

# Effect of the thermal gas component on the stability of vortices in trapped Bose–Einstein condensates

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## Abstract

We study the stability of vortices in trapped single-component Bose–Einstein condensates within self-consistent mean-field theories; in particular, we consider the Hartree–Fock–Bogoliubov–Popov theory and its recently proposed gapless extensions. It is shown that for sufficiently repulsively interacting systems the anomalous negative-energy modes related to vortex instabilities are lifted to positive energies due to partial filling of the vortex core with noncondensed gas. Such behaviour implies that within these theories the vortex states are eventually stable against the transfer of condensate matter to the anomalous core modes. This self-stabilization of vortices, shown to occur under very general circumstances, is contrasted to the predictions of the non-self-consistent Bogoliubov approximation, which is known to exhibit anomalous modes for all vortex configurations and thus implying instability of these states. In addition, the shortcomings of these approximations in describing the properties of vortices are analysed, and the need for a self-consistent theory taking properly into account the coupled dynamics of the condensate and the noncondensate atoms is emphasized.

## 1. Introduction

The realization of Bose–Einstein condensation in trapped, dilute atomic gases [1–3] has opened up unique ways to investigate how weak particle interactions affect the quantum-statistical phenomenon of bosonic condensation. Especially interesting is the question of whether the interactions can sustain superfluidity in such systems. A characteristic property of superfluids is the ability to support dissipationless flow, a feature which is manifested in microscopic length scales as the stability of quantized vortices even in fluids confined by nonrotating vessels. This aspect on the superfluid properties of dilute atomic Bose–Einstein condensates (BECs) has been under vigorous theoretical investigation during recent years [4–29] (for a review, see [30]), and has been further motivated by the experimental advances in creating and observing vortex structures and their dynamics in these systems [31–37]. However, the majority of the analysis

has been carried out within the zero-temperature Bogoliubov approximation (BA), which neglects the effects of the noncondensed gas component that is always present in interacting systems. In particular, in addition to providing a dissipative mechanism for the condensate vortex state [38], the noncondensate component also modifies the excitation spectrum of the system, and may thus in principle affect the stability of the vortices.

In discussing the energetic stability, one has to distinguish between global, thermodynamic stability and stability against small deformations, or local stability. A physical system is locally energetically stable if its state is a local minimum of the free-energy functional under the given external constraints; if the state is also a global minimum of energy, it is thermodynamically stable. In studying the superfluid properties of dilute atomic BECs, it is crucial to find out whether these systems support persistent currents without the application of any external drives; in particular, whether vortices are stable in condensates confined by nonrotating, stationary traps. In this context, the relevant stability criterion is the local energetic stability, because such vortex states are not thermodynamically stable even in superfluids; the global energy minimum in a stationary trap always corresponds to a nonrotating state. On the other hand, local stability of bosonic systems is associated with the positivity of the excitation energies. Our interest, hence, is to investigate the local stability, or metastability, of vortex states by studying their normal mode excitation spectra.

The local stability of vortices in harmonically trapped atomic BECs has been addressed mainly using the zero-temperature Bogoliubov scheme, in which the noncondensed gas is neglected. It has been shown that within the BA, vortex states in nonrotating traps always have anomalous excitations with positive norms but negative energy (with respect to the condensate state), which essentially correspond to a translational displacement of the vortex from the centre of the condensate and/or bending of the vortex line [6–8, 19, 21, 24]. The negative energies of these excitations imply that vortex states are energetically unstable: in the presence of any initial perturbation and dissipative mechanism, these modes would start to be excited through the transfer of matter from the condensate state. In addition to possible bending and deformation of the vortex, the negative-energy displacement instability mode implies that the vortex is prone to start spiralling away from the centre of the condensate [8]. By computing the energy of the condensate particles as a function of the vortex displacement from the condensate centre [9, 20, 25], it has been verified that it is energetically favourable for the vortex to eventually spiral out of the condensate and be annihilated. In conclusion, when the thermal gas component in the system is neglected, the mean-field theory predicts the vortices to be energetically unstable.

At temperatures comparable to the condensation temperature  $T_{\text{BEC}}$ , a substantial portion of the gas is noncondensed and the validity of the instability prediction given by the BA is not obvious. However, for vortex states its validity is also questionable in the zero-temperature limit. This is due to the existence of the negative-energy modes within the BA for vortex states, because in the presence of dissipation a non-negligible portion of the condensate ‘leaks’ to noncondensate populations of these anomalous modes. The consequent noncondensate density concentrated in the vortex core region, in turn, may in principle affect the local stability of the vortex state by changing the anomalous mode energies themselves. A similar phenomenon is observed for two-component condensate systems, in which an irrotational condensate component can stabilize vortex states [25]. The crucial property determining whether such a stabilization is also possible due to finite noncondensate density is the amount of noncondensate required to lift the anomalous modes to positive energies: if the lifting effect is too weak, the whole condensate vortex state will leak to macroscopic occupations of the anomalous modes, and the vortex will be destroyed.

Previously, the self-stabilization mechanism described above has been demonstrated in the case of an infinitely long, axisymmetric, static vortex line located in the centre of a harmonically trapped condensate. It has been shown that within the self-consistent Popov approximation (PA) [39] and its recently proposed gapless extensions G1 and G2 [40, 41], the noncondensate in the vortex core lifts the anomalous modes to positive energies for thermally equilibrated vortex states even in the zero-temperature limit [12, 27]. However, even slightly off-axis vortices are not stationary, but rather move about the condensate typically at velocities, for which the assumption of adiabaticity implicitly made when applying stationary self-consistent mean-field theories to describe time-dependent phenomena is highly questionable [28]. This raises the question of applicability of the results obtained for axisymmetric vortex states to the general off-axis vortex case. In this paper we generalize the previous results for the PA, G1 and G2 by showing that the anomalous modes of a circularly precessing off-axis vortex line penetrating a sufficiently strongly interacting, trapped condensate are also lifted to positive energies as the noncondensate partially fills the vortex core region.

In conclusion, the BA predicts vortices in harmonically trapped BECs to be energetically unstable, while the self-consistent Popov, G1 and G2 mean-field theories (and other theories having a similar structure; we refer to them with the term Popov-like approximations (PLAs)) predict them to be stable. This discrepancy can be contrasted to the experimental observations of vortex precession in spherical condensates [34]. Theoretically, it has been shown that the precession direction of a vortex is related to the energy of the vortex displacement mode: if the mode is anomalous with a negative energy, the vortex precesses in the direction of the condensate flow, and vice versa [21–23]. Except for a minority of so-called rogue vortices, the precession direction observed supports the predictions of the BA rather than those of the self-consistent approximations. As the results derived in this paper suggest that the nonadiabaticity in the time-development of moving vortices is not responsible for the discrepancy between the experimental observations and the predictions of the PLAs, we are essentially left with two possibilities: either dissipation in the experimental conditions has been too weak for the vortex states to be sufficiently properly thermalized, or these self-consistent approximations fail in describing the low-energy collective modes of vortex configurations. These possibilities are discussed in more detail in section 4 where we point out the shortcomings of the above-mentioned models in describing vortex states, and stress the need for an analysis within a self-consistent formalism that takes into account the dynamics of the condensate and the noncondensate on an equal footing.

The structure of this paper is as follows. In section 2 the general formalism of time-dependent Hartree–Fock–Bogoliubov–Popov (HFBP) equations in a rotating frame of reference is presented. In section 3 this formalism is applied to the case of a condensate containing a precessing off-axis vortex line, and a lower bound for the quasiparticle energies is derived. Furthermore, by applying this lower bound to the anomalous modes localized in the vortex core region, we show that such modes are lifted to positive energies as the noncondensate density in the core exceeds a certain value. Finally, the implications and proper interpretation of these results are discussed in section 4.

## 2. General formalism

We model a dilute bosonic gas with the usual effective grand-canonical Hamiltonian, which takes the second-quantized form

$$\hat{H} = \int d\mathbf{r} \psi^\dagger(\mathbf{r}, t) \mathcal{H}_\Omega(\mathbf{r}) \psi(\mathbf{r}, t) + \frac{g}{2} \int d\mathbf{r} \psi^\dagger(\mathbf{r}, t) \psi^\dagger(\mathbf{r}, t) \psi(\mathbf{r}, t) \psi(\mathbf{r}, t) \quad (1)$$

in a frame rotating with the angular velocity  $\Omega$  w.r.t. an inertial coordinate system. Above,  $\psi(\mathbf{r}, t)$  denotes the boson field operator, and

$$\mathcal{H}_\Omega(\mathbf{r}) \equiv -\frac{\hbar^2}{2m}\nabla^2 + V_{\text{tr}}(\mathbf{r}) - \mu - \Omega \cdot \mathbf{L} \quad (2)$$

is the one-particle Hamiltonian, with  $V_{\text{tr}}(\mathbf{r})$  denoting the external trapping potential,  $\mu$  the chemical potential, and  $\mathbf{L}$  the angular momentum operator. The interaction between the particles is modelled by the effective contact potential  $V_{\text{int}}(\mathbf{r} - \mathbf{r}') = g\delta(\mathbf{r} - \mathbf{r}')$ , with the coupling constant  $g$  related to the vacuum s-wave scattering length  $a$  through  $g = 4\pi\hbar^2 a/m$ . The dynamics of the system is determined by the Heisenberg equation of motion

$$i\hbar \frac{\partial}{\partial t} \psi(\mathbf{r}, t) = \mathcal{H}_\Omega(\mathbf{r})\psi(\mathbf{r}, t) + g\psi^\dagger(\mathbf{r}, t)\psi(\mathbf{r}, t)\psi(\mathbf{r}, t). \quad (3)$$

Following the spontaneous symmetry breaking approach, we use the Bogoliubov decomposition

$$\psi(\mathbf{r}, t) = \Phi(\mathbf{r}, t) + \tilde{\psi}(\mathbf{r}, t) \quad (4)$$

to split the field operator to the sum of the  $c$ -number condensate wavefunction  $\Phi(\mathbf{r}, t) = \langle \psi(\mathbf{r}, t) \rangle$  (with  $\langle \dots \rangle$  we signify the time-dependent nonequilibrium average) and the noncondensate, i.e., excitation field operator  $\tilde{\psi}(\mathbf{r}, t)$ . The expectation value of equation (3) yields, when the terms containing the cubic noncondensate operator product or the anomalous average  $\langle \psi\psi \rangle$  are neglected according to the Popov mean-field scheme, the generalized Gross–Pitaevskii (GP) equation

$$i\hbar \frac{\partial}{\partial t} \Phi(\mathbf{r}, t) = \mathcal{L}_\Omega(\mathbf{r}, t)\Phi(\mathbf{r}, t) - gn_c(\mathbf{r}, t)\Phi(\mathbf{r}, t) \quad (5)$$

for the condensate wavefunction. Above,  $\mathcal{L}_\Omega(\mathbf{r}, t) \equiv \mathcal{H}_\Omega(\mathbf{r}) + 2gn(\mathbf{r}, t)$ , and the condensate, noncondensate and total gas densities are, respectively, denoted as

$$n_c(\mathbf{r}, t) = |\Phi(\mathbf{r}, t)|^2, \quad (6a)$$

$$\tilde{n}(\mathbf{r}, t) = \langle \tilde{\psi}^\dagger(\mathbf{r}, t)\tilde{\psi}(\mathbf{r}, t) \rangle, \quad (6b)$$

$$n(\mathbf{r}, t) = n_c(\mathbf{r}, t) + \tilde{n}(\mathbf{r}, t); \quad (6c)$$

the chemical potential  $\mu$  is implicitly determined by the condition

$$N = \int d\mathbf{r} n(\mathbf{r}, t), \quad (7)$$

where  $N$  is the total number of particles in the system. Furthermore, equations (3)–(6) imply in the Popov scheme the equation of motion

$$i\hbar \frac{\partial}{\partial t} \tilde{\psi}(\mathbf{r}, t) = \mathcal{L}_\Omega(\mathbf{r}, t)\tilde{\psi}(\mathbf{r}, t) + g\Phi^2(\mathbf{r}, t)\tilde{\psi}^\dagger(\mathbf{r}, t) \quad (8)$$

for the noncondensate operator. This equation can be diagonalized by expressing the field operator in terms of bosonic quasiparticle operators  $\alpha_n$  and  $\alpha_n^\dagger$ . The Bogoliubov transformation

$$\tilde{\psi}(\mathbf{r}, t) = \sum_n [u_n(\mathbf{r}, t)\alpha_n - v_n^*(\mathbf{r}, t)\alpha_n^\dagger] \quad (9)$$

converts the operator equation (8) to the time-dependent HFBP equations

$$i\hbar \frac{\partial}{\partial t} u_n(\mathbf{r}, t) = \mathcal{L}_\Omega(\mathbf{r}, t)u_n(\mathbf{r}, t) - g\Phi^2(\mathbf{r}, t)v_n(\mathbf{r}, t) \quad (10a)$$

$$-i\hbar \frac{\partial}{\partial t} v_n(\mathbf{r}, t) = \mathcal{L}_{-\Omega}(\mathbf{r}, t)v_n(\mathbf{r}, t) - g\Phi^{*2}(\mathbf{r}, t)u_n(\mathbf{r}, t) \quad (10b)$$

for the quasiparticle amplitudes  $u_n(\mathbf{r}, t)$  and  $v_n(\mathbf{r}, t)$ . In addition, for the Bogoliubov transformation to be canonical, we must impose the condition

$$\int d\mathbf{r} [u_n^*(\mathbf{r}, t)u_{n'}(\mathbf{r}, t) - v_n^*(\mathbf{r}, t)v_{n'}(\mathbf{r}, t)] = \delta_{nn'} \quad (11)$$

for the amplitudes; this time-independent normalization can be straightforwardly shown to be consistent with equation (10). Furthermore, equations (6b) and (9) imply for the noncondensate density the self-consistency relation

$$\tilde{n} = \sum_{nn'} [f_{nn'}(u_n^*u_{n'} + v_n^*v_{n'}) - 2\text{Re}\{g_{nn'}u_nv_{n'}\} + \delta_{nn'}|v_n|^2], \quad (12)$$

where

$$f_{nn'}(t) = \langle \alpha_n^\dagger \alpha_{n'} \rangle, \quad g_{nn'}(t) = \langle \alpha_n \alpha_{n'} \rangle \quad (13)$$

denote the normal and anomalous quasiparticle distribution functions; here and henceforth we often do not explicitly denote the arguments of functions, unless they are needed for clarity.

Neglect of the anomalous average in the Popov scheme is a rather crude method of obtaining a theory with gapless spectrum in the homogeneous limit, as required by Goldstone's theorem. The effects of the anomalous average can be taken into account by renormalizing the two-body interaction potential to a  $T$ -matrix including many-body effects. Effectively, this can be done by replacing the coupling constant  $g$  by suitable spatially varying coupling functions in the mean-field equations [40, 41]. In the following analytic treatment we assume, for simplicity, the effective interaction to be constant, and only finally briefly discuss the more general situation.

### 3. Precessing vortex line

Consider a repulsively ( $g > 0$ ) interacting boson condensate containing a precessing vortex line, trapped by a static, axisymmetric potential. For simplicity, we assume the vortex to move in a circular orbit about the trap symmetry axis, with a constant angular velocity  $\Omega$  directed along the trap axis. Such a precessional motion has been observed in the experiments, and it is in agreement with vortex dynamics given by equation (5) when the noncondensate density is neglected [24, 25, 34]. In any case, our main objective in extending the stability analysis for moving vortices is to investigate the effect of nonadiabaticity due to finite velocity of the vortex line, and for this purpose it is sufficient to restrict ourselves to considering only circular vortex motion. In addition, we assume the system to be thermalized in the sense that the quasiparticle distribution functions  $f_{nn'}$  and  $g_{nn'}$  are constant in time. In such a case, the mean-field Hamiltonian is time-independent in the frame rotating with the angular velocity  $\Omega$  w.r.t. the laboratory coordinate system. Consequently, in such a frame equations (5) and (10) reduce to the stationary equations

$$\mathcal{L}_\Omega(\mathbf{r})\Phi(\mathbf{r}) - gn_c(\mathbf{r})\Phi(\mathbf{r}) = 0 \quad (14)$$

for the condensate wavefunction and

$$\mathcal{L}_\Omega(\mathbf{r})u_n(\mathbf{r}) - g\Phi^2(\mathbf{r})v_n(\mathbf{r}) = E'_n u_n(\mathbf{r}) \quad (15a)$$

$$\mathcal{L}_{-\Omega}(\mathbf{r})v_n(\mathbf{r}) - g\Phi^{*2}(\mathbf{r})u_n(\mathbf{r}) = -E'_n v_n(\mathbf{r}) \quad (15b)$$

for the quasiparticle eigenenergies  $E'_n$  and amplitudes  $u_n(\mathbf{r})$ ,  $v_n(\mathbf{r})$ . Although stationary, these equations are not to be confused with the adiabatic approximation for systems with slow time dependence [28]. The adiabatic approximation fails if the quasiparticle states do not follow the moving vortex core rigidly, but the eigenequations (15) describe the time dependence of

the system in principle exactly. The essential difference between these formalisms is the terms  $\boldsymbol{\Omega} \cdot \mathbf{L}$  contained in equations (15), but absent in the equations corresponding to the adiabatic approximation; it is these terms that account for the deformation of the vortex structure due to its motion [28].

Although considerably less cumbersome than the general time-dependent HFBP formalism, the stationary equations (15) have not been solved so far for self-consistent solutions describing off-axis vortices. However, in the following we show that the essential question concerning the local energetic stability of such vortex states can be studied analytically under very general circumstances. It is to be noted that the thermalization and the energetic stability of the system are determined by its excitation spectrum in the nonrotating, i.e., laboratory frame (LF) of reference, since we assume that the gas is confined with a static external potential and that the thermal component as a whole is nonrotating [42]. Hence, in order to investigate the issue of local stability, we have to study the quasiparticle energies in the LF, instead of the rotating frame eigenenergies  $E'_q$ . Strictly speaking, as the mean-field Hamiltonian in the LF has explicit time dependence via the mean fields, it does not possess stationary solutions, and, consequently, a well defined spectrum; in the LF only the *expectation values* of the quasiparticle energies are well defined. For the same reason, such a system can not attain exact thermodynamic equilibrium. However, if the thermalization rate of the gas substantially exceeds the kinetic rates related to the motion of the vortex, the system is thermalized in the sense that its dynamics becomes adiabatic to a high degree of accuracy, and the quasiparticle distribution functions approach time-independent values (note that here adiabaticity refers to its thermodynamic meaning, in contrast to the adiabaticity of the nonlinear quantum dynamics referred to earlier in this paper).

### 3.1. Lower bound for quasiparticle energies

In order to investigate the positivity of the energies of the lowest quasiparticle excitations, we first derive a lower bound for the spectrum of a trapped BEC within the PA.

Let  $(\Phi, u, v)$  be a self-consistent solution of equations (14) and (15), corresponding to an eigenenergy  $E'$  and a given set of time-independent quasiparticle distribution functions  $f_{nn'}$  and  $g_{nn'}$ . Equations (1) and (2) imply that the expectation value  $\langle E \rangle$  of the quasiparticle energy in the nonrotating LF is related to the eigenenergy  $E'$  in the rotating frame by the relation  $\langle E \rangle = E' + \boldsymbol{\Omega} \cdot \langle \mathbf{l} \rangle$ , where  $\langle \mathbf{l} \rangle$  denotes the expectation value of the angular momentum of the quasiparticle state. By expanding the expectation value  $\int d\mathbf{r} \langle \tilde{\psi}^\dagger \mathbf{L} \tilde{\psi} \rangle$  of the angular momentum related to the noncondensate in terms of the quasiparticle amplitudes and distribution functions, one finds that the angular momentum of the system is changed by the amount  $\int d\mathbf{r} (u^* \mathbf{L} u + v \mathbf{L} v^*)$  as the corresponding quasiparticle occupation number is increased by one. Thus, we find

$$\langle E \rangle = E' + \int d\mathbf{r} [u^* (\boldsymbol{\Omega} \cdot \mathbf{L}) u + v (\boldsymbol{\Omega} \cdot \mathbf{L}) v^*]. \quad (16)$$

On the other hand, integration of the sum of equation (15a) multiplied by  $u^*(\mathbf{r})$  and the complex conjugate of equation (15b) multiplied by  $v(\mathbf{r})$  yields

$$\int d\mathbf{r} [u^* (\mathcal{L}_0 - \boldsymbol{\Omega} \cdot \mathbf{L}) u + v (\mathcal{L}_0 - \boldsymbol{\Omega} \cdot \mathbf{L}) v^* - 2g \Phi^2 u^* v] = E' \int d\mathbf{r} (|u|^2 - |v|^2) = E', \quad (17)$$

where we have used the relation  $\mathcal{L}_\Omega = \mathcal{L}_0 - \boldsymbol{\Omega} \cdot \mathbf{L}$ , the fact that  $\mathbf{L}^* = (\mathbf{r} \times \hbar \nabla / i)^* = -\mathbf{r} \times \hbar \nabla / i = -\mathbf{L}$ , and the normalization condition (11) for the quasiparticle amplitudes. Now, combination of equations (16) and (17) yields the relation

$$\langle E \rangle = \int d\mathbf{r} (u^* \mathcal{L}_0 u + v \mathcal{L}_0 v^* - 2g \Phi^2 u^* v) \quad (18)$$

for the quasiparticle energy in the LF. It is to be noted that this expression has only an implicit dependence on the vortex precession velocity  $\Omega$ , via the fact that the amplitudes  $u(\mathbf{r})$  and  $v(\mathbf{r})$  satisfy equations (15).

In order to derive a simple lower-bound expression for the quasiparticle energy, we first note that  $\int d\mathbf{r} \Phi^2 u^* v$  is real-valued; this can be inferred from equation (18), where all the other terms are real: quasiparticle states which obey the normalization condition (11) can be shown to have real energies (we do not consider the complex frequency states occurring for multiquantum vortices due to their dissociation instability [15]). Thus we have

$$\int d\mathbf{r} \Phi^2 u^* v \leq \int d\mathbf{r} |\Phi|^2 |u||v| \leq \frac{1}{2} \int d\mathbf{r} n_c (|u|^2 + |v|^2), \quad (19)$$

since  $(|u| - |v|)^2 = |u|^2 + |v|^2 - 2|u||v| \geq 0$ . Combination of this upper bound with equation (18) yields, due to the positivity of the coupling constant  $g$ , the inequality

$$\begin{aligned} \langle E \rangle &\geq \int d\mathbf{r} [u^* \mathcal{L}_0 u + v \mathcal{L}_0 v^* - (|u|^2 + |v|^2) g n_c] \\ &= \int d\mathbf{r} [u^* (-\hbar^2 \nabla^2 / 2m) u + v (-\hbar^2 \nabla^2 / 2m) v^*] \\ &\quad + \int d\mathbf{r} (|u|^2 + |v|^2) (V_{\text{tr}} + g n_c + 2g\tilde{n} - \mu). \end{aligned} \quad (20)$$

The kinetic energy terms may be discarded from this lower bound due to their positivity, and we find

$$\langle E \rangle > \int d\mathbf{r} \rho (V_{\text{tr}} + g n_c + 2g\tilde{n} - \mu), \quad (21)$$

where  $\rho(\mathbf{r}) = |u(\mathbf{r})|^2 + |v(\mathbf{r})|^2$  denotes the density distribution of the quasiparticle state. Using the Thomas–Fermi (TF) approximation

$$n_c^{(\text{TF})} = \frac{1}{g} (\mu - V_{\text{tr}} - 2g\tilde{n}) \quad (22)$$

for the condensate density in the strong-interaction limit, we can write the second factor in the integral of the lower bound (21) as

$$V_{\text{tr}} + g n_c + 2g\tilde{n} - \mu = g(n_c - n_c^{(\text{TF})}). \quad (23)$$

Since we are mainly interested in the anomalous quasiparticle states which are localized in the vicinity of the vortex core, it is useful to split the spatial integral in the inequality (21) to contributions from the core region  $C$  and its complement  $C'$ . To be specific, we define the vortex centre line to consist of points in which  $\Phi(\mathbf{r}) = 0$ , and the core region  $C$  to contain points whose distance from the centre line is less than the healing length  $\xi = (8\pi n_0 a)^{-1/2}$ , where  $n_0$  denotes the local total density of a corresponding irrotational system. Especially, in the strong interaction TF limit we have

$$n_0 \simeq \frac{1}{g} (\mu - V_{\text{tr}}), \quad (24)$$

provided that the temperature is not so close to  $T_{\text{BEC}}$  that the thermal gas component would dominate over the condensate. Utilizing the positivity of  $\rho$ ,  $n_c$  and  $g$ , we find equations (21) and (23) to imply

$$\langle E \rangle > \int_C d\mathbf{r} \rho (V_{\text{tr}} + 2g\tilde{n} - \mu) - g \max_{C'} \{n_c^{(\text{TF})} - n_c\} \int_{C'} d\mathbf{r} \rho, \quad (25)$$

where  $\max\{\cdot\cdot\cdot\}_{C'}$  denotes the maximum value in the region  $C'$ . Furthermore, by defining the quantity

$$w = \frac{\int_C d\mathbf{r} \rho}{\int_{C'} d\mathbf{r} \rho} \quad (26)$$

describing the degree of localization of the quasiparticle state  $(u, v)$  into the vortex core region, we can write the lower bound (25) for the quasiparticle energy in the more compact form of

$$\langle E \rangle > \int_C d\mathbf{r} \rho [V_{\text{tr}} + 2g\tilde{n} - \mu - g \max_{C'} \{n_c^{(\text{TF})} - n_c\} / w]. \quad (27)$$

It is to be noted that this lower bound holds for all quasiparticle states, when the corresponding density distributions  $\rho$  and localization degrees  $w$  are used in the integrand.

### 3.2. Anomalous modes

Suppose that the condensate containing a vortex line has an anomalous quasiparticle mode (AM) with a negative energy in the LF. In such a case the system is energetically unstable, because it can, in the presence of dissipative mechanisms, lower its free energy by exciting the AM with matter originating from the condensate. It has been argued that such a ‘drain’ of matter into the AM would lead to the formation of a binary condensate from the vortex core, and a simultaneous spiralling of the vortex out of the condensate [8]. However, it is to be noted that when the noncondensate mean field is taken into account in a self-consistent manner, the quasiparticle energies depend on the distribution functions  $f_{nn'}$  and  $g_{nn'}$ . In particular, the excitation of an AM increases the corresponding occupation number, and thus changes the energy of the mode itself. If the change in the energy is large enough, the anomalous mode may attain positive energies, in which case the ‘collapse’ of the condensate would cease. Consequently, the vortex would be stabilized by the self-interaction of the noncondensate in the core.

More specifically, since the quasiparticle density distribution  $\rho$  is positive, equation (27) implies that a given mode can be a negative-energy AM only if the expression in the brackets is negative somewhere in the core region, i.e.

$$\tilde{n} < \frac{1}{2g} (\mu - V_{\text{tr}} + g \max_{C'} \{n_c^{(\text{TF})} - n_c\} / w) \quad (28)$$

somewhere in  $C$ .

In order to estimate the vortex core localization degrees  $w$  of the anomalous modes, we note that the quasiparticle motion is essentially determined by the potential

$$V_{\text{eff}} = V_{\text{tr}} + 2g(n_c + \tilde{n}) - \mu \quad (29)$$

—this is the effective Hartree potential appearing in equations (10) and (15). For noncondensate density distributions not too far from the form corresponding to a thermalized state,  $V_{\text{eff}}$  is negative in the vortex core region, and positive elsewhere [8, 12]. Consequently, for the energy of the mode to be negative, its density has to be concentrated in the core region, thus yielding a localization degree  $w \gtrsim 1$ . This heuristic reasoning for the localization degree is supported by explicit numerical computations for axisymmetric vortex states of axially harmonically trapped condensates, resulting in values  $w \approx 0.7$  in the large- $N$  limit, and  $w \approx 1.0$  for systems in the noninteracting limit. (For the numerical methods used in computing the wavefunctions and energies of the anomalous modes in axially trapped condensates, see [27].) It is to be noted that for vortex precession velocities that substantially exceed the adiabaticity limit for rigid vortex core motion [28], the quasiparticles may be deformed and the localization degree can



be smaller than the estimates given above. However, for the anomalous modes it always has to be of order unity.

For harmonically trapped gases, the effective strength of the interactions relative to the kinetic energy in the system is characterized by the dimensionless quantity  $Na/a_{\text{ho}}$ , where  $a_{\text{ho}} = (\hbar/m\bar{\omega})^{1/2}$  is the harmonic oscillator length, and  $\bar{\omega} = (\omega_x\omega_y\omega_z)^{1/3}$  denotes the geometric mean of the harmonic oscillator frequencies. The TF approximation for the condensate density is accurate in the interaction-dominated regime  $Na/a_{\text{ho}} \gg 1$ . However, as the corrections to the chemical potential given by the TF value  $\mu_{\text{TF}} = gn_{\text{peak}}$ , where  $n_{\text{peak}}$  denotes the maximum density of the gas, are of the small fractional order  $(15Na/a_{\text{ho}})^{-4/5} \ln(15Na/a_{\text{ho}})^{1/5}$  [43, 44], its accuracy is in general of the order of 10% when  $Na/a_{\text{ho}} \gtrsim 1$ . Thus, in this parameter regime we may use the estimate (24) in the condition (28). Furthermore, as the localization degree  $w$  of the anomalous mode is of the order of unity, the contribution given by the last term in inequality (28) is small, and we get an approximate condition

$$\tilde{n}(\mathbf{r}) \lesssim \frac{1}{2}n_0(\mathbf{r}) \quad (30)$$

for some  $\mathbf{r} \in C$ , when  $Na/a_{\text{ho}} \gtrsim 1$  and the temperature is not too close to  $T_{\text{BEC}}$ . In other words, since  $n_0(\mathbf{r})$  is approximately the local maximum density in the vicinity of the vortex, the anomalous modes are lifted to positive energies as the noncondensate in the core fills half of the core volume.

This result, which also applies to unharmonically trapped condensates provided they interact sufficiently repulsively, shows that within the PA even partial filling of the vortex core with noncondensate is sufficient to ‘suppress’ the anomalous modes by lifting their energies to positive values. In fact the approximate inequality (30) is rather pessimistic in the sense that, according to exact numerical solutions, even much lower noncondensate core densities suffice to lift the anomalous modes to positive energies [27]. The analysis can also in principle be straightforwardly generalized to cover the G1 and G2 approximations, if and when the effective spatially dependent coupling  $g$  is not substantially reduced in the vortex core region. Returning to the question of vortex stability, these results show that the ‘collapse’ of the condensate to the anomalous core modes ceases as the noncondensate density due to these modes partially fills the vortex core and the energies of the instability modes rise above the energy of the condensate state. In the presence of strong enough dissipation, this self-stabilization is a direct consequence of thermalization. All in all, we find that Popov-like self-consistent approximations predict vortices to become locally energetically stable in the course of thermalization.

#### 4. Discussion and conclusions

The BA consisting of the mean-field GP equation and the Bogoliubov normal-mode equations has been widely used to describe the properties of vortices and, especially, to investigate their stability in dilute atomic BECs. Predictions derived from it for the vortex precession direction and angular velocity, together with the critical trap rotation frequencies for vortex nucleation, agree well with experimental results. However, the BA neglects the effects of the noncondensate. At temperatures  $T \gtrsim T_{\text{BEC}}/2$  a considerable fraction of the gas is noncondensed and the BA is insufficient to model the system. Moreover, we argue that for vortex states its accuracy is questionable even at lower temperatures. This is due to the tendency of the condensate matter to be transferred to noncondensate occupations of the anomalous negative-energy core modes. Consequently, as the system is thermalized, the vortex core becomes occupied by a substantial noncondensate density even in the zero-temperature limit.

The PLAs, on the other hand, are finite-temperature mean-field theories that self-consistently take into account the thermal gas component. Their predictions for the excitation frequencies of irrotational condensates are in excellent agreement with experimental results at temperatures  $T \lesssim T_{\text{BEC}}/2$ , but at higher temperatures discrepancies become obvious, as the noncondensate density becomes more dominant. This failure of the PLAs near  $T_{\text{BEC}}$  has been attributed to the fact that they do not properly treat the *dynamics* of the noncondensed gas.

On the basis of explicit numerical solutions for axisymmetric vortex states [12, 27] and the analysis presented in this paper (see also [30, p R177]), we conclude that PLAs imply vortices in sufficiently properly thermalized and repulsively interacting condensates to be locally energetically stable in the sense that the excitation spectrum of the system is positive definite. Especially, these approximations imply that the vortex precession mode has a positive energy, which in turn seems to imply that off-axis vortices precess in the direction opposite to the vortex flow. However, this prediction contradicts experimental results, which rather show that the majority of vortices precess in the direction of the condensate flow. In the light of the success of the mean-field approximations in describing the physics of dilute condensates, the failure of the mean-field theory itself is an unlikely reason underlying this discrepancy. Thus we are essentially left with two alternatives: either the vortex states observed in the experiments have not been properly thermalized, or the PLAs drastically fail in describing the lowest collective modes of the vortex states. In fact the observations of vortex precession have so far been conducted under conditions in which dissipation in the gas is very weak, and the degree of thermalization probably low [34, 38]. It is to be noted that, due to improper thermalization, the anomalous mode occupations may be negligible and the vortex core almost void of noncondensate; in such a case the BA should naturally yield more accurate results than the PLAs.

In any case, there is also a reason to expect PLAs to fail in predicting the lowest quasiparticle modes for vortex states even in the zero-temperature limit. As the vortex state is thermalized, the noncondensate concentration in the vortex core region becomes non-negligible and, as we have seen, it substantially modifies the energies of the lowest quasiparticle states which are essentially localized in the core. However, when describing the collective modes of the system, the PLAs do not correctly take into account the *dynamics* of this noncondensate, but treat the thermal gas peak in the core as a static pinning potential for the vortex. The stabilizing effect of such an ‘external’ potential is naturally stronger than that of a realistic, dynamical noncondensate peak, and the lifting effect of the anomalous modes is consequently exaggerated within the PLAs. For the same reason, the prediction for the precession direction given by the PLAs is compromised, because the lowest mode rather describes precession of the vortex about a static noncondensate peak than a combined motion of the vortex core and its thermal gas filling (see also [30]). Such a failure of PLAs may be somewhat surprising, as one in general would expect these approximations to be unreliable only near  $T_{\text{BEC}}$ , where the noncondensate fraction and the resonant contributions to the self-energies are substantial [45]. When vortex states are considered, however, the dynamics of the thermal gas should be taken into account at all temperatures, due to the filling of the vortex core with noncondensate in the course of thermalization.

Consequently, the effects of the thermal gas concentrated in the vortex core are not properly modelled within the BA nor the PLAs. In order to reliably determine the structure, spectrum and stability of quantized vortices in dilute BECs one has to use a model which treats the dynamics of the condensate and the thermal gas on an equal footing. Such fully dynamic approximations for inhomogeneous gases have been developed in recent years [45–48], but they have not yet been applied to study vortex states. Also due to the advances in experimental techniques and measurement accuracies there is a need for investigating vortex states beyond

the Bogoliubov level. Due to the strong inhomogeneities involved, vortices are potentially quite efficient testbenches for thermal field theories in modelling atomic BECs.

In conclusion, we have studied the stability of precessing off-axis vortices in dilute, trapped BECs within the self-consistent PA and its extensions G1 and G2. By deriving a lower bound for the excitation energies, it has been shown that within these approximations the vortex states are locally energetically stabilized as the vortex core is partially filled with thermal gas. As such filling of the vortex core with noncondensate is an inherent consequence of thermalization and the existence of anomalous modes for vortex states in pure condensates, we argue that the effects of the thermal gas for the physics of vortices may not be neglected even in the zero-temperature limit. Furthermore, the shortcomings of the BA and the Popov-like self-consistent approximations in modelling vortex states have been pointed out and analysed. Essentially, the BA may be too ‘pessimistic’ in predicting vortices to be definitely unstable, and the PLAs too ‘optimistic’ in predicting them to be self-stabilized by the thermal gas component. We argue that a reliable analysis of the stability of quantized vortices in dilute BECs requires a fully dynamic formalism; such an investigation remains a challenge for further studies.

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