Helsinki University of Technology
Department of Electrical Engineering

Report B 34

# Signatures of superfluidity in atomic Fermi gases

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# Abstract

After the experimental realization of Bose-Einstein condensation in dilute gases of alkali atoms, experimentalists started to trap the fermionic isotopes. The degenerate state for fermions was reported in 1999. The main objective of these experiments is to obtain superfluidity of fermionic gases. When there are attractive interactions between the fermions, the Fermi sea becomes unstable with respect to the formation of atomic Cooper pairs and the system becomes a superfluid. It turns out that the existing experimental cooling techniques allow minimum temperatures for fermions of the order of the Fermi temperature. Using Feshbach resonances induced by magnetic fields enhances the effective interactions between the atoms leading to superfluid transition temperatures of the order of the Fermi energy. This is a completely new regime of fermionic superfluidity far from the BCS superconductors, $^3$He and even high-$T_c$ superconductors. The achievement of superfluidity on gases of fermionic alkalis is currently being pursued in many experimental groups.

In this thesis, different signatures of the superfluid transition have been considered. The use of almost on-resonant laser light for coupling between the different internal states of the atoms as a method for probing superfluidity has been analyzed. Coupling between the paired states has been proposed as a way to directly detect the Cooper pair size. The Josephson effect, related to the phases of two coupled superfluids, is shown to present an asymmetry when the internal states of the atoms forming the pairs are coupled with different detunings. Vortices, intimately related to superfluidity, have also been considered. The single vortex solution of the Ginzburg-Landau equation for the superfluid order parameter has been numerically computed and a new vortex core size reflecting the trapping geometry has been obtained. Bloch oscillations have been analyzed for fermionic atoms both in the degenerate regime and in the superfluid regime. Superfluidity is found to supress the amplitude of the Bloch oscillations.

# Preface

The studies presented in this thesis are the result of the research work on degenerate Fermi gases carried out in the Laboratory of Computational Engineering at Helsinki University of Technology during the years 2000-2003 and the Department of Physics at University of Jyväskylä during the years 2001-2003.
I am most grateful to my supervisor professor Päivi Törmä from Jyväskylä University. She has been more than a supervisor to the work I present here and I am glad to acknowledge her significant contribution to the research we have been conducting during these years. Working with her has been very enlightening and I wish to heartfully thank her for everything she has taught me and for the friendly relationship we have had.
I want to thank all the people I have collaborated with during these years, also those involved in work not included in this thesis. Concerning the work presented here, I am especially indebted to Dr. Sorin Paraoanu from whom I learnt a lot during very interesting discussions.
I wish to thank all the friends I have met while studying physics, especially Anu Huttunen for the good and bad moments we have spent together.

Finally, I want to thank my parents for their continuos support and understanding and, of course, many thanks to Esa.

Espoo, October 19, 2003
*Mirta Rodríguez Pinilla*

# List of publications

This Dissertation is a review of the author's work in the field of degenerate Fermi gases. It consists of an overview and the following publications:

I. G. M. Bruun, P. Törmä, M. Rodríguez and P. Zoller, *'Laser probing of Cooper-paired trapped atoms'*, Physical Review A **64**, 033609 (2001).

II. M. Rodríguez and P. Törmä, *'Laser-induced collective excitations in a two-component Fermi gas'*, Physical Review A **66**, 033601 (2002).

III. Gh.-S. Paraoanu, M. Rodríguez and P. Törmä, *'Cooper-pair coherence in a superfluid Fermi-gas of atoms'*, Journal of Physics B **34**, 4763 (2001).

IV. Gh.-S. Paraoanu, M. Rodríguez and P. Törmä, *'Josephson effect in superfluid atomic Fermi gases'*, Physical Review A **66**, 041603(R) (2002).

V. M. Rodríguez, G.-S. Paraoanu and P. Törmä, *'Vortices in trapped superfluid Fermi gases'*, Physical Review Letters **87**, 100402 (2001).

VI. M. Rodríguez and P. Törmä, *'Bloch oscillations in Fermi gases'*, LANL cond-mat/0303634 (submitted).

Throughout the overview these publications are referred to by their Roman numerals.

# Author's contribution

The author has played a central role in all aspects of the research work. She has performed the analytical calculations and developed the theoretical methods in papers II, V and VI. She has implemented the numerical simulations in papers II, III, IV, V and VI. She has contributed to the analytical calculations in papers I, III and IV and to the numerical simulations in paper I. The author has initiated the work that lead to papers II and V. The author has written the papers II, V and VI and participated in writing the papers I, III and IV.

Other publications to which the author has contributed:

- J. Ortigoso, M. Rodriguez, M. Gupta and D. Friedrich, *' Time evolution of pendular states created by the interaction of molecular polarizability with a pulsed nonresonant laser field'*, J. Chemical Physics **110**, 3870 (1999).
- M. Rodriguez, K.A. Suominen and B. M. Garraway, *'Tailoring of vibrational state populations with light-induced potentials in molecules'*, Phys. Rev. A **62**, 053413 (2000).
- M. Rodríguez, P. Pedri, P. Törmä and L. Santos, *'Scissors modes of two-component degenerate gases : Bose-Bose and Bose-Fermi mixtures'*, LANL cond-mat/0310498 (submitted).

# Contents

# Chapter 1

# Introduction

Superconductivity was discovered in 1911 by H. Kamerlingh Onnes [1] just three years after he had first liquified $^4$He. He was interested in studying the electrical properties of pure metals in this new range of temperatures. He observed that the resistance of a thread of mercury dropped almost to zero at $T \approx 4.2$ K and he named this new state a superconductor. Although at that time there were hints that something was peculiar about liquid helium near 2 K it was not until 1938 that superfluidity was discovered and named.

In 1924 S. Bose derived the Planck distribution for photons using a new type of statistics [2]. Einstein generalized the statistics to ideal monoatomic gases assuming that if the particles were waves, they would have the same statistics as the photons [3]. This work led to the prediction that below a certain critical temperature, there would be a macroscopic occupation of the ground state of the system. This discovery was made at a time when not much was known about the distinction of bosons and fermions. The Fermi statistics was discovered in 1926 [4], and the relation between spin and statistics elucidated later. These theories on bosons and fermions were considered as a mere mathematical artefact until in late 30's Fritz London [5] connected the macroscopic occupation of the ground state in the Bose-Einstein statistics with the superfluid properties of $^4$He. He had also introduced a new theory of superconductivity based on the idea of a "macroscopic wave function". The ideas of London could not develop until many body theory was developed. During the 40's Landau and Bogoliubov developed their theories and in the 50's Feymann developed the diagram techniques.

It was 1957 when J. Bardeen, N. Cooper and J. R. Schrieffer developed the BCS theory (named after the authors' names) [6] explaining the physics of superconductors. One year later Bogoliubov extended his method of understanding superfluidity to superconductivity [7].

BEC (Bose Einstein condensation) is closely related to the BCS mechanism of superconductivity which can be viewed as a simultaneous formation and condensation of pairs of fermions. Ginzburg and Landau introduced a time and space dependent macroscopic wave function for studying spatially inhomogeneous superconductors [8]. During the 1950's and 1960's a remarkably complete and satisfactory theoretical picture of the classic superconductors emerged. This situation was overturned in 1986, when a new class of high-temperature superconductors was discovered by Bednorz and Müller [9]. These new superconductors seem to obey the same general phenomenology as classic superconductors but the basic microscopic mechanism remains an open question. Superfluidity in $^3$He was not discovered until 1972 by D. Osheroff, R. Richardson and D. Lee because it requires even lower temperatures, the transition temperature being $T_c = 2.6$ mK.

In helium the interaction length is of the order of the distance between atoms and that makes it very difficult to study it theoretically. In the 1960's began an extensive search for BEC in a dilute, weakly interacting bosonic gas in which the phenomenon would not be smeared by strong correlation effects. After pioneering attempts with H atoms that did not turn into BEC until very recently [10], the improvement in the laser cooling techniques made alkali atoms better candidates for condensation. In 1995, both $^{87}$Rb and $^{23}$Na were succesfully driven into the Bose Einstein condensation transition at JILA and MIT [11]. Experiments with bosonic atoms have gone beyond the achievement of Bose Einstein condensation generating a quantum systems revolution ranging from the development of atom lasers [12] to the study of quantum phase transitions as the Mott-Insulator transition [13].

For $^4$He and alkali bosonic atoms the fermionic neutrons, protons and electrons bind to form a composite boson at temperatures much higher than the superfluid transition temperature. There are other physical systems in which the binding energy of the fermions to form composite bosons and the condensation of bosons into a macroscopically occupied state occurs simultaneously like in the BCS mechanism. Prior to the observation of BEC in dilute atomic gases, the laboratory system which most closely realized the original idea of Einstein was *excitons* in cuprous oxide [14]. Actually excitons are bound pairs of electrons and holes formed by pulsed laser excitation.

Fermionic alkali atoms have been predicted [15] to undergo a superfluid transition analogous to the BCS transition in superconductors if atoms are trapped in different hyperfine states with attractive interactions between them. Due to the attraction between the atoms the Fermi sea becomes unstable with respect to the formation of atomic Cooper pairs with zero momentum if the attraction is weak. If the attraction is very strong, the fermions form composite bosons (pairs in real space not in momentum space) that Bose condense at low enough temperatures.

It was 1999 when fermionic atoms of $^{40}$K were cooled below the Fermi temperature at JILA [16] and it begun an experimental outbreak comparable to the one provoked by the BEC of bosonic atoms, now degenerate Fermi gases of both $^{40}$K [17, 18] and $^{6}$Li [19, 20, 21, 22, 23] being realized in several experimental groups. The ultimate goal of these experiments is to obtain superfluidity for fermions. It turns out that the existing cooling techniques are not sufficient to obtain the superfluid transition temperature for the natural scattering lengths in the system. The controllability of these gases allows to manipulate the interaction strength between the different states and one can tune the interactions using laser fields [24] and magnetic fields [25] taking advantage of resonances that occur when the collision energy of two free atoms coincides with that of a quasi-bound molecular state. Because it is possible to tune the interaction strength, it will be experimentally possible to study the BCS-BEC crossover in fermionic superfluidity. Note that for $^{3}$He, $T_c/T_F \sim 10^{-3}$ while for high-$T_c$ superconductors that ratio is at most $10^{-2}$. With Feshbach resonances, the maximum $T_c$ considered are of $T_c \approx 0.2T_F$ and one is in a completely new regime of superfluidity.

The creation and trapping of molecules out of a degenerate gas of atomic fermions trapped in two different hyperfine states using Feshbach resonances has been reported [17, 19, 22, 23] very recently. The molecules remain trapped in the optical trap and the Bose Einstein condensation of these molecules is being pursued. This molecular BEC would be the first step in obtaining a strongly interacting fermionic superfluid in the BEC part (or effective repulsive interactions) of the crossover. By adiabatically sweeping the magnetic field along the resonance, it might be possible to obtain a weakly interacting superfluid Fermi gas in the BCS regime [26].

The superfluidity in atomic Fermi gases will reproduce and help to understand the physics of other systems. Many properties of the *high temperature superconductors* may be understood in the Bose Einstein condensation of pairs of holes in states with $d$ symmetry. The Bose Einstein condensation of pairs of fermions is observed experimentally in *atomic nuclei* where the effect of pairing may be seen in the excitation spectrum and in the reduced momentum of inertia. The pair size in nuclear matter is large compared to the nuclear size. Pairing is expected to take place also in the interiors of *neutron stars* and observation of glitches in the rate of pulsars have been interpreted in terms of neutron superfluidity. Moreover the superfluid fermions present new physics on their own due to the controllability of the geometry, interactions and internal states of the atoms.

The trapping and cooling of atomic gases has opened a new field in fundamental physics. The cold atomic clouds behave as coherent matter and one might expect the same applications as for coherent light. Cold atomic gases have also been proposed as systems for developing and testing quantum information [27] and quantum communication [28].

The objective of this thesis is to study and analyze different signatures of superfluidity in dilute atomic Fermi gases. In chapter 2, the trapping and cooling techniques are reviewed as well as some basic scattering theory. The interaction of light with atoms is especially considered because it has been used in publications I to IV. In chapter 3 the theory describing the system is summarized both for the degenerate gas and for the superfluid state. We review the BCS algebra used to describe the system in the superfluid state also in the presence of a supefluid current and we show the solutions of the BCS equation both in the weak and strong coupling limits. These techniques have been used to describe the fermionic cloud in publications I to VI. For completeness and due to its experimental relevance we also compile the recent studies of superfluidity in the presence of a Feshbach resonance.

In the publications I to VI we have mainly concentrated on the superfluid state of the fermionic cloud. We have proposed different methods to probe the superfluid transition in the fermionic systems. In the first papers I to IV we have used an analogy to electron current between two superconductors and between a superconductor and a normal state. In the atomic case, the coupling is done with lasers or radio frequency fields. In these systems there are more degrees of freedom for coupling between the states and this leads to new results. In publication II we have analyzed the possibility of excitation of collective modes that would conceal the results in papers I, III and IV. In paper V we have studied vortices within the Ginzburg Landau approach of superfluidity. The difference to any other existing fermionic superfluids is the trapping geometry and we show in paper V that it leads to a different healing length. In publication VI we consider fermions in an optical lattice. By applying a constant force one could produce Bloch oscillations and we analyze them for the fermionic clouds both in the superfluid and in the normal case. We show how the amplitude of the oscillations is changed through the BCS-BEC crossover.

# Chapter 2

# Dilute atomic gases

If one looks at the equilibrium phase diagram of any atomic substance, the Bose condensed region is thermodynamically forbidden except at such high densities that the equilibrium configuration is crystalline. Thus, a dilute quantum gas at equilibrium is impossible but at sufficiently low densities a *metastable* quantum gas can be created. Solidification is achieved through three-body collisions whereas kinetic or thermal equilibrium is obtained via two-body collisions. If the densities are low enough, two-body collisions dominate and one can have a dilute quantum gas before it finds its solid state equilibrium configuration. The cold alkali atoms remain in the gas phase for a few minutes, time long enough to perform measurements.

Atomic alkali are robust systems easy to manipulate that provide an unique opportunity for exploring quantum phenomena on a macroscopic scale. Collective phenomena are expected to take place when the interparticle distance $n^{-1/3}$ is comparable to the thermal de Broglie wavelength $\lambda_T = (2\pi\hbar^2/mk_BT)^{1/2}$. Bosons condense in the lowest single-particle state while fermions tend towards a state with a filled Fermi sea.

Setting the interparticle distance $n^{-1/3}$ and the thermal deBroglie wavelength equal provides a rough estimate for the Bose Einstein transition temperature of a non-interacting Bose system to

$$T_c \approx \frac{2\pi\hbar^2 n^{2/3}}{mk_B}. \tag{2.1}$$

For alkali atoms the achieved densities range from $10^{13}$ cm$^{-3}$ to $10^{15}$ cm$^{-3}$ corresponding to transition temperatures from 100 nK to few $\mu$K. In trapped systems the density can be estimated like $n \sim N/R^3$ with the radius given by $R \approx (k_B T/m\omega_0^2)^{1/2}$. Inserting them in Eq. (2.1) sets the BEC transition temperature for a trapped ideal gas to $k_B T_c = C\hbar\omega_0 N^{1/3}$ where $\omega_0$ is the trapping frequency and $C = 3.3$ is obtained with the exact calculation [29]. Typical trap frequencies are of the order of $10^2$ Hz and the number of atoms $N$ range from $10^5$ to $10^8$. Typical cloud sizes are of the order of 1 $\mu$m.

For degenerate fermions the interparticle distance $k_F^{-1}$ is determined by the Pauli exclusion principle. For a three-dimensional symmetric trap, the degeneracy of the energy levels gives the Fermi temperature $k_B T_F = (6N)^{1/3}\hbar\omega_0$. Note that this is of the order of $T_c$ for BEC but in the fermionic case there is no phase transition. To have a fermionic superfluid transition one needs interactions.

The total spin of a boson particle must be an integer, whilst a fermion particle has half integer total spin. Neutral atoms are made up of fermions (neutrons, electrons and protons) and contain equal number of protons and electrons. Therefore the statistics of an atom is determined only by the number of neutrons. If the number of neutrons is even the atom is a boson, and if it is odd, the atom is a fermion. Alkalis have odd number of protons, isotopes with odd number of neutrons are less abundant than those with even, since they have both an unpaired neutron and an unpaired proton which increases the energy.

Alkali atoms have only one electron out of the closed shells that is in an orbital $s$ (angular momentum $L = 0$). The total electronic spin $\mathbf{J} = \mathbf{S} + \mathbf{L}$ is $J = \frac{1}{2}$. The nuclear spin is $I$ and the total spin of the atoms is $\mathbf{F} = \mathbf{I} + \mathbf{J}$ that for alkalis gives $F = I \pm \frac{1}{2}$. The electron and nuclear spin are coupled by the hyperfine interaction that splits the atomic levels in the absence of a magnetic field $H_{hf} = A_{hf}\mathbf{I} \cdot \mathbf{J}$. The hyperfine splitting between the levels $F = I + \frac{1}{2}$ and $F = I - \frac{1}{2}$ is $\Delta E_{hf} = (I + 1/2)A_{hf} \sim 10^3$ MHz, where $A_{hf} \propto \mu$ . When an external magnetic field is applied, the energy levels are even further splitted and for small magnetic fields the atomic states can be labeled by the quantum numbers $|F, M_F\rangle$. The two fermionic isotopes currently being trapped are $^6$Li and $^{40}$K that have $I = 1$ and $I = 4$ respectively. $^6$Li has positive magnetic moment $\mu$ and the lowest energy state has $F = 1/2$. For $^{40}$K $A_{hf} < 0$ and the lowest energy corresponds to total spin $F = 9/2$.

## 2.1 Atom-light interaction

In this section we consider the influence of light on an atom. Light is used to trap and cool the atomic cloud. This is reviewed in next section. Furthermore, electric fields can be used to manipulate the trapped atoms and we have proposed the coupling of laser light between different hyperfine states to observe different phenomena in publications I-IV.

Absorption and emission of photons by an atom irradiated by a laser beam can be analyzed for times scales longer than the spontaneous emission time in terms of radiative forces. For an atom moving slowly or at rest the radiative force can be split in two parts, the *radiation pressure* related to real atomic transitions and the *dipole force* associated with virtual transitions between the atomic states that in the dressed atom picture can be regarded as an effective potential in which the atom moves. We will now derive the latter using the dipole and rotating-wave approximations for a two-level atom.

The interaction between an atom and the electric field is given in the dipole approximation (valid when the size of the atomic wave packet is small compared to the laser wavelength) by

$$H_{\text{light}} = -\mathbf{d} \cdot \mathbf{E}, \tag{2.2}$$

where $\mathbf{d}$ is the atomic dipole moment and $\mathbf{E}$ the electric field.

Let us consider a two-level atom with the internal ground state $|g\rangle$ and the excited state $|e\rangle$ coupled by a monochromatic classical laser field of frequency $\omega$, $\mathbf{E}(\mathbf{r}) = E(\mathbf{r})\cos(\omega t + \varphi)\hat{\mathbf{e}}$ . From an appropriate reference energy, the Hamiltonian of the two level system reads

$$\hat{H} = \frac{\hbar\omega_{eg}}{2}(\hat{c}_e^\dagger \hat{c}_e - \hat{c}_g^\dagger \hat{c}_g) - \hbar \mathbf{d}_{eg} \cdot \hat{\mathbf{e}} E(\mathbf{r}) \cos{(\omega t + \varphi)} (\hat{c}_e^\dagger \hat{c}_g + \hat{c}_g^\dagger \hat{c}_e), \tag{2.3}$$

where $\hat{c}_e$, $\hat{c}_e^\dagger$ are the destruction and creation operators of atoms in the state $|e\rangle$. The evolution of $|\Psi(t)\rangle$ is given by the Schrödinger equation

$$i\hbar \frac{d}{dt}|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle. \tag{2.4}$$

One can change into the interaction picture and use $|\tilde{\Psi}(t)\rangle = e^{\frac{i\omega}{2}(\hat{c}_e^\dagger \hat{c}_e - \hat{c}_g^\dagger \hat{c}_g)t}|\Psi(t)\rangle$. Inserting it into Eq.(2.4) one obtains

$$i\hbar \frac{d}{dt}|\tilde{\Psi}(t)\rangle = \tilde{H}|\tilde{\Psi}(t)\rangle - \frac{\hbar\omega}{2}(\hat{c}_e^\dagger \hat{c}_e - \hat{c}_g^\dagger \hat{c}_g)|\tilde{\Psi}(t)\rangle \tag{2.5}$$

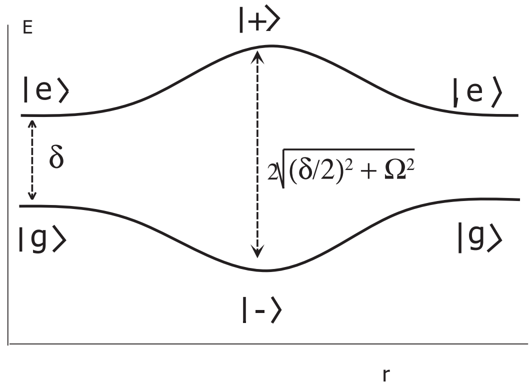


Figure 2.1: Energy level scheme for the dressed-states in the RWA. At $\Omega = 0$, the excited and ground states are separated by energy $\delta$. For $\Omega \neq 0$, the splitting between the two states is given by $2\sqrt{\Omega^2 + (\delta/2)^2}$. As explained in the text, $\delta < 0$ implies $|-\rangle \approx |g\rangle$ at $\Omega \sim 0$ while $|-\rangle \approx |e\rangle$ for $\delta > 0$.

where $\tilde{H} = e^{\frac{i\omega}{2}(\hat{c}_e^\dagger \hat{c}_e - \hat{c}_g^\dagger \hat{c}_g)t} \hat{H} e^{-\frac{i\omega}{2}(\hat{c}_e^\dagger \hat{c}_e - \hat{c}_g^\dagger \hat{c}_g)t}$. Using the quantum commutation rules one obtains the following Hamiltonian

$$\tilde{H} = \frac{\hbar\omega_{eg}}{2}(\hat{c}_e^\dagger \hat{c}_e - \hat{c}_g^\dagger \hat{c}_g) - \frac{\hbar \mathbf{d}_{eg} \cdot \hat{\mathbf{e}} E(\mathbf{r}) e^{i\varphi}}{2}[(e^{2i\omega t} + 1)\hat{c}_e^\dagger \hat{c}_g + (e^{-2i\omega t} + 1)\hat{c}_g^\dagger \hat{c}_e] \quad (2.6)$$

One should notice that the optical frequencies ($\sim 10^{15} Hz$) available with lasers are much bigger than any other energies in the system. The hyperfine splitting is of the order $10^9 Hz$. Trapping frequencies are of the order of $10^2 Hz$ and this gives Fermi energies of $10^5 Hz$. Collective excitations in the harmonic trap are of the order of the trapping energy. Therefore, when using optical light one can use the *rotating wave approximation* (RWA) that consists on time averaging the above Hamiltonian.

The effective Hamiltonian for the state $|\tilde{\Psi}(t)\rangle$ is thus

$$\tilde{H} - \frac{\hbar\omega}{2}(\hat{c}_e^\dagger \hat{c}_e - \hat{c}_g^\dagger \hat{c}_g) = \hbar\Omega(\mathbf{r})\hat{c}_e^\dagger \hat{c}_g + \hbar\Omega^\dagger(\mathbf{r})\hat{c}_g^\dagger \hat{c}_e + \frac{\hbar\delta}{2}(\hat{c}_e^\dagger \hat{c}_e - \hat{c}_g^\dagger \hat{c}_g), \quad (2.7)$$

where the laser detuning $\delta = \omega_{eg} - \omega$ and the Rabi frequency $\Omega(\mathbf{r}) = -\langle e|\mathbf{d} \cdot \hat{\mathbf{e}} E(\mathbf{r}) e^{i\varphi(\mathbf{r})}|g\rangle/2$ were introduced. Thus, in the rotating wave approximation the interaction of the laser light with the matter fields can be described by a time independent Hamiltonian in which the detuning $\delta$ plays the role of an externally imposed difference in the energies of the two states. One can diagonalize this two level matrix and obtain for the rotated states

$$\begin{aligned} |+\rangle &= e^{i\varphi(\mathbf{r})/2} \cos\theta(\mathbf{r})|e\rangle + e^{-i\varphi(\mathbf{r})/2} \sin\theta(\mathbf{r})|g\rangle \\ |-\rangle &= -e^{i\varphi(\mathbf{r})/2} \sin\theta(\mathbf{r})|e\rangle + e^{-i\varphi(\mathbf{r})/2} \cos\theta(\mathbf{r})|g\rangle, \end{aligned} \quad (2.8)$$

where the angle $\theta(\mathbf{r})$ is defined as $\cos 2\theta(\mathbf{r}) = \delta/(2\sqrt{(\frac{\delta}{2})^2 + |\Omega|^2})$ and $\sin 2\theta(\mathbf{r}) = |\Omega|/(\sqrt{(\frac{\delta}{2})^2 + |\Omega|^2})$. The eigenenergies are $E_\pm = \pm\hbar\sqrt{(\frac{\delta}{2})^2 + |\Omega|^2}$. This is depicted in Fig 2.1. The states that diagonalize the atom+laser Hamiltonian are called "dressed states". One can derive them including the photon field operators in the formalism; see e.g. [30]. If the laser is *far detuned*, i.e. $|\delta| \gg |\Omega|$ one can see that the energies $E_\pm \approx \pm\frac{|\delta|}{2}(1 + \frac{2\Omega^2}{\delta^2})$. For blue detuned light $\delta > 0$, $|+\rangle \approx |g\rangle$ and $|-\rangle \approx |e\rangle$ and when the light is red detuned $\delta < 0$, $|+\rangle \approx |e\rangle$ and $|-\rangle \approx |g\rangle$. An atom initially in the state $|g\rangle$ and following the field interaction adiabatically remains in the ground state and feels the optical potential

$$V(\mathbf{r}) = \frac{\hbar\delta\Omega(\mathbf{r})^2}{\delta^2}. \tag{2.9}$$

Thus, when the laser is far detuned one can adiabatically eliminate the excited state. Note that the dipole potential is $\mathbf{r}$-dependent and a periodic configuration constitutes an optical lattice.

To include the spontaneous emission from the excited state $\Gamma_e$, the term $(i\Gamma_e/2)\hat{c}_e^\dagger\hat{c}_e$ has to be added to the initial Hamiltonian (2.3). One can then just replace $\delta$ by the complex quantity $\delta + i\Gamma_e/2$ in the optical potential Eq.(2.9) and obtain the complex potential $V_g - i\hbar\Gamma_g$ where

$$V_g = \frac{\hbar\Omega^2\delta}{\delta^2 + \Gamma_e^2/4} \tag{2.10}$$

$$\Gamma_g = \Omega^2\frac{\Gamma_e/2}{\delta^2 + (\Gamma_e/2)^2}. \tag{2.11}$$

The complex part $\Gamma_g$ is the finite lifetime of the ground state due to transitions to the excited state. It is is proportional to $1/\delta^2$. Far detuned potentials have negligible spontaneous emission from the ground state. Higher intensities imply also higher spontaneous emission rates and therefore a red detuned optical lattice where the potential minima coincides with the points of higher laser intensity has higher spontaneous emission than the blue detuned ones.

Usually in experiments with alkali atoms one wants to couple different atomic hyperfine levels. The typical frequency separation is of the order of GHz. Such frequencies can be provided by a time-dependent electric field (rf-field) or by using two lasers in a *Raman scheme* as showed in Fig. 2.2.

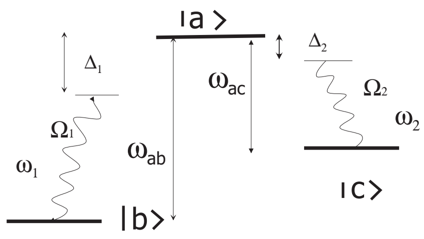


Figure 2.2: Raman laser scheme. The $|c\rangle$ and $|b\rangle$ states are coupled through an intermediate state $|a\rangle$ using two lasers of frequencies $\omega_1$ and $\omega_2$ and detunings $\Delta_1$ and $\Delta_2$.

The Hamiltonian for such a scheme reads

$$\hat{H} = \hbar\omega_{cb}\hat{c}_c^\dagger\hat{c}_c + \hbar\omega_{ab}\hat{c}_a^\dagger\hat{c}_a - [\hbar\mathbf{d}_{ba}\cdot\mathbf{E}_1\cos(\omega_1 t)\hat{c}_b^\dagger\hat{c}_a + h.c.] \\ -[\hbar\mathbf{d}_{ca}\cdot\mathbf{E}_2\cos(\omega_2 t)\hat{c}_c^\dagger\hat{c}_a + h.c.]. \quad (2.12)$$

We define slowly varying fields in the interaction picture by $\hat{s}_a = e^{i\omega_1 t}\hat{c}_a$, $\hat{s}_c = e^{i(\omega_1-\omega_2)t}\hat{c}_c$ and $\hat{s}_b = \hat{c}_b$. For the electric fields $\tilde{E}_1^+ = e^{i\omega_1 t}E_1^+$ and $\tilde{E}_2^+ = e^{i\omega_2 t}E_2^+$ where $+$ denotes the positive frequency component of the electric field. Inserting this in the Hamiltonian, and in the limit of large laser detuning one can adiabatically eliminate the intermediate state $|a\rangle$ like $\hat{s}_a = \frac{1}{\hbar\Delta_1}[d_{ab}\tilde{E}_1^+\hat{s}_b + d_{ac}\tilde{E}_2^+\hat{s}_c]$. Using this $\hat{s}_a$ the Hamiltonian becomes

$$-\tilde{H} = \hbar\delta_1\hat{s}_b^\dagger\hat{s}_b + \hbar\delta_2\hat{s}_c^\dagger\hat{s}_c + [\frac{\hbar\Omega_1^\dagger\Omega_2}{\Delta_1}\hat{s}_b^\dagger\hat{s}_c + h.c.] \quad (2.13)$$

where $\delta_1 = \frac{\Omega_1^2}{\Delta_1}$ and $\delta_2 = \Delta_1 - \Delta_2 + \frac{\Omega_2^2}{\Delta_1}$ are the effective detunings. This solution for the two level system is asymmetric but it can be made symmetric just by adding a phase to the solution.

## 2.2 Trapping and cooling

The trapping and cooling of atoms is done by the application of external fields. Ions can be trapped by electric and magnetic fields due to their charge. External fields interact with neutral atoms only through their permanent magnetic moments or induced electric dipole moments resulting in smaller forces and trap depths at most few Kelvins.

When an *external magnetic field* is applied, the energy of the atom changes due to the interaction of the magnetic moments of the electron and the nucleus with the magnetic field. This is called the Zeeman effect and for a magnetic field $B$ in the $z$ direction, the spin Hamiltonian is

$$H_{\text{spin}} = A_{hf}\mathbf{I}\cdot\mathbf{J} + g_L\mu_B B\cdot J_z - \frac{\mu}{I}B\cdot I_z, \tag{2.14}$$

where $g_L$ is the Lande factor for the electron ($g_L \approx 2$) and $\mu$ the magnetic momentum of the nucleus. The magnetic moment of the nucleus is a factor of $\mu_N = (m_e/m_p)\mu_B \ll \mu_B$ and the interaction with the nucleus can be neglected. The Hamiltonian

$$H_{\text{spin}} = A_{hf}\left[I_z J_z + \frac{1}{2}(I_+J_- + I_-J_+)\right] + g_L\mu_B B\cdot J_z \tag{2.15}$$

couples only states with the same value of the sum $m_I + m_J$. This reflects the invariance of interactions under rotations about the $z$ axis. For low magnetic fields it can be shown [29] that the energy may be written as $E(F, m_F) = E(F) - \mu(F, m_F)B$. If the magnetic moment of the state $\mu(F, m_F)$ is positive, the atom experiences a force that drives it to regions of high field and if the magnetic moment is negative the force is towards regions of lower field. States with positive $\mu$ are refered to as *high-field seekers* and those with a negative one as *low-field seekers*. The Earnshaw's theorem forbids a local maximum in the magnetic field if there are no electrical currents. Thus the case of interest is a local minimum in the magnetic field $B$ and the only states to be trapped magnetically are the low field seekers. The energy depth of magnetic traps is given by the Zeeman energy. Atomic magnetic moments are of the order of the Bohr magneton that in temperature units is approximately 0.67K/T giving trap depths of much less than a Kelvin for magnetic fields less than one Tesla.

The simplest *magnetic trap* is the quadrupole trap in which the magnetic field varies linearly with distance in all directions. Because $\nabla \cdot \mathbf{B} = 0$, $\mathbf{B} = B'(x, y, -2z)$. An atom is assumed to remain in the same quantum state and follow adiabatically the magnetic field variations. However, a moving atom experiences a time-dependent magnetic field that induces transitions to other hyperfine states. In particular low field seeking states may change into high field seeking states that will leave the trap. This effect becomes important if the frequency of the time dependent magnetic field is comparable or greater than the frequencies of transitions between magnetic sublevels

$(\mu B)$. When the magnetic field vanishes this effect is thus very important and the trap has a "hole". One way of removing the hole is applying an oscillating magnetic field $\mathbf{B} = (B'x + B_0 \cos\omega t, B'y + B_0 \sin\omega t, -2B'z)$ that moves the instantaneous position of the node in the magnetic field. This is known as the time-averaged orbiting potential (TOP) trap. The frequency of the bias field $\omega$ is taken such that $\omega < |\mu B|$ for the atom to remain in the same hyperfine state and such that the frequencies of atomic motion $\ll \omega$ to average on time. Close to the origin $\mathbf{r} = 0$ the atom feels the magnetic field $\langle B\rangle_t \approx B_0 + \frac{B'^2}{4B_0}(x^2 + y^2 + 8z^2)$ [31]. Another way of removing the hole in the quadrupole trap is using a blue detuned laser that expels the atoms from the $B = 0$. To avoid the disadvantage of the node in the origin, there are other magnetic configurations like the Ioffe-Pritchard traps that have a finite value at the minimum.

The influence of *light on an atom* results in two types of forces, the *radiation pressure* and the *dipole force*. The expression of the optical dipole potential was derived in Section 2.1, and the lifetime of the ground state $\Gamma_g$ was obtained when considering the influence of spontaneous emission.

The *dipole force* is the force exerted on an atom due to coherent redistribution of photons

$$F_{\text{dip}} = -\frac{dV_g(\mathbf{r})}{d\mathbf{r}} \tag{2.16}$$

where $V_g$ is the optical dipole potential Eq. (2.10). A red detuned $(\delta < 0)$ laser attracts atoms to the low intensity region and a blue detuned laser $\delta > 0$ expels atoms from the high intensity regions in space. The simplest *optical trap* is a single focused red detuned laser and the depth is given by (2.10). In optical traps one can trap simultaneously different hyperfine states but they have the difficulty of inducing heating due to optical excitation of the trapped atoms.

The *radiation force* is the force on an atom corresponding to absorption of a photon of momentum $\mathbf{q}$ followed by spontaneous emission

$$F_{\text{rad}}(\mathbf{v}) = \hbar\mathbf{q}\Gamma_g, \tag{2.17}$$

that depends on the velocity of the atom $\mathbf{v}$. The radiation pressure can be also used to confine atoms. This is done in the *magneto-optical trap (MOT)* that combines laser fields with spatially varying magnetic fields. The magnetic fields induce space dependent energy levels that induce a radiation pressure depending on the position.

The primary application of the radiation force is *Doppler cooling*. When an moving atom is subjected to a red detuned laser field the kinetic energy can provide the extra energy needed to achieve the transition to the excited state. Spontaneous decay to the ground state means that kinetic energy is lost during the process. Let us consider two counterpropagating plane waves acting on an atom moving with velocity $v_z$. The radiation forces are

$$F_{\pm}(v_z) = \pm\hbar q\Omega^2 \frac{\Gamma_e/2}{(-\omega \mp qv_z + \omega_{eg})^2 + (\Gamma_e/2)^2}. \quad (2.18)$$

Due to the Doppler effect, a moving atom "sees" the counterpropagating running wave more than the other. It experiences then a force opposed to its velocity and the atom loses energy. The total force exterted on the atom reads

$$F_{tot} = F_+ + F_- = -\alpha v_z = \frac{dp_z}{dt}, \quad (2.19)$$

$$\alpha = -4\hbar q^2\Omega^2 \frac{\Gamma_e/2}{\left[\delta^2 + (\Gamma_e/2)^2\right]^2}\delta \quad (2.20)$$

If the laser is red detuned, $\alpha$ is positive and the force damps the velocity at a rate $1/\tau_{\text{fric}} = -1/v_z dv_z/dt = \alpha/m$. The cooling rate is then $\frac{d}{dt}\langle p^2\rangle|_{\text{cool}} = 2\frac{\langle p_z^2\rangle}{\tau_{\text{fric}}}$. On the other hand there is diffusion heating due to the random nature of spontaneous emission. This fact limits the lowest temperatures obtainable with Doppler cooling. The radiation force Eq. (2.17) is the average force on an atom in the laser field. It arises from discrete photon scattering and must fluctuate around its average. These fluctuations result in heating of the atoms. Considering a random walk of the atomic momentum, the heating rate results in $\frac{d}{dt}\langle p^2\rangle|_{\text{heat}} = 4\Gamma_g(\hbar q)^2 \equiv D_p$ . An estimate of the lowest temperatures obtainable with Doppler cooling can be obtained by equating the heating and cooling rates that results in $\langle p^2\rangle = \frac{1}{2}D_p\tau_{\text{fric}}$. Using the equipartition theorem the lowest achievable temperature

$$k_B T = \hbar\frac{\delta^2 + (\Gamma_e/2)^2}{-2\delta} \quad (2.21)$$

is obtained. It has a minimum $k_B T_D = \frac{\hbar\Gamma_e}{2}$, that is known as the Doppler limit. Typical widths $\Gamma_e$ give temperatures of the order of hundreds of $\mu$K.

Temperatures below the Doppler limit can be realized using other laser cooling techniques. Two counterpropagating lasers with different polarizations result in a standing wave whose polarization varies in a sub-wavelength distance scale. The atomic alkalis are not simple two level systems and their ground state is degenerate at zero magnetic field. The optical potential perturbing the states depends on position and is different for the degenerate ground states. The variation of polarization with space is such that at each point atoms are transferred to the state with less effective energy at that point. This results in an energy difference between the absorbed and emited photons of the order of the dipole potential light shift (2.10). This process is called *Sisyphus cooling*. It leads to temperatures of the order of the recoil energy (energy imparted to an atom at rest when it absorbs a photon of momentum $\hbar q$) which is of the order of $0.1 - 1\mu K$.

In order to achieve quantum degeneracy for bosons and fermions, one has to obtain even lower temperatures. This is realized using *evaporative cooling*. Particles with higher energy are forced to leave the trap and the remaining particles rethermalize into a lower average temperature. This is done by applying an rf-field that induces transitions from a low-field seeking state into a high-field seeking state. The magnetic trap makes the resonant frequency depend on position. By choosing the frequency of the rf-field one can selectively remove the atoms at the edges of the trap that have higher energy. Another way of directly evaporatively cool the cloud is by lowering the potential height, e.g. the optical dipole potential is reduced by lowering the intensities of the lasers. Evaporative cooling relies on the thermalization of the cloud of atoms. This is done mainly through 2-body collisions and a high enough scattering rate is necessary to achive low temperatures in short enough times. This is a problem for identical fermions that do not interact at low temperatures. One needs either fermionic atoms in at least two different hyperfine states trapped in an optical trap or bosonic and fermionic clouds overlapping in a significant region of space. This process is called *sympathetic cooling* and it has been demonstrated to be a very efficient way of cooling fermions. The bosonic cloud can be cooled into a big Bose Einstein condensate that cools the fermionic cloud by rethermalization. The process is however only effective while the specific heat of the Bose cloud is bigger than the specific heat of the fermions. This sets a limit for sympathetic cooling and the lowest temperatures achieved for fermions at the time of this writing are of $\sim$ 90nK [21].

The rate of collisions is influenced not only by the statistics of the atoms but also by the degree of degeneracy. Due to stimulated emission, degeneracy increases collisions rates for bosons (factors $1 + f_p$ in rates of processes). For fermions the sign in the corresponding factors is the opposite $(1 - f_p)$ because transitions to occupied final states are blocked by Pauli blocking. Thus, it is the last step in cooling (evaporative cooling) what makes the achievement of low temperatures difficult for fermions.

One can create **optical lattices** using standing waves and superpose this potential to the harmonic trapping potential. Two counterpropagating, equal-intensity linearly polarized laser beams with the same frequency create a standing wave with intensity $I = I_0 \cos^2(\mathbf{k} \cdot \mathbf{r})$. The lattice spacing $a$ is given by $a = \lambda/2$. For a sufficiently large detuning between the laser and the atomic transition frequency, as shown in Section 2.1, the atom remains in the ground state and feels a potential proportional to the intensity Eq.(2.9). Thus, the intensity of the laser beam characterizes the well depth of the lattice $V_0$. The well depth of the lattice is typically several times the one photon recoil energy $E_R = \hbar^2 k^2/2m$ ($E_R$ is the energy of an atom if it acquires the momentum of a photon from the light field). One can easily accelerate optical lattices. Instead of forming the standing wave with two counterpropagating waves having equal frequencies $\nu_0$, the wave coming from the left is upshifted in frequency by a small amount $\delta\nu$, while the wave coming from the right is downshifted by the same amount. In the laboratory frame, the lattice moves with velocity $v = \lambda\delta\nu$. One can ramp the frequency shift linearly with time and the potential is then effectively accelerated with an acceleration proportional to $d(\delta\nu)/dt$. This method has been used to observe Bloch oscillations [32] and Wannier states [33] in bosonic clouds.

## 2.3 Interatomic interactions

Atomic interactions are of crucial importance to experimentally obtain quantum degeneracy in trapped gases because at sufficiently low temperatures the two body collision rate dominates the three body rate and the gas reaches kinetic equilibrium (thermal equilibrium=metastable) long before it reaches its chemical equilibrium into a stable solid state. Atomic interactions are important to dilute BEC also in the mean self-energy $U = 4\pi\hbar^2 a_s n/m$ that determines the size and shape of the condensate as well as its excitation spectrum. Further, the sign of $a$ determines whether the condensate stabilizes or collapses. However, interactions are not essential to Bose condensation (it happens also in an ideal gas) but having interactions between fermions is an essential prerequisite for a phase transition to a superfluid state. Especially important are Feshbach Resonances [25] that consists of tuning the value of the scattering length using magnetic fields when a quasi-bound molecular state couples resonantly to the free state of colliding atoms. The use of Feshbach resonances brings up the possibility of realizing strongly correlated systems and the tuning from the weakly correlated to the strongly correlated regime.

Neutral atoms interact as a consequence of the electric dipole-dipole interactions between atoms. This results in the Van der Waals interaction $V(r) \sim r^{-6}$ where $r$ is the distance between the interacting atoms. Two colliding alkali atoms have spins $s = 1/2$ that result in total spin $S = 0, 1$. The interaction depends dramatically on the total spin $S$ and one can separate the interaction potential in the contributions from the triplet state $S = 1$ and the singlet state $S = 0$ as $V = P_T V_T + P_S V_S$. To understand this we recall that the interaction between atoms with electrons outside closed shells have an attractive contribution because two electrons with opposite spin can occupy the same orbital. If the electrons are in the same spin state they can not have the same spatial wave function and there is no attractive contribution.

Let us consider the scattering of two atoms in the relative reference frame. This is equivalent to the scattering of a particle of relative mass $m_r = m/2$ (for identical scattering particles) against a fixed potential $V(\mathbf{r})$ that for simplicity we consider to be spherically symmetric $V(r)$. The asymptotic solution of the wave function of the relative particle is given by

$$\psi = e^{ik\cdot r} + \psi_{sc}(r). \tag{2.22}$$

One can define a scattering amplitude $f(k, \theta)$ as $\psi_{sc}(r) = f(k, \theta)e^{ik\cdot r}/r$ when the scattering potential between two particles $V(r)$ goes to zero when $r$ goes to infinity. The total scattering cross-section is given by $\sigma(k) = \int |f(k, \theta)|^2 d\Omega$. A scattering process between two identical particles with a spherically symmetric potential gives a scattering cross section $\sigma(k) = 8\pi/k^2 \sum_l (2l + 1) \sin^2 \delta_l(k)$ with $l$ even for bosons and odd for fermions. The phase shifts $\delta_l(k)$ follow from the asymptotic solution of the radial Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2 u_l}{dr^2} + \left(V(r) + \frac{\hbar^2 l(l+1)}{2m_r r^2} - E\right) u_l = 0 \tag{2.23}$$

where the radial wave function has the asymptotic form $(r \to \infty)$ $u_l \sim \sin(kr - l\pi/2 + \delta_l)$. When the scattering potential decreases faster than $\frac{1}{r^3}$, the potential created by the centrifugal part $\hbar^2 l(l + 1)/2m_r r^2$ creates a centrifugal barrier that will reflect the particle for low enough energies. Only the partial wave $l = 0$ will lead to non-zero scattering. A careful treatment of the limiting case $k \to 0$ shows that the phase shifts must vary at low energy as $\delta_l \sim k^{(2l+1)}$ and the scattering length is defined as

$$a_l = -\lim_{k\to 0} \frac{\delta_l(k)}{k^{2l+1}}. \tag{2.24}$$

The total cross section for identical boson s-wave scattering reaches the finite value $\sigma_0 = 8\pi a^2$ as $k \to 0$. The s-wave scattering is supressed for identical fermions and the first nonzero contribution to scattering is through p-wave scattering. The scattering length can only be calculated from the knowledge of the interaction potential for atomic hydrogen. For the rest of atoms it is determined from spectroscopic mesurements [34]. It can also be determined by the thermalization rate of trapped atomic gases.

In order to treat the renormalization problem of the scattering length we explicitely write the two particle scattering problem in the momentum representation. The wave function describing the relative motion of the two particles (2.22) written in momentum representation is

$$\psi(\mathbf{k}') = (2\pi)^3\delta(\mathbf{k}' - \mathbf{k}) + \psi_{sc}(\mathbf{k}'). \tag{2.25}$$

One can define the scattering T matrix as $f(\mathbf{k}', \mathbf{k}) = -\frac{m}{4\pi\hbar^2}T(\mathbf{k}', \mathbf{k}; E = \frac{\hbar^2k^2}{m})$. The scattering T matrix satisfies the Lippmann-Schwinger (LS) equation

$$T(\mathbf{k}', \mathbf{k}; E) = U(\mathbf{k}', \mathbf{k}) + \frac{1}{V}\sum_{k''} U(\mathbf{k}', \mathbf{k}'')(E - \frac{\hbar^2 k''^2}{m} + i\delta)^{-1} T(\mathbf{k}'', \mathbf{k}; E), \tag{2.26}$$

where $U(\mathbf{k}' - \mathbf{k})$ is the Fourier transform of the atom-atom interaction potential $V(r)$, $V$ is the system volume and $\delta$ is an infinitesimal that ensures that only outgoing waves ares included. The LS equation follows from the Schrödinger equation for the wave the function (2.25). The T matrix is related to the scattering length as

$$T(0, 0; 0) = \frac{4\pi\hbar^2 a}{m}. \tag{2.27}$$

A priori the simplest approximation to the interaction potential between two particles is the contact interaction potential

$$V(\mathbf{r}_1 - \mathbf{r}_2) = g\delta(\mathbf{r}_1 - \mathbf{r}_2). \tag{2.28}$$

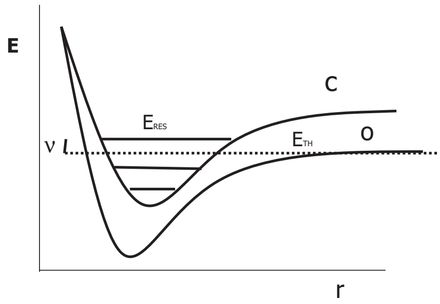


Figure 2.3: Energy level scheme of a resonance between the open (O) channel and the closed channel (C). The detuning $\nu$ is the difference between the threshold energy $E_{\text{th}}$ and the energy of the resonance $E_{\text{res}}$.

As sketched above, given the scattering potential one can determine the scattering length by the asymptotic behaviour of the solution. Doing so for the $\delta(\mathbf{r})$ potential leads to $a \to 0$. This approximation leads to an ultraviolet divergent theory reflecting the fact that the contact interaction is an effective low energy interaction invalid for high energies. One way to deal with this divergence is to introduce an energy cut-off in the interaction. This approach has been used to describe the superconductivity in metals where the Debye frequency is a natural cut-off. Another method is to express the coupling constant in terms of the two-body scattering matrix obtained from the Lipmann-Schwinger equation (2.26)

$$T(E) = \frac{U}{1 - UG_0(E)} \tag{2.29}$$

$$\frac{m}{4\pi\hbar^2 a} = -\frac{1}{g} + \sum_{|\mathbf{k}|<\Lambda} \frac{1}{2\epsilon_k}, \tag{2.30}$$

where $G_0$ is the free particle Green's function and $\epsilon_k = k^2/(2m)$. Note that the scattering length increases from $-\infty$ for a very weak attraction to $+\infty$ for strongly attractive interaction. In the inhomogeneous case, the divergency can be removed by the use of a pseudo-potential [35, 36]

$$V(\mathbf{r})\phi(\mathbf{r}) = g\delta(\mathbf{r})\frac{\partial}{\partial r}(r\phi(\mathbf{r})). \tag{2.31}$$

This type of potential removes the problem at $r = 0$ for wave funtions of the type $u(r)/r$.

We have so far neglected the internal degrees of freedom of the atoms due to the nuclear and electronic spins. Including these degrees of freedom makes the scattering of atoms a multichannel process. The coupling of channels has two effects. First, the atoms can be scattered between different magnetic substates inducing trap losses. Second, the elastic scattering amplitude and the effective interaction are altered. The main effect is that the elastic scattering in one channel can be altered dramatically if there is a low energy bound state in a second channel (closed channel). To first order in the coupling between $O$ (open) and $C$ (closed) channels the scattering is unaltered. However two particles in the channel $O$ can scatter to an intermediate state in a closed channel $C$ which subsequently decays to give two particles in channel $O$. Such a second order process can be treated with perturbation theory and one expects terms of the form $a \sim 1/(E - E_{\mathrm{res}})$ that give huge contributions for energies close to resonance. The effective interaction in the open channel includes the coupling with the closed channel as a non-local potential $H_{OC}$. The Lippmann-Schwinger equation to first order in $U_O$ [29] gives

$$\frac{4\pi\hbar^2 a}{m} = \frac{4\pi\hbar^2 a_O}{m} + \frac{|\langle\Psi_{\mathrm{res}}|H_{OC}|\Psi_0\rangle|^2}{E_{\mathrm{th}} - E_{\mathrm{res}}}. \tag{2.32}$$

Atomic interactions can be tuned by exploiting the fact that the energies of the states depend on external parameters like the strength of the magnetic field. Assuming an external magnetic field and $E_{\mathrm{res}} - E_{\mathrm{th}} \propto (B - B_0)$, the scattering length is therefore given by

$$a = a_O \left(1 - \frac{|\Delta B|}{B - B_0}\right) \tag{2.33}$$

where $a_O$ is the nonresonant or background scattering length. For magnetic fields around $B_0$ the effective scattering length diverges to $\pm\infty$. A resonance between an open channel and an closed channel is depicted in Figure 2.3. In the closed channel there are bound states of the two atoms. By tuning the magnetic field one can vary the energy difference between the two channels $\nu = E_{\mathrm{th}} - E_{\mathrm{res}}$. If the energy of the bound state $E_{\mathrm{res}}$ is lower than the dissociation threshold energy, the bound state can be effectively populated and bound molecules are created. This corresponds to the side of the resonance with repulsive effective atom-atom interactions.

If one traps fermionic atoms in two different hyperfine states with attractive interactions between them, they can form atomic Cooper pairs and the system becomes superfluid. In the weak coupling regime the transition temperature is orders of magnitude smaller than the Fermi temperature. One can exploit the resonances between the different hyperfine states to increase the interaction strength to values that predict experimentally accesible temperatures. Feshbach resonances in fermionic alkali atoms have also been predicted to rise the transition temperature $T_c$ to values comparable to the Fermi temperature $T_F$ [37, 38, 39]. This has been called *Resonance superfluidity*. For $^6$Li there is a extremely broad Feshbach resonance between the states $|\frac{1}{2}, \frac{1}{2}\rangle$, $|\frac{1}{2}, -\frac{1}{2}\rangle$ around $B_0 = 81$mT [22, 23]. A much narrower resonance occurs for the same states at $B_0 \sim 54.3$mT [19]. For $^{40}$K, the states $|\frac{9}{2}, -\frac{5}{2}\rangle$ and $|\frac{9}{2}, -\frac{9}{2}\rangle$ present a narrow resonance of width $\sim 0.8$mT around $B_0 = 22.4$mT [17].

Apart form elastic collisions there are also inelastic collision processes that make the experiments more difficult as they can induce trap losses. There are three types inelastic processes and they can be measured by atom losses

$$\frac{dN}{dt} = -\Sigma N - \int K n^2 d^3\mathbf{r} - \int L n^3 d^3\mathbf{r}. \tag{2.34}$$

The first term accounts for collisions with background atoms at density independent rate $\Sigma$, the second term is due to two-body inelastic processes like two-body spin dipolar relaxation (spin flips) and the third one corresponds to three body recombination processes. Since the density will always decrease with time at sufficiently long times, a log plot of N versus time is a straight line whose slope gives $\Sigma$. At short times the slope might be due to $K$ or $L$ which can be determined from a fit to the plot.

# Chapter 3

# Description of the System.

In this chapter we review the *microscopic* theory of dilute Fermi gases. We include here the methods that we have used to describe the system in publications I to VI.

Another way of describing the system is the *macroscopic* or semiclassical formalism in which one introduces a distribution function $f(\mathbf{r}, \mathbf{p})$ [35] in the phase space $\Gamma = \int d^3\mathbf{r} d^3\mathbf{k}$. The semiclassical approximation holds provided the temperature $T$ is much larger than the single particle energy spacing $\hbar\omega$. We will review the static semiclassical (or Thomas-Fermi) approximation in the next section. Dynamics can be also described with the semiclassical formalism by introducing a time-dependent distribution function $f(\mathbf{r}, \mathbf{v}, t)$. The equation describing the dynamics of fermions in the normal state [40] is the Landau-Vlasov equation that is a Boltzmann equation with an interaction term. It is valid in the collisionless regime and one can add to it a collisional term to treat interatomic interactions . When interactions are very strong one can apply the hydrodynamic description that holds provided the collisional relaxation time $\tau$ is much smaller than the time scale of the typical frequencies characterizing the dynamic phenomena (in this case the trapping frequency $\omega$). The same hydrodynamic equations hold for the the superfluid case [41], assuming that the relevant physical quantities change slowly on distances larger than the healing length. This implies that $\hbar\omega \ll \Delta$ (superfluid gap). It further assumes that the local density approximation (or semiclassical approximation) to the equation of state is justified. This macroscopic description of the system is specially useful for analyzing the collective excitations of the trapped system and has been extensively used by S. Stringari's group in Trento.

## 3.1 Degenerate Fermi gas

We begin by considering $N$ fermions in the same hyperfine state of the atom. At low temperatures identical fermions do not interact and we consider a *non-interacting degenerate Fermi gas*. Fermi particles obey the Pauli exclusion principle that prevents two fermions from being in the same quantum state. The ground state of an ideal Fermi gas is the antisymmetrized product of the lowest $N$ single particle states.

To calculate the thermodynamic properties of the system we have to convert the sums over states into integral using the density of states. For results in the homogenous case see Ref. [42]. We also consider non-interacting fermions in a three-dimensional harmonic trap, because it is the typical geometry in experiments with alkali atoms.

The density of states $g(\epsilon)$ gives the number of states with energy within $\epsilon$ and $\epsilon + d\epsilon$. It is given by

$$g(\epsilon) = C_\alpha \epsilon^{\alpha-1} \tag{3.1}$$

$$C_{3/2} = \frac{Vm^{3/2}}{\sqrt{2}\pi^2\hbar^3} \quad C_3 = \frac{1}{2(\hbar\bar{\omega})^3} \tag{3.2}$$

where $\alpha = 3/2$ corresponds to the three dimensional homogenous gas and $\alpha = 3$ to the three dimensional harmonic trap where $\bar{\omega} = (\omega_x\omega_y\omega_z)^{1/3}$ is the average trapping frequency.

The number of particles is given by

$$N = \int_0^\infty g(\epsilon) f(\epsilon) d\epsilon \tag{3.3}$$

where $f(\epsilon) = 1/(e^{(\epsilon-\mu)} + 1)$ is the Fermi distribution function and $\mu$ the chemical potential. The normalization condition (3.3) is used to obtain the chemical potential.

At $T = 0$, $f(\epsilon) = \theta(\epsilon - \mu)$ and Eq.(3.3) reduces to $N = C_\alpha \frac{\mu^\alpha}{\alpha}$. For the 3D harmonic trap, and defining $\mu(T = 0) \equiv E_F$,

$$E_F = (6N)^{1/3}\hbar\bar{\omega}. \tag{3.4}$$

The characteristic size of the trapped degenerate Fermi gas is defined as $R_F \equiv \sqrt{2E_F/M\bar{\omega}}$.

The chemical potential at $T \neq 0$ is obtained solving numerically Eq.(3.3)[43]. One can find analytic expressions for the chemical potential in the low temperature limit $k_B T \ll E_F$ using the Sommerfeld expansion of the above integral

$$\mu(T,N) = E_F \left[1 - (\alpha - 1)\frac{\pi^2}{6}\left(\frac{T}{T_F}\right)^2\right] + O((T/T_F)^4). \tag{3.5}$$

In the high temperature limit ($k_B T \gg E_F$),

$$\mu(T,N) = -k_B T \ln\left[\alpha! \left(\frac{T}{T_F}\right)\right]. \tag{3.6}$$

The total energy of the system $U = C_\alpha \int_0^\infty \epsilon^\alpha f(\epsilon) d\epsilon$ can be calculated in a similar way and for low temperatures ($k_B T \ll E_F$) $U/N = \left(\frac{\alpha}{\alpha+1}\right) E_F[1 + (\alpha+1)\frac{\pi^2}{6}\left(\frac{T}{T_F}\right)^2] + O((T/T_F)^4$.

Since all the exact eingenstates and eigenenergies of the harmonic trap are known, one can calculate all the observables of the system by summing up over the $N$ states [44]. The density can be obtained by squaring the antisymetrized combination of single particle eigenfunctions. One can also use approximations. The semiclassical or Thomas-Fermi approximation for the dilute trapped Fermi gas [43] consists of characterizing each atom by a position $\mathbf{r}$, and a momentum $\mathbf{k}$, and energy $H(\mathbf{r},\mathbf{k})$, where $H$ is the classical Hamiltonian. One should integrate

$$f(\mathbf{r},\mathbf{k}) = \frac{1}{(2\pi)^3}\frac{1}{\exp[\beta(H(\mathbf{r},\mathbf{k})) - \mu] - 1} \tag{3.7}$$

over the six dimensional phase space $d^3\mathbf{k} d^3\mathbf{r}$. This approximation is valid when one can divide the sample in cells with a big number of particles in which the potential energy is nearly constant so that one can define a local Fermi sea. This is the case provided the number of particles $N$ is large enough. Note that (3.7) is one of the static solutions of the Boltzmann equation (also the Bose and the Boltzmann distribution functions are). The density is given by

$$n(\mathbf{r},T) = \int d^3\mathbf{k} f(\mathbf{r},\mathbf{k}) \tag{3.8}$$

where the chemical potential is obtained by normalization of (3.7) to $N$. At $T = 0$ one may define a "local" Fermi wave number $k_F(\mathbf{r})$ as

$$\frac{\hbar^2 k_F(\mathbf{r})^2}{2m} + V(\mathbf{r}) = E_F \tag{3.9}$$

The density is then $(2\pi)^3 n(\mathbf{r}, T=0) = \frac{4}{3}\pi k_F(\mathbf{r})^3$ which gives

$$n(\mathbf{r}, T=0) = \frac{N}{R_F^3}\frac{8}{\pi^2}\left[1 - \frac{\sum_i(\lambda_i x_i)^2}{R_F^2}\right]^{3/2} \tag{3.10}$$

where $\lambda_i \equiv (\omega_i/\bar{\omega})$. The absorption measurements when releasing the trap reflect the momentum distribution that a $T = 0$ is given by $n(\mathbf{k}, T = 0) = \int d^3\mathbf{r} f(\mathbf{r}, \mathbf{k}) = 8N/(\pi^2 K_F^3)\left[1 - \mathbf{k}^2/K_F^2\right]^{3/2}$ where $K_F$ is given by $E_F$. The momentum distribution for a non-interacting Fermi gas is isotropic even for anisotropic potentials due to the isotropy of the mass but the density in space $n(\mathbf{r})$ is anisotropic when the trapping potential is different in the different directions.

The *two component system of fermions* has been studied in [45] treating the attractive interactions within the mean field approximation $-gn(\mathbf{r})$ with $g = \frac{4\pi\hbar^2 a_s}{m}$. In [45] the authors show good agreement between the numerical calculation using the sums of harmonic oscillator eigenstates and the Thomas-Fermi solutions given by the simultaneous self-consistent equations

$$E(\mathbf{k}, \mathbf{r}) = \frac{\hbar^2\mathbf{k}^2}{2m} - gn(\mathbf{r}) - \mu(\mathbf{r}) \tag{3.11}$$

$$\mu(\mathbf{r}) = \mu - \frac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2) \tag{3.12}$$

$$n(\mathbf{r}) = \int d^3\mathbf{k} f(E(\mathbf{k}, \mathbf{r})) \tag{3.13}$$

where $f$ is the Fermi distribution function (3.7). Attractive interactions lower the chemical potential and the energy $U$ with respect to the non-interacting gas and the momentum distribution is more spread than for the non-interacting case.

## 3.2 Superfluid transition

A superfluid phase transition has been predicted [15] for fermionic atoms with attractive interatomic interactions. It was proposed that a BCS transition would take place and the fermions would form atomic Cooper pairs due to the weak attractive interactions. Due to the tuneability of the interatomic interactions one can go to stronger regimes of fermionic superfluidity [37, 38, 39] that provide superfluid transition temperatures within the limits of the cooling techniques currently available and the possibility of scanning the BCS-BEC crossover.

Let us consider a two-component fermionic gas of atoms in two different hyperfine states and equal densities for both components. As shown in section 2.3, one needs fermions in different hyperfine states to have s-wave scattering between them[1]. The equal density configuration is the optimal for pairing. For a detailed calculation of the superfluid gap for nonequal densities see Ref. [47]. The Hamiltonian of the system including only two particle interactions in the second quantization formalism reads

$$\hat{K} = \int d\mathbf{r} \sum_{\sigma=\uparrow,\downarrow} \left[\hat{\psi}^\dagger_\sigma(\mathbf{r}) \left(-\frac{\hbar^2\nabla^2}{2m} + V_\sigma(\mathbf{r}) - \mu_\sigma\right) \hat{\psi}_\sigma(\mathbf{r})\right]$$
$$+\frac{1}{2} \sum_{\sigma,\beta=\uparrow,\downarrow} \int d\mathbf{r} \int d\mathbf{r}' u_{\sigma\beta}(\mathbf{r}-\mathbf{r}') \hat{\psi}^\dagger_\sigma(\mathbf{r}) \hat{\psi}^\dagger_\beta(\mathbf{r}') \hat{\psi}_\beta(\mathbf{r}') \hat{\psi}_\sigma(\mathbf{r}), \tag{3.14}$$

where the two trapped hyperfine states are denoted by $\uparrow$ and $\downarrow$ and $\hat{\psi}^\dagger_\sigma$, $\hat{\psi}_\sigma$ are field operators. Assuming point-like interactions in 3 dimensions, the interaction coefficient is given by $u_{\uparrow\downarrow} = \frac{4\pi\hbar^2}{m} a_s \equiv -|U|$. Thus, the interaction term reads

$$-\frac{1}{2}|U| \sum_{\sigma,\beta=\uparrow,\downarrow} \int d\mathbf{r} \hat{\psi}^\dagger_\sigma(\mathbf{r}) \hat{\psi}^\dagger_\beta(\mathbf{r}) \hat{\psi}_\beta(\mathbf{r}) \hat{\psi}_\sigma(\mathbf{r})$$
$$= -|U| \int d\mathbf{r} \hat{\psi}^\dagger_\uparrow(\mathbf{r}) \hat{\psi}^\dagger_\downarrow(\mathbf{r}) \hat{\psi}_\downarrow(\mathbf{r}) \hat{\psi}_\uparrow(\mathbf{r}). \tag{3.15}$$

We have included the chemical potentials $\mu_\sigma$ and we are working with the grand-canonical Hamiltonian valid for the description of a system in contact with a thermal bath and fixed average number of particles. We consider the general case of an inhomogeneous single particle Hamiltonian with potential $V_\sigma(\mathbf{r})$. This is the experimental case due to the harmonic trap or optical lattice. The use of a periodic potential has been predicted [48] to increase the superfluid transition temperature. In an optical lattice the effective interaction depends on the potential barrier and one can scan the BCS-BEC crossover just by controlling the laser intensity.

During the thesis we have worked mainly on the superfluid state of the Fermi gas. We have assumed that the above Hamiltonian can be solved

[1] Superfluidity for identical fermions through p-wave pairing [46] leads to lower transition temperature but avoids the experimental problem of having almost equal atomic densities in both states ($\frac{n_\uparrow - n_\downarrow}{n_\uparrow + n_\downarrow} < \frac{T_c}{T_F}$).

within the BCS approximation. During the next sections we review the methods that we have used during the thesis. Experiments are turning into the strong coupling regime or intermediate regime of superfluidity. As first shown by Nozières and Schmitt-Rink [49], the BCS algebra can be used both in the weak coupling regime as well as in the strong coupling regime. Due to its importance in the field we review here also the BCS-BEC crossover and resonance superfluidity (i.e. superfluidity in the presence of a Feshbach resonance). The last section is devoted to the solution of the BCS algebra with a current. This is necessary for our description of Bloch oscillations in publication VI.

It is well known that the static Hartree-Fock-Bogoliubov theory does not fullfil the sum rules obtained from particle current conservation [50]. The density-density response function obtained with this approximation shows an unphysically large response at small wavevectors. The mean field solution assumes that the particle spectrum has a gap and therefore it ignores the possible below gap excitations. One can go beyond the BCS approximation by using the time dependent Hartree-Fock-Gorkov approximation or Random-Phase-Approximation (RPA) to obtain the collective excitations and obtain a collective mode [50] with phonon dispersion relation that lies in the gap. We only review here the static theory but the RPA was used in publication II.

## 3.3 Mean field solution

We review here the mean field solution of the Hamiltonian $\hat{K}$ following Ref. [51]. To simplify the interaction term, let us use the following mean field approximation in the spirit of the Wick's theorem

$$\begin{aligned}\hat{\psi}^\dagger_\uparrow(\mathbf{r})\hat{\psi}^\dagger_\downarrow(\mathbf{r})\hat{\psi}_\downarrow(\mathbf{r})\hat{\psi}_\uparrow(\mathbf{r}) &= \\ \langle\hat{\psi}^\dagger_\uparrow(\mathbf{r})\hat{\psi}^\dagger_\downarrow(\mathbf{r})\rangle\hat{\psi}_\downarrow(\mathbf{r})\hat{\psi}_\uparrow(\mathbf{r}) + \langle\hat{\psi}_\downarrow(\mathbf{r})\hat{\psi}_\uparrow(\mathbf{r})\rangle\hat{\psi}^\dagger_\uparrow(\mathbf{r})\hat{\psi}^\dagger_\downarrow(\mathbf{r}) &+ \\ \langle\hat{\psi}^\dagger_\uparrow(\mathbf{r})\hat{\psi}_\uparrow(\mathbf{r})\rangle\hat{\psi}^\dagger_\downarrow(\mathbf{r})\hat{\psi}_\downarrow(\mathbf{r}) + \langle\hat{\psi}^\dagger_\downarrow(\mathbf{r})\hat{\psi}_\downarrow(\mathbf{r})\rangle\hat{\psi}^\dagger_\uparrow(\mathbf{r})\hat{\psi}_\uparrow(\mathbf{r}) \end{aligned} \tag{3.16}$$

that leads to linear equations of motion. This is called the Hartree-Fock-Bogoliubov approximation where we have defined the *pairing* (gap) and the *Hartree* fields as

$$\Delta(\mathbf{r}) \equiv -|U|\langle\hat{\psi}_\downarrow(\mathbf{r})\hat{\psi}_\uparrow(\mathbf{r})\rangle = |U|\langle\hat{\psi}_\uparrow(\mathbf{r})\hat{\psi}_\downarrow(\mathbf{r})\rangle \tag{3.17}$$

$$W_\sigma(\mathbf{r}) \equiv -|U|\langle\hat{\psi}^\dagger_\sigma(\mathbf{r})\hat{\psi}_\sigma(\mathbf{r})\rangle. \tag{3.18}$$

Note that we have neglected the *Fock* terms $\langle\hat{\psi}^\dagger_\sigma(\mathbf{r})\hat{\psi}_\beta(\mathbf{r})\rangle$ $\sigma \neq \beta$. The effective Hamiltonian reads

$$\hat{K}_{\text{eff}} = \int d\mathbf{r}\left[\hat{\psi}^\dagger_\uparrow(\mathbf{r})K_{o\uparrow}\hat{\psi}_\uparrow(\mathbf{r}) + \hat{\psi}^\dagger_\downarrow(\mathbf{r})K_{o\downarrow}\hat{\psi}_\downarrow(\mathbf{r}) + \Delta^\dagger(\mathbf{r})\hat{\psi}_\downarrow(\mathbf{r})\hat{\psi}_\uparrow(\mathbf{r}) + \Delta(\mathbf{r})\hat{\psi}^\dagger_\uparrow(\mathbf{r})\hat{\psi}^\dagger_\downarrow(\mathbf{r})\right]. \tag{3.19}$$

Using the commutation relations for the fermionic operators, one obtains the following equations

$$\begin{aligned}\left[\hat{\psi}_\uparrow, \hat{K}_{eff}\right] &= K_{o\uparrow}\hat{\psi}_\uparrow + \Delta\hat{\psi}^\dagger_\downarrow \\ \left[\hat{\psi}_\downarrow, \hat{K}_{eff}\right] &= K_{o\downarrow}\hat{\psi}_\downarrow - \Delta^\dagger\hat{\psi}^\dagger_\uparrow.\end{aligned} \tag{3.20}$$

where $K_{o\sigma} = H_{o\sigma} - \mu_\sigma$ and $H_{o\sigma} = -\frac{\hbar^2\nabla^2}{2m} + V_\sigma(\mathbf{r}) + W_\sigma(\mathbf{r})$.

One can expand the field operators in the **Bogoliubov operators** $\hat{\gamma}_{n\sigma}$

$$\hat{\psi}_\uparrow(\mathbf{r}) = \sum_k \left[\hat{\gamma}_{k\uparrow}u_k(\mathbf{r}) - \hat{\gamma}^\dagger_{k\downarrow}v^\dagger_k(\mathbf{r})\right] \tag{3.21}$$

$$\hat{\psi}_\downarrow(\mathbf{r}) = \sum_k \left[\hat{\gamma}_{k\downarrow}u_k(\mathbf{r}) + \hat{\gamma}^\dagger_{k\uparrow}v^\dagger_k(\mathbf{r})\right] \tag{3.22}$$

Using the commutation relations for the field operators, one can see that the commutation relations for the Bogoliubov operators must be

$$\begin{aligned}\{\hat{\gamma}^\dagger_{k\sigma}, \hat{\gamma}_{l\beta}\} &= \delta_{kl}\delta_{\sigma\beta} \\ \{\hat{\gamma}_{k\sigma}, \hat{\gamma}_{l\beta}\} &= 0.\end{aligned} \tag{3.23}$$

The Bogoliubov transformation is chosen in such a way that it diagonalizes the effective Hamiltonian

$$\hat{K}_{eff} = E_G + \sum_{k\sigma} E_k\hat{\gamma}^\dagger_{k\sigma}\hat{\gamma}_{k\sigma}. \tag{3.24}$$

Using Eqs.(3.23), one can see that

$$\left[\hat{\gamma}_{k\sigma}, \hat{K}_{eff}\right] = E_k\hat{\gamma}_{k\sigma} \tag{3.25}$$

$$\left[\hat{\gamma}^\dagger_{k\sigma}, \hat{K}_{eff}\right] = -E_k\hat{\gamma}^\dagger_{k\sigma}. \tag{3.26}$$

Inserting the transformation (3.22) in eqs (3.20), and equating the coefficients in front of the Bogoliubov operators one obtains the Bogoliubov-deGennes [51] equations

$$\begin{aligned}E_k u_k(\mathbf{r}) &= K_{o\uparrow}(\mathbf{r})u_k(\mathbf{r}) + \Delta(\mathbf{r})v_k(\mathbf{r}) \\ E_k v_k(\mathbf{r}) &= -K^\dagger_{o\downarrow}v_k(\mathbf{r}) + \Delta^\dagger(\mathbf{r})u_k(\mathbf{r}).\end{aligned} \tag{3.27}$$

that one has to solve self-consistently with the gap equation

$$\Delta(\mathbf{r}) = |U| \sum_k \left((1 - 2f(E_k)\right) v_k^\dagger u_k, \tag{3.28}$$

where we have used that $\langle \hat{\gamma}_{k\sigma}^\dagger \hat{\gamma}_{l\beta} \rangle = \delta_{kl}\delta_{\sigma\beta} f_k$ and $\langle \hat{\gamma}_{k\sigma} \hat{\gamma}_{l\beta} \rangle = 0$. The self-consistent solution of equations (3.27) and (3.28) gives $u_k$, $v_k$ and $E_k$.

### 3.3.1 BdG Equations

For simplicity we assume the same single particle Hamiltonian $H_o$ for both spin states

$$\begin{pmatrix} H_o(\mathbf{r}) - \mu & \Delta(\mathbf{r}) \\ \Delta^\dagger(\mathbf{r}) & -(H_o(\mathbf{r}) - \mu) \end{pmatrix} \begin{pmatrix} u_k(\mathbf{r}) \\ v_k(\mathbf{r}) \end{pmatrix} = E_k \begin{pmatrix} u_k(\mathbf{r}) \\ v_k(\mathbf{r}) \end{pmatrix}. \tag{3.29}$$

Note that we keep the $\mathbf{r}$ dependence because experimentally these atomic systems are confined in a harmonic or a periodic trap. We use the following ansatz

$$\begin{aligned} u_k(\mathbf{r}) &= e^{ik\mathbf{r}} \phi_k(\mathbf{r}) u_k \\ v_k(\mathbf{r}) &= e^{ik\mathbf{r}} \phi_k(\mathbf{r}) v_k. \end{aligned} \tag{3.30}$$

where $\phi_k(\mathbf{r})$ are the single particle eigenstates. Multiplying Eq.(3.29) by $\begin{pmatrix} u_k^\dagger(\mathbf{r}) & v_k^\dagger(\mathbf{r}) \end{pmatrix}$ and integrating in space , one obtains

$$\begin{aligned} \xi_k |u_k|^2 + u_k^\dagger v_k \Delta_k &= E_k |u_k|^2 \\ -\xi_k |v_k|^2 + v_k^\dagger u_k \Delta_k^\dagger &= E_k |v_k|^2 \end{aligned} \tag{3.31}$$

where we have introduced the *space-independent gap* $\Delta_k \equiv \langle k|\Delta|k\rangle = \int \Delta(\mathbf{r}) \phi_k^\dagger(\mathbf{r}) \phi_k(\mathbf{r}) d^3\mathbf{r}$. These two equations lead to the well known result

$$\begin{aligned} u_k^2 &= \frac{1}{2}\left(1 + \frac{\xi_k}{E_k}\right) \\ v_k^2 &= \frac{1}{2}\left(1 - \frac{\xi_k}{E_k}\right). \end{aligned} \tag{3.32}$$

where $E_k = \pm\sqrt{\xi_k^2 + \Delta_k^2}$. Note that for the Bogoliubov transformation to make sense we need $E_k > 0$ for the quasiparticle energies. The - branch is the "hole" branch. Using (3.32) one can easily obtain the relation

$$2u_k v_k^\dagger = \frac{\Delta_k}{\sqrt{\xi_k^2 + \Delta_k^2}}. \tag{3.33}$$

### 3.3.2 Gap Equation

One can thus obtain $u_k(\mathbf{r})$ and $v_k(\mathbf{r})$ inserting the single particle energies $\xi_k$ and eigenstates $\phi_k(\mathbf{r})$ and $\Delta_k$. The gap is given by the following equation

$$\Delta(\mathbf{r}) = |U| \sum_k ((1 - 2f(E_k)) \, v_k^\dagger(\mathbf{r}) u_k(\mathbf{r}) \tag{3.34}$$

where $u_k(\mathbf{r})$, $v_k(\mathbf{r})$ solve the BdG equations (3.27). One can simplify $1 - 2f(E_k) = \tanh\left(\frac{\beta E_k}{2}\right)$ and using the ansatz (3.30), the gap equation reads

$$\Delta(\mathbf{r}) = |U| \sum_k \tanh\left(\frac{\beta E_k}{2}\right) v_k^\dagger u_k \phi_k(\mathbf{r}) \phi_k^\dagger(\mathbf{r}). \tag{3.35}$$

One can define $\Delta(\mathbf{r}) \equiv \sum_k \Delta_k(\mathbf{r})$ where

$$\Delta_k(\mathbf{r}) = |U| |\phi_k(\mathbf{r})|^2 \tanh\left(\frac{\beta E_k}{2}\right) \frac{\Delta_k}{2\sqrt{\xi_k^2 + \Delta_k}}. \tag{3.36}$$

so that $\int d\mathbf{r} \Delta_k(\mathbf{r}) \neq \Delta_k$. Note that the $\mathbf{r}$-independent gap used in equations (3.31) is given by

$$\Delta_k = |U| \sum_n \tanh\left(\frac{\beta E_n}{2}\right) \frac{\Delta_n}{2\sqrt{\xi_n^2 + \Delta_n}} \int d\mathbf{r} |\phi_k(\mathbf{r})|^2 |\phi_n(\mathbf{r})|^2. \tag{3.37}$$

In case of **no k-dependence of the gap** $\Delta_k \equiv \Delta$,

$$\Delta = |U| \sum_n \tanh\left(\frac{\beta E_n}{2}\right) \frac{\Delta}{2\sqrt{\xi_n^2 + \Delta^2}} \int d\mathbf{r} |\phi_n|^4. \tag{3.38}$$

where we have used that $\int d\mathbf{r} |\phi_k(\mathbf{r})|^2 |\phi_n(\mathbf{r})|^2 = \int d\mathbf{r} |\phi_n(\mathbf{r})|^4 \equiv A$ that must be momentum independent. Then, the self consistent equation for $\Delta$ is Eq.(3.38)

$$1 = A|U| \sum_k \tanh\left(\frac{\beta E_k}{2}\right) \frac{1}{2\sqrt{\xi_k^2 + \Delta^2}}. \tag{3.39}$$

In the **homogeneous case**, the single particle eigenstates reduce to plane waves and Eq. (3.37) reduces to the familiar equation

$$\Delta_k = |U| \sum_n \tanh\left(\frac{\beta E_n}{2}\right) \frac{\Delta_n}{2\sqrt{\xi_n^2 + \Delta_n}} \tag{3.40}$$

that implies $k$-independence of the gap $\Delta_k \equiv \Delta$. The self consistent equation reduces to $1/|U| = \sum_n \tanh\left(\frac{1}{2}\beta E_n\right) 1/(2\sqrt{\xi_n^2 + \Delta^2})$. Using the renormalized interaction (2.30) to remove the ultraviolet divergence, the **gap equation** reads

$$-\frac{m}{4\pi\hbar^2 a_s} = \sum_k \left[\tanh\left(\frac{\beta E_k}{2}\right)\frac{1}{2\sqrt{\xi_k^2 + \Delta^2}} - \frac{1}{2\xi_k}\right], \qquad (3.41)$$

where one takes the maximum contribution to the single particle Green's function $G_0(E = 2\mu)$. In general, the chemical potential $\mu$ is also unknown and apart from equation (3.41) one needs to solve simultaneously the **number equation**

$$n = \sum_k u_k^2(1 - f(E_k)) + v_k^2 f(E_k) = \sum_k \left(1 - \frac{\xi_k}{E_k}\tanh\left(\frac{\beta E_k}{2}\right)\right). \qquad (3.42)$$

## 3.4 Weak coupling superfluidity

In the case of weak-coupling superfluidity one can solve analytically both the gap and the number equation. By weak coupling we mean that the attraction will just affect a few states close to the Fermi energy, $\Delta_k \ll \epsilon_F$. One can also assume that the gap is $k$-independent $\Delta_k \equiv \Delta$.

The **transition temperature** is given by setting $\Delta = 0$ in (3.41) which reduces to

$$\int_0^\infty d\epsilon N(\epsilon)\frac{f(\epsilon)}{\epsilon - \mu} = \frac{m}{4\pi\hbar^2 a_s} \qquad (3.43)$$

where $N(\epsilon) = m^{3/2}\epsilon^{1/2}/(\sqrt{2}\pi^2\hbar^3)$ is the density of states for the homogeneous case. For $N(\epsilon_F)4\pi\hbar^2 a_s/m \ll 1$, $k_B T_c \ll \epsilon_F$ and one can evaluate the above integral at low temperatures $k_B T_c \ll \mu$ which implies $\mu \approx \epsilon_F$. Thus, one does not need the number equation. This gives [29],

$$k_B T_c = \frac{8\gamma}{\pi e^2}\epsilon_F e^{-\pi/2k_F|a_s|} \qquad (3.44)$$

where $\gamma = e^C \sim 1.781$ and C is the Euler constant. The **superfluid gap** at $T = 0$ is calculated in a similar way (note that $U_0 = 4\pi\hbar^2 a_s/m$)

$$1 = \frac{|U_0|}{2}\sum\left(\frac{1}{\sqrt{\xi_p^2 + \Delta^2}} - \frac{1}{\xi_p}\right) \qquad (3.45)$$

which gives

$$\Delta = \frac{8}{e^2}\epsilon_F e^{-\frac{1}{N(\epsilon_F)|U_0|}}. \tag{3.46}$$

The ratio between the zero-temperature gap and the transition temperature is given by

$$\frac{\Delta(T=0)}{k_B T_c} = \frac{\pi}{\gamma} \approx 1.76. \tag{3.47}$$

Until now we have considered only the bare scattering length. As shown in the context of dilute alkali Fermi gases by Heiselberg et al. [52] the medium produces induced interactions. The induced interactions correspond to one fermion polarizing the medium and the second fermion being influenced by this polarization. The induced interaction by the medium corresponds to density-density fluctuations

$$U_{\text{ind}}(\mathbf{p},\mathbf{p}') = -\frac{1}{V}\sum_{p''} U_o^2 \frac{f_{p''} - f_{p+p'+p''}}{\epsilon_{p+p'+p''} - \epsilon_{p''}} = -U_o^2 \chi(|p'+p|), \tag{3.48}$$

where $\chi(q)$ is the density-density response function that we have used in publication II. Inserting $1/(U_0 + \langle U_{\text{ind}}\rangle) \sim 1/U_0 + \langle\chi(q)\rangle$ in (3.44) one can see that fermionic induced interactions decrease the transition temperature by a factor $(4e)^{1/3}$. In the Bose-Fermi mixture case the Bose cloud creates an induced interaction in the fermions because a moving fermion can excite a sound wave in the Bose gas, which will perturb at a large distance another fermion, $U_{ind}(q,0) = -U_{BF}^2/U_{BB}/(1+(\xi_B q)^2)$ where $\xi_B$ is the coherence length of the condensate. If the boson-boson interaction and the fermion-boson interaction are of the same size, the induced interaction is of the same order of magnitude as the bare interaction and it will increase the transition temperature (for a Bose gas such that $U_{BB} > 0$). This effect has been shown to increase dramatically the p-wave transition temperature [53].

## 3.5 BCS-BEC crossover

The weak coupling or BCS limit corresponds to a Fermi sea undergoing a pairing instability at a temperature $T_c \ll \epsilon_F$. The formation of Cooper pairs and their condensation occurs simultaneously at $T_c$. The opposite limit corresponds to the Bose condensation of preformed bosons, i.e. pairs. The composite particles form at some very high temperature scale of the order

$T_{\rm dissoc}$ and then they condense at the BEC $T_c \ll T_{\rm dissoc}$. The possibility of using the BCS algebra, Eqs.(3.41) and (3.42) to study the crossover was suggested by Leggett [54] for $^3$He. Nozières and Schmitt-Rink [49] took up the same formalism for concerning superconductivity of excitons and a recent review has been given by M. Randeria [55]. To show that the BEC limit can be studied with the BCS algebra, they propose the creation operator for the bosons $\hat{b}_0^\dagger = \sum_k \phi_k \hat{c}_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger$ with $\phi_k = (N/2)^{-1/2} v_k/u_k$ where $u_k$ and $v_k$ are the BCS functions (3.32). They show that the gap equation (3.41) rewritten in terms of $\phi_k$ reduces in the strong coupling limit to the Schrödinger equation for a single bound pair. They solve the Eqs.(3.41) and (3.42) in the strong coupling limit to obtain the transition temperature and the gap at $T = 0$. For $1/a \to \infty$ the solutions are $\mu \sim -E_b/2$, $\Delta = (16/3\pi)^{1/2}\epsilon_F/\sqrt{k_F a_s}$, and $T_0 \sim E_b/2 \ln{(E_b/\epsilon_F)}^{3/2}$ where $E_b = \hbar^2/(m a_s^2)$ is the binding energy of the tightly bound pairs. However, this temperature is of the order of the dissociation energy and does not correspond to the superfluid transition temperature. This temperature is the BEC transition temperature for bosons of mass $2m$ and density $n/2$. This can be seen when including quantum fluctuations to the mean field $\Delta$ [49, 55]. The transition from BCS to BEC regime is believed to be a smooth one. Analyzing the quasiparticle excitation energies, one can see that for $\mu > 0$, the gap is just $\Delta$. For $\mu < 0$ the energy gap is $(|\mu|^2 + \Delta^2)$ [i.e. $\min(E_k = \sqrt{\Delta^2 + (\xi_k - \mu)^2}]$ that tends to the dissociation energy in the limit of tightly bound pairs. Thus, the qualitative transition from diatomic molecule to Cooper pairs takes place at $\mu = 0$, which corresponds to the value $k_F a_s \approx 1$ [54].

### Periodic potential

The BCS-BEC crossover problem for a lattice potential was first considered in [49]. In the case of a periodic potential, one can use the Hubbard model.

When the potential is periodic, the one-particle Hamiltonian is solved using the Bloch ansatz. For single atoms the energy eigenstates are Bloch wave functions, and an appropiate superposition of Bloch states yields a set of Wannier functions $\{w_n(x - x_i)\}_{i,n}$ ($i$ is lattice site and $n$ band index) well localized on the individual lattice sites. One can expand the field operators in terms of Wannier functions and on-site fermionic annihilation operators $\hat{c}_{j\sigma}$, $\hat{\psi}_\sigma(\mathbf{r}) = \sum_j \hat{c}_{j\sigma} w(\mathbf{r} - \mathbf{r_j})$. Note that we have considered only the first band Wannier functions. Including more bands leads to the degenerate Hubbard model. The one band approximation is valid as long as the energies involved in the system dynamics are small compared to the excitation energies to the second band and when the density is low such that the average number of particles per well does not exceed 2. The Hamiltonian (3.14) becomes

$$H_0 = J \sum_{\langle i,j \rangle \sigma} c_{i\sigma}^{\dagger} \hat{c}_{j\sigma} + U \sum_{j} c_{j\uparrow}^{\dagger} c_{j\downarrow}^{\dagger} \hat{c}_{j\downarrow} \hat{c}_{j\uparrow} \tag{3.49}$$

where $\langle i, j \rangle$ denotes summation over nearest neighbours and

$$J = -\int d\mathbf{r} w^{\dagger}(\mathbf{r} - \mathbf{r_i}) \left[ -\frac{\hbar^2 \nabla^2}{2m} \right] w(\mathbf{r} - \mathbf{r_j}) \tag{3.50}$$

is the tunneling matrix element between adjacent sites $i$, $j$ and

$$U = \frac{4\pi a_s \hbar^2}{m} \int d\mathbf{r} |w(\mathbf{r})|^4 \tag{3.51}$$

corresponds to the strength of the onsite repulsion (or attraction depending on the sign of $a_s$) of two atoms in the lattice site $i$. When using the Hubbard model two approximations are implied : first, the excitation energies to the second band are high enough and secondly that hopping to next nearest neighbours can be neglected. Using the WBK approximation [48], $U = E_R |a_s| / a\sqrt{8\pi} \left( V_0/E_R \right)^{3/4}$ and $J = \frac{2}{\sqrt{\pi}} E_R \left( V_0/E_R \right)^{3/4} \exp\left( -2\sqrt{V_0/E_R} \right)$ where $V_0$ is the potential height barrier, and $E_R = \hbar^2 k^2/(2m)$ where $k = \pi/a$ and $a$ is the lattice period.

The Hubbard Hamiltonian (3.49) does not need any renormalization of $U$ because it implies a momentum cutoff given by the band width $W = 2zJ$ where $z$ is the coordination number. In the continuum model there is just one dimensionless parameter $1/k_F a_s$ but in the Hubbard Hamiltonian there are two, the interaction strength $U/J$ and the filling factor $n = \frac{1}{M} \sum_{\sigma} \langle n_{\sigma} \rangle$, $n \in [0, 2]$ and $M$ is the number of lattice sites. Note that there is a symmetry between filled and empty spaces around $n = 1$ and one can just consider $n \in [0, 1]$.

In the *weak coupling limit*, the lattice and the continuum model describe the same physics, since the Cooper pair size is much larger than the lattice spacing $a$. This is true as long as there are enough free spaces $n \ll 1$. The effect of the band filling comes as factors $\sqrt{n(n-2)}$ [56] in the expression for the gap (3.46) and the transition temperature $k_B T_c$ (3.44). In the *strong coupling limit*, the physics is quite different because when the pair size becomes comparable to $a$, lattice effects become very important. These pairs

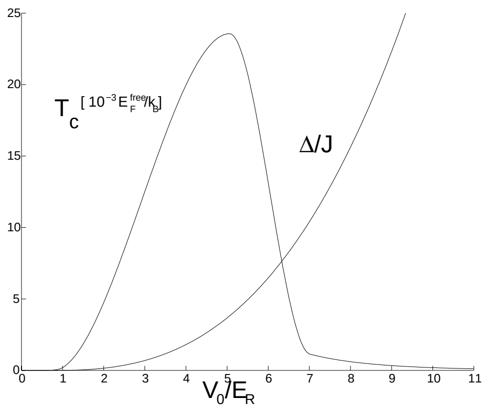


Figure 3.1: The transition temperature for a half filled $CO_2$ laser lattice with $^6$Li atoms with scattering length $a_s = -2.5 \cdot 10^3 a_0$. The $E_R$ is the recoil energy and $E_F^f$ is the Fermi energy for free fermions with the same density. The other line corresponds to the gap in tunneling rate units as a function of the lattice depth $V_0/E_R$.

behave as hard core bosons [56] (the hard core reflecting the Fermi exclusion principle) with an effective hopping $J^2/U$. In the case of *small density* $n \ll 1$, the boson gas is dilute and the effect of the hard core is weak. One recovers the same result as in the continuum, the superfluid transition temperature is given by the BEC transition temperature for bosons of mass $2m_{\text{eff}}$ with the effective mass given $m_{\text{eff}} \sim U/J^2$. For very strong coupling the transition temperature tends to zero (in the continuum it was constant). On the contrary, hard core correlations are very important for densities close to *half filling* $n \sim 1$ and e.g the gap reads $\Delta = \frac{1}{2}|U|\sqrt{n(2-n)}$. For a given band filling, the transition temperature is small for both the BCS and the BEC limits. The maximum of $T_c$ and $\Delta$ is achieved for some value of $U/J$ in between. In three-dimensional simple cubic lattice at half filling, numerical simulations [57] give a maximum of $T_c \sim 0.55J$ for $U/J \sim 10$ and $\Delta \sim 0.9U$. The transition temperature and the superfluid gap for $^6$Li atoms with scattering length $a_s = -2.5 \cdot 10^3 a_0$ in a half-filled 3D $CO_2$ laser lattice $a = 10\mu$m are depicted in figure 3.1 as a function of the lattice depth $V_0$.

## 3.6 Superfluidity with a Feshbach resonance

A Feshbach resonance occurs when the collision energy of two free atoms coincides with that of a molecular state in a closed channel. Atoms populating this closed channel form composite bosons and they can be included in the formalism describing the process. As discussed in Section 2.3 the scattering length changes dramatically in the vicinity of a Feshbach resonance. The model Hamiltonian including explicitely the coupling between the atomic

($\hat{a}_k$) and molecular gases ($\hat{b}_q$) reads [37, 39, 38]

$$\hat{K} = \sum_q (\nu + E_q^0 - 2\mu)\, \hat{b}_q^\dagger \hat{b}_q + \sum_k \xi_k (\hat{a}_{k\uparrow}^\dagger \hat{a}_{k\uparrow} + \hat{a}_{k\downarrow}^\dagger \hat{a}_{k\downarrow}) + U_{bg} \sum_{k,k'} \hat{a}_{k\uparrow}^\dagger \hat{a}_{-k\downarrow}^\dagger$$
$$+ g \sum_{q,k} \left( \hat{b}_q^\dagger \hat{a}_{k+q/2\uparrow} \hat{a}_{-k+q/2\downarrow} + \hat{b}_q \hat{a}_{-k+q/2\downarrow}^\dagger \hat{a}_{k+q/2\uparrow} \right) \tag{3.52}$$

where the chemical potentials $\mu = \mu_{F\uparrow} = \mu_{F\downarrow} = \mu_B/2$ have been included. The coupling between bosons and fermions is characterized by the parameter $g$ and $\nu$ is the threshold energy of the composite Bose particle energy band.

Kokkelmans et al. [58] considered the resonances for the experimentally trapped hyperfine states of $^6$Li where a double resonance occurs. They explicitely obtain

$$a = a_{bg} - \frac{m}{4\pi\hbar^2} \left( \frac{|g_1|^2}{\nu_1} + \frac{|g_2|^2}{\nu_2} \right) \tag{3.53}$$

where $\nu_1$ and $g_1$ are the detuning and the coupling strength to state 1. In Eq.(2.32), $\nu = E_{\text{res}} - E_{\text{th}}$. Note that this is equivalent to defining an effective potential

$$U_{eff} = U_{bg} - g^2 \frac{1}{\nu - 2\mu}, \tag{3.54}$$

where we include the effective interaction between the fermions mediated by a molecular boson.

One should also note that the ultraviolet divergences of the Hamiltonian (3.52) where $g$, $U_{bg}$ and $\nu$ are taken to be constants have to be removed. The scattering equations of both channels (open and closed channels) are used to obtain the Lippman-Schwinger equation [58]

$$T(0) = V^R - \alpha V^R T(0) + \frac{|g_1|^2 (1 - \alpha T(0))}{-\nu_1} + \frac{|g_2|^2 (1 - \alpha T(0))}{-\nu_2} \tag{3.55}$$

where $\alpha = mK/(2\pi^2\hbar^2)$ and $K$ is the arbitrary momentum cut-off and $U^R$ is the effective renormalized potential in the open channel.
Using $U_{bg} = 4\pi\hbar^2 a_{bg}/m$ and $\Gamma = (1 - \alpha U_{bg})^{-1}$ one can see that the renormalization of the coefficients is done by [58] $U^R = \Gamma U_{bg}$, $g_1^R = \Gamma g_1$, $\nu_1^R = \nu_1 + \alpha g_1 g_1^R$, $g_2^R = g_2/(\alpha g_1^2/\nu_1 + \Gamma^{-1})$ and $\nu_2^R = \nu_2 + \alpha g_2 g_2^R$.

The Hamiltonian (3.52) was solved using the Hartree-Fock-Bogoliubov (mean field) theory by the JILA theory group [37] and E. Timmermans et al. [38]. This approximation takes both the pairing field $\Delta = U_{bg} \sum_k \langle a_{-k\downarrow} a_{k\uparrow} \rangle$

and the molecule field $\phi_m = \langle b_{k=0} \rangle$ to be classical fields. The value $\phi_m$ is obtained from the static solution of the bosonic equation which gives

$$\phi_m = \frac{g}{\nu - 2\mu} \frac{\Delta}{U_{bg}}. \tag{3.56}$$

The fermionic fields satisfy the BCS algebra, with a composite order parameter $\tilde{\Delta} \equiv \Delta - g\phi_m$ that satisfies the following gap equation

$$\tilde{\Delta} = U_{\text{eff}} \sum_p \frac{\tilde{\Delta}}{2E_p} \tanh\left(\frac{\beta E_p}{2}\right). \tag{3.57}$$

The theory group at JILA has studied the thermodynamic properties of the mean field Hamiltonian. Timmermans et al. [38] calculated the dynamics and showed that the system responds to changes in the threshold energy $\nu$ by means of the Josephson-like oscillations of the atomic and molecular populations. This coherent oscillation between the molecular condensate and the atomic pairs would happen on times of the order $\hbar/\epsilon_F$. In the first publications [37, 38] the number equation $N = 2\Phi_m^2 + \sum_p \left(1 - \xi_p/E_p \tanh \frac{\beta E_p}{2}\right)$ was used.

In the homogeneous case, it was shown by Nozières and Schmitt-Rink [49] that one should go beyond the mean field approximation and that including Gaussian or second order fluctuations around the mean field order parameter $\Delta$ and $\phi_m$ leads to a considerable correction in the transition temperature $T_c$. These second order fluctuations correspond to considering Cooper pairs and molecules of momentum $\mathbf{q} > 0$ (non condensed bosons). They leave the gap equation unchanged but there will be new terms in the number equation [49]. Including these terms sets the maximum transition temperature for the homogenoeus case around $T_c = 0.218\epsilon_F$ corresponding to the condensation of bosons of mass $2m$ and density $n/2$. The Feshbach resonances have been included in this scheme by Griffin et al. [59] and later in [60] giving the maximum transition temperature of $T_c/T_F \approx 0.25$.

The first experimental observation of molecules created out of fermionic atoms has recently been reported [17, 19, 22, 23]. This corresponds to the repulsive part of the Feshbach resonance where the energy level of the bound molecular state is below the energy of the free atoms. The Bose Einstein condensation of these molecules would constitute the first step in achieving fermionic superfluidity. Adiabatically sweeping the magnetic field across the resonance might be the way to obtain a weakly coupled superfluid Fermi gas of alkali atoms [26].

The creation of bosonic molecules out of fermionic atoms is reversible. The molecules remain trapped in the optical trap and the phase space densities and temperatures are just a factor of two [22] from Bose condensation at the time of this writing. The inelastic molecule-molecule and atom-molecule collision rates depend on $\sim 1/a^3$ [61], where $a$ is the effective scattering length. Thus, the molecule life time can be very long in the center of the resonance and longest times reported are 10 s [23]. The estimated molecule-molecule elastic collision rate is of the order of 1 $\mu$s and therefore it should be possible to evaporate the molecules further to reach the Bose Einstein condensation threshold.

## 3.7 Current carrying state

In this section we summarize the solution of the BCS equations for a current carrying state. The current carrying superconducting state for a homogeneous system [62] was considered just after the derivation of the BCS theory. We generalize here the BCS algebra for a current carrying state for a space dependent potential. This algebra has been used when considering the Bloch oscillations of superfluid Fermi gas in publication VI.

In the BCS ansatz, a common momentum $\mathbf{q}$ can be added to all particles, and the momentum of one pair becomes $2\mathbf{q}$ in the homogeneous case. This can be formally done solving (3.29) with the ansatz

$$\begin{aligned} u_{\mathbf{k}}(\mathbf{r}) &= e^{i(\mathbf{k}+\mathbf{q})\cdot\mathbf{r}}\phi_{\mathbf{k}+\mathbf{q}}(\mathbf{r})\tilde{u}_{\mathbf{k}}^{\mathbf{q}} \\ v_{\mathbf{k}}(\mathbf{r}) &= e^{i(\mathbf{k}-\mathbf{q})\cdot\mathbf{r}}\phi_{\mathbf{k}-\mathbf{q}}(\mathbf{r})\tilde{v}_{\mathbf{k}}^{\mathbf{q}}. \end{aligned} \tag{3.58}$$

One can define a current dependent superfluid gap

$$\Delta_{\mathbf{k}}^{\mathbf{q}} \equiv \int \Delta(\mathbf{r})e^{-i2\mathbf{q}\mathbf{r}}\phi_{\mathbf{k}+\mathbf{q}}^{\dagger}(\mathbf{r})\phi_{\mathbf{k}-\mathbf{q}}(\mathbf{r}). \tag{3.59}$$

Defining the average energy $E_A = \frac{1}{2}(\xi_{\mathbf{k}+\mathbf{q}} + \xi_{\mathbf{k}-\mathbf{q}})$ and the energy difference $E_D = \frac{1}{2}(\xi_{\mathbf{k}+\mathbf{q}} - \xi_{\mathbf{k}-\mathbf{q}})$, one obtains

$$\begin{aligned} E_{\mathbf{k}}^{\pm\mathbf{q}} &= -E_D \pm \sqrt{E_A^2 + |\Delta_{\mathbf{k}}^{\mathbf{q}}|^2} \\ |\tilde{u}_{\mathbf{k}}^{\pm\mathbf{q}}|^2 &= \frac{1}{2}(1 \pm \frac{E_A}{\sqrt{E_A^2 + |\Delta^{\mathbf{q}}|^2}}) \end{aligned}$$

$$
\begin{aligned}
|\tilde{v}_{\mathbf{k}}^{\pm\mathbf{q}}|^2 &= \frac{1}{2}(1 \mp \frac{E_A}{\sqrt{E_A^2 + |\Delta^{\mathbf{q}}|^2}}) \\
\tilde{u}_{\mathbf{k}}^{\pm\mathbf{q}} \tilde{v}_{\mathbf{k}}^{\pm q\dagger} &= \frac{\Delta_{\mathbf{k}}^{\mathbf{q}}}{\sqrt{E_A^2 + |\Delta_{\mathbf{k}}^{\mathbf{q}}|^2}}.
\end{aligned} \tag{3.60}
$$

as solutions of the BdG equations (3.29). Note that the $^+$ solutions correspond to the "particle branch" and the $^-$ solutions to the "hole" branch.

It is well known that a BCS-superconductor can carry a persistent current $\mathbf{q}$ until a $T = 0$ critical velocity, $v_c = \Delta/p_F$. For higher current values, even at $T = 0$, it is energetically favorable to break Cooper pairs and create a pair of quasiparticles [62]. This costs $2\Delta$ in binding energy and decreases the quasiparticle energy by $|\xi_{\mathbf{k}+\mathbf{q}} - \xi_{\mathbf{k}-\mathbf{q}}| = 2|E_D|$. Therefore, the condition for the current to be stable is

$$
|E_D| < \Delta. \tag{3.61}
$$

This is the *Landau criterion of superfluidity.* One has to impose $E_{\mathbf{k}}^{+\mathbf{q}} > 0$, to obtain sensible solutions of the BCS equations because the transformed Hamiltonian (3.24) has to be positive defined. This is the same condition as the Landau criteria i.e. $\min(\sqrt{E_A^2 + |\Delta^{\mathbf{q}}|^2}) = |\Delta^{\mathbf{q}}| > E_D$.

Inserting the ansatz (3.58) in the gap equation (3.28) gives

$$
\Delta^{\mathbf{q}}(\mathbf{r}) = |U| \sum_{\mathbf{n}} \tanh\left(\frac{\beta E_{\mathbf{n}}^{\mathbf{q}}}{2}\right) \frac{\Delta_{\mathbf{n}}^{\mathbf{q}}}{\sqrt{E_A^2 + |\Delta_{\mathbf{n}}^{\mathbf{q}}|^2}} \phi_{\mathbf{n}-\mathbf{q}}^{\dagger}(\mathbf{r}) \phi_{\mathbf{n}+\mathbf{q}}(\mathbf{r}) e^{+2i\mathbf{q}\mathbf{r}}, \tag{3.62}
$$

which leads to the space independent gap

$$
\Delta_{\mathbf{k}}^{\mathbf{q}} = |U| \sum_{\mathbf{n}} \tanh\left(\frac{\beta E_{\mathbf{n}}^{\mathbf{q}}}{2}\right) \frac{\Delta_{\mathbf{n}}^{\mathbf{q}}}{\sqrt{E_A^2 + |\Delta_{\mathbf{n}}^{\mathbf{q}}|^2}} B_{\mathbf{n}\mathbf{q}\mathbf{k}}, \tag{3.63}
$$

where $B_{\mathbf{nqk}} = \int d\mathbf{r} \phi_{\mathbf{n}-\mathbf{q}}(\mathbf{r}) \phi_{\mathbf{n}+\mathbf{q}}^{\dagger}(\mathbf{r}) \phi_{\mathbf{k}+\mathbf{q}}(\mathbf{r}) \phi_{\mathbf{k}-\mathbf{q}}^{\dagger}(\mathbf{r})$.

The quasiparticle velocity (current) is defined for the BCS state in terms of $u$ and $v$ [63] like $\langle \mathbf{v} \rangle = \sum_{\mathbf{k}} f(\mathbf{k}) [\langle u_{\mathbf{k}} | \mathbf{v} | u_{\mathbf{k}} \rangle - \langle v_{\mathbf{k}} | \mathbf{v} | v_{\mathbf{k}} \rangle]$. Using the particle and hole velocities the quasi-particle current for the current carrying BCS state is given by

$$
\langle \mathbf{v} \rangle = \sum_{\mathbf{k}} f(\mathbf{k}) \frac{1}{\hbar} \left[ \frac{d\xi_{(\mathbf{k}+\mathbf{q})}}{d\mathbf{k}} |\tilde{u}_{\mathbf{k}}^{\mathbf{q}}|^2 - \frac{d\xi_{(\mathbf{k}-\mathbf{q})}}{d\mathbf{k}} |\tilde{v}_{\mathbf{k}}^{\mathbf{q}}|^2 \right]. \tag{3.64}
$$

Actually, Eq.(3.64) gives the same result as $\langle v_x \rangle = \sum_{\mathbf{k}} f(\mathbf{k}) \frac{1}{\hbar} \frac{dE_{\mathbf{k}}^{+\mathbf{q}}}{dk_x}$. The distribution function for quasi-particles is $f(\mathbf{k}) = f(E_{\mathbf{k}})$.

### 3.7.1 Superfluid velocity

When solving the BCS algebra for a current carrying state a momentum dependent order parameter wave function $\Delta^{\mathbf{q}}(\mathbf{r})$ was introduced. In the homogenoeus case [51] one can introduce $\Delta^{\mathbf{q}}(\mathbf{r}) = e^{i2\mathbf{q}\cdot\mathbf{r}}C$, where $C$ is a constant in $\mathbf{r}$. Expectation values like momentum ($\mathbf{p} = -i\partial/\partial\mathbf{r}$) can be calculated: $\langle\mathbf{p}\rangle = \langle\Delta^{\mathbf{q}}(\mathbf{r})| - i\partial/\partial\mathbf{r}|\Delta^{\mathbf{q}}(\mathbf{r})\rangle/\langle\Delta^{\mathbf{q}}(\mathbf{r})|\Delta^{\mathbf{q}}(\mathbf{r})\rangle = 2\mathbf{q}$. This is the momentum of one pair in the superfluid. The order parameter wave function is defined in the spirit of the Ginzburg-Landau [8] theory with a space dependent wave function whose absolute value equals the gap. We generalize the above formalism to define a superfluid velocity in the inhomogeneous case

$$\langle\mathbf{v}\rangle = \mathcal{N}\langle\Delta^{\mathbf{q}}(\mathbf{r})|\dot{\mathbf{r}}|\Delta^{\mathbf{q}}(\mathbf{r})\rangle \tag{3.65}$$

where $\mathcal{N} = \langle\Delta^{\mathbf{q}}(\mathbf{r})|\Delta^{\mathbf{q}}(\mathbf{r})\rangle^{-1}$. The gap Eq. (3.62) can be rewritten as $\Delta^{\mathbf{q}}(\mathbf{r}) = \sum_{\mathbf{k}} \Delta^{\mathbf{q}}_{\mathbf{k}}(\mathbf{r})$, where

$$\Delta^{\mathbf{q}}_{\mathbf{k}}(\mathbf{r}) = F(\mathbf{k},\mathbf{q})\phi_{\mathbf{k}+\mathbf{q}}e^{i(\mathbf{k}+\mathbf{q})\cdot\mathbf{r}}\phi^{\dagger}_{\mathbf{k}-\mathbf{q}}e^{-i(\mathbf{k}-\mathbf{q})\cdot\mathbf{r}} \tag{3.66}$$

and $F(\mathbf{k},\mathbf{q}) = |U|[1-2f(E^{+\mathbf{q}}_{\mathbf{k}})]\tilde{u}^{\mathbf{q}}_{\mathbf{k}}\tilde{v}^{\mathbf{q}*}_{\mathbf{k}}$. One can interpret Eq. (3.66) as an internal wave funtion of a pair carrying momentum $2\mathbf{q}$ in the composite boson limit, resembling the internal wave funtions introduced by Leggett [54] and Nozières and Schmitt-Rink [49]. Using $\langle\phi^{\dagger}_{\mathbf{k}-\mathbf{q}}e^{-i(\mathbf{k}-\mathbf{q})\cdot\mathbf{r}}|\dot{\mathbf{r}}|\phi^{\dagger}_{\mathbf{k}-\mathbf{q}}e^{-i(\mathbf{k}-\mathbf{q})\cdot\mathbf{r}}\rangle = -\frac{1}{\hbar}\frac{d}{d\mathbf{k}}\xi_{\mathbf{k}-\mathbf{q}}$, the superfluid velocity becomes

$$\langle\mathbf{v}\rangle = \mathcal{N}\sum_{\mathbf{k}}|F(\mathbf{k},\mathbf{q})|^2\left[-\frac{d\xi_{\mathbf{k}-\mathbf{q}}}{d\mathbf{k}} + \frac{d\xi_{\mathbf{k}+\mathbf{q}}}{d\mathbf{k}}\right]. \tag{3.67}$$

# Chapter 4

# Signatures of superfluidity

In recent years many different theoretical proposals for detecting experimentally the superfluidity in fermionic gases have been made. They mainly involve the response of the atomic cloud to a modulation of the trapping frequency [64] and the excitation of low energy collective modes in a trap including the monopole, quadrupole and spin dipole modes [65]. Other type of proposals point out signatures in the far off-resonant light scattered by the atomic cloud [66] or in the absorption spectra [67]. The density-density correlations in the image of an expanding gas have also been predicted to indicate the superfluid behaviour [68]. The momentum of inertia of the Fermi cloud has been shown to decrease for a superfluid gas [69] with respect to the rigid body value in the normal case. The difference in the density profile [70] of a normal gas and a superfluid one close to a Feshbach resonance has been pointed out too. The anisotropy in the expansion [40] of a superfluid gas can be contrasted to the isotropy of the momentum distribution of collisionless normal Fermi gas. Nevertheless, a strongly interacting Fermi gas close to a Feshbach resonance and a superfluid Fermi gas can be both described with the hydrodynamic equations and both expand anisotropically [41]. The only difference between a superfluid and a strongly interacting Fermi gas is that the velocity field for the former is irrotational for angular velocities below the creation of vortices. One can exploit this difference to detect superfluidity and the splitting of the quadrupole modes [41] for a rotating normal hydrodynamic gas has been proposed as a signature.

In publications I to VI we have studied different aspects of Fermi gases related to their superfluidity. Mainly we have been looking for signatures of superfluidity in two component superfluid Fermi gases (denoted by $\uparrow$ and $\downarrow$

in the following ) in the weak coupling regime. In papers I to IV we consider almost on-resonant laser coupling between different hyperfine states. This is reviewed in section 4.1. One of the main phenomena inherent to superfluids is the existence of quantized vortices. We have studied the singly quantized vortex state in trapped fermionic gases in publication V. An introduction to the paper is given in section 4.2. In publication VI we consider the amplitude of Bloch oscillations (BO) as a signature of superfluidity in a Fermi gas in an optical lattice. We analyze the Bloch oscillations in both the degenerate and the superfluid state and show how the amplitude varies along the BCS-BEC superfluid crossover. Section 4.3 reviews the main results of the publication.

## 4.1 Laser probing

Electron tunneling has given the most detailed experimental examination of the density of states in metallic superconductors. This powerful technique was pioneered by Giaver [71] who used it to confirm the density of states and temperature dependence of the energy gap predicted by BCS. We have considered the analogy of electron tunneling from a normal metal (or superconductor) to a superconductor induced by an external voltage [72] in Fermi gases. We consider atoms in different internal hyperfine states coupled with almost on-resonant laser light. The coupling of a normal and a superfluid state was considered in publication I. Publications III and IV deal with the coupling of two superfluid states. For small intensities, the laser interaction can be treated as a perturbation. In our case the tunneling is between two internal states rather than two spatial regions, resembling the idea of internal Josephson oscillations in two component Bose Einstein condensates [73]. A major advantage of fermionic atoms compared to electrons in condensed matter is the richness of their internal energy structure and the possibility to accurately and efficiently manipulate these energy states by laser light. This leads to new effects, unique to atomic Fermi gases like an asymmetry in the Josephson current (see publication III) or the possibility of directly probing the spatial coherence of the superfluid (see publication IV).

In publication II we have analyzed the linear density response of a perturbation created by laser light to see if it excites any below-gap mode that would wash out the results in [74] and publications I, III and IV. Those publications rely on the validity of the BCS or static mean field theory. A time dependent Hartree-Fock-Bogoliubov approach shows the existence of below gap excitations with a phonon-like dispersion relation [50]. We have shown that because the BCS Hamiltonian is preserved under rotation, most of the considered laser schemes can be transformed into a density perturbation. In this form the perturbation potential is proportional to $|\Omega|^2$ where $\Omega$ is the (effective) Rabi frequency. Therefore, even when the laser light provides momentum and energy ($\Omega \propto e^{i\mathbf{k_L}\cdot\mathbf{r}}e^{i\omega_L t}$), the transformed potential acting as a density perturbation is not time- and space-dependent. This leads to absence of a density response whenever $|\Omega|^2$ is a constant spatially and temporally. For Bragg scattering [75], $|\Omega|^2 \propto f(\mathbf{r},t)$. In this case Anderson-Bogoliubov phonons [50] can be excited, in general. The exception is the case when the laser(s) couple between the two paired components of the gas (the states $\uparrow$ and $\downarrow$). We have shown that, in the homogeneous case, the density response becomes zero because the contributions of the components cancel each other. In a harmonic trap, spin-dipole response is predicted for temperatures near $T_c$. Therefore, the presence or absence of low-energy collective excitations under a perturbation of the type $U(\omega,k)(n_\uparrow - n_\downarrow)$ could be used to observe whether the trapped system can be approximated by a homogeneous system (local density approximation) or whether the trapping effects are dominant.

### 4.1.1 Normal-Superfluid Interface

The idea of using a normal-superfluid interface for probing superfluidity in atomic Fermi gases was proposed in [74]. It consists on transferring atoms from one internal hyperfine state for which the atoms are Cooper paired to another state for which the interatomic interaction is not strong enough to lead to a BCS state. This effectively creates a superconducting-normal state interface across which the atomic population can move.

The basic result in the proposal [74] is that the absorption peak is shifted and becomes asymmetric because of the existence of the gap – the laser has to provide energy for breaking the Cooper pairs in order to transfer atoms from the paired state to the unpaired one. The assymetry is due to the different density of states in the superfluid case in contrast to a mean field shift that would leave the symmetry unchanged. This behaviour is, however, strongly influenced by the specific physical situation. In publication I we investigated in detail how the choice of the chemical potentials for the superfluid and the normal state, and the choice of the interaction strengths and laser profiles affect the absorption. We also compared the results in the cases of a homogeneous system and a trapped gas.

We consider atoms with three internal states available, say $|e\rangle$, $|g\rangle$, and $|g'\rangle$. They are chosen so that the interaction between atoms in states $|g\rangle$ and $|g'\rangle$ is relatively strong and their chemical potentials are nearly equal so that $|g\rangle$ and $|g'\rangle$ can be assumed to be Cooper-paired. All other interactions are small enough and/or the chemical potentials of the corresponding states are different enough in order to assume the $|e\rangle$ atoms to be in a normal state. The laser frequency is chosen to transfer population between $|e\rangle$ and $|g\rangle$, but is not in resonance with any transition which could move population away from the state $|g'\rangle$. We treat the atom-light interaction within the rotating wave approximation as explained in section 2.1.

The observable carrying essential information about the superfluid state is the change in the population of the state $|e\rangle$, we call this the current

$$I(t) = -\langle \dot{N}_e \rangle = \frac{\partial}{\partial t} \int d\mathbf{x} \langle \Phi(t) | \psi_e^\dagger(\mathbf{x}) \psi_e(\mathbf{x}) | \Phi(t) \rangle.$$

The current $I(t)$ is calculated considering the laser coupling part of the Hamiltonian as a perturbation; it becomes the first order response to the external perturbation caused by the laser.

In the homogenoeus case and equal chemical potentials $\mu_g = \mu_{g'} = \mu_e$ and assuming the laser field to be a constant (i.e. a running wave $\Omega(\mathbf{r}) = \Omega e^{i\mathbf{k_L}\cdot\mathbf{r}}$ with very small momentum $k_L$ compared to the momentum of the atoms that is the case for a Raman process), the current reads

$$I = -\pi \Omega^2 \frac{\Delta^2}{\delta^2} \rho(\delta) [\theta(-\delta - \Delta) - \theta(\delta - \Delta)], \tag{4.1}$$

where $\Omega$ is the Rabi frequency of the laser, $\delta$ the laser detuning, $\Delta$ the BCS gap and the density of states $\rho(\delta) = \frac{V}{2\pi^2}\sqrt{(\Delta^2 - \delta^2)/\delta + 2\mu_g}$. The term with the $-$ sign corresponds to $\delta < 0$ i.e. current from $|g\rangle$ to $|e\rangle$, and the positive term to current from $|e\rangle$ to $|g\rangle$. In order to transfer one atom from the state $|g\rangle$ to $|e\rangle$ the laser has to break a Cooper pair. The minimum energy required for this is the gap energy $\Delta$, therefore the current does not flow before the laser detuning provides this energy – this is expressed by the first step function in (4.1). As $|\delta|$ increases further, the current will decrease quadratically. This is because the case $|\delta| = \Delta$ corresponds to the transfer of particles with $p = p_F$, whereas larger $|\delta|$ means larger momenta, and there are simply fewer Cooper-pairs away from the Fermi surface. This behaviour is very different from the electron tunneling where the current grows as $\sqrt{(eV)^2 - \Delta^2}$ [72] (the voltage $eV$ corresponds to the detuning $\delta$ in our case) because all momentum states are coupled to each other. The second step function in (4.1) corresponds to tunneling into the superconductor. In this case one has to provide extra energy because a single particle tunneling into a superconductor becomes a quasi-particle excitation with the minimum energy given by the gap energy.

The assumption of spatial homogeneity is appropriate when the atoms are confined in a trap potential which changes very little compared to characteristic quantities of the system, such as the coherence length and the size of the Cooper pairs. This assumption is also valid when the laser profile is chosen so that it only probes the middle of the trap where the order parameter is nearly constant in space.

In the case of harmonic confinement, the current is

$$\begin{aligned} I &= -2\pi \sum_{n,m} \left| \int d^3x \Omega(\mathbf{x}) u_n(\mathbf{x}) \phi_m(\mathbf{x}) \right|^2 [n_F(\omega_n) - n_F(\xi_m)] \delta(\xi_m + \tilde{\delta} - \omega_n) \\ &\quad + \left| \int d^3x \Omega(\mathbf{x}) v_n^*(\mathbf{x}) \phi_m(\mathbf{x}) \right|^2 [n_F(-\omega_n) - n_F(\xi_m)] \delta(\xi_m + \tilde{\delta} + \omega_n), \quad (4.2) \end{aligned}$$

where $n_F$ is the Fermi distribution function at temperature $T$, $\omega_n$, $u_n(\mathbf{r})$ and $v_n(\mathbf{r})$ are given by the BdG equations (3.27) for the state $|g\rangle$ and $\xi_m = E_m - \mu_e$, where $E_m$ are the single particle energies for a trap potential for the state $|e\rangle$ and $\tilde{\delta} = \delta + \mu_e - \mu_g$. This is the standard Fermi Golden rule result. Due to the non-orthogonality of the trap and the BdG wavefunctions transitions between many quantum numbers are allowed and the total current is the sum of all these. For a harmonic trapping potential there are in-gap low-energy excitations that make it more difficult to resolve the gap energy. In publication I we show that even for a harmonic trapping configuration one can still see clearly the effect of the superfluidity for different configurations. In the case of a constant laser profile the most clear case is when the Hartree fields seen by the $|g\rangle$ and $|e\rangle$ atoms are the same ($g_{gg'} = g_{ge} + g_{g'e}$). In this case the normal-normal current is a Lorentzian centered at $\delta = 0$. The superconductor-normal current is asymmetric and is shifted to negative $\delta$. In order to enhance the effect of pairing on the observed signal one could also trap initially some atoms in the state $|e\rangle$ taking care that $\mu_g - \mu_e \gg \Delta$ so that $|e\rangle$ and $|g\rangle$ atoms do not pair, even if $g_{eg} < 0$. By having the lower states of the $|e\rangle$ atoms filled, there will be only transitions around the Fermi energy, the states most influenced by pairing. It is important to have the Hartree fields of $|e\rangle$ and $|g\rangle$ atoms approximately the same because otherwise not even the normal-normal current is a Lorentzian. To avoid the problems arising from nonhomogenous trapping potential, we propose to probe only the middle of the trap. The order parameter is effectively homogenoeus in the middle and the in-gap excitations are located away from the center. In practice this kind of probing can be done by using two orthogonal Raman beams that intersect in the middle of the trap.

It is worth noting that recently a similar scheme has been used experimentally [76] to measure the mean field shift in strongly interacting Fermi gases. The three level scheme is such that when applying a magnetic field there is a Feshbach resonance between $g$ and $g'$. Initially, a mixture of atoms in the $g$ and $e$ states are trapped and a rf-field is applied to transfer atoms from the $e$ to the $g'$ state. When the Feshbach resonance is on there is a shift in the rf spectra due to the different mean field interactions $\delta\nu = \frac{2\hbar}{m} n_g(a_{gg'} - a_{eg})$ that can be used for direct measuring the scattering length $a_{gg'}$ in the presence of a resonance. Similar methods have been used in $^6$Li atomic clouds [77]. One can infer from Ref. [76, 77] that the scheme considered in publication I can be experimentally realized for probing the superfluid state in Fermi gases. We expect the asymmetry in the shifted absorption curves to signal superfluidity within the $g$ and $g'$ states, in contrast to the symmetric mean field shift.

### 4.1.2 Superfluid-Superfluid Interface

#### Josephson effect in superfluid Fermi gases

In 1962, Josephson predicted that at zero voltage, the flow between two weakly coupled superconductors is given by $I = I_c \sin \Delta\phi$, where $\Delta\phi$ is the phase difference between the two superconducting order parameters. When a voltage was applied he further predicted that the phase difference would evolve according to $d(\Delta\phi)/dt = 2eV/\hbar$ leading to an electron current between the superconductors that oscillates on time.

To have an analogous effect in atoms we need fermionic atoms in four different hyperfine states (we label them $|g\rangle$, $|g'\rangle$, $|e\rangle$ and $|e'\rangle$) trapped simultaneously. This can be done in an optical trap. The s-wave scattering lengths are assumed to be large and negative between atoms in states $|g\rangle$ and $|g'\rangle$, as well as between those in $|e\rangle$ and $|e'\rangle$, and the chemical potentials $\mu_g \simeq \mu_{g'}$

and $\mu_e \simeq \mu_{e'}$. For all other combinations of two atoms in different states the scattering length is assumed to be small and/or the chemical potentials unequal. This leads potentially to the existence of two superfluids, one consisting of Cooper pairs of atoms in the states $|g\rangle$ and $|g'\rangle$, and the other of $|e\rangle$–$|e'\rangle$ pairs. The two superfluids are coupled by driving laser-induced transitions between the states $|g\rangle$ and $|e\rangle$ with the laser Rabi frequency $\Omega$ and detuning $\delta$, and between the states $|g'\rangle$ and $|e'\rangle$ with the Rabi frequency $\Omega'$ and detuning $\delta'$. For metallic superconductors the d.c. Josephson current is driven by applying a voltage over the junction – here the role of the voltage is played by the laser detunings. The difference is that the detunings can be different for the two states forming the pair; in the metallic superconductor analogy this would mean having a different voltage for the spin-up and spin-down electrons, a situation which has not been investigated in the context of metallic superconductors.

We assume again that the interaction with the laser can be treated as a perturbation and the current (e.g. of atoms in the state $|e\rangle$ ) is calculated using linear response theory.

We split the current $I_e$ into a part corresponding to the Josephson current $I_{eJ}$ and the single particle current $I_{eS}$. Using the BCS approximation for the superfluid states, the result for the single particle current for positive detunings at $T = 0$ is

$$I_{eS} = -2\pi \sum_{n,m} \left| \int d\vec{r}\Omega(\vec{r}) v_n^e(\vec{r}) u_m^g(\vec{r}) \right|^2 \delta(\epsilon_n^e + \epsilon_m^g - \tilde{\delta}). \tag{4.3}$$

Again the triplet $(u_n, v_n); \epsilon_n$ is a solution of the (nonuniform) Bogoliubov-de Gennes equations for superconductors (3.27) and $\tilde{\delta} = \mu_e - \mu_g + \delta$. The current $I_{eS}$ is zero when $\tilde{\delta} < \Delta + \Delta'$ since pair breaking is required for single particle excitations.
The Josephson current reduces to

$$I_{eJ} = I_0(\tilde{\delta}') \sin[(\tilde{\delta} + \tilde{\delta}')t] \tag{4.4}$$

$$I_{e'J} = I_0(\tilde{\delta}) \sin[(\tilde{\delta} + \tilde{\delta}')t] \tag{4.5}$$

for a homogenous geometry and constant laser profile. Both partners of the pair thus oscillate in phase, with the same frequency $\tilde{\delta} + \tilde{\delta}'$. But the amplitudes are different whenever the detunings $\tilde{\delta}$ and $\tilde{\delta}'$ differ. This means that more atoms are transferred, say, in the $|g\rangle - |e\rangle$ oscillation than in the

$|g'\rangle - |e'\rangle$ one. As shown in publication III, the asymmetry is more pronounced in the time scale of the Cooper pairs.

In the conventional intuitive picture of the Josephson effect, the particles forming a Cooper-pair tunnel "together" through the junction. Therefore our result seems counterintuitive at first glance. We show in publication III that the asymmetry is related to the fact that the intermediate states of the transfer processes for "spin up" and "spin down" atoms have different energies and the asymmetry is a result of the dynamics of the superfluid state. An estimate of the atoms involved in the process shows that this asymmetry is within the present experimental detection limits.

### Coupling between the paired states

Atomic Fermi gases provide different degrees of freedom than solid state systems. For example one can study the single particle current between two superfluids just by coupling with a rf field or a Raman transition the states $\uparrow$ and $\downarrow$ forming the pairs. This can be used to directly probe the Cooper pair coherence across different regions in the superfluid by using an interferometric scheme as depicted in figure 1 of publication IV. The laser beam is split and focused into two regions with separation $z$. After the beams have passed through the gas they are recombined and the amount of absorption is measured. As shown in the previous section, one obtains two parts of the current when coupling two superfluids. For $\Omega(\mathbf{r}) = \Omega\,[\delta(z - z_1) + \delta(z - z_2)]$

$$I_{\uparrow,\mathcal{G}} = -\frac{|\Omega|^2}{2} \int u_k^2 v_q^2 \,[1 + \cos(k_z + q_z)z]\, \delta(E_q + E_k - \delta) \tag{4.6}$$

$$I_{\uparrow,\mathcal{F}} = \frac{|\Omega|^2}{2} \int u_k v_k u_q v_q \,[1 + \cos(k_z + q_z)z]\, \delta(E_q + E_k - \delta). \tag{4.7}$$

Here $k = \sqrt{\rho^2 + k_z^2}$, $q = \sqrt{\rho^2 + q_z^2}$, $z = z_1 - z_2$, and the integral symbol means $\int \equiv \int_0^\infty d\rho\rho \int_0^\infty dk_z \int_0^\infty dq_z$. Note that we do not use the same notation as in the previous section but the $\mathcal{G}$ corresponds to the single particle current and the $\mathcal{F}$ to the Josephson current. If the superfluid state is not coherent, $u_k(\mathbf{r})$ and $v_k(\mathbf{r})$ would have phase factors to describe the random space- and time-dependent fluctuations. Those factors make the cosine-dependent term in $I_{\uparrow,\mathcal{G}}$ as well as the whole current $I_{\uparrow,\mathcal{F}}$ to disappear. By varying $z$ and checking the dependence of the current on $z$ one can see if coherence is preserved or not. If the current is constant, oscillating terms are not present and coherence is not preserved for that distance. As shown in publication

IV, the current oscillates at distances of the order of the Cooper pair size $x_{CP} = 1/\Delta\sqrt{\mu/(2m)}$ providing a direct measure of the size of the pairs.

## 4.2 Vortices

The main macroscopic effect of superfluidity is the observation of quantized vortices. Bose Einstein condensates of alkali atoms have been shown to support even vortex lattices becoming giant quantum objects with sizes of $R \sim 1\mu$ m. Once the superfluid state of the fermionic atoms is obtained, the creation of vortices on it seems to be a natural step forward. In publication V we have studied the single vortex solution for a superfluid Fermi gas in the weak coupling regime using the Ginzburg-Landau equation [8]. Baranov and Petrov [78] derived this equation for the trapped atoms in two different hyperfine states using the Thomas-Fermi approximation. Close to $T_c$, the order parameter fulfills a nonlinear Schrödinger equation with effective frequency $\omega^* = \sqrt{(1+2\lambda)/(\lambda m^*)}$, effective mass $m^* = \frac{1}{2}\frac{48\pi^2}{7\zeta(3)}(k_B T_c/\Omega)^2\,(T/T_c)^2$ and effective chemical potential.

The nonlinear term is small compared to the kinetic and potential energies and as elucidated for BEC [79] the healing length (or the vortex core size) is in this case given by the first excited state size or the oscillator ground state length

$$\xi^2 = \frac{1}{m^*\omega^*} = \sqrt{\frac{7\zeta(3)}{24\pi^2}}\left(\frac{T_c}{T}\right)\left(\frac{\Omega}{k_B T_c}\right)\sqrt{\frac{\lambda}{2\lambda+1}}. \tag{4.8}$$

where $\lambda = 2p_F^{(0)}|a|/(\hbar\pi)$ and $\Omega$ is the trapping frequency. Note that this leads to a healing length that does not diverge as $T \to T_c$ in contrast to what happens for helium and superconductors.

## 4.3 Bloch oscillations

Bloch oscillations are a pure quantum phenomenon that occurs when a particle in a periodic potential moves under the influence of a constant force. The particle oscillates instead of moving uniformly. Bloch oscillations were predicted [80] when analyzing the electrical conductivity in crystal lattices.

A periodic potential of period $a$ leads to a band structure in the energy spectrum $\varepsilon_n(k)$ of the particle. The eigenstates (Bloch states) can be

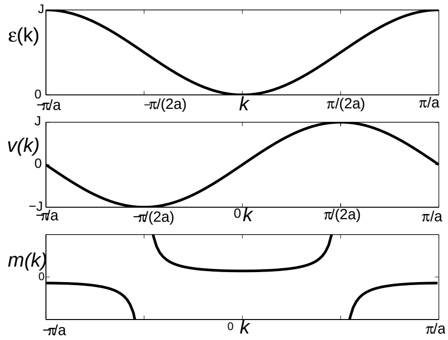


Figure 4.1: Energy, velocity and effective mass of an atom in a periodic potential as a function of the momentum using the tight-binding dispersion relation for the energy.

represented by $|n, k\rangle$, where $n$ labels the band index and $k$ is a continuous quasimomentum. Both $\varepsilon_n(k)$ and $|n, k\rangle$ are periodic functions of $k$ with period $2\pi/a$ and therefore $k$ is conventionally restricted to the first Brillouin zone $]-\pi/a, \pi/a]$.

Semiclassically, when a particle in a given Bloch state $|n, k_0\rangle$ is affected by a constant external force $F$ weak enough not to induce interband transitions it evolves up to a phase factor into the state $|n, k(t)\rangle$ according to

$$k(t) = k_0 + \frac{Ft}{\hbar}. \tag{4.9}$$

This time evolution of the momentum is periodic with a period $\tau_B = h/|F|a$, corresponding to the time required for the quasimomentum to scan the whole Brillouin zone. Thus, when a constant force $F$ is applied, a wave packet with a well defined quasimomentum $k$ (the band index is omitted from now on), will have a time dependent velocity with zero mean. If the force is applied adiabatically and so that there are no interband transitions, the constant force provides momentum to the system but not energy because the effective mass is not always positive as shown in Fig. 4.1.

Bloch oscillations have never been observed in a natural lattice for electrons because the scattering time of the electrons by lattice defects or impurities is much shorter than the Bloch period. However, they have been recently observed in semiconductor superlattices where the lattice period has been increased (from less than a nm to few tens of nm) and the band width decreased [81]. Impurity scattering can be made negligible for atomic gases. Atomic Bose gases have been proved to support Bloch oscillations both at temperatures above condensation [82] and in the superfluid regime [83]. In publication VI we have considered Bloch oscillations in fermionic atomic gases both in the degenerate and the superfluid regime. We consider atoms in a 3 dimensional simple cubic lattice and a constant force $\mathbf{F} = F_x\hat{\mathbf{x}}$ applied in one of the directions. The potential barrier is high enough to use the tight-binding approximation [84] for the lowest energy band dispersion relation $\varepsilon(\mathbf{k}) = J[3 - \cos(k_x a) - \cos(k_y a) - \cos(k_z a)]$. The Hamiltonian of the system can be mapped into the attractive Hubbard model with only onsite interactions $U$. For the experimental regime the interaction $U$ is smaller than the band gap. Thus, the energies involved are not enough to induce interband transitions and the one band approximation is valid.

In the degenerate state, fermions occupy many states and we calculate the velocity by averaging over the whole Fermi sea and show that increasing temperatures and band fillings decrease the oscillation amplitude. In the superfluid case the Landau criterion Eq.(3.61) imposes Cooper pair sizes or the order or smaller than the lattice spacing for observing robust bloch oscillations in the presence of momentum changing collisions. We calculate the superfluid velocity in the periodic potential and show that pairing supresses Bloch oscillations. Inserting the tight-binding dispersion relation in the superfluid velocity Eq.(3.67) we obtain

$$\langle v_{xS} \rangle = \frac{Ja}{\hbar} \sin(qa) \mathcal{N} \sum_{\mathbf{k}} \left| \frac{[1 - 2f(E_{\mathbf{k}}^{\mathbf{q}})]\,\Delta^{\mathbf{q}}}{\sqrt{E_A^2 + |\Delta^{\mathbf{q}}|^2}} \right|^2 \cos k_x a.$$

The $\mathbf{k}$-dependent prefactor in the sum over momentum states reflects the occupation over the Fermi sea. For very strong interactions the wave function is a $\delta$-function in space and integration extends to the whole $\mathbf{k}$ space and the velocity goes to zero. This case corresponds to no empty spaces available for the atoms to move. On-site wave functions produce non-vanishing velocity amplitudes and the amplitude of the oscillations varies dramatically along the BCS-BEC crossover as shown in publication VI.

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