Square-Matrix Embeddable Space–Time Block Codes for Complex Signal Constellations

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***Abstract*—Space–time block codes for providing transmit diversity in wireless communication systems are considered. Based on the principles of linearity and unitarity, a complete classification of linear codes is given in the case when the symbol constellations are complex, and the code is based on a square matrix or restriction of such by deleting columns (antennas). Maximal rate delay optimal codes are constructed within this category. The maximal rates allowed by linearity and unitarity fall off exponentially with the number of transmit antennas.**

***Index Terms*—Clifford algebras, multiple antennas, space–time block codes, transmit diversity.**

I. Introduction

MULTIANTENNA techniques have received a lot of attention in the scientific community after Foschini [1] and Telatar [2] showed that the capacity of the system increases linearly with the number of uncorrelated transmit and receive antennas. With a restricted number of receive antennas, a part of this capacity increase can be realized using transmit diversity. Within a short time, several schemes have been proposed for a number of wireless communication systems [3]–[6].

One implementation of transmit diversity, called space–time trellis coding, was developed by Tarokh, Seshadri, and Calderbank [4]. It performs well in slowly fading environments, but it has the drawback that decoding complexity grows exponentially with the number of antennas.

Recently, the alternative multiantenna transmit diversity concept of space–time block coding[1] emerged in the work of Alamouti [5]. It was further developed and put into a theoretical framework by Tarokh, Jafarkhani, and Calderbank in [6]. The essential feature of these schemes is their inherent orthogonality. This guarantees that linear decoding provides the maximal likelihood result. Even in some systems with frequency-selective fading, orthogonality is almost preserved by the communication channel, as long as the ensuing intersymbol interference can be reliably equalized. This is the case, e.g., in wideband code-division multiple-access (CDMA) systems, where the signaling period is $\gg$ delay spread, and the RAKE receiver separates the multipaths. For each path, the orthogonality of the multiantenna transmission is almost preserved. In such systems, space–time block coding is a viable candidate for providing transmit diversity. Accordingly, the two-antenna complex modulation space–time block code proposed by Alamouti has already been accepted as an open-loop transmit diversity scheme for a third-generation wireless communication system, namely, the wideband CDMA standardized by 3GPP (3rd Generation Partnership Project) [7].

The channel model used in this paper is uncorrelated frequency flat block fading channels between a multitude of transmit and at least one receive antenna. The length of the fading block equals (or is a multiple of the) space–time block code length. The receiver has complete channel state information, the transmitter has none.

In space–time block code design, the essential design criteria are the provided transmit (Tx) diversity, the (symbol) rate of the code, and the delay. The degree of Tx diversity is characterized by the number of independently decodable channels. For full diversity it equals the number of transmit antennas. If multiple receive (Rx) antennas are deployed, the total diversity degree is the product of the Tx and Rx diversity degrees. The number of Rx antennas is, however, irrelevant for the design of orthogonal space–time block codes. The (symbol) rate of the code is the number of symbols transmitted by the code per time epoch. The delay is the length of the space–time block code frame. Depending on the underlying modulation scheme, space–time block codes can be divided into real and complex codes. In this paper, the goal for designing space–time block codes is to maximize the rate and minimize the delay, keeping full diversity.

In [6], the problem of designing rate 1, full diversity space–time block codes was completely solved. Rate-1 real codes were found for any number of Tx antennas. For up to eight antennas, these were constructed from orthogonal designs in two, four, and eight dimensions. An orthogonal design is an orthogonal matrix with entries $\pm x_j$, where x_j are real symbols. For rate 1, the number of symbols equals the matrix dimension. For example, the four-antenna real block code constructed in [6] is based on the orthogonal design

$$C(x_1, x_2, x_3, x_4) = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ -x_2 & x_1 & -x_4 & x_3 \\ -x_3 & x_4 & x_1 & -x_2 \\ -x_4 & -x_3 & x_2 & x_1 \end{bmatrix}. \quad (1)$$

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[1]Here, we adopt the convention of [6] and use the name space–time block codes only for codes that are unitary independently of the modulation scheme. In this terminology, e.g., codes of the kind presented in [3] would not be space–time block codes.

For transmission, the orthogonal design $C(x_1, x_2, x_3, x_4)$ is interpreted as a code matrix defining the connection of the information symbols x_1, x_2, x_3, x_4 and the channel symbols transmitted from a given antenna during a given time epoch. Each row is transmitted at a specified time epoch from four different antennas, and each column is transmitted from a specified antenna at four different time epochs. If the number of Tx antennas N is not equal to the orthogonal design dimensions two, four, or eight, delay optimal codes for $N < 8$ can be constructed by deleting a column from a higher dimensional orthogonal design, i.e., by switching off antennas. Thus, for three transmit antennas, the optimal code can be written by deleting a column from the 4×4 matrix above, and optimal five-, six-, or seven-antenna schemes can be constructed from an 8×8 orthogonal design. For more than eight Tx antennas, generalized orthogonal designs were constructed in [6], which give rate-1 real space time block codes for any number of Tx antennas. For these, the delay grows exponentially with the number of Tx antennas N. The minimal delay for a real symbol generalized orthogonal design with rate 1 is

$$T = 16^{\lfloor (N-1)/8 \rfloor} 2^{\lceil \log_2 (1+(N-1) \bmod 8) \rceil}. \tag{2}$$

Allowing complex signal constellations restricts the available rate-1 codes severely; rate-1 complex space–time block codes exist only for two transmit antennas [6]. This code was found by Alamouti [5], and it is based on complex orthogonal designs (or unitary designs, for short) of the form

$$C(z_1, z_2) = \begin{bmatrix} z_1 & z_2 \\ -z_2^* & z_1^* \end{bmatrix}. \tag{3}$$

In addition to this unitary design, [6] presents rate-$1/2$ complex codes based on the rate-1 real codes; transmitting first a rate–1 orthogonal design with complex symbols, followed by transmitting the same orthogonal design with the symbols complex conjugated, produces a complex rate-$1/2$ block code. This gives rate-$1/2$ codes for any number of antennas, with exponentially growing delay, twice the one given by (2). For example, the rate-$1/2$ code for four transmit antennas, corresponding to the rate-1 real code for four antennas (1), is

$$C(z_1, z_2, z_3, z_4) = \begin{bmatrix} z_1 & z_2 & z_3 & z_4 \\ -z_2 & z_1 & -z_4 & z_3 \\ -z_3 & z_4 & z_1 & -z_2 \\ -z_4 & -z_3 & z_2 & z_1 \\ z_1^* & z_2^* & z_3^* & z_4^* \\ -z_2^* & z_1^* & -z_4^* & z_3^* \\ -z_3^* & z_4^* & z_1^* & -z_2^* \\ -z_4^* & -z_3^* & z_2^* & z_1^* \end{bmatrix}. \tag{4}$$

For comparing to specific codes constructed in this paper, the delays of these rate-$1/2$ codes for up to 16 antennas are of interest. These can be calculated from (2). For five to eight antennas, the rate-$1/2$ code has delay 16, for nine antennas 32, for 10 antennas 64, and for 11 to 16 antennas, the delay is 128. We shall see that for up to eight antennas, these codes are not delay optimal, and for three or four Tx antennas, they are not rate optimal either. This can be seen from a rate-$3/4$ code for three and four transmit antennas which was presented in [6] to inspire future work

$$C = \begin{bmatrix} z_1 & z_2 & \frac{1}{\sqrt{2}} z_3 & \frac{1}{\sqrt{2}} z_3 \\ -z_2^* & z_1^* & \frac{1}{\sqrt{2}} z_3 & -\frac{1}{\sqrt{2}} z_3 \\ \frac{1}{\sqrt{2}} z_3^* & \frac{1}{\sqrt{2}} z_3^* & -x_1 + \mathrm{j}y_2 & -x_2 + \mathrm{j}y_1 \\ \frac{1}{\sqrt{2}} z_3^* & -\frac{1}{\sqrt{2}} z_3^* & x_2 + \mathrm{j}y_1 & -x_1 - \mathrm{j}y_2 \end{bmatrix}. \tag{5}$$

Here, x_i and y_i are the real and imaginary parts of the complex modulation symbols z_i. A three-antenna code is again simply constructed from the code above by deleting one column. This code shows an important disadvantage of square matrix space–time block codes with rate less than one. Even for equal power constellations (QPSK, M-PSK), they are power-unbalanced in that the power transmitted from any given antenna fluctuates in time. For nonequal power constellations (M-QAM etc.), this property of the codes makes the power fluctuations worse. These (amplified) power fluctuations are problematic in view of power-amplifier design, as the region of linear amplification has to be correspondingly extended. Thus, an additional criterion may be added for designing space–time block codes: the power spectrum should be as balanced as possible.

The complex modulation space–time block codes presented above can be seen as examples of the two categories of space–time block codes that have been introduced in the literature. These are the rate-halving codes and square matrix embeddable codes.

In this paper, the theory of square matrix embeddable space–time block codes shall be completed. This theory is based on square matrix maximal rate complex space–time block codes. We shall see that for square matrix embeddable space–time block codes, the delay is minimal, and the maximal rate falls off exponentially with the number of antennas. This is in contrast to the rate-halving codes, where the rate is fixed to $1/2$, and the minimal delay increases exponentially. The simplest forms of square matrix embeddable space–time block codes are unitary (complex orthogonal) designs, i.e., square matrices with elements either $\pm$ a symbol or its complex conjugate, or 0.[2] We shall see that irreducible[3] unitary designs exist only in dimensions that are powers of two.

In Section II, we clarify what is required from complex modulation space–time block codes in this paper; linearity and unitarity. We show that these properties imply linear maximum-likelihood (ML) decoding, and that the rank and determinant criteria for space–time code design are saturated. In Section III, we restrict ourselves to square matrix embeddable codes. In the main theorem of the paper, Theorem 1, the maximal rates for a given number of transmit antenna are stated. In Section IV, we give an explicit construction for unitary designs, the simplest square matrix space–time block codes that achieve the maximal rates for a given diversity. In Section V, we consider unitarily transformed codes, and prove Theorem 2 which states the

[2]With this terminology, the code matrix (3) is a unitary design, whereas (5) is not. Both are square matrix complex modulation space–time block codes. The unitary design corresponding to (5) shall be constructed in Section IV.

[3]Irreducible means not equivalent to a block-diagonal matrix.

unitary equivalence of all possible square matrix embeddable space–time block codes with the unitary designs constructed in Section IV. Section VI rounds off the paper with a discussion. The representation theory of Clifford algebras, which is the mathematical foundation of the results presented in this paper, is developed in the Appendix.

II. Linear Space–Time Block Codes

We consider a wireless communication system consisting of N transmit antennas and M receive antennas. During T time epochs, the system transmits K symbols z_k, $k = 1, \ldots, K$, which are taken from a complex signal constellation. The transmitted symbols are distributed between the antennas and time epochs by a $T \times N$ matrix $C(z)$, the code matrix. The matrix elements of $C(z)$ are linear combinations of the symbols z_k. We shall only be interested in schemes with $T \geq N$, as schemes with $T < N$ are not linearly decodable.

The characteristics of the block code are compared to single-antenna transmission of the same symbol stream The ratio of the transmission rate of the code to this "uncoded" scheme K/T is the symbol rate of the code. It should be noted that this rate is a relative modulation rate; a rate less than one does not imply any coding, nor does it imply any increase in Euclidean distance. It is just a measure of the efficiency of the use of the antenna resource.

We shall adopt two conditions for the design of space–time block codes. These are as follows.

- Unitarity (complex-orthogonality), with the inner products proportional to the sum of the squared amplitudes of the symbols
$$C^\dagger C = \sum_k |z_k|^2 \mathbb{1}_N. \tag{6}$$
Here $\mathbb{1}_N$ is the $N \times N$ unit (identity) matrix, and $C^\dagger$ is the Hermitian conjugate (complex conjugate transpose) of the matrix C. Pseudounitarity in the form (6) is also required for nonsquare matrices with $T > N$.
- Linearity in the symbols z_k. The code matrix can be expanded as
$$C = \sum_{k=1}^{K} (z_k \beta_k^- + z_k^* \beta_k^+) \tag{7}$$
$$= \sum_{k=1}^{K} (x_k \beta_{2k-2} + y_k \beta_{2k-1}). \tag{8}$$
Here $\{\beta_k\}_{k=0}^{2K-1}$ is a set of $2K$ constant $T \times N$ matrices with complex entries. The real variables x_k and y_k are the real and imaginary parts of the symbols $z_k = x_k + \mathrm{j} y_k$,[4] and the matrices $\beta_k^\pm$ are linear combinations of the β_k
$$\beta_k^\pm = \tfrac{1}{2}(\beta_{2k-2} \pm \mathrm{j}\beta_{2k-1}). \tag{9}$$

The definition of space–time block codes adopted in this paper is based on these criteria.

[4]We denote $\sqrt{-1} \equiv \mathrm{j}$.

Definition 1: Complex modulation space–time block codes are linear mappings of a set of complex symbols z_k to a code matrix of transmitted symbols at different antennas and time epochs. The code matrix is proportional to a unitary matrix, with the proportionality coefficient $\sqrt{\sum_k |z_k|^2}$.

Everything that follows is based on the criteria of unitarity and linearity. Combining them, one gets the following algebraic restriction on the coefficient matrices β_k:
$$\beta_k^\dagger \beta_j + \beta_j^\dagger \beta_k = 2\delta_{jk} = \mathbb{1}_N. \tag{10}$$
Any solution of these equations defines a maximal diversity, rate K/T unitary space–time block code. In [6], the real version of this equation, with Hermitian conjugations replaced by transposes, was used in the spirit of Radon and Hurwitz (see, e.g., [8]) to find orthogonal designs, pertinent for real modulation space–time block codes. In Sections III and IV, we shall find all possible solutions of the complexified Radon–Hurwitz equations (10). First, however, we shall take a look on why these criteria lead to ML linear decoding, and to optimal space–time block codes in terms of all design criteria presented in the literature.

A. Decoding

Using the unitarity (6) and linearity (8) properties of space–time block codes, it is easy to devise a maximal likelihood linear decoding scheme for the transmitted symbols.

We assume perfect channel state information at the receiver, and one-tap channels with the same propagation delay between all pairs of Tx and Rx antennas.

The channel between the nth transmit and mth receive antenna is denoted by α_{nm}. All channels are collected into the $N \times M$ channel matrix
$$\alpha = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1M} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{N1} & \alpha_{N2} & \cdots & \alpha_{NM} \end{bmatrix}. \tag{11}$$

Correspondingly, the received signal at time epoch t and receive antenna m is denoted by r_{tm}. The $T \times M$ matrix of these signals is given by
$$R = C(z)\alpha + n \tag{12}$$
where n is a $T \times M$ matrix of additive complex Gaussian white noise.

The complex symbols z_k can be decoded using the matrices $\beta_k^\pm$ of (9) and the channel estimates $\hat{\alpha}$. Indeed, it is a direct consequence of relation (10) that the metric given by the absolute square of the noise
$$\mathcal{M}(z) = \mathrm{Tr}[\hat{\alpha}^\dagger \hat{\alpha}]\, \mathrm{Tr}[(R - C(z)\hat{\alpha})^\dagger (R - C(z)\hat{\alpha})] \tag{13}$$
gives a symbol-wise linear ML metric for the symbols z_k
$$\mathcal{M}(z) = \sum_k \mathcal{M}_k + \mathcal{M}_o$$
$$\mathcal{M}_k(z_k) = |\mathrm{Tr}[R^\dagger \beta_k^+ \hat{\alpha} + \hat{\alpha}^\dagger (\beta_k^-)^\dagger R] - \mathrm{Tr}[\hat{\alpha}^\dagger \hat{\alpha}] z_k|^2.$$

$\mathcal{M}_o$ does not depend on z_k. This metric generalizes the metric presented in [5] for 2×2 complex block codes to arbitrary matrix dimensions. Using (10), it is easy to see that

$$\mathrm{Tr}[R^\dagger \beta_k^+ \alpha + \alpha^\dagger (\beta_k^-)^\dagger] = \mathrm{Tr}[\alpha^\dagger \alpha] z_k + \text{noise}.$$

This means that any unitary (6) and linear (8) space–time block code maximum ratio combines all channels between the transmit and receive antennas. From a diversity point of view, this is the optimal result. For the modulation rates and dimensions allowed by the construction, space–time block codes are thus optimal space–time codes. In the next section, we shall see this in terms of design criteria for space–time codes.

B. Space–Time Code Design Criteria and Space–Time Block Codes

The main design criteria for space–time codes presented in the literature are formulated in terms of the codeword difference matrix $D_{ce} = C_c - C_e$. Here C_c and C_e are the code matrices corresponding to the encoded and possibly erroneously detected sets of information bits, respectively. Minimizing the pairwise error probability of deciding in favor of C_e when transmitting C_c leads to the following design criteria.

- The rank criterion [3]. The diversity gained by the multiple transmitter scheme is

$$\text{diversity} = \min_{e\neq c} \mathrm{Rank}[D_{ce}^\dagger D_{ce}] \leq \min[T, N]. \qquad (14)$$

  To achieve maximal diversity, $D_{ce}^\dagger D_{ce}$ should have full rank for all distinct codewords c and e.
- The determinant criterion [4]. To optimize performance in a (Rayleigh) fading environment, C should be designed to maximize

$$\min_{e\neq c} \det{}'[D_{ce}^\dagger D_{ce}]. \qquad (15)$$

  The prime in the determinant indicates that zero eigenvalues should be left out from the product of eigenvalues when computing the determinant.

Now we can investigate space–time block codes in terms of these criteria. From linearity (8) it follows that the codeword difference matrix D_{ce} inherits the unitarity property (6) of the code matrix C

$$D_{ce}^\dagger D_{ce} = \sum_k |z_{k,c} - z_{k,e}|^2 \mathbb{1}_N. \qquad (16)$$

Thus, the design criteria are fulfilled.

- Rank criterion. As a unitary matrix, D_{ce} is full-rank for all distinct codeword pairs. Thus, all space–time block codes give full diversity, equaling the number of Tx antennas.
- Determinant criterion. As D_{ce} is unitary

$$\det[D_{ce}^\dagger D_{ce}] = \sum_k |z_{k,c} - z_{k,e}|^{2N}.$$

  This is the maximum given a fixed transmit power.

The defining relations (6) and (8) of space–time block codes thus give codes that are optimal with respect to the rank and determinant criteria. In the previous section it was seen that the structure (10), following from unitarity (6) and linearity (8), guarantees a linear decoding scheme which maximal ratio combines all channels. From diversity point of view this is the optimal, so it is not too surprising that the space–time code design criteria are saturated.

III. Square Matrix Block Codes and Clifford Algebras

All complex modulation space–time block codes presented in the literature belong to one of two categories. They are either rate-halving codes, like (4), or square matrix embeddable codes, like (3), (5) and their restrictions to lesser numbers of antennas.

Definition 2: The category of rate-halving complex modulation space–time block codes [6] consists of codes built from an orthogonal design and its complex conjugate, halving the rate of the real modulation code based on the orthogonal design.

Definition 3: The category of square matrix embeddable complex modulation space–time block codes [5], [6] consists of codes based on a square code matrix, or a column-wise restriction of one.

Here, we concentrate on square matrix embeddable codes. Thus, when proceeding with the analysis of (10), we specialize to square matrices, i.e., $T = N$.

If we redefine

$$\gamma_k = \beta_0^\dagger \beta_k, \qquad k = 0, \ldots, 2K-1 \qquad (17)$$

we have $\gamma_0 = \mathbb{1}_N$, and the form of the algebra (10) remains unchanged for the γ's. From the relation between γ_0 and the other γ's we then see that these should be anti-Hermitian

$$\gamma_k^\dagger = -\gamma_k, \qquad k = 1, \ldots, 2K-1. \qquad (18)$$

The algebra of these remaining γ's is now

$$\gamma_k\gamma_j + \gamma_j\gamma_k = -2\delta_{jk}\mathbb{1}_N, \qquad j, k = 1, \ldots, 2K-1. \quad (19)$$

This is the defining relation of generators of the Clifford algebra (see, e.g., [9]), which thus is a core concept in constructing space–time block codes. It is an amusing coincidence that much of the work on Clifford algebras during the past century has been related to their applicability to describing matter in space–time.

We have found the following generic prescription for finding a complex modulation space–time block code.

Proposition 1: Any square matrix embeddable space–time block code for N transmit antennas, transmitting K complex modulation symbols during $T \geq N$ time epochs, can be constructed by the following procedures.

– Finding a representation of the Clifford algebra (19) in terms of anti-Hermitian $T \times T$ matrices γ_k, $k = 1, \ldots, 2K-1$.

— Taking a unitary matrix $\beta_0 \in U(T)$.

— Defining $\beta_k = \beta_0\gamma_k$, $k = 1, \ldots, 2K-1$.

— Using the matrices β_k, $k = 0, \ldots, 2K-1$ to create a code matrix $C(z)$ according to prescription (8).

— If $N < T$, choosing N columns of the constructed $T\times T$ matrix.

Proof: This follows directly from unitarity (6) and linearity (8), as explained when deriving (10) and (18). □

TABLE I
MAXIMAL RATES FOR FULL DIVERSITY LINEAR SPACE–TIME BLOCK CODES BASED ON SQUARE CODE MATRICES

Tx antennas	delay	symbols	rate
1	1	1	1
2	2	2	1
3 to 4	4	3	3/4
5 to 8	8	4	1/2
9 to 16	16	5	5/16
$2^{K-2}+1$ to 2^{K-1}	2^{K-1}	K	$K/2^{K-1}$

By construction, these complex linear space–time codes have full diversity, and rate K/N. The codes with minimal dimensions for a given rate are thus delay optimal codes.

In [6], a very strict constraint on the existence of rate-1 complex block codes was found; they exist only for $N = 2$. Similarly, the representation theory of Clifford algebras give very stringent conditions on the existence of block codes with arbitrary rate K/N. These restrictions are derived in the Appendix. For any given number of symbols K to be transmitted, there is a corresponding minimal dimension N for the block code matrix. The result is the following theorem.

Theorem 1: The maximal achievable rate of a square matrix embeddable space–time block code with N transmit antennas is

$$\frac{\lceil \log_2 N \rceil + 1}{2^{\lceil \log_2 N \rceil}}.$$

Proof: Follows immediately from Proposition 1 and Theorem A.1 proved in the Appendix. □

Here, $\lceil \circ \rceil$ is the integer greater or equal to $\circ$. The results for the maximal rates are collected in Table I. It is worth noting that the minimal dimension grows and the maximal rate decreases exponentially in the number of transmitted symbols. The corresponding codes will be constructed presently.

Comparing to the rate-halving codes constructed along the principles of [6], the square matrix embeddable codes for five to eight antennas have the same rate ($1/2$), but half the delay. The square matrix embeddable codes for nine to 16 antennas have a smaller rate ($5/16$ compared to $1/2$), but the delay is shorter (16 compared to 32, 64, 128, or 256).

IV. UNITARY DESIGNS

In [6], the space–time block codes constructed for real signal constellations were (generalized) orthogonal designs. These are $T \times N$ matrices with orthogonal columns and $N \leq T$, where all entries come from the set $\{\pm x_k\}_{k=1}^K \in \mathbb{R}^K$, and the norm of all columns is $\sum_{k=1}^K x_k^2$. Rate-1 designs have $K = T$. For lower rate designs, one should allow some of the matrix entries to take the value 0. For square matrix embeddable complex modulation space–time block codes, square matrix complex orthogonal designs, or unitary designs for short, are needed.

Definition 4: A rate-K/N unitary design is an $N \times N$ matrix C with entries from the set $\{0\} \cup \{\pm z_k, \pm \mathrm{j} z_k\}_{k=1}^K$, with $z_k \in \mathbb{C}$, and

$$C^\dagger C = \sum_{k=1}^K |z_k|^2 \mathbb{1}_N.$$

Thus, unitary designs are a subset of the set of all square matrix space–time block codes constructed as in Proposition 1, with particularly simple matrix elements. The anti-Hermitian irreducible representations of the Clifford generators were constructed in (A15) in terms of tensor products of 2×2 matrices.[5] Unitary designs are readily constructed from these representations, choosing $\beta_0 = \mathbb{1}_{2^{K-1}}$, $\beta_k = \gamma_k$, $k = 1, \ldots, 2K-1$ in (8).

Proposition 2: The $2^{K-1} \times 2^{K-1}$-matrices

$$\begin{aligned} C(\vec{z}) = {} & z_1(\mathbb{1}_{2^{K-1}} + \otimes^{K-1}\sigma_3)/2 \\ & + z_1^*(\mathbb{1}_{2^{K-1}} - \otimes^{K-1}\sigma_3)/2 \\ & + \sum_{k=2}^{K} (\otimes^{K-k}\mathbb{1}_2) \\ & \otimes \begin{bmatrix} 0 & z_k \\ -z_k^* & 0 \end{bmatrix} \otimes (\otimes^{k-2}\sigma_3) \end{aligned} \tag{20}$$

with $\sigma_3 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$, are rate $K/2^{K-1}$ unitary designs saturating the maximal rates of Theorem 1. They are delay optimal complex modulation orthogonal designs for the number of antennas being a power of two.

Proof: This follows directly from Theorem A.2 proved in the Appendix, Proposition 1, and (8). □

Other unitary designs may be constructed from (20) by applying some of the discrete transformations of

- permuting rows and/or columns of the matrix (20),
- permuting the symbols $\{z_k\}_{k=1}^K$ in (20),
- multiplying a symbol with -1 or $\pm$j in (20),
- conjugating a symbol in (20),

or any combination of these. These transformations correspond to permutations of the symbols z_k in (8), or to exchanging the complex and imaginary parts of a symbol, i.e., complex conjugating and multiplying it with j, or to multiplying with a nontrivial unitary matrix β_0 when constructing βs from γ's in Proposition 1.

Still more unitary designs may be constructed by using select versions of the unitary similarity transform (A16). For example, a unitary transformation with (A21) that changes roles of the σ_j in a tensor product space, combined with a reordering of the β_k in (8) would produce different unitary designs.

To be explicit, for $K = 2$, (20) yields the 2×2 matrix

$$C = z_1(\mathbb{1}_2 + \sigma_3)/2 + z_1^*(\mathbb{1}_2 - \sigma_3)/2 + \begin{bmatrix} 0 & z_2 \\ -z_2^* & 0 \end{bmatrix} \tag{21}$$

[5] Representations that are not equivalent to a block-diagonal representation are irreducible. The tensor product of a 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

with a $m \times m$ matrix B is a $2m \times 2m$ matrix which reads in block form

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix}.$$

The mth tensor power of a matrix A is defined as

$$\otimes^m A = \underbrace{A \otimes A \otimes \cdots \otimes = A}_{m\text{times}}.$$

which is exactly the Alamouti code (3). For $K=3$, employing the four-dimensional identity matrix and the five γ_k of (A20), one gets the rate-$3/4$ unitary design

$$C=\begin{bmatrix} z_1 & z_2 & z_3 & 0 \\ -z_2^* & z_1^* & 0 & -z_3 \\ -z_3^* & 0 & z_1^* & z_2 \\ 0 & z_3^* & -z_2^* & z_1 \end{bmatrix}. \tag{22}$$

The design above has two versions of the Alamouti code on the block diagonal. Compared to code (5), this is of simpler form.

For $K=4$, one gets the rate-$1/2$ unitary design

$$C=\begin{bmatrix} z_1 & z_2 & z_3 & 0 & z_4 & 0 & 0 & 0 \\ -z_2^* & z_1^* & 0 & -z_3 & 0 & -z_4 & 0 & 0 \\ -z_3^* & 0 & z_1^* & z_2 & 0 & 0 & -z_4 & 0 \\ 0 & z_3^* & -z_2^* & z_1 & 0 & 0 & 0 & z_4 \\ -z_4^* & 0 & 0 & 0 & z_1^* & z_2 & z_3 & 0 \\ 0 & z_4^* & 0 & 0 & -z_2^* & z_1 & 0 & -z_3 \\ 0 & 0 & z_4^* & 0 & -z_3^* & 0 & z_1 & z_2 \\ 0 & 0 & 0 & -z_4^* & 0 & z_3^* & -z_2^* & z_1^* \end{bmatrix}. \tag{23}$$

This has the $3/4$ design (22) in the upper left and a complex conjugate inverted version in the lower right corner.

For practical purposes, it would be beneficial to have as much self-similarity as possible in the code matrix. Thus, it would be desirable to, e.g., have a 4×4 code, where two Alamouti blocks would be on the block diagonal, and a third symbol would be encoded on the block off-diagonal. This, however, is impossible.

Proposition 3: No unitary design with maximal symbol rate exists, which would have only copies of the same 2×2 design, possibly multiplied with ±1 or $\pm$j, on the block diagonal.

Proof: First consider the case of 4×4 designs. According to Proposition 1, no generality is lost if one coefficient matrix is taken to be unity. Thus, a matrix of the proposed form could be transformed to the form $\mathbb{1}_2\otimes C_2$, where C_2 is a design constructed from $\mathbb{1}_2$ and three Cliff_3 generators. According to Theorem A.2, these three are unitarily equivalent to $\sigma_1,\ \sigma_2,\ \pm\mathrm{j}\sigma_3$. Without loss of generality, they may be supposed to be exactly of this form. From this and Proposition 1 it follows that the coefficient matrices encoding the third symbol should anticommute with $\mathbb{1}_2\otimes\sigma_1,\ \mathbb{1}_2\otimes\sigma_2,\ \mathbb{1}_2\otimes\sigma_3$. As a consequence of Proposition A.8 and Theorem A.2, no such matrix exists. The proof for unitary designs of higher dimensions follows inductively, using Proposition A.8 and Theorem A.2. □

Nonsquare complex orthogonal designs (or pseudounitary designs) can be constructed by eliminating antennas (columns) from the designs above. These designs transmit K complex symbols during 2^{K-1} time epochs from $N\le 2^{K-1}$ antennas. For $2^{K-2}+1\le N\le 2^{K-1}$, they are delay optimal within the categories of complex space–time block codes presented in the literature; the rate-halved and the square matrix embeddable codes. For example, the rate-$1/2$ schemes for five to eight antennas, constructed from (23), have only half the delay when compared to corresponding rate-halving codes presented in [6].

V. Unitarily Transformed Block Codes

Space–time block codes based on unitary designs, treated above, are codes where all code matrix entries depend only on one of the signals. A useful generalization is to lift these restrictions. In [6], the concept of linear processing block codes was presented, where the elements were linear combinations of the symbols to be transmitted, and the matrix remained unitary. Here such codes are just called space–time block codes.

A constructive way to choose any viable linear combination is inherent in the construction of block codes in Section III and the Appendix. Combining the freedom to choose β_0 in (17) to the unitary symmetry of the Clifford algebra in (A16), one can define the generic concept of unitarily transformed block codes.

Theorem 2: All square matrix embeddable space–time block codes can be constructed from the unitary designs $C(\vec{z})$ of (20) by possibly deleting rows from a matrix of the form

$$\tilde{C}(\vec{z})=UC(\vec{z})V \tag{24}$$

where U is a $2^{K-1}\times2^{K-1}$ unitary matrix and V is a $2^{K-1}\times 2^{K-1}$ special unitary matrix, up to permutations and possibly one sign change in the set of real and imaginary parts of the symbols.

Proof: This is a direct consequence of Proposition 1 and Corollary A.2. The possible sign change corresponds to the two equivalence classes of irreducible representations in Theorem A.2, and the permutations correspond to permutations in the set $\{\beta_k\}_{k=0}^{2K-1}$ in Proposition 1. □

For each block length $T=2^{K-1}$, this construction gives a family of block codes with $2^{2K-1}-1$ continuous parameters. This family encompasses all unitary designs of the same dimensions.

The unitary transformations do not change the performance of the code. They can be used, e.g., to optimize properties of secondary importance, e.g., the power spectra of the antennas.

As an example, consider the rate-$3/4$ code for four transmit antennas, (22). A generic special unitary 4×4 matrix with unit determinant can be written as

$$V=\exp\{\mathrm{j}\,W\} \tag{25}$$

where the $\exp$ operation is a matrix exponential, and W is a traceless Hermitian 4×4 matrix, which can be linearly parameterized by 15 real parameters. U is of the same form, with an added overall phase factor. All in all, that makes 31 real parameters.

All possible generalizations of the code (22) can be constructed by applying transformation (24), with V and U of the form (25), accompanied by possible (real) symbol permutations and a sign change. Within this family, one can, for example, optimize the power distribution of the code so that the average power transmitted from each antenna at each time epoch is the same. This problem was addressed in [10].

One member of this family is the example found in [6], quoted above as (5). The unitary equivalence of (5) and (22) is given by the matrices

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad V = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

VI. Conclusion

We have proposed a digital communication system which utilizes any number of transmit and receive antennas, with full diversity provided by space–time block coding, and generalized the linear decoding principle of [6] to them. We have classified all possible linear and unitary space–time block codes with complex symbol constellations that are square matrices, or can be constructed from square matrices by deleting columns. The families of codes found include new complex modulation full diversity codes for more than four antennas. For five to eight antennas, these codes have shorter delay than the codes previously presented in the literature, with the same rate.

Also, we have made a concrete construction generalizing space–time block codes to unitarily transformed block codes. This gives a $T^2 + N^2 - 2$-dimensional continuously parameterized family of $T \times N$ codes. Within this family, one may, e.g., optimize the power distribution between antennas. Within such a family, we have found the simplest form of a space–time block code for three and four antennas.

From the construction of square matrix embeddable codes it follows that the only place remaining to look for better space–time block codes than the ones presented in [6] and here is to investigate block codes based on nonsquare pseudounitary matrices that are not columnwise restrictions of a square unitary matrix. This is an open problem. For example, the construction in [6] yielding rate-1 real codes for any number of antennas cannot be updated to the complex number domain.

The main result of the paper is Theorem 1, which states that the maximal rate of a square matrix embeddable space–time block code falls off exponentially with the number of transmit antennas. This is in constrast to rate-halving codes, which may have rate $1/2$ for any number of antennas, with an exponentially increasing delay. These facts imply that linearity and unitarity are too strict requirements when designing space–time modulations for future high-rate wireless communication systems.

Appendix
Representation Theory of Clifford Algebras

Recall the defining relation of a Clifford algebra (19).

Definition A.1: Cliff_L is the algebra over $\mathbb{R}$ generated by L objects γ_k, $k = 1, \ldots, L$, which are anticommuting

$$\gamma_k \gamma_j = -\gamma_j \gamma_k, \qquad \forall\, k \neq j \tag{A1}$$

square roots of -unity

$$\gamma_k^2 = -\mathbb{1}, \qquad \forall\, k = 1, \ldots, L. \tag{A2}$$

The basis of Cliff_L is

$$\mathcal{B}_L = \{\mathbb{1}\} \cup \{\gamma_k | k = 1, \ldots, L\} \bigcup_{m=2}^{L} \left\{ \prod_{i=1}^{m} \gamma_{k_i} \,\middle|\, 1 \leq k_i < k_{i+1} \leq L \right\}. \tag{A3}$$

The number of basis elements is the number of all nonordered combinations of L objects

$$|\mathcal{B}_L| = \sum_{i=0}^{L} \binom{L}{i} = 2^L.$$

The basis can be constructed by observing that a product of more than L generators γ_k can be reduced to $\pm$ the product of at most one of each generator by using the defining relations (A1) and (A2).

Due to the inherent role of complex numbers in communications, we are interested in representations in terms of complex numbers.

Definition A.2: An N-dimensional representation of an algebra is a homomorphism from the abstract algebra to a linear algebra of operators acting on $\mathbb{C}^N$.

A representation of an algebra is completely specified by a representation of its basis, which again is completely specified by a representation of the generators. For a Clifford algebra, we are thus interested in representations of the generators γ_k. In an N-dimensional representation $\mathbb{1}$ is represented by $\mathbb{1}_N$, the N-dimensional identity matrix, and the generators are anticommuting matrices that square to $-\mathbb{1}_N$.

To start, we recall some basic concepts in representation theory.

R1: Irreducible representations are representations with no invariant subspaces.

R2: Completely reducible representations can be decomposed to a direct sum of irreducible representations. They are equivalent to block-diagonal representations, with irreducible representation matrices on the block diagonal.

R3: Two representations $\mathcal{R}$ and $\mathcal{R}'$ of the algebra $\mathcal{A}$ are equivalent, if a similarity transform V exists so that

$$\mathcal{R}'(a) = V^{-1}\mathcal{R}(a)V, \qquad \forall\, a \in \mathcal{A}.$$

In the sequel, the expression "irreducible representation" is taken to mean an equivalence class of irreducible representations, according to R3. There is a subtle point regarding the concept of equivalence of representations, and irreducibility, when comparing to the treatment of Clifford algebras in the mathematical literature, see, e.g., [11]. We shall return to this when the results are stated.

To find a complete classification of the representations of Clifford algebras, we shall use the fact that a double cover of the basis of a Clifford algebra (all elements of $\mathcal{B}$ multiplied by ± 1), can be interpreted as a finite group.

Proposition A.1: The set of elements

$$\mathcal{G}_L = \mathcal{B}_L \cup \{-b | b \in \mathcal{B}_L\} \tag{A4}$$

is a finite group with respect to (w.r.t.) the multiplication in Cliff_L. The order of $\mathcal{G}_L$ is 2^{L+1}.

Proof: The multiplication is associative, the unit is $\mathbb{1}$, and the inverse of the element $\pm\prod_{i=1}^{m}\gamma_{k_i}$ is

$$\pm(-1)^{\lceil \frac{m}{2} \rceil}\prod_{i=1}^{m}\gamma_{k_i}.$$

The number of elements in the group is twice the number of basis elements. □

Note that when $\mathcal{G}_L$ is interpreted as a finite group, the representation of $-\mathbb{1}$ does not necessarily have anything to do with -1 times $\mathbb{1}_N$, and similarly, for a generic $-b$, $b\in\mathcal{B}_L$. Symbols $-\mathbb{1}$ and $-b$ are just symbols denoting group elements with some specified multiplication rules with the other group elements, of the form

$$\begin{aligned}(-\mathbb{1})b &= (-b), \quad \forall\, b\in\mathcal{B}_L\\ (-\mathbb{1})(-b) &= b, \quad \forall\, b\in\mathcal{B}_L.\end{aligned}$$

The element $-\mathbb{1}$ may be represented by any matrix (or number) so that the multiplication table of the group is fulfilled. Nondegenerate representations where $\mathcal{R}(-\mathbb{1}) = -\mathcal{R}(\mathbb{1}) = -\mathbb{1}_N$ can be constructed from representations of Cliff_L.

Corollary A.1: A restriction of a representation of Cliff_L to its basis gives a representation of $\mathcal{G}_L$. A restriction of an irreducible representation gives an irreducible representation. Restrictions of two nonequivalent representations give nonequivalent representations. These representations of $\mathcal{G}_L$ are nondegenerate, $\mathcal{R}(\mathbb{1}) = -\mathbb{1}_N$. Inversely, nonequivalent nondegenerate irreducible representations of $\mathcal{G}_L$ may be extended to nonequivalent irreducible representations of Cliff_L.

Proof: These are trivial consequences of the relation of an algebra and its basis, and Proposition A.1. The inverse property is proved by a trivial embedding of $\{1, -1\}$ into $\mathbb{R}$. □

Now we may concentrate on finding all irreducible representations of $\mathcal{G}_L$, which thus include all irreducible representations of Cliff_L. First, recall the following.

R4: Unitary group representations are representations in terms of unitary matrices.

Due to the square root of $-\mathbb{1}$ property (A2), unitary representations of the Clifford generators are anti-Hermitian

$$\gamma_k^\dagger = \gamma_k^{-1} = -\gamma_k, \qquad k = 1, \ldots, = L. \tag{A5}$$

Moreover, any two of the three properties of unitarity, anti-Hermiticity and squaring to -1, imply the third. Here we are interested in Clifford representations due to their relation to a set of $L+1$ complexified Radon–Hurwitz matrices satisfying (10), i.e., due to the unitarity property of the code matrix (6). For this, the representations we are looking for are exactly the anti-Hermitian, and thus unitary representations of $\mathcal{G}_L$. These are generated by collections of L anticommuting, anti-Hermitian, unitary matrices with complex entries.

This is the crucial difference between complex and real modulation space–time block codes. For real modulation codes, we restrict to antisymmetric $(\gamma^T = -\gamma)$ representations of the Clifford algebra, and thus look for a family of anticommuting, antisymmetric, orthogonal matrices. This is a restriction, as a Hermitian antisymmetric matrix may be made anti-Hermitian by multiplying with j, whereas a symmetric anti-Hermitian matrix cannot be made antisymmetric. With this restriction, the theory of Clifford algebra representations reproduce the rates found in [6], see [12].

A. The Representation Theory of $\mathcal{G}_L$

To construct the representations of the finite group $\mathcal{G}_L$, recall some basic theorems pertaining to representations of finite groups [13].

FG1: Every representation of a finite group is equivalent to a unitary representation.

FG2: Every representation of a finite group is completely reducible.

FG3: The order of the group is the sum of the squares of the dimensions of the irreducible representations.

FG4: The number of equivalence classes of irreducible representations equals the number of conjugacy classes (subsets $S\subset\mathcal{G}$ of the group with $g^{-1}Sg\subset S = \forall\, g\in\mathcal{G}$).

Finally, we shall need Schur's lemma.

SCH: If $\mathcal{R}(g)A = A\mathcal{R}(g)\ \forall\, g \in \mathcal{G}$, where $\mathcal{R}$ is a finite-dimensional representation, then $A\sim\mathbb{1}$.

First observe that for odd L, Cliff_L, and $\mathcal{G}_L$ have a central element (an element that commutes with everything) in addition to $\mathbb{1}$:

Proposition A.2: For odd L, the product of all generators is a central element, it commutes with all elements in $\mathcal{G}_L$.

Proof: First consider the commutation of the product of all generators with a generator

$$\begin{aligned}\prod_{i=1}^{L}\gamma_i\gamma_k &= (-1)^{(L-k)}\prod_{i=1}^{k-1}\gamma_i\gamma_k^2\prod_{j=k+1}^{L}\gamma_j\\ &= (-1)^{(L-k+k-1)}\gamma_k\prod_{i=1}^{L}\gamma_i.\end{aligned}$$

For odd L, the product commutes with all generators, and thus also with all products of generators, i.e., with all elements in the group $\mathcal{G}_L$. □

Now we proceed with finding the conjugacy classes of $\mathcal{G}_L$.

Proposition A.3: For even L, $\mathcal{G}_L$ has 2^L+1 conjugacy classes, for odd L, it has 2^L+2 conjugacy classes.

Proof: Two elements of $\mathcal{G}_L$ either commute or anticommute, as they are associative products of elements that anticommute or commute. If an element commutes with all elements in a group, it forms a conjugacy class by itself. If it anticommutes with some and commutes with the rest, it forms a conjugacy class together with $-$ itself. By definition $\mathbb{1}$ and $-\mathbb{1}$ commute with all elements in $\mathcal{G}_L$ and are alone in their respective conjugacy classes. According to Proposition A.2, $\prod_{i=1}^{L}\gamma_i$ and $-\prod_{i=1}^{L}\gamma_i$ for odd L commute with all elements, and are thus alone in their respective conjugacy classes. If $L=1$, there are no further elements. For $L>1$, the remaining elements anticommute with some other elements. Consider $g_m = \prod_{i=1}^{m}\gamma_{k_i}$,

$k_i < k_{i+1}$. If $1 \leq m < L$, the element $g'_m = \prod_{i=1}^{m-1} \gamma_{k_i}\gamma_{k'_m}$ with $k'_m \notin \{k_i\}_{i=1}^m$ exists, and it anticommutes with g_m

$$\begin{aligned} g_m g'_m &= \prod_{i=1}^{m-1} \gamma_{k_i}\gamma_{k_m} \prod_{j=1}^{m-1} \gamma_{k_j}\gamma_{k'_m} \\ &= (-1)^m \prod_{i=1}^{m-1} \gamma_{k_i} \prod_{j=1}^{m-1} \gamma_{k_j}\gamma_{k'_m}\gamma_{k_m} \\ &= (-1)^{2m-1} \prod_{i=1}^{m-1} \gamma_{k_i}\gamma_{k_m} \prod_{j=1}^{m-1} \gamma_{k_j}\gamma_{k_m} = -g'_m g_m. \end{aligned}$$

It remains to consider the product of all generators g_L for even L. Take $g' = \gamma_L$

$$\begin{aligned} g_L g' &= \prod_{i=1}^{L-1} \gamma_i \gamma_L \gamma_L = (-1)^{L-1}\gamma_L \prod_{i=1}^{L-1} \gamma_i \gamma_L \\ &= (-1)^{L-1} g' g_L. \end{aligned} \tag{A6}$$

Consequently, for even L, the conjugacy class consists of $\{g_L, -g_L\}$. Thus, there is one conjugacy class for each element in $\mathcal{B}_L$, one for $-\mathbb{1}$, and for odd L, one extra for $-g_L$. □

The finite group $\mathcal{G}_L$ has a host of irreducible one-dimensional (1-D) representations, where $\pm\mathbb{1}$ are both represented by 1. The simplest is the trivial representation, where all generators, and thus all elements are represented by 1. These degenerate representations are clearly not representations of Cliff$_L$, but they have to be taken into account when classifying the irreducible representations of $\mathcal{G}_L$.

Proposition A.4: The finite group $\mathcal{G}_L$ has at least 2^L 1-D irreducible representations, which are degenerate, $\mathcal{R}(-\mathbb{1}) = 1$.

Proof: Take $\mathcal{R}(\pm\mathbb{1}) = 1$, for each of the generators $\mathcal{R}(\pm\gamma_k) = 1$ or $\mathcal{R}(\pm\gamma_k) = -1$, and

$$\mathcal{R}\left(\pm\prod_{i=1}^{m} \gamma_{k_i}\right) = \prod_{i=1}^{m} \mathcal{R}(\gamma_{k_i}).$$

This choice is an irreducible 1-D representation of $\mathcal{G}_L$. The number of different representations is the number of choosing $+1$ or -1 for each of the L generators, i.e., 2^L. □

This leads to an immediate classification of all representations of $\mathcal{G}_L$ with even L.

Proposition A.5: The finite group $\mathcal{G}_{2K}$ has $2^{2K} + 1$ irreducible representations. One is 2^K-dimensional, 2^{2K} are 1-D.

Proof: From Proposition A.3 and FG4 it follows that $\mathcal{G}_{2K}$ has $2^{2K} + 1$ irreducible representations. From Proposition A.4, one has 2^{2K} 1-D irreducible representations. This leaves one additional irreducible representation. From Proposition A.1, the order of $\mathcal{G}_{2K}$ is 2^{2K+1}. Applying FG3 one has for the dimension N of the remaining irreducible representation

$$2^{2K+1} = 2^{2K} \times 1^2 + N^2. \tag{A7}$$

This means that $N = 2^K$. □

The existence of a central element in $\mathcal{G}_L$ with odd L leads to a straightforward connection between nondegenerate representations of $\mathcal{G}_{2K-2}$ and $\mathcal{G}_{2K-1}$, and thus for Cliff$_{2K-2}$ and Cliff$_{2K-1}$.

Proposition A.6: A nondegenerate representation $\mathcal{R}_{2K-2}$ of $\mathcal{G}_{2K-2}$ can be extended to two nonequivalent nondegenerate representations $\mathcal{R}^{\pm}_{2K-1}$ of $\mathcal{G}_{2K-1}$. If $\mathcal{R}_{2K-2}$ is irreducible, $\mathcal{R}^{\pm}_{2K-1}$ are irreducible.

Proof: From Proposition A.2, the product of all generators is a central element. From Schur's lemma it follows that in all finite-dimensional (and thus in this case, all irreducible) representations of $\mathcal{G}_{2K-1}$

$$\prod_{i=1}^{2K-1} \mathcal{R}(\gamma_i) = \lambda \mathbb{1}_N \Leftrightarrow \mathcal{R}(\gamma_1) = \lambda(-1)^{K-1} \prod_{i=2}^{2K-1} \mathcal{R}(\gamma_i)$$

where λ is a proportionality constant, and N is the dimension of the representation. In a nondegenerate representation, λ may be fixed by checking that the representation respects $\mathcal{R}(\gamma_{2K-1})^2 = -\mathbb{1}_N$. This gives two solutions, $\lambda = \pm \mathrm{j}^K$, which correspond to two representations. The representations $\mathcal{R}^{\pm}_{2K-1}$ may now be generated from a nondegenerate representation $\mathcal{R}_{2K-2}$ of the subgroup generated by γ_k, $k = 2, \ldots, 2K-1$, isomorphic to $\mathcal{G}_{2K-2}$, by taking[6]

$$\mathcal{R}^{\pm}_{2K-1}(\gamma_k) = \mathcal{R}_{2K-2}(\gamma_k), \qquad k = 2, \ldots, 2K-1 \tag{A8}$$

$$\mathcal{R}^{\pm}_{2K-1}(\gamma_1) = \pm \mathrm{j}^K \prod_{i=2}^{2K+1} \mathcal{R}_{2K-2}(\gamma_i). \tag{A9}$$

These two representations are clearly nonequivalent. Suppose that a similarity transform would exist that would transform one representation to the other. It would change the sign of $\mathcal{R}(\gamma_1)$. Also, it would leave all $\mathcal{R}(\gamma_i)$, $i = 2, \ldots, 2K-1$ invariant, and thus also their product. This, however, should be proportional to $\mathcal{R}(\gamma_1)$. This is a contradiction. From the definition of irreducible representations, it follows that if $\mathcal{R}_{2K-2}$ is irreducible, so are $\mathcal{R}^{\pm}_{2K-1}$. □

Finally, one may classify all representations of $\mathcal{G}_L$ with odd L.

Proposition A.7: The finite group $\mathcal{G}_{2K-1}$ has $2^{2K-1} + 2$ irreducible representations. Two are 2^{2K-2} dimensional, 2^{2K-1} are 1-D.

Proof: From Proposition A.3 and FG4 it follows that $\mathcal{G}_{2K-1}$ has $2^{2K-1}+2$ irreducible representations. From Proposition A.4 one has 2^{2K-1} 1-D irreducible representations. This leaves two additional irreducible representations. These are the two 2^{K-1}-dimensional irreducible representations constructed according to Proposition A.6 from the 2^{K-1}-dimensional irreducible representation of $\mathcal{G}_{2K-2}$ that exists according to Proposition A.5. □

As a consistency check, consider FG3. From Proposition A.1, the order of $\mathcal{G}_{2K-1}$ is 2^{2K}. The sum of the squared dimensions of the irreducible representations found in Proposition A.7 is

$$2^{2K-1} \times 1^2 + 2 \times (2^{K-1})^2 = 2^{2K}$$

consistent with FG3.

[6]The slightly awkward numbering of the generators here is chosen to streamline the notations in the main body of this paper.

B. The Representations of Cliff_L

The dimensions and numbers of representations of Cliff_L follow directly from the results proved in the previous subsection.

Theorem A.1: Cliff_L has one equivalence class of irreducible representations for even L, and two equivalence classes for odd L. These representations are $2^{\lfloor L/2 \rfloor}$-dimensional.

Proof: This is a direct consequence of Propositions A.5 and A.7, and Corollary A.1. □

According to FG1, each of the equivalence classes of irreducible representations of $\mathcal{G}_L$ constructed in the previous subection includes at least one unitary irreducible representation. Due to the anti-Hermiticity requirement (18), we are interested in those representations.

The attentive reader might notice a discrepancy between the number of irreducible representations in Theorem A.1 and results in the mathematical literature, see, e.g., [11]. There, two irreducible representations exist only for even K in $L = 2K-1$. The reason is that here, representations as matrix algebras over $\mathbb{C}$ are considered (Definition A.2), and equivalence is defined in R3 only up to automorphisms that can be realized as similarity transformations. Equivalence of representations of algebras are usually up to all automorphisms. In Clifford algebras, some outer automorphisms, e.g., reflections, may not be realized as similarity transformations. The argument relating Theorem A.1 to the results presented in [11] goes as follows. For even K, the proportionality constant λ in the central element in Proposition A.6 squares to 1, so that the representations of γ_1 and $\prod_{i=2}^{2K+1}$ are the same up to a sign, see (A9). For odd K, there is an additional factor of j. Thus, the dimension of the irreducible representations of Cliff_{2K-1} as matrix algebras over $\mathbb{R}$ are 2^{2K-2} for even K, but 2^{2K-1} for odd K. Due to this, considered as matrix algebras over $\mathbb{R}$, the two representations of Cliff_{2K-1} discussed above are equivalent w.r.t. nonsimilarity outer automorphisms if K is odd, and nonequivalent if K is even.

In constructing explicit representations of Clifford algebras, we shall use the following anticommuting 2×2 matrices:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \sigma_2 = \begin{bmatrix} 0 & \mathrm{j} \\ \mathrm{j} & 0 \end{bmatrix} \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad \text{(A10)}$$

For later convenience, these have been chosen so that σ_1 and σ_2 are anti-Hermitian, and σ_3 is Hermitian. All three are unitary. Their products are given by

$$\gamma_1 \gamma_2 = \mathrm{j}\sigma_3. \quad \text{(A11)}$$

First consider the case of two anticommuting objects $L = 2$. An anti-Hermitian representative of the 2-D irreducible representation of Cliff_2 can be constructed as

$$\mathcal{R}(\gamma_1) = \sigma_1 \qquad \mathcal{R}(\gamma_2) = \sigma_2. \quad \text{(A12)}$$

The anti-Hermiticity and unitarity of these matrices guarantee that they square to $-\mathbb{1}$.

Now consider representations of Clifford algebras with an even number of generators $L = 2K$.

Proposition A.8: Cliff_{2K-2} can be represented as a tensor product of $K-1$ copies of representations of Cliff_2.

Proof: For each $k = 1, \ldots, K-1$, define $\widetilde{\mathrm{Cl}}_2^{(k)} = \{\tilde{\gamma}_{2k-1}, \tilde{\gamma}_{2k}\}$, where

$$\begin{aligned} \tilde{\gamma}_{2k-1} &= (-\mathrm{j})^{k-1} \prod_{i=1}^{2k-2} \gamma_i \, \gamma_{2k-1} \\ \tilde{\gamma}_{2k} &= (-\mathrm{j})^{k-1} \prod_{i=1}^{2k-2} \gamma_i \, \gamma_{2k}. \end{aligned} \quad \text{(A13)}$$

The inverse of this mapping is constructed by exchanging the $\tilde{\gamma}_j$ with γ_j. Due to the even number of elements in $\prod_{i=1}^{2k-2} \gamma_i$, the two elements in each $\widetilde{\mathrm{Cl}}_2^{(k)}$ anticommute

$$\begin{aligned} \tilde{\gamma}_{2k-1}\tilde{\gamma}_{2k} &= \prod_{j=1}^{2k-1} \gamma_j \, \gamma_{2k-1} \prod_{i=1}^{2k-2} \gamma_i \, \gamma_{2k} \\ &= \prod_{j=1}^{2k-2} \gamma_j \prod_{i=1}^{2k-2} \gamma_i \, \gamma_{2k}\gamma_{2k-1} \\ &= -\prod_{j=1}^{2k-2} \gamma_j \, \gamma_{2k} \prod_{i=1}^{2k-2} \gamma_i \, \gamma_{2k-1} = -\tilde{\gamma}_{2k}\tilde{\gamma}_{2k-1} \end{aligned}$$

and square to $-\mathbb{1}$. Further, $\widetilde{\mathrm{Cl}}_2^{(k)}$ and $\widetilde{\mathrm{Cl}}_2^{(l)}$ commute for $k \neq l$. Without loss of generality, take $l > k$. Consider one of the four commutations

$$\begin{aligned} \tilde{\gamma}_{2k-1}\tilde{\gamma}_{2l-1} &= \prod_{i=1}^{2k-1} \gamma_i \prod_{j=1}^{2k-1} \gamma_j \prod_{j=2k}^{2l-1} \gamma_j \\ &= \prod_{i=1}^{2k-1} \gamma_i \prod_{j=2k}^{2l-1} \gamma_j \prod_{j=1}^{2k-1} \gamma_j = \tilde{\gamma}_{2l-1}\tilde{\gamma}_{2k-1}. \end{aligned}$$

The other three commutations are similar. The commutation relations of these $K-1$ commuting pairs of two anticommuting elements may thus be realized as a $K-1$-fold tensor product of two representations of Cliff_2, e.g., (A12)

$$\begin{aligned} \mathcal{R}(\tilde{\gamma}_{2k-1}) &= (\otimes^{K-1-k}\mathbb{1}_2) \otimes \sigma_1 \otimes (\otimes^{k-1}\mathbb{1}_2) \\ \mathcal{R}(\tilde{\gamma}_{2k}) &= (\otimes^{K-1-k}\mathbb{1}_2) \otimes \sigma_2 \otimes (\otimes^{k-1}\mathbb{1}_2). \end{aligned} \quad \text{(A14)}$$

Inverting (A13), one gets a representation of Cliff_{2K-2}. □

This leads to a explicit characterization of all representations of Cliff_L.

Theorem A.2: The matrices

$$\begin{aligned} \mathcal{R}(\gamma_2) &= \underbrace{\mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \ldots \otimes \mathbb{1}_2}_{K-2 \text{ times}} \otimes \sigma_1 \\ \mathcal{R}(\gamma_3) &= \underbrace{\mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \ldots \otimes \mathbb{1}_2}_{K-2 \text{ times}} \otimes \sigma_2 \\ &\vdots \\ \mathcal{R}(\gamma_{2k}) &= \underbrace{\mathbb{1}_2 \otimes \ldots \otimes \mathbb{1}_2}_{K-1-k \text{ times}} \otimes \sigma_1 \otimes \underbrace{\sigma_3 \otimes \ldots \otimes \sigma_3}_{k-1 \text{ times}} \\ \mathcal{R}(\gamma_{2k+1}) &= \underbrace{\mathbb{1}_2 \otimes \ldots \otimes \mathbb{1}_2}_{K-1-k \text{ times}} \otimes \sigma_2 \otimes \underbrace{\sigma_3 \otimes \ldots \otimes \sigma_3}_{k-1 \text{ times}} \\ &\vdots \\ \mathcal{R}(\gamma_{2K-2}) &= \sigma_1 \otimes \underbrace{\sigma_3 \otimes \ldots \otimes \sigma_3}_{K-2 \text{ times}} \end{aligned}$$

$$\mathcal{R}(\gamma_{2K-1}) = \sigma_2 \otimes \underbrace{\sigma_3 \otimes \ldots \otimes \sigma_3}_{K-2 \text{ times}}$$
$$\mathcal{R}(\gamma_1) = \pm \mathrm{j} \underbrace{\sigma_3 \otimes \sigma_3 \otimes \ldots \otimes \sigma_3}_{K-1 \text{ times}}. \tag{A15}$$

are representatives in terms of anti-Hermitian generators of the two equivalence classes of irreducible representations of Cliff_{2K-1}. An irreducible representation of Cliff_{2K-2} may be constructed by dropping $\mathcal{R}(\gamma_1)$ from the representations above.

Proof: The representation of Cliff_{2K-2} constructed in Proposition A.8 is nondegenerate. According to Theorem A.1, it must be a representative of the only equivalence class of irreducible representations of Cliff_{2K-2}. Inverting (A14) using (A13), (A11) one gets the representation matrices $\mathcal{R}(\gamma_k)$, $k = 2, \ldots, 2K-1$ in (A15). The representation of γ_1 is constructed as in (A9). According to Proposition A.6 and Theorem A.1, this gives the two irreducible representations of Cliff_{2K-1}. From the anti-Hermiticity of σ_1, σ_2 and the Hermiticity of σ_3 it follows that the matrices (A15) are anti-Hermitian by construction. □

Corollary A.2: All irreducible representations of Cliff_L with anti-Hermitian generators may be constructed from (A15) by a similarity transformation with a $2^{\lfloor L/2 \rfloor} \times 2^{\lfloor L/2 \rfloor}$ unitary matrix V with determinant 1

$$\mathcal{R}(\gamma_k) \mapsto V^\dagger \mathcal{R}(\gamma_k) V. \tag{A16}$$

Proof: This follows from the definition R3 of equivalence of representations, and that only unitary similarity transformations preserve anti-Hermiticity. The restriction to special unitary matrices ($\det V = 1$) comes from the fact that a possible overall phase factor in V commutes with all matrices, and thus cancels in a similarity transformation. □

C. Examples

For constructing unitary designs, Clifford algebras with an odd number of generators are relevant. The representations of Cliff_3 are generated by (A15) with $K = 2$

$$\begin{aligned}\mathcal{R}(\gamma_2) &= \sigma_1\\ \mathcal{R}(\gamma_3) &= \sigma_2\\ \mathcal{R}(\gamma_1) &= \pm\gamma_2\gamma_3 = \pm \mathrm{j}\sigma_3.\end{aligned} \tag{A17}$$

Further, the representations of Cliff_5 are given by (A15) with $K = 3$. The two commuting pairs of anticommuting combinations of the generators are given by (A13) with permuted indexes

$$\begin{aligned}\tilde{\gamma}_2 &= \gamma_2\\ \tilde{\gamma}_3 &= \gamma_3\\ \tilde{\gamma}_4 &= -\mathrm{j}\gamma_2\gamma_3\gamma_4\\ \tilde{\gamma}_5 &= -\mathrm{j}\gamma_2\gamma_3\gamma_5.\end{aligned} \tag{A18}$$

Their representation in terms of 2×2 matrices (A13) is

$$\begin{aligned}\mathcal{R}(\tilde{\gamma}_2) &= \mathbb{1}_2 \otimes \sigma_1\\ \mathcal{R}(\tilde{\gamma}_3) &= \mathbb{1}_2 \otimes \sigma_2\\ \mathcal{R}(\tilde{\gamma}_4) &= \sigma_1 \otimes \mathbb{1}_2\\ \mathcal{R}(\tilde{\gamma}_5) &= \sigma_2 \otimes \mathbb{1}_2\end{aligned} \tag{A19}$$

and, consequently, $\mathcal{R}(\gamma_2\gamma_3) = \mathbb{1}_2 \otimes \mathrm{j}\sigma_3$. From (A18), (A19) we then get

$$\begin{aligned}\mathcal{R}(\gamma_4) &= \mathrm{j}(-\mathbb{1}_2 \otimes \mathrm{j}\sigma_3)(\sigma_1 \otimes \mathbb{1}_2) = \sigma_1 \otimes \sigma_3\\ \mathcal{R}(\gamma_5) &= \mathrm{j}(-\mathbb{1}_2 \otimes \mathrm{j}\sigma_3)(\sigma_2 \otimes \mathbb{1}_2) = \sigma_2 \otimes \sigma_3.\end{aligned}$$

The remaining generator may be represented by

$$\begin{aligned}\mathcal{R}(\gamma_1) &= \pm \mathrm{j}\mathcal{R}(\gamma_2\gamma_3)\mathcal{R}(\gamma_4)\mathcal{R}(\gamma_5)\\ &= \mp(\mathbb{1}_2 \otimes \sigma_3)(\sigma_1 \otimes \sigma_3)(\sigma_2 \otimes \sigma_3) = \mp \mathrm{j}\sigma_3 \otimes \sigma_3.\end{aligned}$$

The constructed representation of Cliff_5 reads in matrix form

$$\gamma_2 = \begin{bmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{bmatrix} \qquad \gamma_3 = \begin{bmatrix} 0 & \mathrm{j} & 0 & 0\\ \mathrm{j} & 0 & 0 & 0\\ 0 & 0 & 0 & \mathrm{j}\\ 0 & 0 & \mathrm{j} & 0\end{bmatrix}$$
$$\gamma_4 = \begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\\ -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix} \qquad \gamma_5 = \begin{bmatrix} 0 & 0 & \mathrm{j} & 0\\ 0 & 0 & 0 & -\mathrm{j}\\ \mathrm{j} & 0 & 0 & 0\\ 0 & -\mathrm{j} & 0 & 0\end{bmatrix}$$
$$\gamma_1 = \pm \begin{bmatrix} \mathrm{j} & 0 & 0 & 0\\ 0 & -\mathrm{j} & 0 & 0\\ 0 & 0 & -\mathrm{j} & 0\\ 0 & 0 & 0 & \mathrm{j}\end{bmatrix}. \tag{A20}$$

Different but equivalent representations may be constructed by applying the unitary similarity transforms (A16). Of particular interest when constructing unitary designs are representations where all matrix elements are $\{\pm 1, \pm \mathrm{j}\}$. Such may be constructed e.g., by transformations that change the basis in some of the tensor product spaces. Thus, e.g.,

$$V = \otimes^{K-1-k} \mathbb{1}_2 \otimes (\sigma_1 + \sigma_2)/\sqrt{2} \otimes^{k-1} \mathbb{1}_2 \tag{A21}$$

exchanges the matrices σ_1 and σ_2 and changes the sign of σ_3 in the kth tensor product space.

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REFERENCES

[1] G. J. Foschini, "Layered space–time architecture for wireless communication in a fading environment when using multi-element antennas," *Bell Labs Tech. J.* , Autumn 1996.

[2] I. E. Telatar, "Capacity of multi-antenna Gaussian channels," AT&T Bell Labs, Internal Tech. Memo, June 1995.

[3] J.-C. Guey, M.P. Fitz, M.R. Bell, and W.-Y. Kuo, "Signal design for transmitter diversity wireless communication systems over Rayleigh fading channels," in *Proc. IEEE Vehicular Technology Conf. (VTC'96}*, pp. 136–140.

[4] V. Tarokh, N. Seshadri, and A. R. Calderbank, "Space–time codes for high data rate wireless communication: Performance criterion and code construction," *IEEE Trans. Inform. Theory*, vol. 44, pp. 744–765, Mar. 1998.

[5] S. M. Alamouti, "A simple transmitter diversity scheme for wireless communications," *IEEE J. Select. Areas Commun.*, vol. 16, pp. 1451–1458, Oct. 1998.

[6] V. Tarokh, H. Jafarkhani, and A. R. Calderbank, "Space–time block codes from orthogonal designs," *IEEE Trans. Inform. Theory*, vol. 45, pp. 1456–1467, July 1999.

[7] 3rd Generation Partnership Project; and Technical Specification Group Radio Access Network. (1999, Dec.) Physical channels and mapping of transport channels onto physical channels (FDD), (3G TS 25.211 version 3.1.0). [Online]. Available: ftp://ftp. 3gpp.org/Specs/December_/25_series/25211-311.zip.

[8] A. V. Geramita and J. Seberry, “Orthogonal designs, quadratic forms and Hadamard matrices,” in *Lecture Notes in Pure and Applied Mathematics*. New Yourk: Marcel Dekker, 1979, vol. 43.
[9] W. E. Baylis, Ed., *Clifford (Geometric) Algebras With Applications to Physics, Mathematics and Engineering*. Boston, MA: Birkhauser, 1996.
[10] O. Tirkkonen and A. Hottinen, “Complex space–time block codes for four Tx antennas,” in *Proc. IEEE GLOBECOM’00*, vol. 2, Nov. 2000, pp. 1005–1009.
[11] J. C. Baez. (2001, May) The Octonions. [Online]. Available: http://xxx.lanl.gov/abs/math/0105155.
[12] O. Tirkkonen and A. Hottinen, “The algebraic structure of space–time block codes,” in *Proc. Finnish Wireless Communications Workshop 2000*, May 2000, pp. 80–84.
[13] A. O. Barut and R. Raczka, *Theory of Group Representations and Applications*. Singapore: World Scientific, 1987.