A Lower Bound for a Ramsey Number

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Abstract

A new lower bound for a Ramsey number, $R(5,9) > 120$, is presented. The bound is obtained by constructing a two-color edge-coloring of the complete graph with 120 vertices such that the coloring contains neither a clique of size 5 in one color nor a clique of size 9 in the other. The construction was found by using tabu search and restricting the search to cyclic edge-colorings.

1 Introduction

For every k and l, the integer $n = R(k,l)$ is the least integer such that all two-color edge-colorings of the complete graph $\mathbf{K}_n$ contain a $\mathbf{K}_k$ in the first color or a $\mathbf{K}_l$ in the second color. In 1930, Ramsey proved in his famous paper [5] that such a least integer always exists.

Ramsey's theorem raised the question of determining the values of $R(k,l)$, the Ramsey numbers. However, evaluating Ramsey numbers is difficult. A few values corresponding to small values of k and l are known exactly, but in most cases only bounds are known. Many results and bounds for small Ramsey numbers may be found in Radziszowski's regularly updated survey [4].

For $R(5,9)$, Calkin, Erdős, and Tovey [1] gave the first non-trivial lower bound $R(5,9) > 113$ by constructing a two-color edge-coloring of $\mathbf{K}_{113}$ with no $\mathbf{K}_5$ in the first color nor a $\mathbf{K}_9$ in the second color. The construction was found by searching exhaustively the two-color edge-colorings of $\mathbf{K}_{113}$ with cyclic symmetry.

According to [4] the bound $R(5,9) > 115$, given by Exoo [2], is the best previously known lower bound. The corresponding construction was found by using a local search method to search for an edge-coloring with cyclic symmetry.

In this article the lower bound $R(5,9) > 120$ is presented. The corresponding construction was found by applying tabu search and, again, restricting the search to edge-colorings with cyclic symmetry.

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2 The search method

We use tabu search in the style of Piwakowski [3] to search for a two-color edge-coloring of $\mathbf{K}_n$ with no $\mathbf{K}_k$ in the first color and no $\mathbf{K}_l$ in the second color as a subgraph. Tabu search is a local search method; in every iteration the current solution is replaced by one of its neighbors. Tabu search is stochastic by nature: it is not guaranteed to always find an optimal solution even if one exists. Nevertheless, tabu search is sometimes very effective in solving combinatorial problems.

The tabu search algorithm used can be described on a general level as follows:

- Take a random solution as the current solution.
- While the current solution has a positive cost, repeat:
  - Calculate the cost of all non-tabu solutions in the neighborhood of the current solution.
  - Make the non-tabu neighbor with the lowest cost the new current solution. If there are several neighbors with the lowest cost, one of them is chosen at random.

To complete the description of the algorithm, we next define the solution space, neighborhood, cost function, and tabu condition used.

The solution space used is not the set of all two-color edge-colorings of the $\mathbf{K}_n$. Instead, we restrict the size of the search space by partitioning the edges into edge sets and requiring that all edges in each edge set be colored with the same color. The solution space is then the set of two-colorings of the edge sets. In particular, the vertices are identified with the integers $1, \ldots, n$, and the edges are partitioned according to the distance of their endpoints into $\left\lfloor \frac{n}{2} \right\rfloor$ edge sets $\mathcal{E}_d$:

$$\mathcal{E}_d = \{\{i, j\} : \text{dist}\,(i, j) = d\}\,,\ 1 \leq d \leq \left\lfloor \frac{n}{2} \right\rfloor,$$
$$\text{where dist}(i, j) = \min\left(|i - j|\,, n - |i - j|\right).$$

This is equivalent to restricting the search to edge-colorings whose automorphism group has the cyclic group acting on the vertices as a subgroup.

The neighborhood of a coloring is the set of colorings obtainable from the coloring by changing the color of one edge set.

The cost of a coloring is the number of $\mathbf{K}_k$'s in the first color plus the number of $\mathbf{K}_l$'s in the second color.

A neighboring coloring is tabu, if the edge set where the current and neighboring colorings differ has been recolored within the past t iterations. Here t, the length of the tabu list, is a search parameter.

3 The result

Using the technique described above with $n = 120$, $k = 5$, $l = 9$, and $t = 12$, we found a two-color edge-coloring of $\mathbf{K}_{120}$ that contains neither a $\mathbf{K}_5$ in the first color nor a $\mathbf{K}_9$ in the second color. With

$$\begin{aligned} S = \{&2, 3, 6, 7, 13, 15, 17, 18, 19, 20, 22, 23, 28, \\ &29, 31, 33, 41, 42, 43, 45, 48, 52, 53, 54, 60\}, \end{aligned}$$

the coloring may be constructed by coloring the edges in edge sets $\mathcal{E}_d$ for which $d \in S$ with the first color and the remaining edges with the second color. Using a computer it is not hard to verify that the resulting coloring contains no $\mathbf{K}_5$ in the first color and no $\mathbf{K}_9$ in the second color. Thus, this construction yields the new lower bound $R(5, 9) > 120$.

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