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ROOT-EXCHANGE PROPERTY OF CONSTRAINED LINEAR PREDICTIVE MODELS

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ABSTRACT

In recent works, we have studied linear predictive models constrained by time-domain filters. In the present study, we will study the one-dimensional case in more detail. Firstly, we obtain root-exchange properties between the roots of an all-pole model and corresponding constraints. Secondly, using the root-exchange property we can construct a novel matrix decomposition $\mathbf{A}^T\mathbf{R}\mathbf{A}^{\#} = \mathbf{I}$, where $\mathbf{R}$ is a real positive definite symmetric Toeplitz matrix, superscript $^{\#}$ signifies reversal of rows and $\mathbf{I}$ is the identity matrix. In addition, there exists also an inverse matrix decomposition $\mathbf{C}^T\mathbf{R}^{-1}\mathbf{C}^{\#} = \mathbf{I}$, where $\mathbf{C} \in \mathbb{C}$ is a Vandermonde matrix. Potential applications are discussed.

1. INTRODUCTION

Let us define the residual e_n of a linear predictive model of order m as

$$e_n = x_n + \sum_{i=0}^{m-1} h_i \tilde{x}_{n-i}, \tag{1}$$

where x_n is the wide-sense stationary input signal, h_i ($0 \leq i \leq m-1$) are the model parameters and $\tilde{x}_n = c_n * x_n$, where $c_n = \sum_{k=0}^{l} c_k \xi_{n-k}$ is the impulse response of a causal FIR filter. In literature, this type of models are sometimes named generalised linear predictive models [1]. The transfer function of the predictor can be written as follows

$$A(z) = 1 + \sum_{i=0}^{m-1}\sum_{k=0}^{l} h_i c_k z^{-i-k} = 1 + C(z)H(z), \tag{2}$$

where $C(z)$ and $H(z)$ are the Z-transforms of c_n and h_n, respectively.

In contrast to conventional linear prediction (LP) (e.g. [2]) this model is *constrained* since it defines a predictor of order $m+l-1$ with m model parameters. Note that according to Eq. 2, $x(n)$ is predicted from the samples of $\tilde{x}_n$ from the *current* to the $m-1$'th delayed sample. One would therefore easily be led to believe that the predictor is non-causal. Fortunately, however, if the FIR filter is non-trivial (i.e. c_0 and at least one of the coefficients c_k for $k \geq 1$ is non-zero) then the residual e_n can be determined, since its computation (Eq. 1) contains terms of x_{n-i}, where $i \in [0, m+l-1]$, and the optimisation problem is unambiguous. The transfer function of the predictor obtained will therefore have a coefficient of z^0 that is generally not equal to one.

In matrix notation, Eq. 1 becomes

$$e_n = \mathbf{b}^T\mathbf{x} + \mathbf{h}^T\mathbf{C}^T\mathbf{x}, \tag{3}$$

where $\mathbf{x} = [x_0 \ \dots \ x_{m+l-1}]^T$ is the input signal, $\mathbf{h} = [h_0 \ \dots \ h_{m-1}]^T$ the parameter vector and $\mathbf{b}_i = [1\, 0 \ \dots \ 0]^T$. The convolution matrix $\mathbf{C} \in \mathbb{R}^{(m+l)\times m}$ is defined such that its elements are $\mathbf{C}_{ij} = c(j-i)$.

Straightforward minimisation of the expected value $E[\cdot]$ of the squared residual, $\partial E[e^2(n)]/\partial\mathbf{h} = 0$, yields $\mathbf{C}^T\mathbf{R}\mathbf{C}\mathbf{h} = -\mathbf{C}^T\mathbf{R}\mathbf{b}$, where $\mathbf{R} = E[\mathbf{x}\mathbf{x}^T]$ is the autocorrelation matrix. This solution is unique since $\mathbf{R}$ is positive definite and $\mathbf{C}$ of full rank, and the solution is thus the global minimum.

Equivalently, the minimum of the residual can be found by defining $\mathbf{a} = \mathbf{b} + \mathbf{C}\mathbf{h}$, which yields $E[e^2(n)] = \mathbf{a}^T\mathbf{R}\mathbf{a}$. Since $\mathbf{a} - \mathbf{b} = \mathbf{C}\mathbf{h}$ and $\mathbf{a} - \mathbf{b}$ is therefore in the column space of $\mathbf{C}$, a suitable constraint is $\mathbf{C}_0^T(\mathbf{a} - \mathbf{b}) = 0$, provided that $\mathbf{b}$ is not in the null-space of $\mathbf{C}$. The null-space $\mathbf{C}_0$ of $\mathbf{C}$ is defined by $\mathbf{C}^T\mathbf{C}_0 = 0$ where $\mathbf{C}_0 \in \mathbb{R}^{(m+l)\times l}$. The objective function is then

$$\eta(\mathbf{a}, \mathbf{g}) = \mathbf{a}^T\mathbf{R}\mathbf{a} - \mathbf{g}^T\mathbf{C}_0^T(\mathbf{a} - \mathbf{b}), \tag{4}$$

and the minimum is at $\mathbf{R}\mathbf{a} = \mathbf{C}_0\mathbf{g}$, where the Lagrange multiplier vector $\mathbf{g} = [\gamma_1, \dots, \gamma_l]^T$ can be solved from equation $\mathbf{C}_0^T\mathbf{R}^{-1}\mathbf{C}_0\mathbf{g} = \mathbf{C}_0^T\mathbf{b}$ [3].

In our earlier work, we have shown that $A^{-1}(z)$ is stable if the zeros ξ_i ($1 \leq i \leq l$) of $C(z)$ are real $\xi_i \in \mathbb{R}$, $|\xi_i| > 1$ and $\gamma_i > 0$ ($1 \leq i \leq l$) [4, 5]. However, in this work we will concentrate on the properties of this model for $l = 1$.

The results presented in this work are mostly of theoretical significance and cannot directly be applied to real-world applications. Nevertheless, gaining information on the relation between Toeplitz matrices and the roots of corresponding polynomials could supply improvement to root-finding algorithms as well as provide us with the ability to control the stability of arbitrary predictors.

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2. MINIMUM-PHASE PROPERTY

Definition In this article we will adopt the following notation:

$\mathbf{d}$	a column vector $\mathbf{d} = [d_0, \ldots, d_m]^T$.
$\mathbf{d}^\#$	vector $\mathbf{d}$ with its rows reversed.
$\mathbf{d}^\pm$	the symmetric and antisymmetric part of vector $\mathbf{d}$, that is, $\mathbf{d}^\pm = \mathbf{d} \pm \mathbf{d}^\#$.
$D(z)$	the polynomial corresponding to the Z-transform of vector $\mathbf{d}$.
T	the set of all real positive definite symmetric Toeplitz matrices.

Lemma 1 (Null-space) *Let polynomial $C(z)$ be defined as in Eq. 2 and the corresponding convolution matrix $\mathbf{C}$ as defined in Eq. 3. If $C(\xi_i) = 0$ then it follows that vector $\mathbf{c}_i = [1, \xi_i^{-1}, \ldots, \xi_i^{-(m+l-1)}]^T$ is in the null-space of $\mathbf{C}$. Consequently, the complete null-space $\mathbf{C}_0$ of $\mathbf{C}$ is the set of vectors $\mathbf{c}_i$ where the ξ_i's ($1 \le i \le l$) are the zeros of $C(z)$, provided that the ξ_i's are distinct.*

Matrices of form $\mathbf{C}_0$ (in Lemma 1) are known as Vandermonde matrices [6]. In the following, we will drop the subscript 0 and by $\mathbf{C}$ denote all Vandermonde matrices that appear.

Lemma 2 (Minimum-phase constraint) *Let $\mathbf{R} \in T$ and predictor $\mathbf{a}$ be the solution to ($l = 1$)*

$$\mathbf{R}\mathbf{a}_i = \left[1, \xi_i, \xi_i^2, \ldots, \xi_i^m\right]^T. \qquad (5)$$

Predictor $\mathbf{a}$ is minimum-phase if $\xi_i \in \mathbb{C}$, $|\xi_i| < 1$. Further, for $|\xi_i| = 1$ predictor $\mathbf{a}$ will have its roots on the unit circle.

Proof of these lemmata was presented in [4, 5].

3. THE ROOT-EXCHANGE PROPERTY

Lemma 3 (Root exchange) *Let $\mathbf{a}_0$ be a solution to Eq. 5 with $\xi_0 \in \mathbb{C}$, $|\xi_0| < 1$, and let ξ_i ($1 \le i \le m$) be the zeros of polynomial $A_0(z)$ (with coefficients $\mathbf{a}_0$). Then the polynomials $A_i(z)$ ($1 \le i \le m$), with coefficients $\mathbf{a}_i$ solving Eq. 5 using ξ_i, have zeros ξ_k for $i \ne k$.*

In other words, exchanging a root ξ_k from $A_i(z)$ for ξ_i, corresponds in Eq. 5 to replacing ξ_i with ξ_k. Then $A_i(z)(1 - \xi_i z^{-1}) = \zeta_{ik} A_k(z)(1 - \xi_k z^{-1})$, where ζ_{ik} is some scalar that depends on i and k.

The root exchange property of one root is illustrated in Fig. 1.

Proof Polynomial $A_i(z)$ can be factored as $A_i(z) = \left(1 - \xi_k z^{-1}\right) B(z)$, or equivalently,

$$\mathbf{a}_i = \mathbf{B} \begin{bmatrix} 1 \\ -\xi_k \end{bmatrix} = \begin{bmatrix} b_0 & 0 \\ b_1 & b_0 \\ \vdots & \vdots \\ 0 & b_{m-1} \end{bmatrix} \begin{bmatrix} 1 \\ -\xi_k \end{bmatrix}. \qquad (6)$$

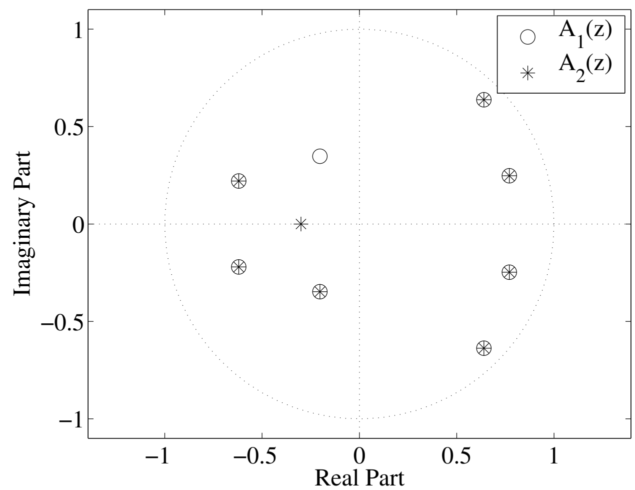

Fig. 1. An example of the root exchange property of constrained LP models $A_i(z)$ ($m = 8$). Coefficient vectors $\mathbf{a}_i$ of $A_i(z)$ solve $\mathbf{R}\mathbf{a}_i = \mathbf{c}_i$ where $\mathbf{c}_i = (\xi_i^k)_{k=0\ldots m}$ and ξ_i is a root of $C_j(z)$ for $i \ne j$.

From $\mathbf{R}\mathbf{a}_i = \mathbf{c}_i$ we obtain

$$\mathbf{c}_i = \mathbf{R}\mathbf{B} \begin{bmatrix} 1 \\ -\xi_k \end{bmatrix} = \begin{bmatrix} d_0 & d_{-1} \\ d_1 & d_0 \\ \vdots & \vdots \\ d_m & d_{m-1} \end{bmatrix} \begin{bmatrix} 1 \\ -\xi_k \end{bmatrix}, \qquad (7)$$

which defines the coefficients d_i uniquely. From 7 we can readily obtain relation

$$\begin{bmatrix} d_0 & d_{-1} \\ d_1 & d_0 \\ \vdots & \vdots \\ d_m & d_{m-1} \end{bmatrix} \begin{bmatrix} 1 \\ -\xi_i \end{bmatrix} = \eta \mathbf{c}_k, \qquad (8)$$

where η is a positive constant. Recall that matrix $\mathbf{B}$ corresponds to $\mathbf{a}_i$ deconvoluted by $(1 - \xi_k z^{-1})$ and Eq. 8 corresponds to convolution of another term $(1 - \xi_i z^{-1})$. Since the result on the right hand side in Eq. 8 is $\eta \mathbf{c}_k$, it must be equal to $\eta \mathbf{c}_k = \eta \mathbf{R}\mathbf{a}_k$. The corresponding polynomials $A_i(z)$ and $A_k(z)$ are thus equal except for a scaling coefficient and an exchanged zero ξ_k for ξ_i. This concludes the proof. □

Lemma 4 (Symmetric root exchange) *Let $\mathbf{a}_0$ be a solution to Eq. 5 with $\xi_0 \in \mathbb{R}$. Then for the symmetric or antisymmetric part $\mathbf{a}_0^\pm$ we have $\mathbf{R}\mathbf{a}_0^\pm = \mathbf{c}_0^\pm$. Let ξ_i with $1 \le i \le m$ be the roots of $A_0(z)$. Then*

$$\mathbf{R}\mathbf{a}_i^\pm = \mathbf{c}_i^\pm \qquad (9)$$

for $0 \le i \le m$. In other words, $A_i(z)$ has zeros ξ_k with $i \ne k$ and $|\xi_i| = 1$ for $i \ne 0$.

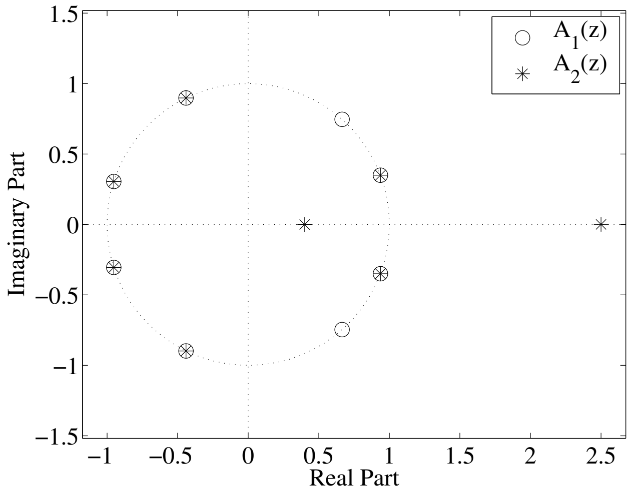

Fig. 2. An example of the symmetric root exchange property of constrained LP models $A_i^+(z)$ ($m = 8$). Coefficient vectors $\mathbf{a}_i^+$ of $A_i^+(z)$ solve $\mathbf{R}\mathbf{a}_i^+ = \mathbf{c}_i^+$ where $\mathbf{c}_i^+$ is the symmetric part of $\mathbf{c}_i = (\xi_i^k)_{k=0\dots m}$ and ξ_i is a root of $C_j(z)$ for $i \neq j$.

The root exchange property of conjugate pair roots on the unit circle is illustrated in Fig. 2.

Proof Since predictor $\mathbf{a}_0$ is minimum-phase due to Lemma 2, its symmetric and antisymmetric parts $\mathbf{a}_0^\pm$ will have their roots on the unit circle [7, 9]. By factoring a conjugate root pair ξ_i and ξ_i^{-1} ($1 \leq i \leq m$) from $\mathbf{a}_0$, that is, $A_0(z) = (1 - (\xi_i + \xi_i^{-1})z^{-1} + z^{-2})B(z)$, we can readily proof the root exchange property similarly as in proof of Lemma 2. □

Note that in the proof above, ξ_0 in $\mathbf{c}_0^\pm$ will appear in $\mathbf{a}_i^\pm$ ($1 \leq i \leq m$) as a root pair ξ_0 and ξ_0^{-1}. Especially, when $\xi_0 \to 0$, the first and last coefficients of $\mathbf{a}_i^\pm$ will tend to zero.

Further, note that by proper scaling, we can make $\mathbf{a}_i^\pm$ and $\mathbf{c}_i^\pm$ real. This is a significant advantage in reduction of complexity (between the root exchange and symmetric root exchange properties) if we are to apply root exchange in a computational algorithm.

An interesting consequence of Lemma 4 is that $\mathbf{a}_i^\pm$ always has distinct roots. To see this, let ξ_k be a double root and of $A_i^\pm(z)$. Then $A_k^\pm(z)$ will also have a root ξ_k by the root exchange rule. It follows that $0 = A_k^\pm(\xi_k) = A_k^\pm(\xi_k^{-1}) = \mathbf{c}_k^{\pm T}\mathbf{a}_k^\pm = \mathbf{a}_k^{\pm T}\mathbf{R}\mathbf{a}_k^\pm$. This is a contradiction since $\mathbf{R}$ is positive definite, and the assumption that ξ_k is a double root is therefore defective. Moreover, if we consider the family of equations Eq. 9 with $\xi_0 \in (-1, +1)$, none of these models will have overlapping roots. In other words, the roots follow monotonic paths on the unit circle as a function of ξ_0.

Lemma 5 (Reverse decomposition) *For a real, positive definite and symmetric Toeplitz $(m+1) \times (m+1)$ matrix $\mathbf{R}$, there exists a decomposition matrix $\mathbf{A} \in \mathbb{C}$ such that $\mathbf{A}^T\mathbf{R}\mathbf{A}^\# = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix.*

Note that $\mathbf{A}$ is not unique and that we indeed use the transposition ^{T} operator and not the complex conjugate transpose ^{H}, also known as the hermitian. Further, equation $\mathbf{A}^T\mathbf{R}\mathbf{A}^\# = \mathbf{I}$ is equivalent with $(\mathbf{A}^\#)^T\mathbf{R}\mathbf{A} = \mathbf{I}$ since $\mathbf{R}$ is symmetric.

Proof Let vectors $\mathbf{a}_i$, ξ_i and $\mathbf{c}_i$ be defined as in Lemma 3. Further, let matrices $\mathbf{A}$ and $\mathbf{C}$ consist of column vectors $\mathbf{a}_i$ and $\mathbf{c}_i$, respectively, such that $\mathbf{R}\mathbf{A} = \mathbf{C}$. We know that the roots of $A_i(z)$ are ξ_k with $i \neq k$ and $0 \leq i, k \leq m$. Consequently, $A_i(\xi_k) = \mathbf{a}_i^T\mathbf{c}_k^\# = 0$ for $i \neq k$ and thus $\mathbf{A}^T\mathbf{C}^\# = \mathbf{D}$, where $\mathbf{D}$ is a diagonal matrix with positive elements.

It follows that $\mathbf{D} = \mathbf{A}^T\mathbf{C}^\# = \mathbf{A}^T\mathbf{R}\mathbf{A}^\#$. Since coefficients of $\mathbf{D}$ are non-zero, we can, with suitable scaling of the columns of $\mathbf{A}$, find an $\hat{\mathbf{A}}$ such that $\hat{\mathbf{A}}^T\mathbf{R}\hat{\mathbf{A}}^\# = \mathbf{I}$. □

As a corollary to Lemma 5, we can readily see that there exists a matrix decomposition for the inverse of $\mathbf{R}$ such that $\hat{\mathbf{C}}^T\mathbf{R}^{-1}\hat{\mathbf{C}}^\# = \mathbf{I}$. The inverse $\mathbf{R}^{-1}$ exists since $\mathbf{R}$ is strictly positive definite.

While Lemma 5 uses Lemma 3, a similar decomposition can be constructed using the result of Lemma 4. Then both the symmetric and antisymmetric parts of $\mathbf{a}_i^\pm$ and $\mathbf{c}_i^\pm$ have to be included in the matrices $\mathbf{A}$ and $\mathbf{C}$, respectively. The advantage of this approach is that the matrices $\mathbf{A}$ and $\mathbf{C}$ can then be scaled to real while the construction above produces matrices $\mathbf{A}$ and $\mathbf{C}$ that are generally complex.

4. DISCUSSION AND SUMMARY

We have presented root-exchange properties between the all-pole form of the constrained linear predictive model and the corresponding constraint. Further, as a corollary we obtained novel matrix decompositions for real positive definite symmetric Toeplitz matrices and its inverse, the latter with a Vandermonde matrix.

Intuitively, the root-exchange property can be explained by the fact that the constraint $\mathbf{c}_i$ is equivalent to requiring that $A_i(z)$ is strictly positive at ξ_i^{-1}, that is, $A_i(\xi_i^{-1}) = \mathbf{a}_i^T\mathbf{c}_i = \mathbf{a}_i^T\mathbf{R}\mathbf{a}_i > 0$. Therefore, $A_i(z)$ cannot have a zero at ξ_i^{-1}. When a zero ξ_k is exchanged for ξ_i then $A_k(\xi_k) > 0$ and it is not a surprise that $A_k(\xi_i) = 0$.

The conventional LP model is a special case of constrained linear predictive models, whereby Eq. 5 with $\xi_0 = 0$ becomes $\mathbf{R}\mathbf{a}_0 = [1, 0, \dots, 0]^T$. If vectors $\mathbf{a}_i$ are defined with the root-exchange property as in Lemmata 3 and 5, then the corresponding $A_i(z)$ with $1 \leq i \leq m$ will have

$A_i(0) = 0$, and the m'th component $a_m^{(i)}$ of $\mathbf{a}_i$ is thus always $a_m^{(i)} = 0$. In other words, using the root-exchange property, we have means to reduce the $(m+1)\times(m+1)$ matrix problem to a $m \times m$ problem. This property could potentially be used for order reduction of polynomial root-finding problems.

Unfortunately, in general, the matrix reverse decompositions are non-unique and we can find several matrices $\mathbf{A}$ and $\mathbf{C}$ that satisfy $\hat{\mathbf{A}}^T\mathbf{R}\hat{\mathbf{A}}^{\#} = \mathbf{I}$ and $\hat{\mathbf{C}}^T\mathbf{R}^{-1}\hat{\mathbf{C}}^{\#} = \mathbf{I}$. However, if we constrain one root, e.g. the zero root in the case of conventional LP, then the decomposition becomes unique. (We have then assumed that $\mathbf{C}$ is constrained to complex Vandermonde matrices and $\mathbf{RA} = \mathbf{C}$.) Moreover, the question of finding the decomposition matrices without solving first the roots of polynomial $A_0(z)$ remains open. On the other hand, assuming that we can, with some convenient criterion, constrain $\mathbf{C}$ to Vandermonde matrices, it could be possible to use this decomposition for iterative root-finding. Such a root-finding algorithm could be applied to any polynomial with distinct roots inside the unit circle, since $\mathbf{Ra}_0 = [1, 0, \ldots, 0]^T$ uniquely defines $\mathbf{R}$.

The presented theory can be partially generalised to infinite dimensional matrices of functional analysis. Then, the Toeplitz and Vandermonde matrices $\mathbf{R}$ and $\mathbf{C}$ become Toeplitz (bi-infinite) and Vandermonde (infinite or bi-infinite) operators, and matrix $\mathbf{A}$ is zero-extended into infinity (infinite or bi-infinite). This extension of the Toeplitz matrix implies (when $\xi = 0$) that we introduce new values of $\mathbf{R}_{ij}$ corresponding to zero reflection coefficient value. Consequently, the extension is quite elegant, since it does not introduce any new information to the problem. In addition, this is a sufficient criterion to make the decomposition unique. However, the row-reversal operation is not as easily transfered to the infinite dimensional case, since it requires taking elements starting from infinity and making them the first elements. [8]

Note that the $(n+k)\times n$ zero extension $\tilde{\mathbf{A}}$ of $n \times n$ matrix $\mathbf{A}$ is not equivalent to $\mathbf{A}$ in the reflection decomposition sense. In other words, since $\mathbf{A}^T\mathbf{R}\mathbf{A}^{\#} = \mathbf{I}$ we obtain $\tilde{\mathbf{A}}^T\mathbf{R}\tilde{\mathbf{A}}^{\#} = \tilde{\mathbf{A}}^T\tilde{\mathbf{C}}^{\#} = \mathbf{D}^k\mathbf{A}^T\mathbf{C}^{\#} = \mathbf{D}^k$ where $\tilde{\mathbf{C}}$ is the extension of $\mathbf{C}$ (which is also a Vandermonde matrix), $\mathbf{D}$ is a diagonal matrix with the exponents of $\mathbf{C}$ and we have assumed that $\mathbf{R}$ has been extended with zero reflection coefficients. This implies that we have a description of the autocorrelation matrix very much alike the Krylov-subspace [6]. The author believes, that there is some potential in this property that could lead to new root-finding algorithms.

In our earlier work [4, 5], we have shown that polynomials $A_i(z)$ whose coefficients solve Eq. 5 with real $|\xi_i| < 1$, form a convex space of polynomials with the minimum-phase property. The root-exchange property presented in this paper, is compatible with the convex space formulation only as long as the exchanged roots are real. The convexity property can be used, for example, in interpolation between polynomials when optimising the model by a frequency domain criterion. Currently, however, the author has not found simple exchange-rules for complex conjugate root pairs, other than those on the unit circle.

Roots of presented models which lie on the unit circle must always be distinct. This can easily be seen using the root-exchange rule, by exchanging one of the multiple zeros with a distinct zero. The obtained model will thus have a zero on the diagonal of matrix $\mathbf{D}$ (in Lemma 5) and $\mathbf{R}$ cannot be positive definite. This property could be used for creation of polynomial pairs with interlacing zeros on the unit circle. Recall that such a property is useful in creation of stable predictors from symmetric/antisymmetric polynomial pairs [9].

While the results of this paper do not present any straightforward applications, at least as far as the author is aware, they still offer theoretical insight into the properties of constraints of linear predictive models as well as the properties of Toeplitz matrices and their relation to Vandermonde matrices.

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