A Time Domain Interpretation for the LSP Decomposition

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Abstract—**The Line Spectrum Pair (LSP) decomposition is a widely used method in speech coding. In this article, we will show that the LSP polynomials, whose trivial zeros have been removed, are equivalent to two optimal (in the mean square sense) predictors in which a sample is predicted from linear combinations of its previous averaged and differentiated values.**

Keywords—**Line Spectrum Pairs, Linear Prediction, symmetric polynomials**

I. Introduction

LINE Spectrum Pairs (LSP) were introduced in [1] as a method to represent LP (Linear Prediction) parameters. In LSP computation, a linear predictive analysis filter $A(z)$ is decomposed into both a symmetric and an antisymmetric polynomial. These polynomials are called the LSP polynomials. It can be shown that the roots of these polynomials, the LSPs, are interlaced on the unit circle, if $A(z)$ is minimum phase [2]. Moreover, it has been found that LSPs behave well when interpolated [3],[4]. These properties have made the LSP decomposition an attractive method to quantize LP information and it is currently used in various speech coders (e.g., [5],[6]). In addition, it is well known that LSPs have an close relation to the split Levinson type algorithms [7].

In spite of the fact that LSPs are so widely applied especially in speech coding, it is disconcerting to notice that the motivation behind this decomposition is rather artificial. LSPs have been explained from the point of view of acoustic tube models as follows. Given a p'th order predictor in the lattice form, it is possible to obtain a symmetric and an antisymmetric LSP polynomial by augmenting an extra stage with its reflection coefficient equal to $+1$ and -1, respectively [2],[8]. This is equivalent to setting the corresponding acoustic tube model either completely closed or completely open at the $(p+1)$'th stage. A similar explanation was presented in [9], where it was shown that LSPs are the pole and zero frequencies of the glottal driving-point impedance of a discrete matched-impedance vocal tract model.

In the present paper, we will show that the LSP decomposition can be interpreted as an optimal time domain prediction, which is a reformulation of the conventional LP analysis. Our study shows that the LSP polynomials are optimal solutions (in the mean square error sense) to two problems of linear prediction, in which a sample is predicted from linear combinations of its previous averaged and differentiated values. The work is a sequel to our prior studies on reformulations of linear prediction in order to express speech information in a compressed form (e.g., [10],[11]).

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II. Background

A. Linear Prediction

The conventional LP predictor for signal $x(n)$ as given by [12] is

$$A(z) = 1 + \sum_{i=1}^{p} a_i z^{-i}. \tag{1}$$

The optimal coefficients a_i ($1 \leq i \leq p$) of this predictor are calculated as follows. The residual (prediction error) $e(n)$ is

$$e(n) = x(n) + \sum_{i=1}^{p} a_i x(n-i). \tag{2}$$

The extreme point of the expected value of the squared error $E[e^2(n)]$ is found by setting $E\left[\partial e^2(n)/\partial a_j\right] = 0$ ($1 \leq j \leq p$), which yields the Yule-Walker equations [12]:

$$\sum_{i=1}^{p} a_i R(i-j) = -R(j), \qquad 1 \leq j \leq p, \tag{3}$$

where signal $x(n)$ is assumed stationary in the wide sense and the autocorrelation $R(i)$ for $x(n)$ is estimated as

$$R(i) = \sum_{n=0}^{L-1-i} w(n)x(n)w(n+i)x(n+i), \quad \forall\, i \in \mathbf{N}, \tag{4}$$

where $w(n)$ is a smooth window of length L.

B. The LSP Decomposition

A polynomial $A_s(z)$ of order p is said to be symmetric if

$$A_s(z) = z^{-p} A_s(z^{-1}). \tag{5}$$

and polynomial $A_a(z)$ is antisymmetric if

$$A_a(z) = -z^{-p} A_a(z^{-1}). \tag{6}$$

For a polynomial $A(z)$ of order p, the symmetric and antisymmetric LSP polynomials are constructed as follows [2]:

$$\begin{aligned} P(z) &= A(z) + z^{-(p+1)} A\left(z^{-1}\right) \\ Q(z) &= A(z) - z^{-(p+1)} A\left(z^{-1}\right). \end{aligned} \tag{7}$$

The polynomial $A(z)$ can be easily reconstructed from $P(z)$ and $Q(z)$ by

$$A(z) = \frac{1}{2}\left[P(z) + Q(z)\right], \tag{8}$$

which holds for any polynomial $A(z)$.

Polynomials $P(z)$ and $Q(z)$ have trivial zeros at $z = \pm 1$ [13]. Canceling the trivial zeros yields symmetric polynomials $R_P(z)$ and $R_Q(z)$ of even order:

$$\begin{aligned} R_P(z) &= \frac{P(z)}{1+z^{-1}},\ R_Q(z) = \frac{Q(z)}{1-z^{-1}}, \quad p \text{ even} \\ R_P(z) &= P(z) \quad\ ,\ R_Q(z) = \frac{Q(z)}{1-z^{-2}}, \quad p \text{ odd.} \end{aligned} \tag{9}$$

In the sequel, an appropriate subscript will be used to denote which polynomial was used when creating $R_P(z)$ or $R_Q(z)$. For example, polynomial $R_P(z)$ computed from the LSP decomposition of a polynomial $B(z)$ would be denoted $R_{P,B}(z)$.

III. Linear Prediction Using Averaged and Differentiated Values

In this section, two time domain signal transformations will be introduced. In Section IV, it will be shown that formulating linear prediction using these two transformations corresponds to the computation of the LSP decomposition.

A. Time Domain Transformations

We will firstly introduce two time domain signal transformations

$$\widehat{x}^{+}(n) = \frac{1}{2}\left[x(n) + x(n+1)\right] \tag{10}$$

$$\widehat{x}^{-}(n) = \frac{1}{2}\left[x(n) - x(n+1)\right]. \tag{11}$$

Note that these transformations are not causal. However, this does not impose problems because the transformations will be combined with linear prediction and, consequently, only delayed samples of $x^{+}(n)$ and $x^{-}(n)$ are needed.

The transformation $\widehat{x}^{+}(n)$ is described in figure 1. It corresponds to an averaging filter yielding the average value of two consecutive values of $x(n)$. The dual transform $\widehat{x}^{-}(n)$ corresponds to a differentiating filter.

B. Optimal Predictors Using the Transformed Signals

Next, we will predict $x(n)$ from the p previous values of $\widehat{x}^{+}(n)$ (and $\widehat{x}^{-}(n)$) by formulating an optimal predictor. To determine the predictor, we need the expression for the residual, which we define for $\widehat{x}^{+}(n)$ as

$$\begin{aligned} e_{h+}(n) &= x(n) + \sum_{i=1}^{p} h_i^{+}\widehat{x}^{+}(n-i) \\ &= x(n) + \sum_{i=1}^{p} \frac{h_i^{+}}{2}\left[x(n-i) + x(n-i+1)\right]. \end{aligned} \tag{12}$$

and for the dual case as

$$e_{h-}(n) = x(n) + \sum_{i=1}^{p} \frac{h_i^{-}}{2}\left[x(n-i) - x(n-i+1)\right]. \tag{13}$$

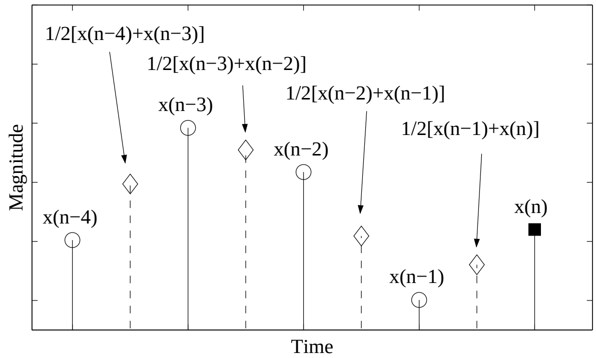

Fig. 1
Selection of data samples to be used in linear prediction of sample $x(n)$ (black square) with $p = 4$: conventional LP analysis (solid lines and circles) and the proposed prediction (dashed lines and diamonds) corresponding to the averaging operation (Eq. 10) used in the LSP decomposition.

The expected value of the squared error $E\left[e_{h+}^2(n)\right]$ is minimized for each h_i^{+} by partial differentiation:

$$\begin{aligned} &E\left[\frac{\partial e_{h^+}^2(n)}{\partial h_j^{+}}\right] = 0 \quad \forall n,\ 1 \le j \le p \\ &\sum_{i=1}^{p} h_i^{+}\left[2R(i-j) + R(i-j+1) + R(i-j-1)\right] \\ &\qquad = -2\left[R(j) + R(j-1)\right], \quad 1 \le j \le p. \end{aligned} \tag{14}$$

These are the Yule-Walker equations corresponding to the transformation equation given in Eq. 10. The corresponding Yule-Walker equation for the transformation equation determined in Eq. 11 can similarly be shown to be

$$\begin{aligned} &\sum_{i=1}^{p} h_i^{-}\left[2R(i-j) - R(i-j+1) - R(i-j-1)\right] \\ &\qquad = -2\left[R(j) - R(j-1)\right], \quad 1 \le j \le p. \end{aligned} \tag{15}$$

When Eqs. 14 and 15 are written in matrix form, the resulting matrices are symmetric and Toeplitz.

C. Transfer Functions of the Optimal Predictors

The transfer functions of the optimal predictors can be readily obtained from Eqs. 12 and 13 by the Z-transform, which yields

$$\begin{aligned} H^{+}(z) &= \sum_{i=0}^{p} \frac{z^{-i}}{2}\left(h_i^{+} + h_{i+1}^{+}\right) \\ H^{-}(z) &= \sum_{i=0}^{p} \frac{z^{-i}}{2}\left(h_i^{-} - h_{i+1}^{-}\right), \end{aligned} \tag{16}$$

where we have introduced artificial variables $h_0^{+} = 2$, $h_0^{-} = 2$, $h_{p+1}^{+} = 0$ and $h_{p+1}^{-} = 0$.

IV. Time Domain Interpretation of the LSP Decomposition

A. Problem Formulation

Our objective is to show that the predictors constructed upon the two transformations given in Eqs. 10 and 11 are equal to the line spectrum polynomials $R_P(z)$ and $R_Q(z)$ (Eq. 9). Figure 2 illustrates the setting.

$$x(n) \xrightarrow{LP} A(z) \xrightarrow{LSP} \begin{cases} R_Q(z) \\ R_P(z) \end{cases}$$

time domain transformation, optimal predictor, Z-transform

$$x(n) \longrightarrow \begin{cases} \widehat{x}^+(n) \\ \widehat{x}^-(n) \end{cases} \longrightarrow \begin{cases} h_i^+ \\ h_i^- \end{cases} \longrightarrow \begin{cases} H^+(z) \\ H^-(z) \end{cases}$$

Fig. 2
Illustration of the problem setting.

We thus need to show that polynomials $R_P(z)$ and $R_Q(z)$ are equivalent to $H^+(z)$ and $H^-(z)$ (not necessarily respectively).

The complete proof is somewhat laborious and some details have been moved to appendices. The proof is structured as follows: Firstly, create criteria for symmetry of the transfer functions in Eq. 16. Secondly, it is shown that solutions to the Yule-Walker equations (Eqs. 14 and 15) satisfy the symmetry criteria and lastly, we prove that the symmetric (or antisymmetric) part of a linear predictor corresponds to the acquired transfer function.

B. Symmetry Criteria for the Optimal Predictors

It is evident that for arbitrary h_i^+ and h_i^- polynomials $H^+(z)$ and $H^-(z)$ are generally neither symmetric nor antisymmetric. However, we will set criteria for coefficients h_i^+ and h_i^- that are necessary and sufficient to yield symmetric or antisymmetric polynomials $H^+(z)$ and $H^-(z)$.

Polynomial $H^+(z)$ (Eq. 16) is symmetric if, and only if, coefficients of z^i and z^{p-i} are equal ($0 \le i \le p$). In other words,

$$h_i^+ - h_{p-i+1}^+ = -\left(h_{i+1}^+ - h_{p-i}^+\right) \;, \quad 1 \le i \le p\,. \tag{17}$$

Since the left hand side of Eq. 17 equals 2 for $i = 0$ (see Eq. 16), it follows that

$$h_i^+ = 2(-1)^i + h_{p-i+1}^+ \;, \quad 1 \le i \le p\,,\ p \text{ even.} \tag{18}$$

By substituting $i = p$ in the expression above, we obtain

$$h_p^+ = 2(-1)^p + h_1^+,$$

implying (by substitution of $2 + h_1^+ = h_p^+$):

$$2 + h_1^+ = 2(-1)^p + h_1^+ \quad \Rightarrow \quad 1 = (-1)^p,$$

which is a contradiction if p is odd. Therefore, Eq. 18 holds for p even only.

In the antisymmetric case, sign on the right hand side of Eq. 17 are changed and following a similar process as above, the final relation (corresponding to Eq. 18) is

$$h_i^+ = 2(-1)^i - h_{p-i+1}^+ \;, \qquad 1 \le i \le p, \tag{19}$$

which holds for p odd only.

Further, by applying reasoning similar to that above, criteria for symmetry of the dual polynomial $H^-(z)$, for both p even and odd, can be shown to be

$$h_i^- = 2 - h_{p-i+1}^- \;, \qquad 1 \le i \le p. \tag{20}$$

However, $H^-(z)$ cannot be antisymmetric. This can easily be seen by deriving a necessary criterion for antisymmetry, similar to that above. In that case inserting $i = p$ results in a contradiction.

It can readily be shown that these criteria of symmetry hold for $h_i^\pm$ solving Eq. 14 and 15, respectively. See appendix II-A for details.

C. Correspondence of the Optimal Predictor $H^+(z)$ and the LSP Decomposition

Continuing, we show that polynomial $H^+(z)$ is in fact, equal to polynomial $R_P(z)$ or polynomial $R_Q(z)$, for p even or odd, respectively.

Let us define an arbitrary polynomial $B(z) = b_0 + \sum_{i=1}^p b_i z^{-i}$. The symmetric LSP polynomial of $B(z)$ is thus $P_B(z) = B(z) + z^{-(p+1)} B\left(z^{-1}\right)$.

Because $B(z)$ was arbitrary, such $B(z)$ can be chosen that its symmetric LSP polynomial $R_{P,B}(z)$ is equal to the symmetric LSP polynomial $R_{P,H^+}(z)$ computed for polynomial $H^+(z)$, that is, $R_{P,H^+}(z) := R_{P,B}(z)$. Thus only the symmetric part of polynomial $B(z)$ has been restricted, and the antisymmetric part remains free.

Using Eq. 9 and from Appendix I-A Eq. 31 we can write

$$\left(1 + z^{-1}\right) H^+(z) \quad = \quad B(z) + z^{-p-1} B(z^{-1}), \tag{21}$$

where p is even and $H^+(z)$ is thus symmetric.

For the two sides to be equal, all coefficients of common order of z^{-i} must be equal:

$$\begin{aligned} z^0 : &\qquad 1 + h_1^+/2 &&= b_0 \\ z^{-i} : &\quad h_i^+ + \left(h_{i-1}^+ + h_{i+1}^+\right)/2 &&= b_i + b_{p-i+1}, \end{aligned} \tag{22}$$

where $1 \le i \le p+1$, $h_0^+ = 2$, $h_{p+1}^+ = 0$ and $b_i = 0$ for $i \notin [0, p]$.

Substituting into Eq. 14 yields

$$\begin{aligned} &\sum_{i=1}^p 2R(i-j)\left(b_i + b_{p-i+1}\right) \\ &\quad -2R(1-j) + 2R(-j)(b_0 - 1) + 2R(p+1-j)b_0 \\ &\qquad\qquad = -2\left[R(j) + R(j-1)\right]. \end{aligned} \tag{23}$$

Canceling common terms on both sides, and dividing by b_0 we have

$$\begin{aligned} &\sum_{i=1}^p 2R(i-j)\widehat{b}_i + \sum_{i=1}^p 2R(p-i-j+1)\widehat{b}_i \\ &\qquad = -2\left[R(j) + R(p+1-j)\right], \quad 1 \le j \le p, \end{aligned} \tag{24}$$

where $\widehat{b}_i = b_i/b_0$.

Comparing Eqs. 3 and 24 we see that if the coefficients $\widehat{b}_i$ solve the Yule-Walker equations (Eq. 3) then they will also solve Eq. 24. The goal in proving the correspondence between the symmetric LSP decomposition (computed from an LP predictor $A(z)$) and $H(z)^+$, is thus completed. That is, for p even it holds that

$$H^+(z) = R_{P,A}(z). \tag{25}$$

However, there is an infinite number of other sets of parameters $\widehat{b}_i$ that solve Eq. 24 as well. The reason is that only the symmetric part of $B(z)$ has been constrained and the antisymmetric part is undefined. Thus, any antisymmetric part and the specified symmetric part of $B(z)$ will solve Eq. 24. There is an infinite number of antisymmetric polynomials, and thus there is an infinite number of solutions to Eq. 24.

In the case of p odd we have

$$\begin{aligned} \frac{H^+(z)}{1-z^{-1}} &:= R_{Q,B}(z) \\ \left(1+z^{-1}\right)H^+(z) &= B(z) - z^{-i-1}B(z^{-1}). \end{aligned} \tag{26}$$

The relation between coefficients is then

$$\begin{aligned} z^0: &\quad 1 + h_1^+/2 &&= b_0 \\ z^{-i}: &\quad h_i^+ + \left(h_{i-1}^+ + h_{i+1}^+\right)/2 &&= b_i - b_{p-i+1}, \end{aligned} \tag{27}$$

which yields, by insertion into the Yule-Walker equation (Eq. 14):

$$\begin{aligned} &\sum_{i=1}^{p} 2R(i-j)\widehat{b}_i - \sum_{i=1}^{p} 2R(p-i-j+1)\widehat{b}_i \\ &\qquad = -2\left[R(j) - R(p+1-j)\right], \quad 1 \le j \le p. \end{aligned} \tag{28}$$

With the same motivation as for p even, equation above will hold if the coefficients $\widehat{b}_i$ obey the Yule-Walker equations of LP (Eq. 3).

D. *Correspondence of the Optimal Predictor $H^-(z)$ and the LSP Decomposition*

With an analogous proof as in Section IV-C, it can be shown that polynomial $H^-(z)$ is equivalent to the LSP decomposition of an LP predictor in the sense that

$$H^-(z) = R_{Q,B}(z), \quad p \text{ even} \tag{29}$$

$$\frac{H^-(z)}{1+z^{-1}} = R_{Q,B}(z), \quad p \text{ odd.} \tag{30}$$

For p even, $H^+(z)$ is equivalent to the symmetric LSP polynomial $R_{P,B}(z)$ and $H^-(z)$ to the antisymmetric LSP polynomial $R_{Q,B}(z)$. In the proof of equivalence of these polynomials, extraordinary care was taken to restrict only either the symmetric or antisymmetric part of polynomial $B(z)$. Hence, it is possible to show that with the constant term scaled to unity for both polynomials and with the trivial zeros included, the sum (as in Eq. 8) of $H^+(z)$ and $H^-(z)$ is equal to the LP polynomial[1]. □

[1] We can readily show that the symmetric LSP decomposition of a polynomial can be defined separately from the antisymmetric LSP, by insertion of Eq. 8 into Eq. 7. The symmetric or antisymmetric LSP decomposition (Eq. 7) of Eq. 8 will always return only the symmetric or the antisymmetric polynomial, respectively. The two can therefore always be separated, and can always be defined separately.

E. *Summary*

Properties of polynomials $H^+(z)$ and $H^-(z)$ provided that their coefficients solve the corresponding Yule-Walker equations (Eqs. 14 and 15), are listed in Tables I and II.

	p even	p odd
$H^+(z)$	symmetric	antisymmetric
$H^-(z)$	symmetric	symmetric

TABLE I
PROPERTIES OF POLYNOMIALS $H^+(z)$ AND $H^-(z)$ WITH THE TRIVIAL ZEROS INCLUDED.

transfer function	LSP polynomial	
	p even	p odd
$H^+(z)$	$R_P(z)$	$R_Q(z)$
$H^-(z)$	$R_Q(z)$	$R_Q(z)$

TABLE II
CORRESPONDENCE OF POLYNOMIALS $H^+(z)$ AND $H^-(z)$ TO LSP POLYNOMIALS WITH THE TRIVIAL ZEROS REMOVED.

V. PROPERTIES

The time domain signal transformation that yields $\widehat{x}^+(n)$ (Eq. 10), is an averaging filter, as shown in Figure 1. This filter is a low-pass filter; it has linear phase and its 3 dB cutoff-point is at half the Nyquist frequency (see Fig. 3). This means that the data, from which the optimal predictor is generated, is emphasized in the low end of the frequency range. There is thus a trend for the corresponding Line Spectral Frequencies (LSFs) to lie at low frequencies.

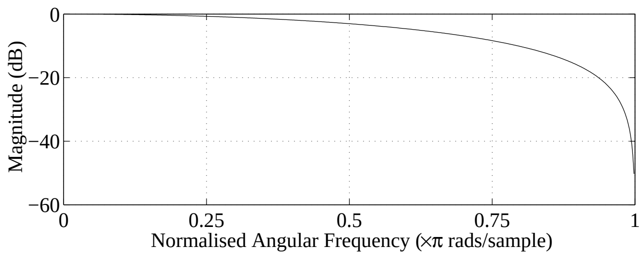

Fig. 3
MAGNITUDE RESPONSE OF THE AVERAGING TRANSFORM CORRESPONDING TO $\widehat{x}^+(n)$.

The dual transformation (Eq. 11) is a differentiating operator, and as such, a high-pass filter. Its frequency response is the mirror image (i.e., shifted by π) that of $\widehat{x}^+(n)$'s frequency response in Figure 3. The LSFs thus have a tendency to lie at high frequencies.

The averaging and differentiation affect the magnitude responses of the corresponding LSP filters as shown in Figure 4 (where $p = 10$). The magnitude response corresponding to the averaging operation (solid line), shows that each of the five line frequencies are farther to the left than those corresponding to the differentiation operation (dashed line). Furthermore, the spectrum of the LSP filter corresponding to the averaging filter decays at high frequencies, whereas the spectrum corresponding to the differentiating filter is almost horizontal.

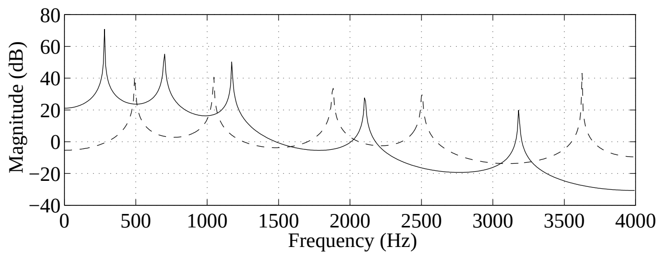

Fig. 4

ILLUSTRATION OF HIGH- AND LOW-END EMPHASIS OF THE AVERAGING AND DIFFERENTIATING FILTERS. THE SOLID LINE DEPICTS THE SYMMETRIC LSP POLYNOMIAL (OPTIMAL PREDICTION OBTAINED USING AVERAGING FILTERING) OF A SUSTAINED VOWEL [A:] (MALE VOICE), AND THE DASHED LINE THE CORRESPONDING ANTISYMMETRIC LSP POLYNOMIAL (DIFFERENTIATING FILTER). POLYNOMIAL ORDER FOR BOTH FILTERS IS $p = 10$, AND THE TRIVIAL ZEROS HAVE BEEN REMOVED FROM BOTH POLYNOMIALS.

Since the polynomial $R_P(z)$ is calculated from the high-pass filtered signal $x^-(n)$, the spectrum of $R_P(z)$ fits the spectrum of $x^-(n)$. Similarly, the spectra of $R_Q(z)$ and $x^-(n)$ fit as well. This phenomenon is illustrated in Fig. 5.

A curious lack of symmetry can be noted in the results. Perhaps contrary to what one could expect, $H^-(z)$ is symmetric for both p even and odd. Indeed, $H^-(z)$ is equivalent to $H^+(z)$ for p odd. Thus, for p odd, there is no emphasizing of low or high frequencies, as claimed above for p even. This feature can be explained intuitively by the fact that for an antisymmetric polynomial with an odd number of terms, the center term is zero. Polynomial $P(z)$ in Eq. 7 has an odd number of terms when p is odd. Therefore, for p odd, we have one less degree of freedom, and consequently only one of the two polynomials $R_P(z)$ and $R_Q(z)$ can be equivalent with our time domain formulation.

VI. CONCLUSIONS

We have shown that the polynomials $R_P(z)$ and $R_Q(z)$ of the LSP decomposition, can be interpreted in the time domain as results of optimal linear predictions, where the signal to be predicted is formed using averaging and differentiating operators. In other words, transfer functions of the proposed predictors (Eq. 16) optimized by solving the corresponding Yule-Walker equations (Eqs. 14 and 15) are equal to the LSP polynomials with their trivial roots removed (Eq. 9).

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APPENDICES

I. MATHEMATICAL PRELIMINARIES

A. Symmetry properties of LSP decompositions

For a symmetric polynomial $A_s(z)$, for both p even and p odd, the following equations hold true (using Eqs. 5 and 7):

$$\begin{aligned}\frac{P_{As}(z)}{1+z^{-1}} &= \frac{A_s(z)+z^{-(p+1)}A_s\left(z^{-1}\right)}{1+z^{-1}} \\ &= \frac{A_s(z)+z^{-1}A_s(z)}{1+z^{-1}} = A_s(z),\end{aligned} \tag{31}$$

and similarly (using Eqs. 6 and 7)

$$\frac{Q_{As}(z)}{1-z^{-1}} = \frac{A_s(z)-z^{-1}A_s(z)}{1-z^{-1}} = A_s(z), \tag{32}$$

where $P_{As}(z)$ and $Q_{As}(z)$ are the LSP polynomials defined in Eq. 7. Further, for an antisymmetric polynomial $A_a(z)$ we have

$$\frac{P_{Aa}(z)}{1-z^{-1}} = A_a(z) \quad \text{and} \quad \frac{Q_{Aa}(z)}{1+z^{-1}} = A_a(z). \tag{33}$$

Notice that Eqs. 31–33 are valid for any symmetric or antisymmetric polynomial.

B. Trivial identities

Beginning with the trivial identity

$$\begin{aligned}0 &= 4(-1)^i - 2(-1)^i - 2(-1)^i \qquad \forall\, i \\ &= 4(-1)^i + 2(-1)^{i-1} + 2(-1)^{i+1}.\end{aligned} \tag{34}$$

Using Eq. 34, we will now write an equation resembling Eq. 14 (with $1 \leq j \leq p$)

$$0 = \sum_{i=1}^{p} R(i-j)\left[4(-1)^i + 2(-1)^{i-1} + 2(-1)^{i+1}\right]$$

$$\begin{aligned}
&= \sum_{i=1}^{p} 2(-1)^i \Big[2R(i-j) \\
&\qquad\qquad + R(i-j+1) + R(i-j-1) \Big] \qquad (35)\\
&\quad +2R(j-1) - 2(-1)^p R(p-j+1) \\
&\qquad\qquad +2R(j) + 2(-1)^{p+1} R(p-j).
\end{aligned}$$

Similarly, with $1 \leq j \leq p$:

$$\begin{aligned}
0 &= 4(-1)^{i+1} - 2(-1)^{i+1} - 2(-1)^{i+1} \\
&= \sum_{i=1}^{p} 2(-1)^{i+1} \Big[2R(i-j) \\
&\qquad\qquad + R(i-j+1) + R(i-j-1) \Big] \qquad (36)\\
&\quad +2R(j-1) - 2(-1)^{p+1} R(p-j+1) \\
&\qquad\qquad +2R(j) + 2(-1)^{p+2} R(p-j).
\end{aligned}$$

Further, with $1 \leq j \leq p$:

$$\begin{aligned}
0 &= 4 - 2 - 2 = \sum_{i=1}^{p} R(i-j)(4-2-2) \\
&= \sum_{i=1}^{p} 2 \Big[2R(i-j) \\
&\qquad\qquad - R(i-j+1) - R(i-j-1) \Big] \qquad (37)\\
&\quad -2R(1-j) + 2R(p-j+1) \\
&\qquad\qquad +2R(j) - 2R(p-j).
\end{aligned}$$

Each of these equations (Eqs. 35, 36 and 37) holds true for any $R(n)$ and p.

II. Symmetry of optimal predictors

A. Symmetry of the Optimal Predictor $H^+(z)$

We will show that polynomial $H(z)^+$ is symmetric or antisymmetric for any set of coefficients that obeys the corresponding Yule-Walker equations (Eq. 14). By substitution of $p-j+1$ for all j in Eq. 14 we obtain

$$\begin{aligned}
&\sum_{i=1}^{p} h_i^+ \Big[2R(p-j-i+1) + R(p-j-i+2) \\
&\quad + R(p-j-i) \Big] = -2\left[R(p-j+1) + R(p-j) \right]. \qquad (38)
\end{aligned}$$

Further, inverting indexing of i by substitution of $p-i+1$ for all i, and the following results:

$$\begin{aligned}
&\sum_{i=1}^{p} h_{p-i+1}^+ \Big[2R(i-j) + R(i-j+1) \\
&\quad + R(i-j-1) \Big] = -2\left[R(p-j+1) + R(p-j) \right]. \qquad (39)
\end{aligned}$$

The expression in Eq. 35 is equal to zero, whereby it can be added to the left hand side in the equation above and merge the summations, which yields (for p even)

$$\begin{aligned}
&\sum_{i=1}^{p} \Big\{ \left[h_{p-i+1}^+ + 2(-1)^i \right] \\
&\qquad \cdot \left[2R(i-j) + R(i-j+1) + R(i-j-1) \right] \Big\} \\
&\qquad\qquad = -2\left[R(j) + 2R(j-1) \right], \quad 1 \leq j \leq p. \qquad (40)
\end{aligned}$$

The equation above is of an identical form as the Yule-Walker equations (Eq. 14), except for the coefficient $h_{p-i+1}^+ + 2(-1)^i$. We have shown that if Eq. 14 holds, then the above equation will also hold. Therefore, for p even, the following holds: $h_i^+ = h_{p-i+1}^+ + 2(-1)^i$. This is the symmetry criterion for p even (Eq. 18) and polynomial $H^+(z)$ is therefore symmetric for p even.

A similar proof for p odd exists, which shows that $H^+(z)$ is antisymmetric. In the proof, Eq. 36 must be added to Eq. 39 similarly as in the case of p even.

B. Symmetry of the Optimal Predictor $H^-(z)$

For polynomial $H^-(z)$, we have (adapting from Eq. 39)

$$\begin{aligned}
&\sum_{i=1}^{p} h_{p-i+1}^- \Big[2R(i-j) - R(i-j+1) \\
&\quad - R(i-j-1) \Big] = -2\left[R(p-j+1) - R(p-j) \right], \qquad (41)
\end{aligned}$$

where $1 \leq j \leq p$. By subtraction of Eq. 37 we have

$$\begin{aligned}
&\sum_{i=1}^{p} \Big\{ \left[2 - h_{p-i+1}^- \right] \\
&\qquad \cdot \left[2R(i-j) - R(i-j+1) - R(i-j-1) \right] \Big\} \\
&\qquad\qquad = -2\left[2R(j) - 2R(1-j) \right], \quad 1 \leq j \leq p. \qquad (42)
\end{aligned}$$

Which (comparing to Eq. 15) implies that $h_i^- = 2 - h_{p-i+1}^-$, and hence, $H^-(z)$ is always symmetric.

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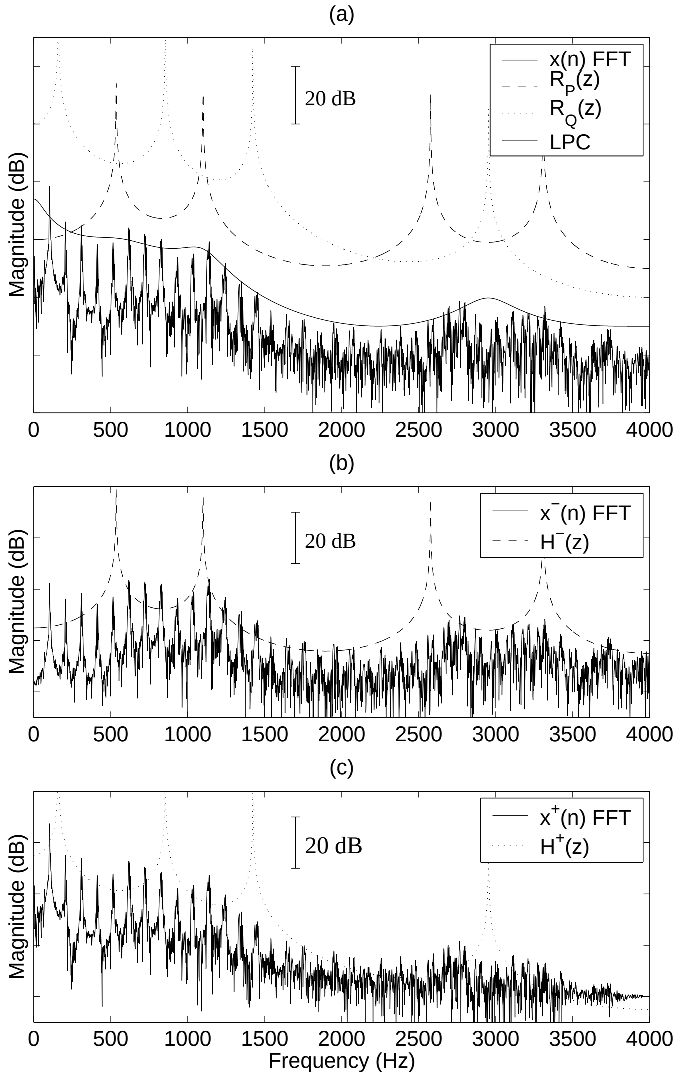

Fig. 5

FREQUENCY DOMAIN VISUALIZATION OF THE EQUIVAILENCE BETWEEN DIFFERENT COMPUTATIONS OF LSP POLYNOMIALS (FOR NOTATIONS, SEE FIG. 2). AN EXAMPLE CALCULATED FROM A MALE VOWEL [A:] WITH $p = 10$. (A) THE POLYNOMIALS $R_P(z)$ AND $R_Q(z)$ AND CONVENTIONAL LP TOGETHER WITH THE FFT-SPECTRUM OF THE ORIGINAL SIGNAL $x(n)$. (B) THE POLYNOMIAL $H^-(z)$ TOGETHER WITH THE FFT-SPECTRUM OF THE HIGH-PASS FILTERED SIGNAL $x^-(n)$. (C) THE POLYNOMIAL $H^+(z)$ TOGETHER WITH THE FFT-SPECTRUM OF THE LOW-PASS FILTERED SIGNAL $x^+(n)$. SIGNAL LEVELS HAVE BEEN ADJUSTED FOR VISUAL CLARITY.