

# An Analysis of a Bilinear Reduced Strain Element in the Case of an Elliptic Shell in a Bending Dominated State of Deformation<sup>1</sup>

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## Summary

We consider a bilinear reduced-strain finite element formulation for a shallow shell model of Reissner-Naghdi type. We estimate the error of this scheme when approximating an inextensional displacement field making strong assumptions on the domain and on the finite element mesh.

## Introduction

By now it is well known that reliable numerical modeling of shells by traditional low-order finite element formulations is not an easy task. The most dramatic failure occurs when approximating nearly inextensional (or bending-dominated) deformations by standard low-order elements. In this case an asymptotic approximation failure, known as *shear-membrane locking*, occurs at the limit of zero shell thickness.

To avoid the locking in parameter dependent problems, it is customary to search for 'simple and efficient' low(est) order elements that are based on some non-standard variational formulation of the problem. Among the (apparently many) possible technical variations within this approach, we choose to consider in our work the well known formulation by Bathe et al. [1] named MITC4.

## The shell problem

We consider a dimensionally reduced shell model for a shell of thickness  $t$  arising from linear shell theory with homogeneous and isotropic material. We assume that the membrane, transverse shear and bending strains  $\beta_{ij}$ ,  $\rho_i$  and  $\kappa_{ij}$  depend on the displacements  $(u, v, w)$  and on the rotations  $(\theta, \psi)$  as follows:

$$\beta_{11} = \frac{\partial u}{\partial x} + aw, \quad \beta_{22} = \frac{\partial v}{\partial y} + bw, \quad \beta_{12} = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + cw = \beta_{21}, \quad (1)$$

$$\kappa_{11} = \frac{\partial \theta}{\partial x}, \quad \kappa_{22} = \frac{\partial \psi}{\partial y}, \quad \kappa_{12} = \frac{1}{2} \left( \frac{\partial \theta}{\partial y} + \frac{\partial \psi}{\partial x} \right) = \kappa_{21}, \quad (2)$$

and

$$\rho_1 = \theta - \frac{\partial w}{\partial x} \quad \rho_2 = \psi - \frac{\partial w}{\partial y}, \quad (3)$$

where we are assuming the shell to be *shallow* so that the parameters  $a$ ,  $b$  and  $c$  defining the shell geometry can be taken constants. We consider only the case  $ab - c^2 > 0$  s.t. the shell is elliptic. We will assume that the computational domain  $\Omega$  (the shell midsurface) is of rectangular shape in the assumed coordinate system, so

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that  $\Omega = \{(x, y) | 0 < x < L, 0 < y < H\}$  with  $c^{-1} \leq \frac{L}{H} \leq c$  for some fixed constant  $c$  and with periodic boundary conditions at  $y = 0, H$ .

It is convenient to define the vector field  $\underline{u} = (u, v, w, \theta, \psi)$  and the bilinear form  $A$

$$A(\underline{u}, \underline{v}) = a(\underline{u}, \underline{v}) + \frac{1}{t^2} b(\underline{u}, \underline{v}) \quad (4)$$

where

$$a(\underline{u}, \underline{v}) = \int_{\Omega} v \{ (\kappa_{11} + \kappa_{22})(\underline{u})(\kappa_{11} + \kappa_{22})(\underline{v}) + (1 - v) \sum_{i,j=1}^2 \kappa_{ij}(\underline{u}) \kappa_{ij}(\underline{v}) \} dx dy \quad (5)$$

and

$$b(\underline{u}, \underline{v}) = 6(1 - v) \int_{\Omega} \{ \rho_1(\underline{u}) \rho_1(\underline{v}) + \rho_2(\underline{u}) \rho_2(\underline{v}) \} dx dy + 12 \int_{\Omega} \{ v(\beta_{11} + \beta_{22})(\underline{u})(\beta_{11} + \beta_{22})(\underline{v}) + (1 - v) \sum_{i,j=1}^2 \beta_{ij}(\underline{u}) \beta_{ij}(\underline{v}) \} dx dy. \quad (6)$$

Here  $v$  is the Poisson ratio of the material. Then the shell problem can be formulated as: Find  $\underline{u} \in U$  such that

$$A(\underline{u}, \underline{v}) = Q(\underline{v}) \quad \forall \underline{v} \in U \quad (7)$$

where  $U$  is the energy space and  $Q(\underline{v})$  is the load potential.

We are interested in the finite element approximation of *inextensional* displacement fields satisfying  $b(\underline{u}, \underline{u}) = 0$ , i.e.

$$\beta_{11}(\underline{u}) = \beta_{12}(\underline{u}) = \beta_{22}(\underline{u}) = 0, \quad \rho_1(\underline{u}) = \rho_2(\underline{u}) = 0 \quad (8)$$

and denote the space of these fields by  $U_0$ . We aim to expand the inextensional modes by the Fourier expansion

$$\underline{u} = \sum_{\lambda \in \Lambda} \varphi_{\lambda}(y) \underline{\phi}_{\lambda}(x), \quad \varphi_{\lambda}(y) = e^{i\lambda y}, \quad \Lambda = \{ \lambda = \frac{2\pi v}{H}, v \in Z \} \quad (9)$$

where, in view of (8),  $\underline{\phi}_{\lambda} = (u_{\lambda}, v_{\lambda}, w_{\lambda}, \theta_{\lambda}, \psi_{\lambda})$  satisfies

$$\begin{cases} u'_{\lambda} + a w_{\lambda} = 0 \\ i\lambda v_{\lambda} + b w_{\lambda} = 0 \\ i\lambda u_{\lambda} + v'_{\lambda} + 2c w_{\lambda} = 0 \\ \theta_{\lambda} - w'_{\lambda} = 0 \\ \psi_{\lambda} - i\lambda w_{\lambda} = 0 \end{cases} \quad (10)$$

### The reduced-strain FE scheme

It is well-known that due to the parametric dependence of the energy norm, the best error bound in case of lowest degree elements and bending-dominated deformation is [5]

$$\| \underline{u} - \underline{u}_h \| \sim \min \{ 1, \frac{h}{t} \| \underline{u} \|_2 \}, \quad (11)$$

where  $|||\cdot||| = \sqrt{A(\cdot, \cdot)}$ . To prevent the error amplification at small  $t$ , we therefore need to consider some modification of the standard formulation. A natural approach is to modify the membrane and transverse stresses  $\beta_{ij}$  and  $\rho_i$  substituting these with  $\tilde{\beta}_{ij} = R_h^{ij} \beta_{ij}$  and  $\tilde{\rho}_i = R_h^i \rho_i$  where the  $R_h^{ij}$ 's and  $R_h^i$ 's are suitably chosen reduction operators.

In the finite element scheme to be studied, we assume that  $\Omega$  is subdivided by a rectangular mesh with maximal side length =  $h$ . We assume that the mesh is uniform in the  $y$ -direction. We write  $h_x^j = x^{j+1} - x^j$  and  $h_x = \max h_x^j$ . Note that we make no assumption on the ratios  $h_x^j/h_y$ . On this mesh we consider a continuous piecewise bilinear representation of each component of the displacement field. In this setup, we define the reduced membrane and shear strains as

$$\tilde{\beta}_{11} = \Pi_h^x \beta_{11}, \quad \tilde{\beta}_{22} = \Pi_h^y \beta_{22}, \quad \tilde{\beta}_{12} = \Pi_h^{xy} \beta_{12} = \tilde{\beta}_{21} \quad (12)$$

and

$$\tilde{\rho}_1 = \Pi_h^x \rho_1 \quad \tilde{\rho}_2 = \Pi_h^y \rho_2, \quad (13)$$

where  $\Pi_h^x$ ,  $\Pi_h^y$  and  $\Pi_h^{xy}$  are  $L^2$ -projection operators defined elementwise as projectors onto the global spaces  $W_h^x$ ,  $W_h^y$  and  $W_h^{xy}$ , where  $W_h^x$  consists of functions that are constant in  $x$  and piecewise linear with respect to  $y$  on each element,  $W_h^y$  is defined analogously and  $W_h^{xy}$  consists of functions that are elementwise constants. The connection between our choice and the engineering tradition is given in [4]. We let further

$$\begin{aligned} b^h(\underline{u}, \underline{v}) = & 6(1 - \nu) \int_{\Omega} \{ \tilde{\rho}_1(\underline{u}) \tilde{\rho}_1(\underline{v}) + \tilde{\rho}_2(\underline{u}) \tilde{\rho}_2(\underline{v}) \} dx dy \\ & + 12 \int_{\Omega} \{ \nu (\tilde{\beta}_{11} + \tilde{\beta}_{22})(\underline{u}) (\tilde{\beta}_{11} + \tilde{\beta}_{22})(\underline{v}) + (1 - \nu) \sum_{i,j=1}^2 \tilde{\beta}_{ij}(\underline{u}) \tilde{\beta}_{ij}(\underline{v}) \} dx dy. \end{aligned} \quad (14)$$

and look for a FE solution  $\underline{u}_h \in U_h$  such that

$$A_h(\underline{u}_h, \underline{v}) = a(\underline{u}_h, \underline{v}) + \frac{1}{t^2} b^h(\underline{u}_h, \underline{v}) = Q(\underline{v}) \quad \forall \underline{v} \in U_h. \quad (15)$$

**Remark 1.** *Our reduced formulation is such that in the FE space the derivate terms in  $\beta_{ii}$ 's and  $\rho_i$ 's are unaffected.*

**Theorem 1.** *Let  $U_{0,h} = \{ \underline{u} \in U_h | b^h(\underline{u}, \underline{u}) = 0 \}$  where  $U_h$  is the bilinear finite element space and  $b^h$  is defined by (14) where the reduced strains are further defined by (12) – (13). Then if  $\underline{u} \in U_0$ , there exists a  $\tilde{\underline{u}} \in U_{0,h}$  such that*

$$||| \underline{u} - \tilde{\underline{u}} |||_h \leq Ch |\underline{u}|_2. \quad (16)$$

where  $|||\cdot|||_h = \sqrt{A_h(\cdot, \cdot)}$  denotes the modified energy norm.

**Remark 2.** *The main idea of the proof is to use the Fourier representation (9) and approximate then  $\underline{u}$  by*

$$\tilde{\underline{u}} = \sum_{\lambda \in \Lambda: |\lambda| \leq \lambda_0} \tilde{\Phi}_{\lambda}(y) \tilde{\Phi}_{\lambda}(x), \quad (17)$$

where  $\lambda_0 = \lambda_0(h)$  is a truncation frequency to be chosen,  $\tilde{\Phi}_{\lambda}$  is the piecewise linear interpolant of  $\Phi_{\lambda}$ , and  $\tilde{\Phi}_{\lambda}$  is a special approximation of  $\Phi_{\lambda}$  to be found.

*Proof of Theorem 1.* Consider first a single Fourier mode. Since  $\tilde{\Phi}_\lambda \tilde{\Phi}_\lambda \in U_{0,h}$  it must satisfy the constraints  $\tilde{\beta}_{ij}(\tilde{\Phi}_\lambda \tilde{\Phi}_\lambda) = \rho_i(\tilde{\Phi}_\lambda \tilde{\Phi}_\lambda) = 0$ . This is equivalent to requiring the nodal values of the components to satisfy

$$\begin{cases} u_{k+1}^n - u_k^n + \frac{a}{2} h_x^k (w_k^n + w_{k+1}^n) = 0 \\ v_k^{n+1} - v_k^n + \frac{b}{2} h_y (w_k^n + w_{k+1}^{n+1}) = 0 \\ \frac{1}{2h_y} (u_{k+1}^{n+1} - u_{k+1}^n + u_k^{n+1} - u_k^n) + \frac{1}{2h_x^k} (v_{k+1}^{n+1} - v_k^{n+1} + v_{k+1}^n - v_k^n) + \frac{c}{2} (w_{k+1}^{n+1} + w_k^{n+1} + w_{k+1}^n + w_k^n) = 0 \\ \frac{h_x^k}{2} (\theta_k^n + \theta_{k+1}^n) - (w_{k+1}^n - w_k^n) = 0 \\ \frac{h_y}{2} (\psi_k^n + \psi_k^{n+1}) + (w_k^{n+1} - w_k^n) = 0 \end{cases} \quad (18)$$

Inspired by the form of  $\phi_\lambda \underline{\Phi}_\lambda$  we seek a solution to these equations in the form  $u_k^n = e^{i\lambda n h_y} U^k$ ,  $v_k^n = e^{i\lambda n h_y} V^k$ ,  $w_k^n = e^{i\lambda n h_y} W^k$ ,  $\theta_k^n = e^{i\lambda n h_y} \Theta^k$  and  $\psi_k^n = e^{i\lambda n h_y} \Psi^k$ . Substituting these expressions and simplifying we get

$$\begin{cases} W^k = \frac{-2i}{bh_y} \tan\left(\frac{1}{2}\lambda h_y\right) V^k \\ \Psi^k = \frac{2}{h_y} \tan\left(\frac{1}{2}\lambda h_y\right) W^k \\ \Theta^{k+1} + \Theta^k = \frac{2}{h_x^k} (W^{k+1} - W^k) \end{cases} \quad (19)$$

and that  $U^k$  and  $V^k$  satisfy

$$\begin{pmatrix} V \\ U \end{pmatrix}_{k+1} - \begin{pmatrix} V \\ U \end{pmatrix}_k = \delta_k M \left[ \begin{pmatrix} V \\ U \end{pmatrix}_{k+1} + \begin{pmatrix} V \\ U \end{pmatrix}_k \right] \quad (20)$$

where  $\delta_k = \frac{h_x^k}{h_y} \tan\left(\frac{1}{2}\lambda h_y\right)$  and

$$M = i \begin{pmatrix} \frac{2c}{b} & -1 \\ \frac{a}{b} & 0 \end{pmatrix} \quad (21)$$

Since  $u_\lambda$  and  $v_\lambda$  satisfy by (10) a system

$$\begin{pmatrix} v_\lambda \\ u_\lambda \end{pmatrix}' = i\lambda \begin{pmatrix} \frac{2c}{b} & -1 \\ \frac{a}{b} & 0 \end{pmatrix} \begin{pmatrix} v_\lambda \\ u_\lambda \end{pmatrix} \quad (22)$$

we see that (20) is a finite difference approximation to (22) with a truncation error  $\gamma_k \leq C(|\lambda|^3 (h_x^k)^3 + |\lambda|^3 h_y^2 h_x^k)$ . If we denote by  $\tilde{v}_\lambda$  the piecewise linear function satisfying  $\tilde{v}_\lambda(x^k) = V^k$  and other components of  $\underline{\Phi}_\lambda$  analogously then by the standard techniques for A-stable difference schemes (cf. [2]) we see that

$$|v_\lambda(x^k) - \tilde{v}_\lambda(x^k)| \leq C \int_0^{x^k} e^{-\beta|\lambda|(x^k-t)} \lambda^3 (h_x^2 + h_y^2) e^{-\alpha|\lambda|t} dt \leq Ch^2 \lambda^2 e^{-\beta|\lambda|x^k} \quad (23)$$

for some  $\alpha > \beta > 0$ . By (20), (22) we have

$$|v'_\lambda(x^{k+1/2}) - \tilde{v}'_\lambda(x^{k+1/2})| \leq Ch^2 |\lambda|^3 e^{-\beta|\lambda|x^k} \quad (24)$$

where  $x^{k+1/2} = \frac{1}{2}(x^{k+1} + x^k)$ . Hence, (23), (24) lead to the bounds

$$\begin{aligned} |\psi_\lambda(x^k) - \tilde{\psi}_\lambda(x^k)| &\leq Ch^2 \lambda^4 e^{-\beta|\lambda|x^k} \\ |\psi'_\lambda(x^{k+1/2}) - \tilde{\psi}'_\lambda(x^{k+1/2})| &\leq Ch^2 |\lambda|^5 e^{-\beta|\lambda|x^k} \end{aligned} \quad (25)$$

by (19).

We note that in this case

$$\| \|\varphi_\lambda \underline{\zeta}_\lambda - \tilde{\varphi}_\lambda \tilde{\underline{\zeta}}_\lambda \|_h^2 = a(\varphi_\lambda \underline{\zeta}_\lambda - \tilde{\varphi}_\lambda \tilde{\underline{\zeta}}_\lambda, \varphi_\lambda \underline{\zeta}_\lambda - \tilde{\varphi}_\lambda \tilde{\underline{\zeta}}_\lambda) \leq C(|\varphi_\lambda \theta_\lambda - \tilde{\varphi}_\lambda \tilde{\theta}_\lambda|_1^2 + |\varphi_\lambda \psi_\lambda - \tilde{\varphi}_\lambda \tilde{\psi}_\lambda|_1^2). \quad (26)$$

where from by a direct evaluation of the norms

$$|\varphi_\lambda \psi_\lambda - \tilde{\varphi}_\lambda \tilde{\psi}_\lambda|_1^2 \leq Ch^2 |\varphi_\lambda \psi_\lambda|_2^2. \quad (27)$$

The other term,  $|\varphi_\lambda \theta_\lambda - \tilde{\varphi}_\lambda \tilde{\theta}_\lambda|_1^2$  can be handled similarly, only in this case we have that the error  $e_k = \tilde{\theta}_\lambda(x^k) - \theta_\lambda(x^k)$  satisfies a recursion

$$\frac{1}{2}(e_{k+1} + e_k) = \omega_k \quad (28)$$

with

$$\begin{aligned} \omega_k = & \frac{2}{b} \left[ \left( \frac{1}{h_y} \tan\left(\frac{1}{2}\lambda h_y\right) \right)^2 - \frac{\lambda^2}{4} \right] \left( \frac{2c}{b} (v_\lambda(x^{k+1}) + v_\lambda(x^k)) - (u_\lambda(x^{k+1}) + u_\lambda(x^k)) \right) \\ & + \left( \frac{1}{h_y} \tan\left(\frac{1}{2}\lambda h_y\right) \right)^2 \left( \frac{2c}{b} (\delta v_\lambda^{k+1} + \delta v_\lambda^k) - (\delta u_\lambda^{k+1} + \delta u_\lambda^k) \right) \end{aligned} \quad (29)$$

where  $\delta v_\lambda^j = \tilde{v}_\lambda(x^j) - v_\lambda(x^j)$  and  $\delta u_\lambda^j$  similarly. This leads to

$$|\theta_\lambda(x^k) - \tilde{\theta}_\lambda(x^k)| \leq Ch^2 \lambda^4 \quad (30)$$

Since the functions  $\varphi_\lambda \underline{\phi}_\lambda$  and  $\tilde{\varphi}_\lambda \tilde{\underline{\phi}}_\lambda$  are orthogonal with respect to the inner product generated by the bilinear form  $a(\cdot, \cdot)$  on  $U$  we can write  $\underline{u}$  as its Fourier-expansion  $\underline{u} = \sum_{\lambda \in \Lambda} \varphi_\lambda(y) \underline{\phi}_\lambda(x)$  and let our approximation be  $\tilde{\underline{u}} = \sum_{|\lambda| \leq \lambda_0} A_\lambda \tilde{\varphi}_\lambda(y) \tilde{\underline{\phi}}_\lambda(x)$  where  $\tilde{\underline{\phi}}_\lambda(x)$  is defined as above. A direct calculation gives

$$\| \|\tilde{\underline{u}} - \underline{u} \|_h^2 = \| \|\sum_{|\lambda| \leq \lambda_0} \tilde{\varphi}_\lambda \tilde{\underline{\phi}}_\lambda - \sum_{\lambda \in Z} \varphi_\lambda \underline{\phi}_\lambda \|_h^2 \leq \sum_{|\lambda| \leq \lambda_0} \| \|\tilde{\varphi}_\lambda \tilde{\underline{\phi}}_\lambda - \varphi_\lambda \underline{\phi}_\lambda \|_h^2 + \sum_{|\lambda| > \lambda_0} \| \|\varphi_\lambda \underline{\phi}_\lambda \|_h^2 \quad (31)$$

We can then set  $\lambda_0 = \frac{c}{h}$  to obtain

$$\| \|\tilde{\underline{u}} - \underline{u} \|_h^2 \leq Ch^2 \sum_{|\lambda| \leq \frac{c}{h}} |\varphi_\lambda \underline{\phi}_\lambda|_2^2 + Ch^2 \sum_{|\lambda| > \frac{c}{h}} \| \|\lambda \varphi_\lambda \underline{\phi}_\lambda \|_h^2 \leq Ch^2 |\underline{u}|_2^2 + Ch^2 |\underline{u}|_2^2 \quad (32)$$

□

## Numerical example

As a numerical example on the performance of our reduced-strain formulation (15) we take the Morley hemispherical shell as in [3]. We parameterize the problem by the angles  $\vartheta$  and  $\phi$  and use a uniform rectangular mesh with respect to these parameters and let  $R = 10$ ,  $t = 0.04$ ,  $\nu = 1/3$  to define the geometry and material. We measure the quality of the results by the magnitude of transverse deflection at the northern edge on the meridian of the load. The results are shown in Table 1 where the reference value is obtained using a  $p$ -version of FEM with  $p = 6$ . In this outset, which is favorable to the reduced-strain formulation, the classical formulation suffers from severe locking and produces essentially a zero solution. We note that the reduced-strain formulation is in this case comparable even to the  $p$ -version of FEM.

DOFs	deflection	absolute relative error
225	0.443469	0.3458
1425	0.653925	0.0331
3625	0.680617	0.0063
6825	0.689944	0.0201

Table 1: Deflection of the radial component for the Morley shell with the reduced-strain formulation. The reference value obtained with a  $p$ -version of FEM having roughly 112000 DOFs and  $p = 6$  is 0.676345. All results were computed on a uniform mesh.

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