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Classification of whist tournaments with up to 12 players

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Abstract

A v-player whist tournament Wh(v) is a schedule of games, each involving two players opposing two others. Every round, the players are partitioned into games, with at most one player left over. Each player partners every other player exactly once and opposes every other player exactly twice during the tournament. Directed whist tournaments DWh(v), and triplewhist tournaments TWh(v), are Wh(v) with certain additional requirements. In this work the nonisomorphic Wh(v), DWh(v), and TWh(v) are enumerated for $v \leqslant 12$. We find an apparently new Wh(9) and establish that there exists no DWh(12)—and thereby no (12, 4, 1)-RPMD—nor a TWh(12).

Keywords: Automorphism group; Resolvable design; Whist tournament

1. Introduction

A v-player whist tournament Wh(v) is a schedule of games, where in each round the v players are partitioned into games of four players each with at most one player left over. All pairs of players must play in the same game exactly three times during the tournament. Additionally, the order of the players in a game is relevant: a whist game is a game of four ordered players (c_1, c_2, c_3, c_4). It is convenient to interpret the tuple as listing, in order, the player sitting on the north, east, south, and west side of the playing table. The unordered pairs $\{c_1, c_3\}$ and $\{c_2, c_4\}$ are partners. In a Wh(v), of

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the three times a pair of players plays in the same game, the players must be partners exactly once.

A directed whist tournament DWh(v) is a Wh(v) with the additional condition that for any pair of players p_1 and p_2, the player p_2 must play once as p_1's left-hand opponent and once as p_1's right-hand opponent. We say that each whist game (c_1, c_2, c_3, c_4) contains the ordered pairs of players (c_1, c_2), (c_2, c_3), (c_3, c_4) and (c_4, c_1). These ordered pairs define the left-hand opponent relation in the game; the inverse relation is the right-hand opponent relation. Each of the $v(v-1)$ ordered pairs of players must occur exactly once in a directed whist tournament. The existence of a DWh(v) is equivalent to the existence of a $(v, 4, 1)$-RPMD, a resolvable perfect Mendelsohn design with block size four [6].

In a triplewhist tournament TWh(v) in each game (c_1, c_2, c_3, c_4) the unordered pairs $\{c_1, c_2\}$ and $\{c_3, c_4\}$ are opponents of the first kind, and the unordered pairs $\{c_1, c_4\}$ and $\{c_2, c_3\}$ are opponents of the second kind. A TWh(v) is a Wh(v) with the additional requirement that each player opposes every other player once as an opponent of the first kind and once as an opponent of the second kind.

In this work we determine all nonisomorphic Wh(v), DWh(v), and TWh(v) for $v \leqslant 12$. In particular, we find an apparently new Wh(9), and we show that the two Wh(12) found by Finizio [8] are the only Wh(12) and that neither a DWh(12) nor a TWh(12) exists.

The paper is outlined as follows. In Section 2 various concepts related to whist tournaments are introduced and earlier results are surveyed. A method for finding whist tournaments from resolved $(v, 4, 3)$-designs is discussed in Section 3. An approach for isomorph rejection by mapping tournaments into graphs and using the graph automorphism program *nauty* is considered in Section 4. The complete classification results for $v \leqslant 12$ are presented in Section 5, and the paper is concluded in Section 6.

2. Preliminaries and previous results

We call the following relations partnership relations: unordered pairs of partners, opponents of the first kind, opponents of the second kind, and ordered pairs of players and their left-hand opponents.

Let the permutation $\pi \in S_4$ permute the positions in a whist game: $\pi((c_1, c_2, c_3, c_4)) = (c_{\pi(1)}, c_{\pi(2)}, c_{\pi(3)}, c_{\pi(4)})$. Those permutations π that preserve the partnership relations form a group: for a whist game it is the dihedral group generated by the permutations $(1, 2, 3, 4)$ and $(1, 3)$, for a directed whist game it is the cyclic group generated by the permutation $(1, 2, 3, 4)$, and for a triplewhist game it is the Vierergruppe generated by $(1, 2)$ $(3, 4)$ and $(1, 4)$ $(2, 3)$. These groups partition the set of whist games with four given players into orbits of games with identical partnership relations.

We consider two Wh(v), two DWh(v), or two TWh(v) isomorphic, if there is a bijection from the players of the first tournament to the players of the second tournament that induces a bijection from the rounds and games of the first tournament to the rounds and games (to be exact: between the orbits of the games under the groups mentioned above) of the second tournament. Such a mapping from a tournament onto itself is an automorphism. The group of all automorphisms is the full automorphism group of a tournament.

Note that the set of nonisomorphic DWh(v) or the set of nonisomorphic TWh(v) cannot be treated as a subset of the set of nonisomorphic Wh(v), as there may be several DWh(v) or TWh(v) with the same pairs of partners in each game, and hence the same underlying Wh(v). A similar situation occurs elsewhere in design theory, for example, in directing triple systems [7].

Some whist tournaments may be conveniently described in terms of orbits of rounds under a group. For a group G with operator $\bigstar$, let the orbit of s under G be $G(s) = \{g(s)\colon g \in G\}$, where $g(s) = g \bigstar s$ for $s \in G$, $g(\infty) = \infty$, $g(\{s_1, \dots, s_k\}) = \{g(s_i)\colon 1 \leqslant i \leqslant k\}$, and $g((s_1, \dots, s_k)) = (g(s_1), \dots, g(s_k))$.

A $\mathbb{Z}$-cyclic Wh(v), DWh(v), or TWh(v) is a Wh(v), DWh(v), or TWh(v) that can be described as the orbit of the initial round of the tournament under a cyclic group. When $v \equiv 0 \bmod 4$, the players are labeled with $\infty, 0, 1, \dots, v-2$ and the group is taken to be $\mathbb{Z}_{v-1}$. Traditionally ∞ is paired with $i-1$ in round i. Similarly when $v \equiv 1 \bmod 4$, the players are labeled with $0, 1, \dots, v-1$, and the group is $\mathbb{Z}_v$. Traditionally player $i-1$ sits out, that is, does not play, in round i. The terms *cyclic* and 1-*rotational* may be used for tournaments with an automorphism group acting cyclically on all players and all but one player, respectively.

For surveys of earlier results on whist tournaments, see [2,3]. We give just a brief overview of the central existence results for the three types of whist tournaments discussed here.

In the 1970s Baker [4], H. Hanani, and R. M. Wilson showed that a Wh(v) exists for all $v \geqslant 4$ with $v \equiv 0 \bmod 4$. In [1] Anderson gives an account of that result and shows that a Wh(v) also exists for all $v \geqslant 5$ with $v \equiv 1 \bmod 4$. Finizio determined all $\mathbb{Z}$-cyclic Wh(v) for $v \leqslant 21$ by computer in [8]. Of particular interest for us is that there are two nonisomorphic $\mathbb{Z}$-cyclic Wh(12) tournaments.

Since the existence of a DWh(n) is equivalent to that of a $(v, 4, 1)$-RPMD, from [5,17] we know that a DWh(v) exists for all $v \equiv 1 \bmod 4$, DWh(4) and DWh(8) do not exist, and a DWh(v) exists for all $v \geq 12$ with $v \equiv 0 \bmod 4$ except possibly for 49 values. In this paper, the value of 12 will be removed from this list of values; there is no DWh(12), and hence no $(12, 4, 1)$-RPMD.

A TWh(v) does not exist when $v = 5$ or 9. In [11] Lu and Zhu show that a TWh(v) exists for all $v \geqslant 12$ with $v \equiv 0$ or $1 \bmod 4$ except possibly for $v \in \{12, 56\} \cup \{13, 17, 45, 57, 65, 69, 77, 85, 93, 117, 129, 133, 153\}$. Ge and Zhu [9] reduce this list by finding a TWh(133), and we prove nonexistence of a TWh(12) in this paper.

3. Whist tournaments from resolved designs

A (v, k, λ)-design is a set system $(\mathscr{X}, \mathscr{B})$ where $|\mathscr{X}| = v$ and $\mathscr{B}$ is a collection of k-subsets—called blocks—of $\mathscr{X}$ such that every pair of elements of $\mathscr{X}$ occurs in exactly λ blocks.

A (v, k, λ)-design is resolvable if the blocks can be partitioned into parallel classes such that the blocks in each parallel class are disjoint and their union is $\mathscr{X}$. The corresponding partitioning is called a resolution of the design.

A (v,k,λ)-design is near resolvable if the blocks can be partitioned such that the blocks in each partition are disjoint and their union contains all elements of $\mathscr{X}$ save one. We call the corresponding partitioning a near resolution of the design.

Disregarding momentarily the order of players in each game, the rounds and games in any Wh(v), DWh(v), or TWh(v) must be the parallel classes and blocks of a resolution (when $v \equiv 0 \bmod 4$) or a near resolution (when $v \equiv 1 \bmod 4$) of a $(v,4,3)$-design. For $v=4$, 5, 8 and 9 there is a unique resolvable or near resolvable design. For $v=4,5$ this is trivial; for $v=8$ see [12]; for $v=9$ we inspected the 11 nonisomorphic $(v,4,3)$-designs by hand. Moreover, recent results by Morales and Velarde [14] show that there are five nonisomorphic resolvable $(12,4,3)$-designs. All these designs have a unique resolution or near resolution.

In order to construct whist tournaments of various types, we take a resolution or near resolution of a $(v,4,3)$-design and impose the additional partnership relations in a computer search.

To find all Wh(v) with a given underlying resolvable or near resolvable design, we find all ways of replacing each block $\{c_1,c_2,c_3,c_4\}$ in the design by a partition of the players into two partnerships $\{\{c_1,c_3\},\{c_2,c_4\}\}$ such that the resulting set system satisfies the criteria for a Wh(v), that is, each pair of players occurs once in the set system as partners. For a DWh(v) we similarly find all ways of replacing each block $\{c_1,c_2,c_3,c_4\}$ with a set of ordered pairs of players and their left-hand opponents $\{(c_1,c_2),(c_2,c_3),(c_3,c_4),(c_4,c_1)\}$. For a TWh($v$) we replace each block $\{c_1,c_2,c_3,c_4\}$ with an ordered triple of distinct partitions of the three players $(\{\{c_1,c_3\},\{c_2,c_4\}\},\{\{c_1,c_2\},\{c_3,c_4\}\},\{\{c_1,c_4\},\{c_2,c_3\}\})$, where the first, second, and third partition represent pairs of partners, opponents of the first kind, and opponents of the second kind, respectively.

We transform the problem of searching for such set systems into instances of the satisfiability problem by generating constraints based on the blocks of the resolved designs (for a similar approach for other types of designs, see [7]). This is done by taking each nonisomorphic resolution or near resolution of a $(v,4,3)$-design, $v \leqslant 12$, in turn, introducing for each pair of players $x<y$ and block b the boolean variables p_{xyb}, l_{xyb}, r_{xyb}, o^1_{xyb}, and o^2_{xyb} to represent whether x is, respectively, the partner, left-hand opponent, right-hand opponent, opponent of the first kind or opponent of the second kind of y in the game that corresponds to block b. With those definitions it is straightforward to formulate logic programs—in our case, essentially instances of the satisfiability problem—whose solutions represent partnership relations in a whist tournament with the given underlying $(v,4,3)$-design. We then use *Smodels* [16] to determine all solutions to the logic programs by exhaustive search. Then we convert the solutions back to set systems.

4. Distinguishing nonisomorphic tournaments

In order to eliminate isomorphic solutions we map the whist tournaments of various types, expressed as set systems, to graphs and examine them with *nauty* [13]. From a given whist tournament, we construct the graph $G=(V,E)$. In V we will have one

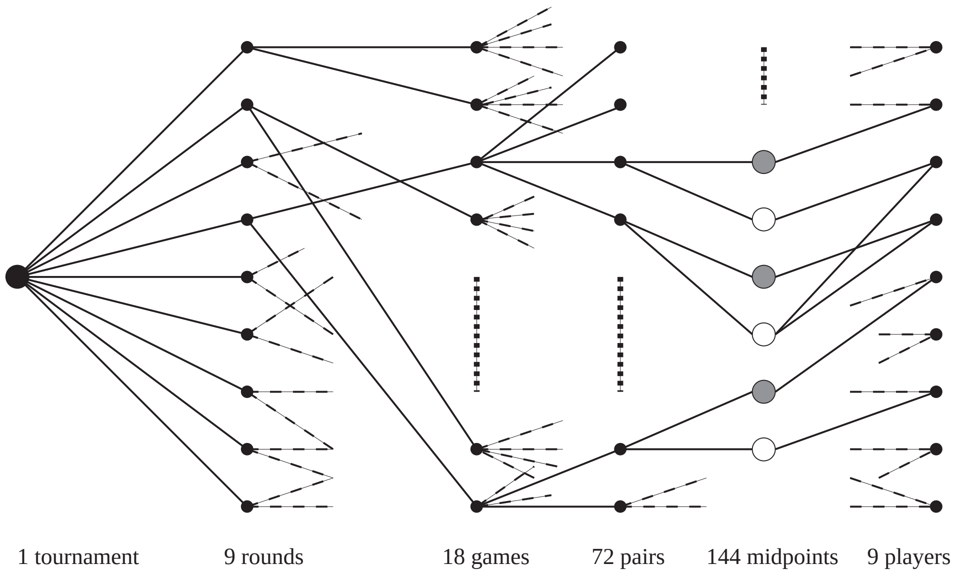

Fig. 1. A DWh(9) as a graph.

vertex for each set, tuple, and element in the set system. For each $u, v \in V$, we have the edge $\{u, v\} \in E$ exactly when $v \in u$ in the set system. When u is a tuple and v is the cth element of u, we color the edge $\{u, v\}$ with color c, while the other edges remain uncolored. The automorphism group of G, restricted to act on the players only, is the automorphism group of the set system. As a technical note, if we only allow coloring of the vertices, the equivalent of coloring an edge can be done by splitting the edge in two and coloring the new vertex with the given color. By also coloring the tournament vertex with a distinct color, one can assure that the automorphism group of the graph maps vertices corresponding to players only within that class.

Fig. 1 illustrates the structure of the graph in the case of a DWh(9). The graph has a total of 253 vertices, which correspond to the tournament, the 9 rounds, the 18 games, the 72 ordered pairs of players, the 144 midpoints on the edges from the pairs to the players, and the 9 players. From the tournament there is an edge to each of the round; from each round there is an edge to two games; from each game there is an edge to four ordered pairs; and from each of the ordered pairs there is an edge to two midpoints, from each of which there is an edge to a player. The two midpoints adjacent to an ordered pair are colored with colors 1 and 2 (in Fig. 1, gray and white) to represent whether the player adjacent to the midpoint is the first or the second player of the ordered pair, respectively.

For every set system we form the corresponding graph and use *nauty* to determine its automorphism group and canonical labeling. Two set systems—and hence, whist tournaments—are isomorphic, if the corresponding canonically labeled graphs are identical. We only consider one tournament from each isomorphism class.

Having determined the nonisomorphic Wh(v), DWh(v), and TWh(v) for some parameters along with their automorphisms, we used GAP [15] to investigate their automorphism groups.

5. The results

The number of nonisomorphic Wh(v), DWh(v), and TWh(v) for $v \leqslant 12$ is displayed in Table 1. As expected, we find no DWh(4), DWh(8), TWh(5) nor TWh(9). There are no DWh(12) nor TWh(12) either.

The nonisomorphic Wh(v), DWh(v), and TWh(v) for $v \leqslant 12$ are listed in Table 2. In Table 2, in the initial rounds of the DWh(9) with automorphism group $\mathbb{Z}_3 \times \mathbb{Z}_3$, the notation ab is shorthand for (a,b), and $\mathbb{Q}_8$ is the quaternion group. Two DWh(v) or TWh(v) are joined by a curly brace whenever they have the same underlying Wh(v); for those v for which DWh(v) or TWh(v) exist, the Wh(v) are not explicitly listed, as they can be obtained by interpreting the DWh(v) or TWh(v) as a Wh(v).

The full automorphism groups of the Wh(v) for $v = 4, 5, 8, 12$ are of order 24, 20, 56, and 11, respectively. The full automorphism groups of the two Wh(9) are of order 144 and 16, in the order the tournaments are listed in Table 2. The full automorphism groups of the TWh(4), DWh(5), and TWh(8) are of order 12, 20 and 56, respectively. The full automorphism groups of the four DWh(9) are of order 72, 72, 8 and 8, in the order the tournaments are listed in Table 2.

When $v = 4t + 1$ is a prime power and θ is a primitive element of the Galois field $\mathbb{F}_v$, then one can obtain a DWh(v) by calculating the orbit of the initial round $\{(\theta^i, \theta^{t+i}, \theta^{2t+i}, \theta^{3t+i})\colon 0 \leqslant i < t\}$ under $\mathbb{F}_v^+$, the additive group of $\mathbb{F}_v$. This construction is attributed to Baker in [2]. The first DWh(9) in Table 2 is isomorphic to $\mathbb{F}_9^+(\{(\theta^0, \theta^2, \theta^4, \theta^6), (\theta^1, \theta^3, \theta^5, \theta^7)\})$. Similarly, the second DWh(9) can be obtained as $\mathbb{F}_9^+(\{(\theta^0, \theta^2, \theta^4, \theta^6), (\theta^7, \theta^5, \theta^3, \theta^1)\})$.

There are no DWh(12) nor TWh(12). There are exactly two Wh(12), both of which are $\mathbb{Z}$-cyclic. Recall that for $v = 12$ there are five nonisomorphic resolvable $(v, 4, 3)$-designs; both Wh(12) have the same underlying resolved $(12, 4, 3)$-design.

Both for directed and triplewhist tournaments, it is possible to enlarge the sets of indistinguishable designs. Two directed whist tournaments are said to be equivalent if they are isomorphic, or if they are isomorphic after simultaneously exchanging positions 2 and 4 in each whist game in one DWh(v). Such a permutation preserves the partner pairs, but exchanges the left-hand opponent relation and the right-hand opponent relation; in a sense, this is a mirror image of the DWh(v). As triplewhist tournaments may be viewed as a composition of three whist tournaments—one for partners, one for opponents of the first kind, and one for opponents of the second kind—two triplewhist tournaments are said to be equivalent if they are isomorphic or if they are isomorphic after applying a fixed permutation to the last three players of each game of one of the tournaments. This operation permutes the partnership relation, first-kind opponent relation, and second-kind opponent relation, and gives a TWh(v). It is easily seen, for example, that the two nonisomorphic DWh(5) and the two nonisomorphic TWh(8) in Table 2 are equivalent.

Table 1
The number of nonisomorphic Wh(v), DWh(v), and TWh(v) for $v \leqslant 12$

	Wh(v)	DWh(v)	TWh(v)
4	1	0	1
5	1	2	0
8	1	0	2
9	2	4	0
12	2	0	0

Table 2
The nonisomorphic Wh(v), DWh(v), and TWh(v) for $v \leqslant 12$

TWh(4)	$\mathbb{Z}_3(\{(\infty,1,0,2)\})$
DWh(5)	$\left\{\begin{array}{l}\mathbb{Z}_5(\{(1,2,4,3)\})\\ \mathbb{Z}_5(\{(1,3,4,2)\})\end{array}\right.$
TWh(8)	$\left\{\begin{array}{l}\mathbb{Z}_7(\{(\infty,1,0,3),(2,4,6,5)\})\\ \mathbb{Z}_7(\{(\infty,3,0,1),(2,5,6,4)\})\end{array}\right.$
DWh(9)	$\left\{\begin{array}{l}\mathbb{Z}_3\times\mathbb{Z}_3(\{(10,01,20,02),(11,21,22,12)\})\\ \mathbb{Z}_3\times\mathbb{Z}_3(\{(10,01,20,02),(12,22,21,11)\})\end{array}\right.$ $\left\{\begin{array}{l}\mathbb{Z}_8(\{(\infty,1,0,3),(2,6,7,5)\})\cup\\ \mathbb{Z}_8(\{(0,2,4,6),(1,3,5,7)\})\\ \mathbb{Q}_8(\{(i,k,-j,-i),(j,-1,-k,\infty)\})\cup\\ \mathbb{Q}_8(\{(1,i,-1,-i),(-k,-j,k,j)\})\end{array}\right.$
Wh(12)	$\mathbb{Z}_{11}(\{(\infty,1,0,4),(2,3,6,8),(5,9,7,10)\})$ $\mathbb{Z}_{11}(\{(\infty,1,0,7),(2,3,5,10),(4,8,6,9)\})$

6. Conclusions

This work completes the classification of whist tournaments, directed whist tournaments, and triplewhist tournaments with up to 12 players. The main results are the nonexistence proofs for DWh(12) and TWh(12), and the discovery of an apparently previously unknown whist tournament for nine players. In the following, we use the last result to develop a real-life tournament.

In duplicate bridge two players are said to play a deal in the same direction, if both of them play north or south, or if both of them play east or west. The score a player

Table 3
A bridge movement for nine individuals

Round	Table 1				Table 2				A	Out
	N	E	S	W	N	E	S	W		
1	8	5	4	7	6	2	3	1		0
2	8	6	5	0	7	3	4	2	*	1
3	8	7	6	1	0	4	5	3		2
4	8	0	7	2	1	5	6	4	*	3
5	8	1	0	3	2	6	7	5		4
6	8	2	1	4	3	7	0	6	*	5
7	8	3	2	5	4	0	1	7		6
8	8	4	3	6	5	1	2	0	*	7
9	0	2	4	6	1	3	5	7		8

obtains in a given deal is compared to the scores of other players who play the deal in the same direction. For fairness, it is desirable that all pairs of players are compared an equal number of times, that is, that all pairs of players play the same number of deals in the same direction. To balance the comparisons, the concept of *arrow switching* is useful. When a table is arrow switched in a given round, that round is played with a quarter-turn offset with regard to the seating at that table. The player seated west will play the north cards, the player seated north will play the east cards, etc. This allows balancing the comparisons without unduly complicating the player movement—an important consideration in practice.

The tournament in Table 3 is such a DWh(9) that when the second table is arrow switched in the rounds marked in the A column, all pairs of players play the same number of rounds in the same direction, and we have a perfectly balanced nine-player individual bridge tournament. In the tournament in Table 3 rounds 2–8 can be obtained from the previous round by adding $1 \bmod 8$ to each element less than eight.

Work is in progress to classify the near resolutions of $(13,4,3)$-designs and whist tournaments for 13 players. We expect the number of resolutions of $(16,4,3)$-designs to be so large that the approach adopted here is no longer feasible.

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