Homogeneous scattering model for impure superfluid ³He

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Abstract

This work is motivated by recent experiments studying superfluid ³He inside porous aerogel. Using the homogeneous scattering model, we calculate the pairing amplitude and the superfluid density for the A and B phases at all temperatures. At high temperatures the results are in agreement with simpler calculations based on the Ginzburg–Landau theory. We also study the effect of large impurities in the B phase. We find that higher scattering channels give essentially the same results as obtained by limiting to s-wave scattering. © 1998 Elsevier Science B.V. All rights reserved.

Keywords: Superfluid; Aerogel; ³He; Impurity scattering; Quasiclassical theory

1. Introduction

Recent experiments [1,2] have shown that the superfluid transition of ³He occurs not only in pure helium, but also in very porous aerogel, where 2% of the volume is filled with aerogel. It was found that the superfluid transition temperature T_c is reduced from the transition temperature in pure ³He, T_{c0}. Other measured quantities include the pairing amplitude Δ and the superfluid density ρ_s. Different models have been studied in order to explain theoretically the experimental observations [3]. The simplest one of these is the homogeneous scattering model (HSM).

The first calculations applying the HSM to ³He were done in the Ginzburg–Landau (GL) region,

i.e. in the temperature region just below T_c [3,4]. It was already visible in these results that the HSM is in disagreement with experiments. There remained, however, two uncertainties. Firstly, it was a priori possible that the results in the GL region were misleading and at lower temperature the HSM would predict something that strongly deviates from the extrapolation of the GL results. Secondly, the first calculations took into account only s-wave scattering ($l = 0$). It is known that the aerogel consists of strands, whose diameter (≈ 3 nm) is roughly four times the Fermi wavelength of the quasiparticles. Thus most of the scattering takes place via higher partial waves $l > 0$, and it was not known to us if this would radically alter the s-wave prediction. The purpose of this paper was to study these two points.

A further motivation for the present calculations is that the HSM, alhough in disagreement

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with experiments, has such basic simplicity that it naturally forms the first step in the process of developing more sophisticated theories [3]. Different applications of the HSM in ³He are given in Refs. [5–8].

Firstly, we calculate the pairing amplitude Δ in the B phase including all scattering partial waves. Secondly, limiting to the case of s-wave scattering, we calculate Δ in the A phase, and the superfluid density ρ_s for both A and B phases. These calculations are made at all temperatures $T < T_c$. We find that the GL theory gives good approximate results under a wide temperature range. Concerning higher partial waves, we conclude that a realistic choice corresponds to a result approximately half way between the Born and unitarity limits of s-wave scattering.

Part of the results in this paper has been published previously. The impure B phase is analogous to the case of magnetic impurities in s-wave superconductors [9]. This has been studied numerically in Ref. [10] limiting to the Born approximation. The anisotropy of the A phase has been studied by Choi and Muzikar in connection with unconventional superfluidity [11]. The superfluid density for both A and B phases has been recently calculated by Higashitani [8]. All these works are limited to s-wave scattering.

The calculations are done using the quasiclassical theory, which is briefly discussed in Section 2. The calculation of the pairing amplitude is presented in Section 3 and the superfluid density in Section 4.

2. Quasiclassical theory

We use the quasiclassical theory for impure weak-coupling p-wave superfluid. Some basic assumptions of this theory are discussed in Ref. [4]. In this section we write down the general equations of this theory. The derivation of these equations is similar as presented by Serene and Rainer [12] in the pure case. In addition one needs to use averaging over impurity locations and some basics of scattering theory [13]. Although the derivation requires some care, it will not be presented here.

An important intermediate quantity in the quasiclassical theory is the propagator $\breve{g}(\hat{\boldsymbol{k}}, \boldsymbol{r}, \varepsilon_m)$. It is

a 4×4 matrix, which can be thought of as a 2×2 Nambu matrix

$$\breve{g} = \begin{pmatrix} g + \boldsymbol{g} \cdot \boldsymbol{\sigma} & (f + \boldsymbol{f} \cdot \boldsymbol{\sigma})\mathrm{i}\sigma_2 \\ \mathrm{i}\underline{\sigma}_2(\tilde{f} + \tilde{\boldsymbol{f}} \cdot \boldsymbol{\sigma}) & \tilde{g} - \underline{\sigma}_2 \, \tilde{\boldsymbol{g}} \cdot \boldsymbol{\sigma}\underline{\sigma}_2 \end{pmatrix}, \tag{1}$$

where each of the four elements is a 2×2 spin matrix. Here $\underline{\sigma}_i$ denote the Pauli matrices in the spin submatrix, and $\breve{\tau}_i$ are the same thing as Nambu matrices. The arguments of the propagator $\breve{g}(\hat{\boldsymbol{k}}, \boldsymbol{r}, \varepsilon_m)$ are the direction of the momentum $\hat{\boldsymbol{k}}$, the location $\boldsymbol{r}$, and the Matsubara frequencies $\varepsilon_m = 2\pi k_B T(m - \frac{1}{2})$, where m is an integer. The propagator is determined from the Eilenberger equation and the normalization condition

$$[\mathrm{i}\varepsilon\breve{\tau}_3 - \breve{v} - \breve{\rho} - \breve{\Delta}, \breve{g}] + \mathrm{i}\hbar v_F \hat{\boldsymbol{k}} \cdot \nabla_r \breve{g} = 0 \tag{2}$$

$$\breve{g}\breve{g} = -1. \tag{3}$$

The quantities $\breve{v}$, $\breve{\rho}$, and $\breve{\Delta}$ are defined below. $\breve{\Delta}(\hat{\boldsymbol{k}}, \boldsymbol{r})$ is the Nambu-off-diagonal self-energy

$$\breve{\Delta} = \begin{pmatrix} 0 & \boldsymbol{\Delta} \cdot \underline{\boldsymbol{\sigma}}\mathrm{i}\underline{\sigma}_2 \\ \mathrm{i}\underline{\sigma}_2\boldsymbol{\Delta}^* \cdot \underline{\boldsymbol{\sigma}} & 0 \end{pmatrix}, \tag{4}$$

where $*$ denotes complex conjugate. In the weak-coupling limit the order parameter $\boldsymbol{\Delta}(\hat{\boldsymbol{k}}, \boldsymbol{r})$ is determined from the self-consistency equation

$$\boldsymbol{\Delta}(\hat{\boldsymbol{k}}, \boldsymbol{r}) \ln \frac{T}{T_{c0}} + \pi k_B T \sum_{\varepsilon_m} \left[\frac{\boldsymbol{\Delta}(\hat{\boldsymbol{k}}, \boldsymbol{r})}{|\varepsilon_m|} \right. $$
$$\left. - 3\langle (\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}}') \boldsymbol{f}(\hat{\boldsymbol{k}}', \boldsymbol{r}, \varepsilon_m) \rangle_{\hat{k}'} \right] = 0 \tag{5}$$

Here T_{c0} is the critical temperature of pure helium, $\sum_{\varepsilon_m}$ denotes the summation over all ε_m, and $\langle \ldots \rangle_{\hat{k}'}$ denotes the angular average with respect to $\hat{\boldsymbol{k}}'$. The Fermi-liquid parameters give rise to a Nambu-diagonal self-energy

$$\breve{v} = \begin{pmatrix} v + \boldsymbol{v} \cdot \boldsymbol{\sigma} & 0 \\ 0 & \tilde{v} - \underline{\sigma}_2\tilde{\boldsymbol{v}} \cdot \boldsymbol{\sigma}\underline{\sigma}_2 \end{pmatrix}, \tag{6}$$

where $\tilde{v}(\hat{\boldsymbol{k}}, \boldsymbol{r}) = v(-\hat{\boldsymbol{k}}, \boldsymbol{r})$ and $\tilde{\boldsymbol{v}}(\hat{\boldsymbol{k}}, \boldsymbol{r}) = \boldsymbol{v}(-\hat{\boldsymbol{k}}, \boldsymbol{r})$ are real functions determined by equations

$$v(\hat{\boldsymbol{k}}, \boldsymbol{r}) = \pi k_B T \sum_{\varepsilon_m} \langle A^s(\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}}') g(\hat{\boldsymbol{k}}', \boldsymbol{r}, \varepsilon_m) \rangle_{\hat{k}'} \tag{7}$$

$$\boldsymbol{v}(\hat{\boldsymbol{k}}, \boldsymbol{r}) = \pi k_B T \sum_{\varepsilon_m} \langle A^a(\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}}') \boldsymbol{g}(\hat{\boldsymbol{k}}', \boldsymbol{r}, \varepsilon_m) \rangle_{\hat{k}'} \tag{8}$$

Here the functions A^s and A^a can be expressed in terms of Legendre polynomials and Fermi-liquid parameters [12].

The effect of impurities in the Eilenberger equation is taken into account by the impurity self-energy $\check{\rho}$. It equals the concentration of the impurities $n(\mathbf{r})$ times the forward scattering T-matrix of the single impurity: $\check{\rho}(\hat{\mathbf{k}}, \mathbf{r}, \varepsilon_m) = n(\mathbf{r})\check{t}(\hat{\mathbf{k}}, \hat{\mathbf{k}}, \mathbf{r}, \varepsilon_m)$. The equation for the T-matrix is

$$\check{t}(\hat{\mathbf{k}}, \hat{\mathbf{k}}', \mathbf{r}, \varepsilon_m) = \check{v}(\hat{\mathbf{k}}, \hat{\mathbf{k}}')$$
$$+ \pi N_F \langle \check{v}(\hat{\mathbf{k}}, \hat{\mathbf{k}}'')\check{g}(\hat{\mathbf{k}}'', \mathbf{r}, \varepsilon_m)\check{t}(\hat{\mathbf{k}}'', \hat{\mathbf{k}}', \mathbf{r}, \varepsilon_m)\rangle_{\hat{\mathbf{k}}'}, \quad (9)$$

where $2N_F = m_{eff}k_F/\pi^2\hbar^2$ is the total density of states at the Fermi surface, m_{eff} is the effective mass and k_F the Fermi wave vector. The mass m_3 of a ³He atom is related to m_{eff} by $m_{eff}/m_3 = 1 + F_1^s/3$. For nonmagnetic scattering the impurity potential is proportional to the unit matrix: $\check{v} = v\check{1}$. A spherically symmetric potential can be represented as

$$v(\hat{\mathbf{k}}, \hat{\mathbf{k}}') = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} P_l(\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}') v_l, \quad (10)$$

where P_l are the Legendre polynomials, for example $P_0(x) = 1$ and $P_1(x) = x$. The coefficients v_l are related to the scattering phase shifts δ_l by $v_l = -(4/N_F)\tan\delta_l$. The transport cross section in the normal state is then given by

$$\sigma_{tr} = \frac{4\pi}{k_F^2} \sum_{l=0}^{\infty} (l+1)\sin^2(\delta_{l+1} - \delta_l). \quad (11)$$

The transport mean free path is defined by $l_{tr} = (n\sigma_{tr})^{-1}$. In the calculation we consider δ_l and l_{tr} as independent variables, which means that the impurity density n is a dependent variable: $n(\delta_l, l_{tr})$. Another important length in the theory is the coherence length $\xi_0 = \hbar v_F/2\pi k_B T_{c0}$. The principal dimensionless parameter in the following is ξ_0/l_{tr}.

Very little is known about the phase shifts δ_l in aerogel. Therefore, we consider the following model cases. One is the Born limit, where all δ_l are assumed small ($\delta_l \ll 1$ for all l). Another limiting case is that only the s-wave scattering phase shift δ_0 is nonzero. The case that $\sin^2\delta_0$ approaches unity is called the unitarity limit. Still one more model is a hard sphere of radius R. There the phase shifts are given by $\tan\delta_l = j_l(k_F R)/n_l(k_F R)$, where j_l and n_l are

the spherical Bessel and Neumann functions, respectively. The transport cross section (11) as a function of $k_F R$ is plotted in Ref. [14].

A nonzero superfluid velocity $\mathbf{v}_s = (\hbar/2m_3)\mathbf{q}$ arises when the phase of the order parameter is not constant, but has the form $\Delta(\hat{\mathbf{k}}, \mathbf{r}) = \exp(i\mathbf{q} \cdot \mathbf{r})\Delta(\hat{\mathbf{k}})$. The superfluid density is defined by $\rho_s = \lim_{v_s \to 0} j_s/v_s$, where $\mathbf{j}_s$ is the superfluid mass current given by

$$\mathbf{j}_s(\mathbf{r}) = 2m_3 v_F N_F \pi k_B T \sum_{\varepsilon_m} \langle \hat{\mathbf{k}} g(\hat{\mathbf{k}}, \mathbf{r}, \varepsilon_m)\rangle_{\hat{\mathbf{k}}}. \quad (12)$$

In pure helium at zero temperature, the superfluid density equals the total density of the liquid $\rho = m_3 k_F^3/3\pi^2$.

The equations above are rather general results of the quasiclassical scattering theory. The principal limitation for the impurity density $n(\mathbf{r})$ is that it is a smooth function on the length scale of the Fermi wavelength $\lambda_F = 2\pi/k_F$ [4]. However, in the following, we restrict to the *homogeneous scattering model*, where $n(\mathbf{r})$ is assumed to be independent of the location $\mathbf{r}$.

3. Pairing amplitude

In this section we calculate the pairing amplitude $|\Delta|$ in the case that the order parameter $\Delta(\hat{\mathbf{k}}, \mathbf{r})$ is independent of $\mathbf{r}$. The pairing amplitude can directly be measured by nuclear magnetic resonance [3,4]. We note that $|\Delta|$ is not simply related to the energy gap of the excitation spectrum, except in pure superfluid.

We study both A and B phases. In the B phase we can take $\Delta(\hat{\mathbf{k}}) = \Delta_B \hat{\mathbf{k}}$. In the A phase $\Delta(\hat{\mathbf{k}}) = \Delta_A \hat{\mathbf{d}}[(\hat{\mathbf{m}} + i\hat{\mathbf{n}}) \cdot \hat{\mathbf{k}}]$, where $\hat{\mathbf{m}} \perp \hat{\mathbf{n}}$. Here Δ_A and Δ_B are real.

We can verify afterwards that $\check{v} = 0$ and the impurity self-energy has the form

$$\check{\rho}(\hat{\mathbf{k}}, \varepsilon_m) = a(\varepsilon_m)\check{1} + ib(\varepsilon_m)\check{\tau}_3 + c(\varepsilon_m)\check{\Delta}(\hat{\mathbf{k}}). \quad (13)$$

The Eilenberger Eqs. (2) and (3) then allow the solution

$$\check{g} = \frac{-i\tilde{\varepsilon}\check{\tau}_3 + (1+c)\check{\Delta}}{\sqrt{\tilde{\varepsilon}^2 + (1+c)^2|\Delta|^2}}, \quad (14)$$

where $\tilde{\varepsilon} = \varepsilon_m - b$.

3.1. B phase

For the B phase the T-matrix Eq. (9) can be solved analytically for general scattering phase shifts [12]. The calculation is similar to the one presented in Refs. [14,15]. Requiring consistency with $\breve{\rho}$ (13), we find that $\tilde{\varepsilon}$ and c have to satisfy the following equations:

$$\tilde{\varepsilon} - \varepsilon_m = -\frac{n}{4\pi} \sum_{j=1/2}^{\infty} \sum_{s=-1/2}^{1/2} (j + \tfrac{1}{2})$$

$$\times \text{Im}\left[\frac{v_{j+s}(1 - \mathrm{i}\zeta\tilde{\varepsilon}v_{j-s})}{d_{js}}\right], \quad (15)$$

$$c = \frac{n}{4\pi}(1 + c)\zeta \sum_{j=1/2}^{\infty} \sum_{s=-1/2}^{1/2} (j + \tfrac{1}{2})\frac{v_{j+1/2}v_{j-1/2}}{d_{js}}. \quad (16)$$

Here j and s take half-integer values ($j = \frac{1}{2}, \frac{3}{2}, \ldots$; $s = \pm\frac{1}{2}$),

$$d_{js} = (1 + \mathrm{i}\zeta\tilde{\varepsilon}v_{j+s})(1 - \mathrm{i}\zeta\tilde{\varepsilon}v_{j-s})$$

$$+ [\zeta\Delta_{\mathrm{B}}(1 + c)]^2 v_{j+1/2}v_{j-1/2} \quad (17)$$

and $\zeta = N_{\mathrm{F}}/4\sqrt{\tilde{\varepsilon}^2 + (1 + c)^2\Delta_{\mathrm{B}}^2}$. The coefficients $\tilde{\varepsilon}(\varepsilon_m) = -\tilde{\varepsilon}(-\varepsilon_m)$ and $c(\varepsilon_m) = c(-\varepsilon_m)$ are real. The program is to solve Eqs. (15) and (16) together with the self consistency equation

$$\ln\frac{T}{T_{\mathrm{c}0}} + 2\pi k_{\mathrm{B}}T \sum_{\varepsilon_m > 0}\left[\frac{1}{\varepsilon_m} - \frac{1 + c}{\sqrt{\tilde{\varepsilon}^2 + (1 + c)^2\Delta_{\mathrm{B}}^2}}\right] = 0, \quad (18)$$

which follows from Eqs. (5) and (14). We note that limiting to s-waves scattering implies $c = 0$ and (15) reduces to

$$\tilde{\varepsilon} - \varepsilon_m = \frac{\hbar v_{\mathrm{F}}}{2l_{\mathrm{tr}}}\frac{\tilde{\varepsilon}\sqrt{\tilde{\varepsilon}^2 + \Delta_{\mathrm{B}}^2}}{\tilde{\varepsilon}^2 + \Delta_{\mathrm{B}}^2\cos^2\delta_0}. \quad (19)$$

The pairing amplitude is nonzero only at temperatures below the superfluid transition temperature T_{c}. This temperature is determined by the condition

$$\ln\frac{T_{\mathrm{c}}}{T_{\mathrm{c}0}} + 2\pi k_{\mathrm{B}}T \sum_{\varepsilon_m > 0}\left(\frac{1}{\varepsilon_m} - \frac{1}{\varepsilon_m + \hbar v_{\mathrm{F}}/2l_{\mathrm{tr}}}\right) = 0 \quad (20)$$

This relation is the same for A and B phases, and it depends on the phase shifts δ_l only via $l_{\mathrm{tr}} = (n\sigma_{\mathrm{tr}})^{-1}$ (11).

We first present results for pure s-wave scattering. Fig. 1 shows the pairing amplitude in the B phase for three values of $\sin^2\delta_0$: 0, 0.5, and 1. The plots of Δ_{B}^2 show a linear dependence on T near T_{c}. In order to show better the scaling of Δ_{B} with T_{c}, we have presented the results in a different way in Fig. 2. There we plot the suppression factor

$$S_{\Delta_{\mathrm{B}}^2} = \frac{\Delta_{\mathrm{B}}^2(tT_{\mathrm{c}})}{\Delta_{\mathrm{B}0}^2(tT_{\mathrm{c}0})}, \quad (21)$$

where $\Delta_{\mathrm{B}0}^2$ is the squared pairing amplitude in the pure case (uppermost curve in Fig. 1). The suppression factor is plotted as a function of squared T_{c} suppression ($T_{\mathrm{c}}^2/T_{\mathrm{c}0}^2$) at different relative temperatures t. We see that Δ_{B} and T_{c} are suppressed nearly equally.

Let us consider the effect of higher partial waves ($l \geqslant 1$). First, we note that the results for the s-wave Born limit are valid also in the general Born limit. In other words, if all the phase shifts are small ($\delta_l \ll 1$), Δ_{B} is the same as plotted in Figs. 1 and 2 for $\sin^2\delta_0 \to 0$. We have also considered the case that
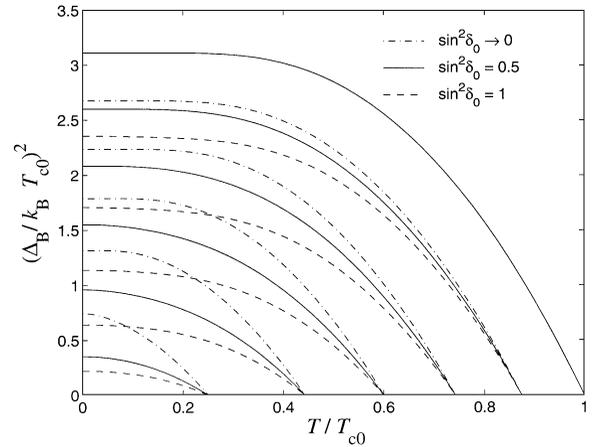

Fig. 1. The B-phase pairing amplitude Δ_{B}^2 vs. temperature. The six bunches of curves correspond to $\xi_0/l_{\mathrm{tr}} = 0.00, 0.05, 0.10, 0.15, 0.20$, and 0.25 in the order of decreasing T_{c}. The three types of lines correspond to three different values of the s-wave scattering phase shift δ_0.
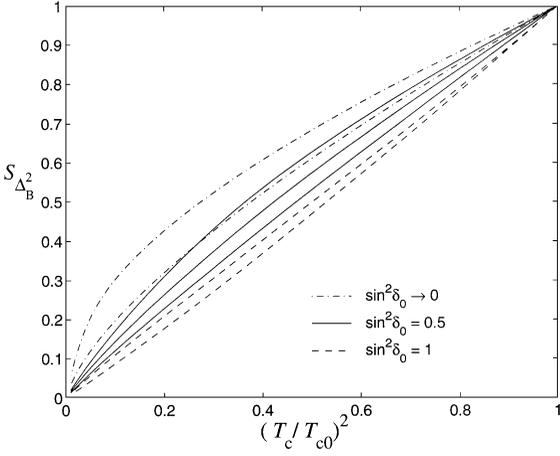
Fig. 2. The suppression factor (21) for Δ_B^2. The upper lines correspond to $t = 0.1$ and the lower ones to $t = 1.0$. The middle solid line is for $t = 0.5$.

several partial waves are in the unitarity limit. Numerical calculations show that in this case Δ_B is rather near the s-wave unitarity limit. Thus it seems that the s-wave Born and unitarity limits represent the general upper and lower limits of Δ_B for a given l_{tr}.

We have calculated Δ_B for hard spheres. For large spheres ($k_F R \gg 1$) the results seem to be independent of the radius. Furthermore, they are in a range corresponding to $\sin^2 \delta_0$ between 0.5 and 0.7 in the s-wave approximation. (The precise correspondence depends on ξ_0/l_{tr} and t.) The fact that the result corresponds to $\sin^2 \delta_0 \approx \frac{1}{2}$ might be understood as follows. The phase shifts δ_l for hard spheres are rather uniformly distributed for $l < k_F R$, and they vanish rapidly for larger l. Assuming that the partial waves with $l < k_F R$ contribute independently from each other [which can be only approximately valid because Eqs. (15) and (16) couple δ_l with $l = j \pm \frac{1}{2}$], they could produce an average that is simulated by random δ_0: $\langle \sin^2 \delta_0 \rangle = \frac{1}{2}$.

The aerogel strands are thick in comparison to k_F^{-1}. Because of the randomness of the aerogel structure, we expect that the scattering phase shifts are nearly randomly distributed. Therefore we expect that the results using $\sin^2 \delta_0 \approx 0.5$ would be the most relevant for comparison between theory

and experiment. We note that calculating Δ^2 for $\sin^2 \delta_0 = \frac{1}{2}$ differs insignificantly from the more complicated procedure of finding the average of Δ^2 for a uniform distribution of δ_0.

3.2. A phase

In the A phase we limit to s-wave scattering. It follows that $c = 0$ and the equation for $\tilde{\varepsilon}$ is

$$\tilde{\varepsilon} - \varepsilon_m = \frac{\hbar v_F}{2l_{tr}} \frac{\tilde{\varepsilon} \langle \Omega_A^{-1} \rangle}{\cos^2 \delta_0 + \tilde{\varepsilon}^2 \langle \Omega_A^{-1} \rangle^2 \sin^2 \delta_0}, \quad (22)$$

where $\Omega_A = \sqrt{\tilde{\varepsilon}^2 + \Delta_A^2 \sin^2 \theta}$ and we use the notation $\langle f(\theta) \rangle = \int_0^1 d(\cos \theta) f(\theta)$. The selfconsistency equation (5) reduces to

$$\ln \frac{T}{T_{c0}} + 2\pi k_B T \sum_{\varepsilon_m > 0} \left(\frac{1}{\varepsilon_m} - \frac{3}{2} \left\langle \frac{\sin^2 \theta}{\Omega_A} \right\rangle \right) = 0, \quad (23)$$

which has to be solved simultaneously with Eq. (22). The different angular integrals in the A phase can be calculated analytically

$$\langle \Omega_A^{-1} \rangle = \Delta_A^{-1} \arctan(\Delta_A/\tilde{\varepsilon}), \quad (24)$$

$$\langle \Omega_A^{-3} \rangle = [\tilde{\varepsilon}(\tilde{\varepsilon}^2 + \Delta_A^2)]^{-1} \quad (25)$$

$$\left\langle \frac{\sin^2 \theta}{\Omega_A} \right\rangle = \frac{\tilde{\varepsilon}}{2\Delta_A^2} - \frac{\tilde{\varepsilon}^2 - \Delta_A^2}{2\Delta_A^3} \arctan \frac{\Delta_A}{\tilde{\varepsilon}}, \quad (26)$$

$$\left\langle \frac{\sin^2 \theta}{\Omega_A^3} \right\rangle = \frac{1}{\Delta_A^3} \arctan \frac{\Delta_A}{\tilde{\varepsilon}} - \frac{\tilde{\varepsilon}}{\Delta_A^2(\tilde{\varepsilon}^2 + \Delta_A^2)}, \quad (27)$$

$$\left\langle \frac{\sin^4 \theta}{\Omega_A^3} \right\rangle = \frac{\tilde{\varepsilon}(3\tilde{\varepsilon}^2 + \Delta_A^2)}{2\Delta_A^4(\tilde{\varepsilon}^2 + \Delta_A^2)} - \frac{3\tilde{\varepsilon}^2 - \Delta_A^2}{2\Delta_A^5} \arctan \frac{\Delta_A}{\tilde{\varepsilon}}. \quad (28)$$

The behaviour of Δ_A^2 is very similar to the B phase. Therefore, we only plot the suppression factor in Fig. 3. The variation between the Born and unitarity limits is slightly smaller than in the B phase.

4. Superfluid density

In order to calculate the superfluid density ρ_s we study the order parameter $\Delta(\hat{k}, r) = \exp(i q \cdot r) \Delta(\hat{k})$. For the propagator we make a gradient expansion $\breve{g} = \breve{g}_0 + \breve{g}_1 + \dots$. Here $\breve{g}_0(\hat{k}, r, \varepsilon_m)$ is the solution calculated in the previous section corresponding to
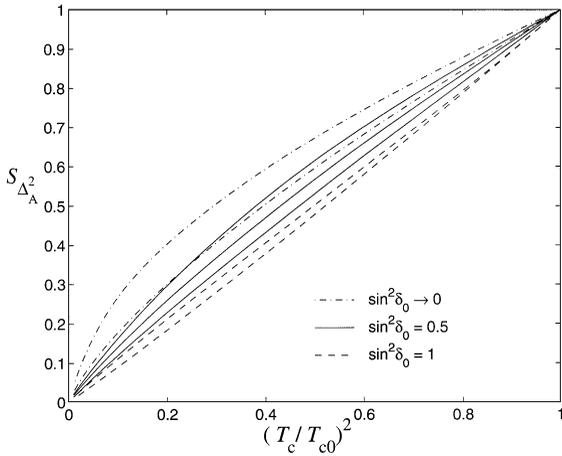
Fig. 3. The suppression factor $S = \Delta_A^2(tT_c)/\Delta_{A0}^2(tT_{c0})$ for the pairing amplitude in the A-phase. The upper lines correspond to $t = 0.1$ and the lower ones to $t = 1.0$. The middle solid line is for $t = 0.5$.

the local order parameter at $\mathbf{r}$. The correction $\breve{g}_1(\hat{\mathbf{k}}, \mathbf{r}, \varepsilon_m)$ is linear with respect to $\nabla_r \breve{g}_0$. We consider only s-wave scattering. It can be verified afterwards that $\breve{\Delta}_1 = 0$ and $\breve{v}_1 = s\hbar(\hat{\mathbf{k}} \cdot \mathbf{q})\breve{\tau}_3$, where s is a scalar. This allows to solve the linearized equations (2) and (3). We get

$$\breve{g}_1 = \frac{1}{2\sqrt{\tilde{\varepsilon}^2 + |\Delta|^2}}\left[\frac{(v_F + 2s)\hbar(\hat{\mathbf{k}} \cdot \mathbf{q})}{\sqrt{\tilde{\varepsilon}^2 + |\Delta|^2}}\breve{g}_0\breve{\tau}_3\breve{\Delta}\right.$$

$$\left. + \breve{g}_0\breve{p}_1\breve{g}_0 + \breve{p}_1\right]. \tag{29}$$

Linearizing the T-matrix equation (9) gives

$$\breve{p}_1 = \frac{\pi N_F}{n}\breve{p}_0\langle\breve{g}_1\rangle\breve{p}_0. \tag{30}$$

In both A and B phases these can be solved by making the ansatz $\breve{p}_1(\mathbf{r}, \varepsilon_m) = \alpha(\varepsilon_m)\breve{\Delta}(\mathbf{q}, \mathbf{r})$, where α is scalar. The parameter s can be solved from Eq. (7).

4.1. B-phase

In the B-phase our assumption about $\breve{v}_1$ is exact. The superfluid density can be written as

$$\rho_s = \frac{\rho_s^b}{1 + \frac{1}{3}F_1^s(1 - \rho_s^b/\rho)} \tag{31}$$

where the *bare* superfluid density ρ_s^b is given by

$$\rho_s^b = 2\pi\rho k_B T \sum_{\varepsilon_m > 0} \frac{\Delta_B^2(1 + \chi)}{(\tilde{\varepsilon}^2 + \Delta_B^2)^{3/2}}. \tag{32}$$

Here χ is a dimensionless parameter which vanishes in the pure limit

$$\chi = \frac{\hbar v_F\tilde{\varepsilon}^2}{6l_{tr}\sqrt{\tilde{\varepsilon}^2 + \Delta^B}}[\tilde{\varepsilon}^2 + \Delta_B^2\cos^2\delta_0] - \hbar v_F(3\tilde{\varepsilon}^2 + 2\Delta_B^2). \tag{33}$$

The same result has been obtained by Higashitani [8].

We show here results only for the bare superfluid density because it is independent of the Fermi-liquid parameters. It is plotted in Fig. 4 as a function of temperature. The suppression factor $S = \rho_s^b(tT_c)/\rho_{s0}^b(tT_{c0})$ is shown in Fig. 5.

4.2. A-phase

In the A-phase the superfluid density depends on an infinite number of Fermi-liquid parameters F_l^s with odd l [16]. In order to avoid too much complication, we cut off all F_l^s with $l > 2$. An additional complication is that there exist two
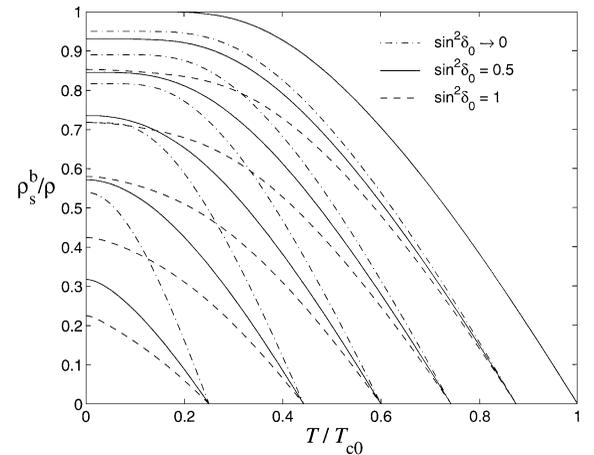

Fig. 4. The bare superfluid density in the B-phase. The curves correspond to $\xi_0/l_{tr} = 0.00, 0.05, 0.10, 0.15, 0.20$, and 0.25 in the order of decreasing T_c.

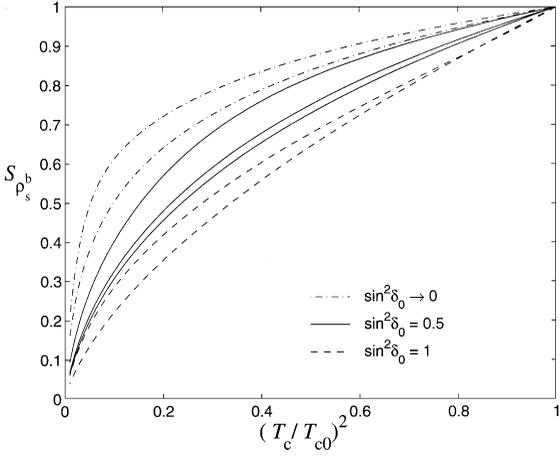Fig. 5. The suppression factor for bare superfluid density in the B-phase. The upper lines correspond to $t = 0.1$ and the lower ones to $t = 1.0$. The middle solid line is for $t = 0.5$.
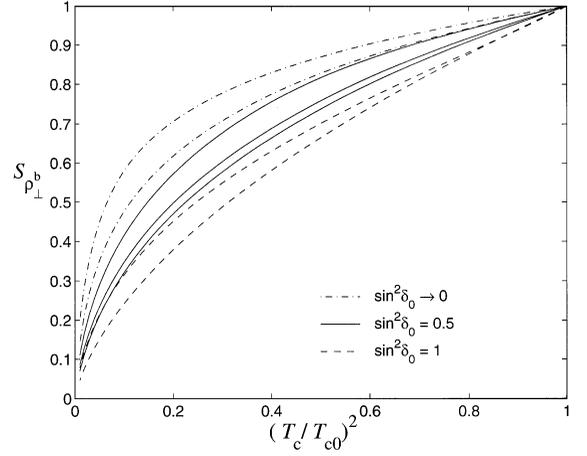

Fig. 6. The suppression factor $S = \rho_{\perp}^{b}(tT_c)/\rho_{\perp 0}^{b}(tT_{c0})$. The upper lines correspond to $t = 0.1$ and the lower ones to $t = 1.0$. The middle solid line is for $t = 0.5$.

eigenvalues for the superfluid density depending on the direction of the superfluid velocity relative to $\hat{l} = \hat{m} \times \hat{n}$. The eigenvalues parallel and perpendicular to $\hat{l}$ are given in terms of the corresponding bare superfluid densities as

$$\rho_{\parallel,\perp} = \frac{\rho_{\parallel,\perp}^{b}}{1 + \frac{1}{3}F_1^s(1 - \rho_{\parallel,\perp}^{b}/\rho)}. \tag{34}$$

The bare superfluid densities are

$$\rho_{\parallel}^{b} = 6\pi\rho k_B T \Delta_A^2 \sum_{\varepsilon_m > 0} \langle \Omega_A^{-3} \sin^2\theta \cos^2\theta \rangle, \tag{35}$$

$$\rho_{\perp}^{b} = 3\pi\rho k_B T \Delta_A^2 \sum_{\varepsilon_m > 0} \langle \Omega_A^{-3} \sin^2\theta (\sin^2\theta + \chi) \rangle, \tag{36}$$

where

$$\chi = \frac{\hbar v_F \tilde{\varepsilon}^2 \langle \Omega_A^{-3} \sin^2\theta \rangle}{4l_{tr}(\cos^2\delta_0 + \tilde{\varepsilon}^2 \langle \Omega_A^{-1} \rangle^2 \sin^2\delta_0) - \hbar v_F(\tilde{\varepsilon}^2 \langle \Omega_A^{-3} \rangle + \langle \Omega_A^{-1} \rangle)}. \tag{37}$$

The same result has been obtained by Higashitani [8].

The suppression factors for bare superfluid densities are shown in Figs. 6 and 7. The parallel superfluid density differs qualitatively from all other quantities studied here. This is probably associated with the singularity of Ω_A^{-3} in Eq. (35) in the limit $\sin\theta \to 0$, $\varepsilon_m \to 0$.

5. Conclusion

We have studied the homogeneous scattering model for superfluid ^{3}He. We find that the Ginzburg–Landau theory (extrapolation from $T = T_c$) yields reliable results above $\approx 0.7\,T_c$ for the pairing amplitude Δ and for the superfluid density ρ_s, similar to the case in pure ^{3}He. This confirms the conclusion reached earlier [3] that the homogeneous scattering model gives roughly by a factor of two larger values for Δ^2 and ρ_s than observed experimentally for superfluid ^{3}He in aerogel. In this connection we note that the comparison in the case of the A-phase should be made to the average superfluid density $\frac{1}{3}\rho_{\parallel} + \frac{2}{3}\rho_{\perp}$ because $\hat{l}$ most likely is oriented randomly by the anisotropy of the aerogel rather than uniformly by the flow in the torsional oscillator experiment.

In principle, the generalization beyond s-wave scattering is essential for ^{3}He in aerogel. However, we find that a realistic assumption about the scattering phase shifts gives essentially similar results as s-wave scattering with random phase shifts. We
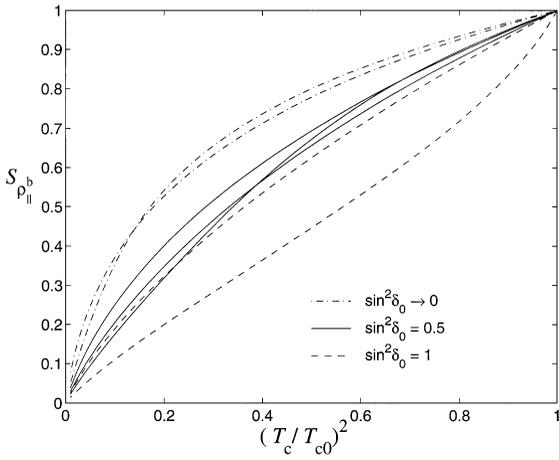
Fig. 7. The suppression factor $S = \rho_{\parallel}^{b}(tT_c)/\rho_{\parallel 0}^{b}(tT_{c0})$. At low T_c/T_{c0} the curves for each $\sin^2 \delta_0$ are just in the opposite order compared to other plots of the suppression factors: the lowest are for $t = 0.1$, the uppermost for $t = 0.1$, and the middle solid line is for $t = 0.5$. At higher T_c/T_{c0} some of the lines intersect.

believe this is a general result, but it should be remembered that we have proven it here only for the pairing amplitude of the B-phase.

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