^3 u}{\partial x\partial y^2}, \frac{\partial v}{\partial y}\right\rangle_{\Omega_1} \\ & + \left\langle \frac{1}{3}h_2^2\frac{\partial^3 u}{\partial x\partial y^2}, \frac{\partial v}{\partial y}\right\rangle_{\Omega_2}\end{aligned}$$

we obtain

$$\begin{aligned}&\left\langle \frac{1}{3}h_1^2\frac{\partial^3 u}{\partial x\partial y^2}, \frac{\partial v}{\partial y}\right\rangle_{\Omega_1} + \left\langle \frac{1}{3}h_2^2\frac{\partial^3 u}{\partial x\partial y^2}, \frac{\partial v}{\partial y}\right\rangle_{\Omega_2} = \\ &-\left\langle \frac{1}{3}h_1^2\frac{\partial^4 u}{\partial x\partial y^3}, \frac{\partial v}{\partial y}\right\rangle_{\Omega_1} - \left\langle \frac{1}{3}h_2^2\frac{\partial^4 u}{\partial x\partial y^3}, \frac{\partial v}{\partial y}\right\rangle_{\Omega_2} \\ &+ \left\langle \frac{1}{3}(h_1^2-h_2^2)\frac{\partial^3 u}{\partial x\partial y^2}, v\right\rangle_{\Gamma},\end{aligned}$$

where now the additional line integral term is bounded as

$$\left|\int_\Gamma \frac{1}{3}(h_1^2 - h_2^2)\frac{\partial^3 u}{\partial x \partial y^2} v dx\right| \leq Ch^2 \|\frac{\partial^3 u}{\partial x \partial y^2}\|_{L^2(\Gamma)} \|v\|_{L^2(\Gamma)}.$$

The asserted bound then follows from the inverse estimate $\|v\|_{L^2(\Gamma)} \leq Ch^{-1/2}\|v\|_{L^2}$ and and stability estimate (4.4). The extension of this argument to an arbitrary piecewise uniform mesh is obvious.

Let us now consider Problem B. In case of a rectangular or a piecewise uniform mesh the non-vanishing boundary term in (6.1) is bounded as

$$\begin{aligned} &\left|\int_0^1 (I - \Lambda_y)\frac{\partial u}{\partial y}(0,y) v(0,y)\, dy\right| \\ &= \left|\int_0^1 (I - \Lambda_y)\frac{\partial u}{\partial y}(0,y)(I - \Lambda_y) v(0,y)\, dy\right| \\ &\leq Ch^2 \|\frac{\partial^2 u}{\partial y^2}(0,\cdot)\|_{L^2(0,1)} \|\frac{\partial v}{\partial y}(0,\cdot)\|_{L^2(0,1)} \\ &\leq C|u|_{2,\partial\Omega} |||v|||_h \epsilon^{-1} h^{3/2} \end{aligned}$$

on a rectangular mesh and as

$$\begin{aligned} &\left|\int_0^1 (I - \Lambda_y)\frac{\partial u}{\partial y}(0,y) v(0,y)\, dy\right| \\ &\leq C(N)|u|_{2,\partial\Omega} |||v|||_h \min\{h^{1/2}, \epsilon^{-1}h^{3/2}\} \end{aligned}$$

on a piecewise uniform mesh, by an inverse estimate and by the stability estimates (4.3), (4.4). The asserted bounds follow using estimate (2.6) with $k = 2$.

The case of Problem C is finally easily covered with the help of the decomposition (2.7). Since $\frac{\partial u^0}{\partial \xi} = 0$ we have that

$$\begin{aligned} (\mathcal{A} - \mathcal{A}_h)(u, v) =& (1 - \epsilon^2)\left\langle (I - R_h)\frac{\partial u^1}{\partial \xi}, \frac{\partial v}{\partial \xi}\right\rangle \\ =& (1 - \epsilon^2)\left\langle \frac{\partial u^1}{\partial \xi}, (I - R_h)\frac{\partial v}{\partial \xi}\right\rangle. \end{aligned}$$

and the same analysis as above can be carried out with u^1 replacing u, except that a factor ϵ is introduced canceling the error growth as $\epsilon \to 0$. □

Remark 6 Again, experiments show that the reduced formulation behaves fairly well on a general quadrilateral mesh, although this was not proved.

Lemma 2 *Let* $K_{ij} = \{(x,y)\,|\, x_{j-1} < x < x_j,\, y_{i-1} < y < y_i\}$ *where* $h_i = y_i - y_{i-1} \le h$*, and let* $y_{i-1/2} = \frac{1}{2}(y_{i-1} + y_i)$*. Further let* $\psi = \psi(x,y)$ *be a given smooth function on* K_{ij} *and let* $\varphi \in L^2(x_{j-1}, x_j)$ *be such that* φ *is independent of* y*. Then*

$$\left| \int_{K_{ij}} \left[\psi - \frac{1}{h_i} \int_{y_{i-1}}^{y_i} \psi(x,y')dy' - (y - y_{i-1/2}) \frac{\partial \psi}{\partial y} \right] (y - y_{i-1/2}) \varphi \, dxdy \right|$$
$$\le Ch^4 \| \frac{\partial^3 \psi}{\partial y^3} \|_{L^2(K_{ij})} \|\varphi\|_{L^2(K_{ij})}.$$

Proof. For the proof, see appendix A.

7 Numerical experiments

In order to test our reduced formulation we have conducted numerical experiments comparing the performance of this formulation to the classical one. In all cases we have assumed homogeneous boundary conditions, i.e. $w = g = 0$. For the load function f we have chosen $f = \sin(2\pi x) = \sin(2\pi(\alpha\xi - \beta\eta))$. Note that f satisfies the condition $\int_\Omega f dx = 0$ as required. We have also set $\alpha = \beta = 1/\sqrt{2}$ for simplicity.

7.1 The cool state

In the cool state we consider only Problem A where the asymptotic solution u^0 satisfies $\frac{\partial^2 u^0}{\partial \xi^2} = f$. It is then easy to compute the exact asymptotic solution for $0 \le \eta \le 1/\sqrt{2}$ as

$$u^0 = \frac{1}{2\pi^2} \left(\sin(\sqrt{2}\pi(\xi - \eta)) - \frac{\xi \sin(2\pi(\sqrt{2}\eta - 1))}{\sqrt{2} - 2\eta} + \frac{\eta \sin(2\pi(1 - \sqrt{2}\eta))}{\sqrt{2} - 2\eta} \right).$$

On the second half of Ω u^0 is obtained via a reflection through the center of Ω as shown in Fig. 2. Experimenting with a few small values of epsilon and a few different mesh parameters we see that our reduced formulation performs very well as compared to the classical one on a uniform mesh, as expected by the error analysis. A typical behavior is shown in Fig. 3. Also the evaluation of the relative error in both H_1 and the modified energy norm shows that the reduced formulation produces a fairly good solution. These graphs are shown in Figs. 4 and 5. On a more general mesh the results were found less encouraging.

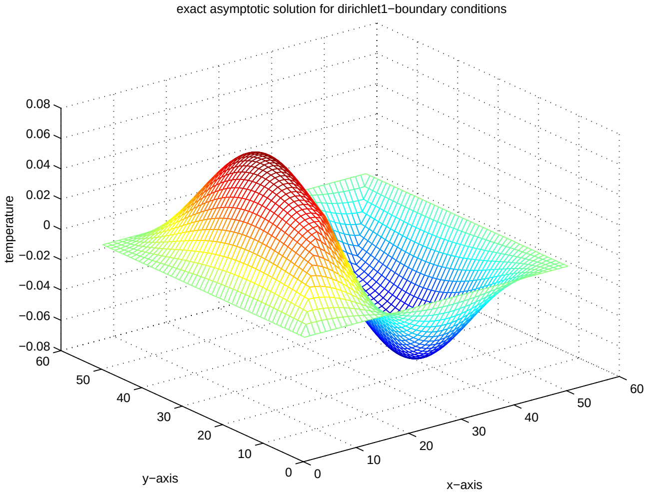

Fig. 2. The exact asymptotic solution in the cool state with the load function $f = \sin(2\pi x)$

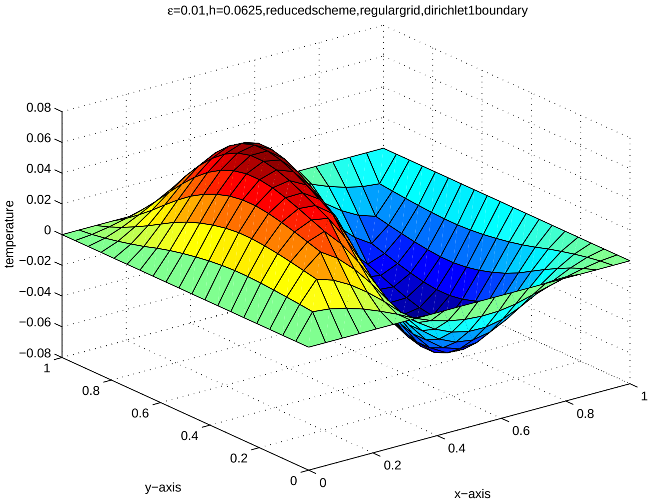

Fig. 3. The finite element solution obtained by the reduced-flux formulation in the cool state with $\epsilon = 10^{-2}$, $h = \frac{1}{16}$ and the load function $f = \sin(2\pi x)$

7.2 The hot state

In the hot state we have the boundary conditions $\partial_\nu u = 0$ on $\partial\Omega$ and the asymptotic solution u^0 satisfies the condition $\frac{\partial u^0}{\partial \xi} = 0$. Using the variational formulation (2.1) with test functions now in $\mathcal{W}$, integrating with respect to ξ, and integrating by parts we obtain that the asymptotic solution u^0 satisfies the equations

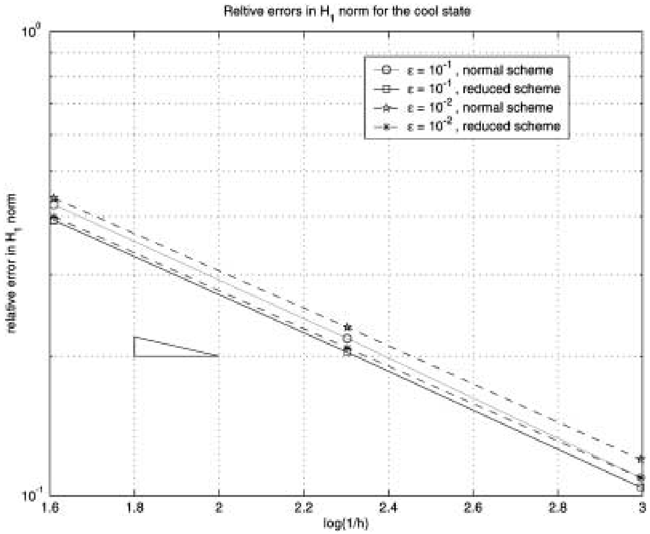

Fig. 4. The relative error in H_1 norm for the cool state (uniform mesh)

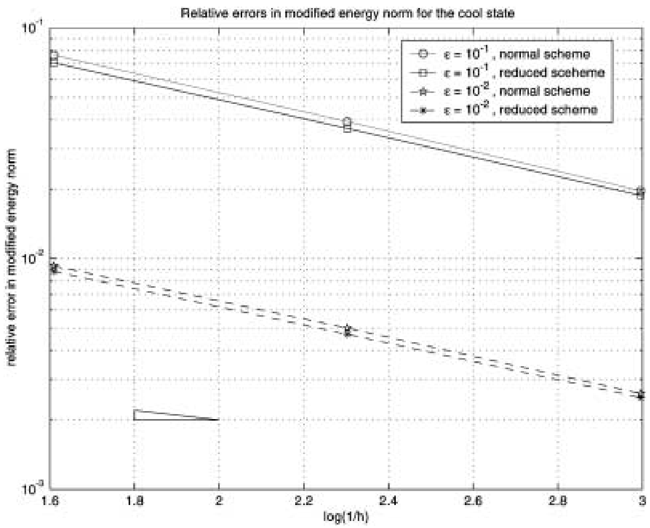

Fig. 5. The relative error in modified energy norm for the cool state (uniform mesh)

$$\begin{cases} 2\frac{\partial u^0}{\partial \eta} + (2\eta - \sqrt{2})\frac{\partial^2 u^0}{\partial \eta^2} = -\frac{\sqrt{2}}{\pi}\sin^2(\sqrt{2}\pi\eta), \ 0 < \eta < 1/\sqrt{2} \\ 2\frac{\partial u^0}{\partial \eta} + (2\eta + \sqrt{2})\frac{\partial^2 u^0}{\partial \eta^2} = \frac{\sqrt{2}}{\pi}\sin^2(\sqrt{2}\pi\eta), \ -1/\sqrt{2} < \eta < 0 \end{cases}$$

Furthermore, u^0 is continuously differentiable on Ω and satisfies the symmetry condition $\frac{\partial u^0}{\partial \eta}|_{\eta=0} = 0$. From these properties the asymptotic solution is found to be

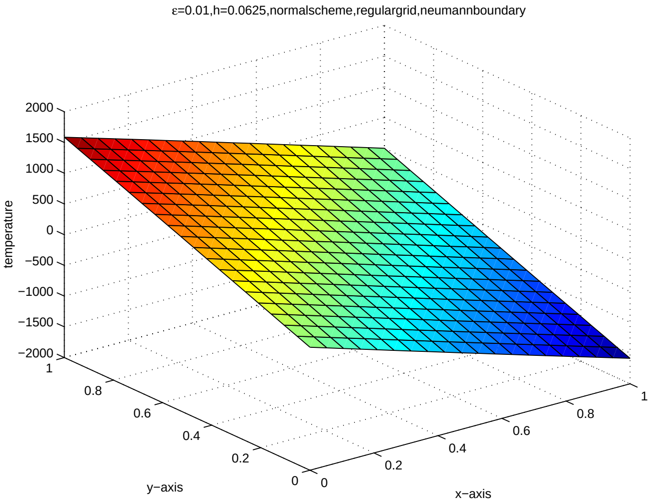

Fig. 6. The finite element solution obtained by the classical formulation in the hot state with $\epsilon = 10^{-2}$, $h = \frac{1}{16}$ and the load function $f = \sin(2\pi x)$

$$u^0 = \begin{cases} \frac{1}{8\pi^2}\int_0^{2\pi}\frac{\sin t}{t}\,dt - \frac{1}{2\sqrt{2}\pi}\eta + \frac{1}{8\pi^2}\int_0^{2\pi(\sqrt{2}\eta-1)}\frac{\sin t}{t}\,dt, \\ \qquad 0 < \eta < 1/\sqrt{2}, \\ \frac{1}{8\pi^2}\int_0^{2\pi}\frac{\sin t}{t}\,dt + \frac{1}{2\sqrt{2}\pi}\eta - \frac{1}{8\pi^2}\int_0^{2\pi(\sqrt{2}\eta+1)}\frac{\sin t}{t}\,dt, \\ \qquad -1/\sqrt{2} < \eta < 0 \end{cases}$$

where the normalization is chosen such that ${u^0}_{|\eta=0} = 0$. Again, experiments with a few small values of epsilon and a few mesh parameters indicate severe locking in the classical formulation whereas the reduced-flux formulation performs very well on a uniform mesh as predicted. Examples can be seen in Figs. 6 and 7. Note that the standard FE solution at $\epsilon = 0$ is actually the projection of u^0 onto the one-dimensional function space $\mathcal{W}_h = \{v = v(\xi, \eta) = c\eta, c \in \mathbb{R}\}$ in accordance with Theorem 1. The relative errors with respect to H_1 and modified energy norms also show that the reduced-flux formulation is superior to the classical one in the hot state. These graphs are plotted in Figs. 9 and 10.

On a more general mesh the results with the reduced-flux formulation are still encouraging, giving a much better solution than with the classical formulation as can be seen in Fig. 8. In view of the above error analysis this indicates that the generalized box-scheme (5.1) may work even in the case of a general mesh, although there is no error analysis for difference schemes on general meshes. In this case we have only the error bound of Remark 5 for the reduced-flux formulation.

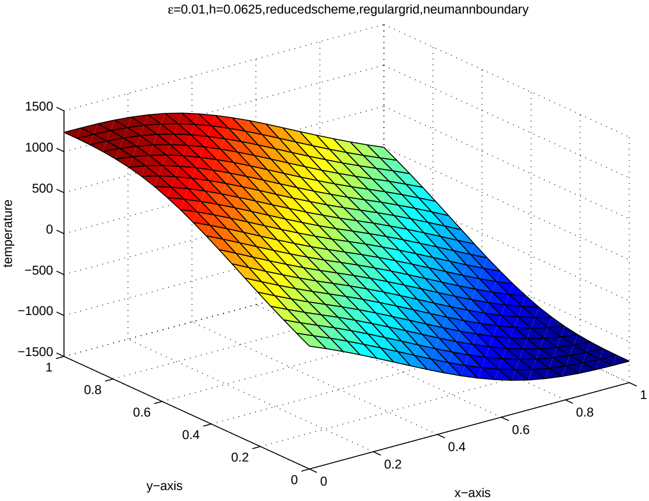

Fig. 7. The finite element solution obtained by the reduced-flux formulation in the hot state with $\epsilon = 10^{-2}$, $h = \frac{1}{16}$ and the load function $f = \sin(2\pi x)$

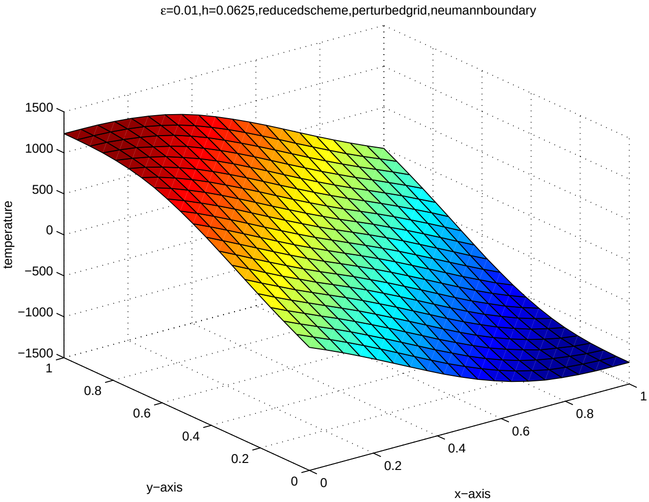

Fig. 8. The finite element solution obtained by the reduced-flux formulation in the hot state with general quadrilateral mesh and with $\epsilon = 10^{-2}$, $h = \frac{1}{16}$ and the load function $f = \sin(2\pi x)$

A Proof of Lemma 2

We present here the postponed proof of lemma 2. Using Taylor's theorem we can write

$$\begin{aligned}\psi(x,y) =& \psi(x,y_{i-1/2}) + \frac{\partial\psi}{\partial y}(x,y_{i-1/2})(y-y_{i-1/2})\\ &+ \frac{1}{2}\frac{\partial^2\psi}{\partial y^2}(x,y_{i-1/2})(y-y_{i-1/2})^2\\ &+ \frac{1}{2}\int_{y_{i-1/2}}^{y}\frac{\partial^3\psi}{\partial y^3}(x,t)(y-t)^2 dt\end{aligned}$$

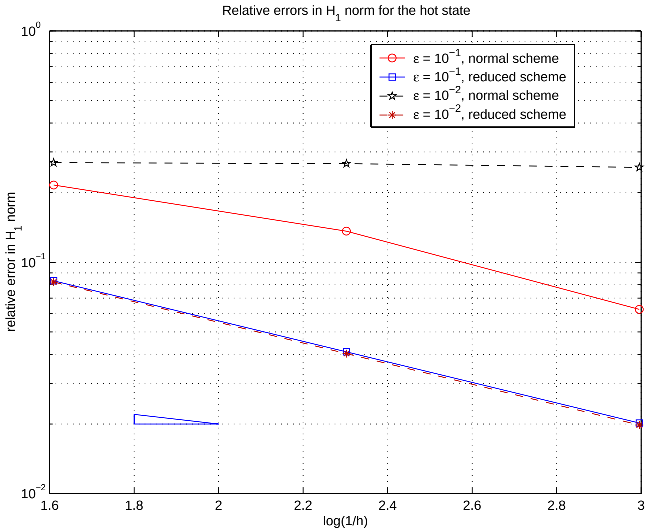

Fig. 9. The relative error in H_1 norm for the hot state (uniform mesh)

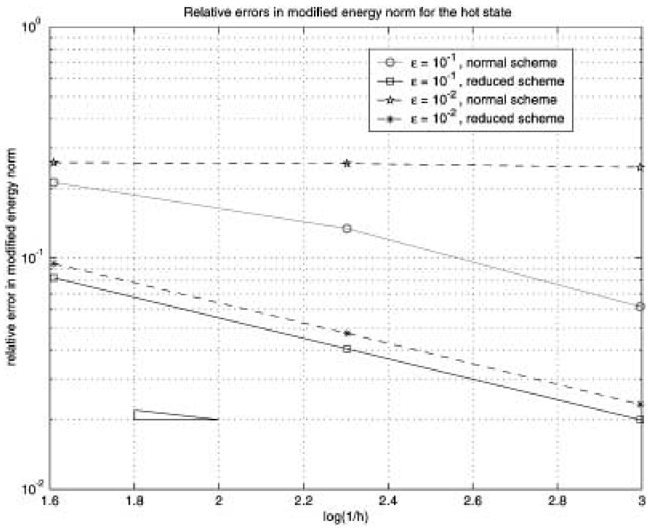

Fig. 10. The relative error in modified energy norm for the hot state (uniform mesh)

and

$$\frac{\partial\psi}{\partial y}(x,y) = \frac{\partial\psi}{\partial y}(x,y_{i-1/2}) + \frac{\partial^2\psi}{\partial y^2}(x,y_{i-1/2})(y-y_{i-1/2}) + \int_{y_{i-1/2}}^{y} \frac{\partial^3\psi}{\partial y^3}(x,t)(y-t)dt.$$

Then

$$\left|\int_{K_{ij}}\left[\psi - \frac{1}{h_i}\int_{y_{i-1}}^{y_i}\psi(x,y')dy' - (y-y_{i-1/2})\frac{\partial\psi}{\partial y}\right](y-y_{i-1/2})\varphi\,dxdy\right|$$
$$= \left|\int_{x_{j-1}}^{x_j}\int_{y_{i-1}}^{y_i}\varphi\int_{y_{1/2}}^{y}\frac{\partial^3\psi}{\partial y^3}(x,t)(y-y_{i-1/2})(y-t)(y_{i-1/2}-t)\,dtdydx\right|$$
$$\leq \int_{x_{j-1}}^{x_j}\int_{y_{i-1}}^{y_i}|\varphi|\int_{y_{i-1}}^{y_i}|\frac{\partial^3\psi}{\partial y^3}(x,t)|$$
$$|(y-y_{i-1/2})||(y-t)||(y_{i-1/2}-t)|\,dtdydx$$
$$= \int_{y_{i-1}}^{y_i}\left(\int_{x_{j-1}}^{x_j}\int_{y_{i-1}}^{y_i}|\varphi|\,|\frac{\partial^3\psi}{\partial y^3}(x,t)|\right.$$
$$\left.|(y-y_{i-1/2})||(y-t)||(y_{i-1/2}-t)|\,dtdx\right)dy$$
$$\leq \int_{y_{i-1}}^{y_i}\|\frac{\partial^3\psi}{\partial y^3}(x,t)\|_{L^2(K_{ij})}\left(\int_{x_{j-1}}^{x_j}\int_{y_{i-1}}^{y_i}|\varphi|^2\right.$$
$$\left.|(y-y_{i-1/2})|^2|(y-t)|^2|(y_{i-1/2}-t)|^2\,dtdx\right)^{1/2}dy$$
$$\leq \|\frac{\partial^3\psi}{\partial y^3}(x,t)\|_{L^2(K_{ij})}\left(\int_{x_{j-1}}^{x_j}|\varphi|^2\,dx\right)^{1/2}$$
$$\int_{y_{i-1}}^{y_i}\left(\int_{y_{i-1}}^{y_i}|(y-y_{i-1/2})|^2|(y-t)|^2|(y_{i-1/2}-t)|^2\,dt\right)^{1/2}dy$$

and since

$$\int_{y_{i-1}}^{y_i}\left(\int_{y_{i-1}}^{y_i}|(y-y_{i-1/2})|^2|(y-t)|^2|(y_{i-1/2}-t)|^2\,dt\right)^{1/2}dy$$
$$= \left(\frac{2}{15}\right)^{3/2}h_i^{9/2}$$

the claim follows. □

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