12
Modeling Electricity Forward Curve Dynamics in the Nordic Market

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ABSTRACT

This chapter considers the modeling of electricity forward curve dynamics with parameterized volatility and correlation structures. We estimate the model parameters by using the Nordic market's price data and show how the model can be implemented into everyday industry practice.

12.1 INTRODUCTION

Electricity markets are different from the usual financial markets and many other commodity markets due to the non-storability of electricity. The spot price of electricity is set by the short-term supply–demand equilibrium, and supply and demand must be in balance at each instance. Because the demand (supply) today does not necessarily have anything to do with the demand (supply) in the future, the spot electricity today is a different asset from the spot electricity in the future. This implies that the relation between the spot price and the forward prices in the electricity markets is not as straightforward as in the usual financial and commodity markets.

In this chapter we develop a simple parameterized model for forward curve dynamics. We estimate the model parameters by using the data from the Nordic electricity market. The Nordic electricity market is hydro-dominated with roughly 50% of the electricity supply being hydro-based. The winters are cold and much of the precipitation comes as snow. In the spring the snow melts causing floods whose timing varies a lot from year to year due to the temperature. There is a significant electricity heating load while the mild summers do not require a lot of air conditioning, so that electricity demand is concentrated on the winter season. The time-dependent variation present in the demand results in a seasonal, weekly and daily profile in the electricity spot price and electricity forward curve. However, these price variations are smoothed to some extent in the Nordic market because of the hydropower production. Some hydro producers have the possibility to optimize their discharge up to one year ahead, and many have the possibility for some months ahead. The short-term, i.e. intra-week and intra-day, variations in the spot prices decrease due to the easily adjustable hydropower. On the other hand, there is high variation in the price level between different years because the total amount of

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hydropower available in the market depends on the amount of yearly precipitation. The forward prices of electricity mostly reflect the market expectations on the future reservoir levels.

Stochastic modeling of the deregulated markets has concentrated on the electricity spot markets. Research is roughly divided into statistical models and fundamental models. Statistical models depend on the set of parameters that describe the properties of the spot process while fundamental electricity price models are based on competitive equilibrium models for the electricity market. Several models are presented in Wallace and Fleten (2002) and Skantze and Ilic (2001). The statistical models easily fall to over-parameterization and are often considered to be "black-boxes" while the fundamental models require a complete set of coherent historical data to be useful. Further, as mentioned earlier, because the electricity supply–demand equilibrium depends on the time of the year and on the development in the hydrological situation, the stochastic process for the spot price changes over time. Thus, it is likely that the form of the stochastic model is not constant and, therefore, the estimation of the model parameters is difficult.

There are also a few models that study the forward price behavior. The benefit from modeling the forward curve directly is that unlike with the spot models there is no problem fitting the model to the current forward curve. This is a similar advantage to when the Heath, Jarrow and Morton (1992) framework is used in interest rate markets. General statistical analysis on electricity forward prices is found in Lucia and Schwartz (2002). The similarities between the interest rate markets and electricity forward markets are studied in detail by Koekebakker and Ollmar (2001). They find that a simple Heath–Jarrow–Morton approach does not explain electricity forward curve dynamics as well as interest rate dynamics. However, we partly utilize their framework and use similar volatility parameterization but we allow the spot volatility to be time-dependent. Further, we use a parameterized correlation structure and a different estimation method. Within an infinite-dimensional Heath–Jarrow–Morton type model, Björk and Landén (2000) study the theoretical properties of futures and forward convenience yield rates in a case where the underlying asset can be non-tradable, like electricity.

In this chapter we study the dynamics of the whole forward curve. Implicitly this also gives the relationship between the spot price and the forward prices. This is important for the market participants in many business applications, such as power plant optimization, risk management and in the pricing of exotic derivative instruments [see e.g. Geman and Vasicek (2001), Keppo (2002a,b), Vehviläinen (2002), Deng *et al.* (2001)]. For example, the understanding of the relation between the spot and forward prices is needed when hedging electricity production with the forward contracts. In this chapter we focus on the modeling of a few key features in the forward price dynamics and they enable the combined analysis of spot and forward markets. The features that we study are the spot volatility curve, the volatility curve's maturity effect and the forward curve's correlation structure. The historical daily quotes of electricity forwards and futures as well as the historical spot prices are used in the estimation of the model parameters.

The rest of the chapter is organized as follows. Section 12.2 introduces the model while the Appendix illustrates the estimation method of the model parameters. Section 12.3 illustrates the model estimation with market data. Then Section 12.4 gives three practical examples and finally Section 12.5 concludes.

12.2 THE MODEL

In this section we introduce our parametric model for electricity forward curve dynamics. We consider an electricity market where forward contracts are traded continuously within a finite time horizon $[0, \tau]$. When describing the probabilistic structure of the market, we refer to an underlying probability space (Ω, F, P), along with the standard filtration $\{F_t : t \in [0, \tau]\}$. Here Ω is a set, F is a σ-algebra of subsets of Ω, and P is a probability measure on F.

We denote by $f(t, T_1,T_2)$ the forward price for the time period $[T_1,T_2]$ at time t. That is, $f(t, T_1,T_2)$ is a constant price for the duration $T_2 - T_1$ and, therefore, it can be viewed as the average price for the period $[T_1,T_2]$. In order to get the forward curve that depends only on one maturity date, we model the following theoretical forward prices:

$$f(t, T) = \lim_{T_2 \to T} f(t, T, T_2) \quad \text{for all} \quad t \in [0, T],\ T \in [0, \tau] \tag{12.1}$$

That is, $f(t, T)$ is the forward price at time t for the time period $[T,T + dt]$. In contrast to the usual financial markets, where a price process is usually given by a one-dimensional Itô process, in electricity markets the corresponding price at time t is the whole forward curve $f(t, \cdot) : [t, \tau] \to \mathbf{R}_+$ [see e.g. Björk and Landén (2000)].

Note that all forward prices are in nominal terms. According to the definition of the forward price, the spot price is given by

$$S(t) = f(t, t) = \lim_{T \to t} f(t, T) \quad \text{for all} \quad t \in [0, \tau] \tag{12.2}$$

That is, the forward price converges to the spot price. Later in this chapter we use weekly average prices for electricity spot price and, therefore, at the expiration date the spot price equals the weekly future price.

The following assumption characterizes the dynamics of the forward prices, i.e., it gives our parameterized forward curve dynamics.

Assumption 12.1. *The forward prices follow an Itô stochastic differential equation*

$$df(t, T) = f(t, T)e^{-\alpha(T-t)}\sigma(T)dB_T(t) \quad \text{for all} \quad t \in [0, T],\ T \in [0, \tau] \tag{12.3}$$

where the forward price $f(t, T) = E[S(T)|F_t]$, α *is a strictly positive constant,* $\sigma : [0, \tau] \to \mathrm{R}_+$ *is a bounded and deterministic spot volatility curve, and* $B_T(\cdot)$ *is a Brownian motion corresponding to the T-maturity forward price on the probability space* (Ω, F, P) *along with the standard filtration* $\{F_t\text{: } t \in [0, \tau]\}$. *The correlation structure of the Brownian motions is given by*

$$dB_{T*}(t)dB_T(t) = e^{-\rho|T-T*|}dt \quad \text{for all} \quad T, T^* \in [0, \tau] \tag{12.4}$$

where ρ *is a strictly positive constant.*

Assumption 12.1 captures the main elements of our model. Specifically it implies:

- Forward prices are equal to the expected future spot prices and, therefore, forward prices are martingale under the objective probability measure P.
- The electricity spot volatility curve $\sigma(\cdot)\text{: } [0, \tau] \to \mathrm{R}_+$ is deterministic.
- A forward price's volatility is lower than the corresponding spot volatility and the parameter α models this effect.
- Forward prices with maturity dates that are close to each other are significantly correlated. Parameter ρ captures this effect.
- The forward prices follow lognormal distributions.

The expectation hypothesis is made for simplicity and if this is not true under the objective measure P, it is true under the pricing measure Q [see e.g. Hull (2000)]. Thus, in this chapter we assume that P equals Q and, therefore, we do not estimate the expected drifts of the forward prices. According to equation (12.1) the stochastic process for a forward price follows an exponential process where $f^2(t,T)e^{-2\alpha(T-t)}\sigma^2(T)$ is the rate of change of the conditional variance of $f(t,T)$. Note that this volatility parameterization is similar to the one used in Koekebakker and Ollmar (2001). The boundedness of the volatility function guarantees the existence and uniqueness of the solution to (12.3). The deterministic spot volatility structure is quite restrictive and it is made in order to ease the estimation and implementation of the model. In practice there are uncertainties in the spot volatility curve due to the changes in the demand and supply. Therefore, the stochastic volatility models are important also in electricity markets [see e.g. Deng (1999)]. With the third and fourth bullets we model the decreasing volatility as a function of maturity and the decreasing correlation as a function of the difference between forwards' expiration dates. The errors from Assumption 12.1 are analyzed in Sections 12.3 and 12.4. Note that since in equation (12.3) we model expected values ($f(t,T) = E[S(T)|F_t]$), the spot process $S(\cdot)$ can be e.g. geometric Brownian motion or mean-reverting [see for instance Schwartz (1997)]. Further, from Assumption 12.1 we get that the distribution of $S(T)$ is given by

$$\begin{aligned} \log(S(T)) - \log(f(t,T)) &= \log(f(T,T)) - \log(f(t,T)) \\ &\sim \phi\left(-\tfrac{1}{2}\hat{\sigma}^2(T-t), \hat{\sigma}\sqrt{T-t}\right) \end{aligned} \tag{12.5}$$

where $\phi(m,s)$ is a normal distribution with mean m and standard deviation equal to s, $T > t$, and the average volatility on $[t,T]$ is according to (12.3) given by

$$\hat{\sigma} = \frac{\sigma(T)}{\sqrt{T-t}}\sqrt{\int_t^T \exp(-2\alpha y)\,dy} = \frac{\sigma(T)}{\sqrt{2\alpha(T-t)}}\sqrt{[\exp(-2\alpha t) - \exp(-2\alpha T)]}$$

Thus, equation (12.5) implies that the electricity prices follow lognormal distributions.

In the Appendix we show how the model parameters are estimated by using a maximum likelihood method. In the next section we apply our forward model to the Nordic market.

12.3 FORWARD MODEL IN THE NORDIC MARKET

In this section, we estimate the model parameters by using the Nord Pool electricity exchange's market data. First we briefly discuss the forward and future contracts in this Nordic market.

12.3.1 Products in the Nordic power market

In the Nordic market around one-quarter of the total physical demand is traded via the Nord Pool electricity exchange and, therefore, the Nord Pool's electricity price is a credible reference index for the whole market. There is an active market for electricity forwards and futures, both in the exchange and in the OTC markets with volumes nearly 10-fold over the size of the total physical market.

Spot prices for physical delivery are set by an equilibrium model where the supply and demand curves of all the market participants are matched day-ahead. The last-minute balance management is done after the spot market has closed.

Nord Pool's electricity futures contracts include weeks and blocks. Week contracts are traded for the nearest four to seven weeks after which there are block contracts for about one year

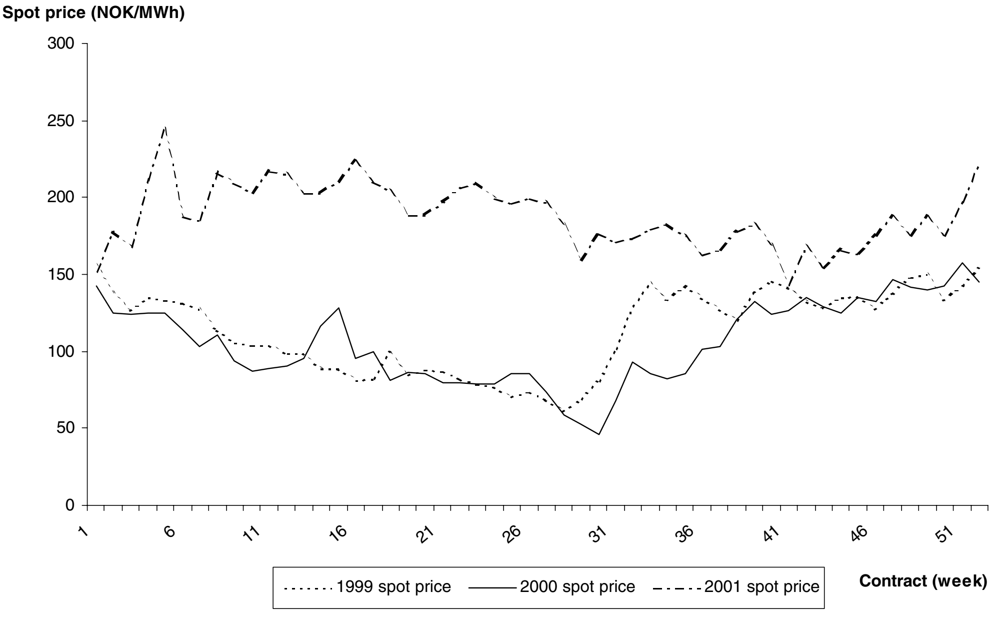

Figure 12.1 Weekly average for the spot price in 1999, 2000 and 2001

forward. Blocks are combinations of four weeks and split to weekly contracts as time passes. The electricity forward contracts are years and seasons that divide the year to three periods: Winter-1, Summer and Winter-2. The closest few years are traded both as seasons and as years. The difference of settlement between futures and forward contracts is ignored here because we only model nominal prices. For further information about Nord Pool see Nord Pool (2002) product information.

We estimate the volatility discount factor, the correlation discount factor and the weekly volatility structure by using Nord Pool's prices for weekly future contracts during the years 1999, 2000 and 2001. Our database consists of prices for the weekly products on each trading day. Figure 12.1 illustrates the realized electricity spot prices in different years. In the Nordic market the electricity spot price is usually high in the winter and low in the summer, as shown in years 1999 and 2000. This is due to the cold winter in the Nordic area and, therefore, high demand during winter. As can be seen from Figure 12.1, year 2001 was different since the price was all the time close to the yearly average price. The hydrological situation changed from relatively wet conditions to dry in the beginning of year 2001. The change was due to cold and relatively dry weather in the first months of 2001 and was reflected as a sharp rise in the spot price. The hydrological situation improved during the year, thus causing the spot price to fall towards autumn.

12.3.2 Estimation of the model parameters

The model parameters are estimated by using the maximum likelihood method. All the parameters (volatility curve, volatility and correlation parameters) are calculated in a single estimation routine. This method is illustrated in the Appendix.

The volatilities of the 52 weekly futures contracts traded during years 1999, 2000 and 2001 are shown in Figure 12.2. This figure indicates that the volatility varies inside the year and also

Table 12.1 Estimated volatility discount parameter as well as its standard deviation in different years

	Year 1999	Year 2000	Year 2001
Volatility parameter α	2.31	6.06	3.67
Standard deviation	0.25	0.32	0.19

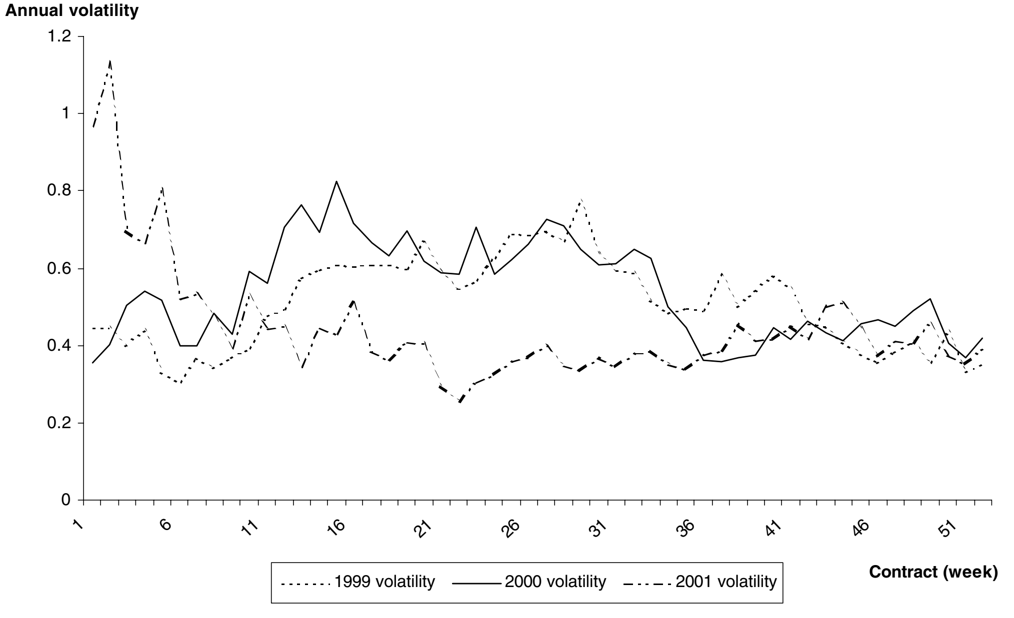

Figure 12.2 Volatilities of the 52 weekly contracts by using the data from 1999, 2000 and 2001

between different years. As in years 1999 and 2000, the volatility is usually high in the summer and low in the winter. This is because in the summer the water reservoirs are usually quite empty and, therefore, small changes in the demand can cause changes in the used production technologies and production marginal costs. On the other hand, because in the winter mainly condensing power is used, there are no major changes in the production marginal costs and the winter volatility is usually lower. The winter 1999–2000 was very snowy whereas during the winter 2000–2001 there was less snow than normally. The accumulation of snow over the normal level during the winter 1999–2000 increased the uncertainty in the spot price level during the possible spring flood period in 2000 and thus caused the high volatility during that period. However, as mentioned earlier the year 2001 is different because in the beginning of year 2001 the hydrological situation changed from relatively wet conditions to dry and this created high spot volatility in the beginning of the year. Because in that year there was not much snow, there was no uncertainty on the spring flood. In Figure 12.2 the average volatility from all the volatility structures is 0.5.

Table 12.1 gives the estimated volatility discount factor and its standard deviation in different years. According to Table 12.1 the volatility of a forward price is lower than the corresponding

Table 12.2 Estimated correlation discount parameter as well as its standard deviation in different years

	Year 1999	Year 2000	Year 2001
Correlation parameter ρ	3.62	5.30	4.61
Standard deviation	0.04	0.26	0.17

spot volatility and the parameter α (volatility discount factor) models this effect. Each year the parameter is stable but due to the different annual hydro-inflow the parameter is different in different years. Since the winter 1999–2000 was very snowy it seems that during snowy years the volatility discount parameter is high. The average value is $\alpha = 4.02$ and it implies that, for instance, if the spot volatility is 50% then a one-month-maturity forward has 36% volatility. Note that similar parameter changes can be observed with the volatility estimates in stock markets [see e.g. Schwert (2002)].

Table 12.2 illustrates the estimated correlation discount factor and its standard deviation in different years. According to Table 12.2 forward prices with maturity dates that are close to each other are significantly correlated and the parameter ρ (correlation discount factor) captures this effect. The average value $\rho = 4.51$ gives, e.g., that two forwards with maturity date one month apart have about a 0.69 correlation. Note that from Tables 12.1 and 12.2 we get that the volatility and correlation discount factors are correlated and their values are close to each other. Therefore, for instance, during snowy years both the parameters are high.

12.4 MODEL USAGE EXAMPLES

In this section we illustrate the usage of the model in everyday industry practice. We consider three cases: conditional forecasting of the forward curve when a forecast for spot price is available, the pricing of forward options and analysis on the accuracy of a forward curve model that describes the forward curve dynamics with a finite number of forward curve points.

In the conditional forecasting we update the initial forward curve with spot price predictions and study whether this kind of forecast model is suited for short or long-term planning. This conditional forecasting can be used in the production optimization and in the pricing and hedging of complex path-dependent electricity derivatives such as swing options [see e.g. Jaillet *et al.* (2001), Thompson (1995) and Keppo (2002a)].

Many options in electricity markets are options on forward prices. Therefore, the Black-76 model [see Black (1976)] is widely used. The only Black-76 model parameter not received directly from the market is the underlying forward price's average volatility during the lifetime of the option. We estimate this volatility from our forward curve model and show a few numerical pricing examples.

In the forward curve accuracy analysis we study the percentage of uncertainties described by a forward curve model that uses a finite number of forward curve points. This is important in the selection of a convenient forward curve model. For instance, frequently production optimization can be carried out by using only a rough estimate of the forward curve dynamics while in the hedging of derivative instruments better description is needed. By using the accuracy analysis the convenient model can be selected and its error can be estimated.

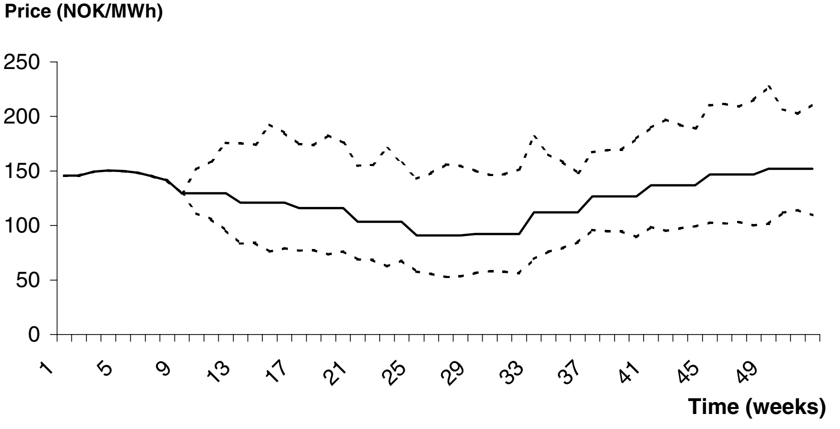

Figure 12.3 Updated forward curve and the corresponding 95% confidence interval

12.4.1 Conditional forecast for the forward curve

Our conditional prediction model uses the estimated parameters of the previous section to calculate the corresponding forward curve from a spot price scenario. In order to analyze our parameterized forward dynamics we assume that the spot price scenario is equal to the realized spot price during the year 2001. Then we compare our predicted forward curve with the corresponding realized forward curve in the market. Thus, the possible forward curve prediction error is from our forward curve model because the spot price scenario is equal to the realized spot price. We use the estimated parameters from year 2000 and test our model with independent data from year 2001. Note that based on Figures 12.1 and 12.2 spot dynamics during years 2000 and 2001 are different. Figure 12.3 illustrates the conditional forecasting. In the figure an initial forward curve has been updated with a spot price scenario during the first nine weeks.

We use 25 initial forward curves and create predictions for two-week, one-month, three-month and six-month horizons. The initial forward curves are updated weekly with the realized spot prices. For all initial forward curves and scenario horizons we compare our updated curves with the corresponding realized forward curves in the market. In measuring the differences between these curves we use the mean square error.

Since the contracts mature weekly, the number of data points in the updated forward curve decreases as our planning horizon increases and, thus, we normalize the mean square error by the number of data points. Considering all of the 25 initial forward curves, we take the average of the mean square errors for each time horizon. Figure 12.4 shows us a summary of how well our updated forward curve fits the realized forward curve. The solid line is the error term from our conditional prediction with different prediction horizons. The dotted line is the corresponding prediction error without the spot price information, i.e., prediction by using directly Assumption 12.1 and the estimated parameters in Tables 12.1 and 12.2. As expected, the dotted line is higher than the solid line because in the conditional forecast more information is used. Further, in practice prediction horizons of less than a month can be used. Thus, Figure 12.4 illustrates that even though one knows the spot price one does not know much about the forward curve. This is because there are many uncertainties in the forward curve and, therefore, knowing only one point in the curve gives an accurate situation only close to this point.

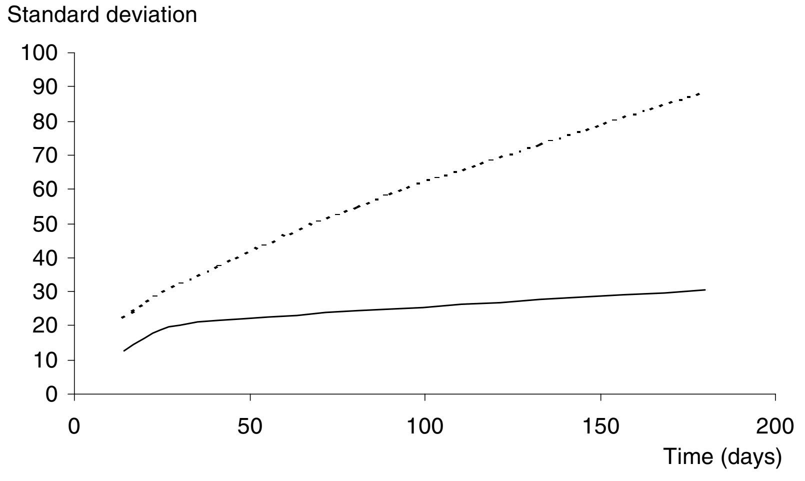

Figure 12.4 Relationship between the forward curve prediction error and the time horizon. The solid line is the error from the conditional forecast and the dotted line is the error from the forecast without the spot price realization

12.4.2 Pricing of forward options

Many options in electricity markets are options on electricity forwards or futures. Therefore, the Black-76 model [see Black (1976)] is widely used in these markets. In the case of a deterministic volatility structure the Black-76 model is used with the average volatility during the lifetime of the option. Therefore, the options on an electricity forward price can be modeled as follows:

$$\begin{aligned} c(t, T_o, T) &= \exp\left(-r(T_o - t)\right)\left[f(t, T)N(d_1) - XN(d_2)\right] \\ p(t, T_o, T) &= \exp\left(-r(T_o - t)\right)\left[XN(-d_2) - f(t, T)N(-d_1)\right] \end{aligned} \tag{12.6}$$

where c is the call option price, p is the put price, T_o is the maturity date of the options, f is the underlying forward price, X is the strike price, T is the maturity date of the forward, t is the current time, $N(\cdot)$ is the cumulative normal distribution function:

$$d_1 = \frac{\ln\left(\frac{f(t,T)}{X}\right) + \frac{1}{2}\hat{\sigma}^2(t, T_o, T)(T_o - t)}{\hat{\sigma}(t, T_o, T)\sqrt{T_o - t}}$$

$$d_2 = d_1 - \hat{\sigma}(t, T_o, T)\sqrt{T_o - t}$$

and $\hat{\sigma}(t, T_o, T)$ is the average volatility of $f(\cdot,T)$ during $t - T_o$.

The problem with the Black-76 model is to find the correct average volatility for different maturities. Because in our model we have a deterministic volatility curve, we get from equation (12.5)

$$\hat{\sigma}^2(t, T_o, T) = \frac{\sigma^2(T)\int_t^{T_o}\exp(-2\alpha y)dy}{T_o - t} = \frac{\sigma^2(T)}{2\alpha(T_o - t)}\left[\exp(-2\alpha t) - \exp(-2\alpha T_o)\right] \tag{12.7}$$

We illustrate this framework through a numerical example, where we calculate the call prices with different maturities. The strike price is assumed to be equal to the underlying forward price

Table 12.3 Call prices on electricity forwards with different maturities

Maturity	Average volatility, %	Call price, % from forward price
2 weeks	46.4	3.6
1 month	42.8	4.9
3 months	33.0	6.4
6 months	24.9	6.8

and the risk-free rate equal to 5% (annual, continuous time). For simplicity, the maturity of each option is equal to the underlying forward contract's maturity. The spot volatility structure is flat and equal to 50%. Using equations (12.6) and (12.7) we calculate the call option prices and Table 12.3 summarizes the results. According to Table 12.3 the average volatility decreases as a function of maturity. However, $g(T_o) = \hat{\sigma}(t, T_o, T_o)\sqrt{T_o - t}$ is an increasing function of T_o even though $\hat{\sigma}(t, T_o, T_o)$ is a decreasing function of T_o. Therefore, time to maturity increases the call prices in Table 12.3.

12.4.3 Accuracy of a forward curve model

In practice the dynamics of the forward curve are modeled by using a finite number of forward curve points. The advantage of this is that it is easier to analyze these points than the whole curve. The drawback is that we lose some accuracy because we do not model all the uncertainties in the curve, i.e., we do not model the area between the points. In this subsection we analyze this error term and similar accuracy analysis is done in Koekebakker and Ollmar (2001).

By using the correlation discount factor we estimate the percentage of the uncertainties captured with the selected forward curve points. Let us denote by Δ the time interval between the forward curve points and assume that this interval is constant. Then we get, by using the correlation parameter ρ, that the cumulative correlation between a forward curve point and the part of the forward curve that is closest to this point is given by

$$2\int_0^{\frac{\Delta}{2}} \exp(-\rho y)dy = \frac{2}{\rho}\left[1 - \exp\left(-\rho\tfrac{\Delta}{2}\right)\right] \tag{12.8}$$

This is because the time interval $\left(T - \frac{\Delta}{2}, T + \frac{\Delta}{2}\right)$ is closest to the maturity T. Since the time length of this part is Δ, we model the uncertainty proportion that the discrete model describes as follows:

$$\frac{2}{\Delta\rho}\left[1 - \exp\left(-\rho\tfrac{\Delta}{2}\right)\right] \tag{12.9}$$

Equation (12.9) models the correlation effect to the forward curve on $(T - \frac{\Delta}{2}, T + \frac{\Delta}{2})$ from the single forward curve point $f(t, T)$. Actually, this equation gives the lower boundary for the proportion since it ignores the independent effects from other forward curve points. However, since the closest point has the strongest correlation, for our purposes the above equation is accurate enough.

We analyze the uncertainty proportion with different discrete time interval Δ by using equation (12.9). Figure 12.5 illustrates the results. According to Figure 12.5 the longer the time interval the less we are able to model the uncertainties. For instance, if we use four

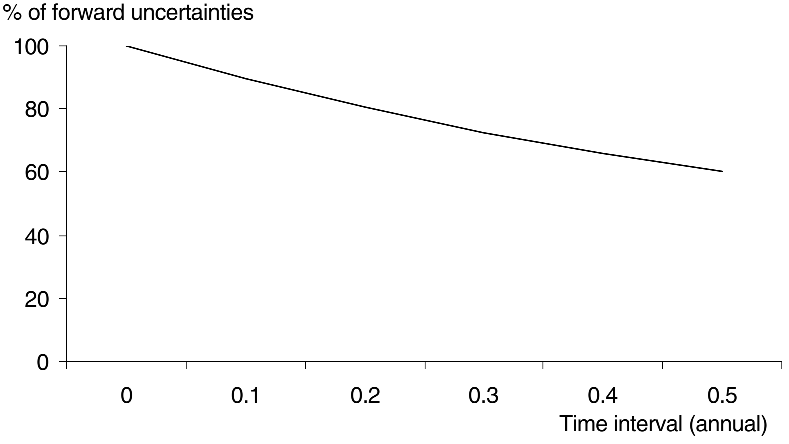

Figure 12.5 The percentage of forward curve uncertainties described by finite number of forward curve points. Time interval is the time period between the points

forward curve points (time interval 0.25, times: 0, 0.25, 0.5, 0.75) to model the whole annual forward curve dynamics we are able to capture at least 76% of the forward curve uncertainties.

12.5 CONCLUSION

In the electricity market the price risk is described by the whole forward curve dynamics. Therefore, the modeling of the curve dynamics is as important as the stock price modeling in the usual financial markets. In this chapter we have proposed a simple parameterized model for forward curve dynamics and estimated the parameters by using the price data from the Nordic market. According to the estimation results a forward's correlation with the spot price and its volatility decrease as a function of maturity. For instance, the volatility of a two-month-maturity future contract is about half of the corresponding spot price's volatility and the correlation between the future price and the current spot price is about 0.5.

We have shown several possible applications for our forward curve model. Firstly, we made a conditional forecast for the forward curve when a perfect forecast for the spot price is available. Secondly, we combined our volatility parameterization with a usual option pricing method. Finally, we estimated the accuracy of a forward curve model by using our correlation parameterization. Further application is, e.g., the modeling of a hydropower production. Because the market is highly competitive we can assume that the producer is a price-taker and, therefore, the power plant can be modeled as a basket of electricity options. As we have seen in this chapter, in the calculation of these option prices the forward curve dynamics are crucial. A similar example is considered in Keppo (2002a), where a power plant is modeled by using a swing option.

Our forward curve model can be extended in several ways. As was noted, there is considerable variation in the model parameters between different years. This is due to the fact that in the Nordic market the forward curve dynamics depend on the hydrological situation, i.e., the contents of the water and snow reservoirs. To develop the forward curve model further the effect of the changes in the water and snow reservoir contents into our forward curve parameters could be modeled.

APPENDIX: ESTIMATION OF MODEL PARAMETERS

In this Appendix we show how the spot volatility curve, the maturity parameter and the correlation parameter can be estimated by using a maximum likelihood estimation method. In this estimation method, we apply an exponential weighting to give more weight to the most recent data points.

Based on Assumption 12.1 and the properties of the lognormal distribution we get

$$\text{In}\left[\frac{f(t+\Delta t, T)}{f(t,T)}\right] = \frac{-1}{4\alpha}\sigma^2(T)\left[e^{-2\alpha(T-(t+\Delta t))} - e^{-2\alpha(T-t)}\right] + \frac{1}{\sqrt{2\alpha}}\sigma(T)\sqrt{\left[e^{-2\alpha(T-(t+\Delta t))} - e^{-2\alpha(T-t)}\right]}\varepsilon \quad \text{(A12.1)}$$

where ε is a random variable that is distributed according to a standard normal distribution.

At given time t, one can observe a set of contracts with maturities $(T_i)_{i=n(t)}^{N(t)}$, where $n(t)$ and $N(t)$ denote the upper and lower indices of the contracts for which the price can be observed at time t. Note that $n(t)$ and $N(t)$ fluctuate with time and the maturities are fixed. Let $f^i(t)$ be the price of the T_i-maturity contract at time t. For our model, we will assume that there are 250 trading days and 52 weeks per year. Consequently, since we use weekly futures we define

$$\Delta t = \frac{1}{250}$$
$$T_i - T_j = \frac{i-j}{52} \quad \text{(A12.2)}$$

Define $v^i(t) = \log f^i(t+\Delta t) - \log f^i(t)$. We know that $v^i(t)$ is Gaussian, with

$$E[v^i(t)] = -\frac{\sigma_i^2}{4\alpha}e^{-2\alpha(T_i-t)}(e^{2\alpha\Delta t} - 1)$$
$$Var[v^i(t)] = \frac{\sigma_i^2}{2\alpha}e^{-2\alpha(T_i-t)}(e^{2\alpha\Delta t} - 1)$$
$$Corr[v^i(t), v^j(s)] = \begin{cases} 0 & t \neq s \\ e^{-p|T_i-T_j|} & s = t \end{cases} \quad \text{(A12.3)}$$

where $\sigma_i = \sigma(T_i)$. Note that the assumption of no correlation between $v(t)$ and $v(s)\,(s \neq t)$ implies that $v(t) = (v^i(t))_{i=n(t)}^{N(t)}$ and $v(s) = (v^i(s))_{i=n(s)}^{N(s)}$ are mutually independent.

Define

$$u^i(t) = \frac{v^i(t) - E[v^i(t)]}{\sqrt{Var[v^i(t)]}} = h(v^i(t)) \quad \text{(A12.4)}$$

Equation (A12.4) is simply a centered-reduced form of v since the expected value of u is 0 and its variance is 1.

We assume that we observe $u^i(t)$ on a set of dates $(\tau_j)_{j=1}^{T}$, with equally spaced observation time, so that $\tau_j = j \cdot \Delta t$. Denote by u the vector of all the observations of all the contract maturities, i.e., the vector of $u^i(\tau_j)$ for all i and j. Using the fact that $u^i(t)$ is Gaussian, their joint density $g_u(u)$ is given by

$$\begin{aligned}
g_u(u) &= \prod_{j=1}^{T} g_j(u(\tau_j)) \\
g_j(u(\tau_j)) &= (2\pi)^{-M(\tau_j)/2}|\Omega_j|^{-1/2}\exp\left(-\frac{1}{2u}(\tau_j)^T\Omega_j^{-1}u(\tau_j)\right) \\
[\Omega_j]_{k,l=1}^{M(\tau_j)} &= \tilde{\rho}^{|k-l|} \\
\tilde{\rho} &= e^{-\rho/52} \\
M(\tau_j) &= N(\tau_j) - n(\tau_j) + 1 \\
\tau_j &= j \cdot \Delta t
\end{aligned} \tag{A12.5}$$

The first equation of (A12.5) is deduced from the assumption of independence of $v(\tau_j)$ and $v(\tau_k)(k \neq j)$ stated in equation (A12.3), which allows us to write the density as the product of marginal densities of the $u(\tau_j)$. The second equation simply expresses the (marginal) density of the variables $u^i(\tau_j)$ as a Gaussian density with mean zero, unit variance and a correlation given by Ω_j. The third and fourth detail the values of the elements of Ω_j, which are obtained from the third equation of (A12.3). The variable $M(\tau_j)$ is the number of observable contracts that are observable at time τ_j.

By expanding the expression for the determinant and the inverse of the correlation matrix Ω_j, it can be shown that g_j may be written as

$$\begin{aligned}
g_j(u(\tau_j)) = {} & (2\pi)^{-M(\tau_j)/2}(1-\tilde{\rho}^2)^{-\frac{M(\tau_j)-1}{2}} \\
& \exp\left[-\frac{1}{2(1-\tilde{\rho}^2)}\left(\sum_{i=n(\tau_j)}^{N(\tau_j)-1}(u^{i+1}(\tau_j)-\tilde{\rho}u^i(\tau_j))^2 + u^{N(\tau_j)}(\tau_j)^2(1-\tilde{\rho}^2)\right)\right]
\end{aligned} \tag{A12.6}$$

We now have an expression for the density of the $u^i(\tau_j)$ as defined by equation (A12.4). The density of $v(\tau_j)$ is simply obtained by multiplying this density by the Jacobian of the transformation (A12.4):

$$d_j(v(\tau_j)) = \frac{g(h(v(\tau_j)))}{\prod_{i=n(\tau_j)}^{N(\tau_j)}\sqrt{Var(v^i(\tau_j))}} \tag{A12.7}$$

where $Var(v^i(\tau_j))$ is given by (A12.3) and, as mentioned earlier, $\tau_j = j \cdot \Delta t$. The denominator is the determinant of the Jacobian of the transformation from v to u. It can be deduced immediately from equation (A12.4).

A maximum likelihood estimator of the parameters can be constructed by solving

$$(\hat{\alpha}_{ML}, \hat{\rho}_{ML}, \hat{\sigma}_{ML}) = \underset{(\alpha,\rho,\sigma)}{\arg\max}\sum_{j=1}^{T}\log d_j(v(\tau_j)) \tag{A12.8}$$

Unfortunately, no closed-form solution exists for these estimators and, therefore, the optimization must be carried out numerically.

In order to reduce the risk of instability in the parameters of the model, we introduce the following objective function:

$$\varphi_j(\alpha, \rho, \sigma) = \log d_j(v(\tau_j)) \cdot \exp(-\gamma(\tau_{T_j} - \tau_j)) \tag{A12.9}$$

The exponential term that is added to the likelihood function acts as a weighting factor that discounts the likelihood of the observations. This technique is a heuristic way of protecting the estimators against parameter instability.

The corresponding estimators are therefore given by

$$(\hat{\alpha}, \hat{\rho}, \hat{\sigma}) = \underset{(\alpha,\rho,\sigma)}{\arg\max} \sum_{j=1}^{T} \varphi_j(\alpha, \rho, \sigma) \tag{A12.10}$$

These estimators are exactly equal to the maximum likelihood estimators if γ is equal to zero. In that case, the estimator is efficient in the sense that it meets the Cramer–Rao bound [see e.g. Greene (1997, pp. 133–138)]. If $\gamma > 0$, the estimator is still a member of the class of consistent and asymptotically Gaussian M-estimators [see e.g. Gourieroux and Monfort (1996, chapter 8)]. Defining $\theta = (\alpha \quad \rho \quad \sigma)^T$, the covariance matrix of our estimator is given by

$$\begin{aligned}
&\sqrt{T}(\hat{\theta} - \theta) \underset{T\to\infty}{\to} N(0, J^{-1}IJ^{-1}) \\
&J = E\left[\frac{\partial^2 \varphi}{\partial\theta\partial\theta^T}\right] \\
&I = E\left[\frac{\partial\varphi}{\partial\theta}\frac{\partial\varphi}{\partial\theta^T}\right]
\end{aligned} \tag{A12.11}$$

and the covariance matrix is estimated as follows:

$$\begin{aligned}
\hat{J} &= \frac{1}{T}\sum_j \left[\frac{\partial^2 \varphi_j}{\partial\theta\partial\theta^T}\right] \\
\hat{I} &= \frac{1}{T}\sum_j \left[\frac{\partial\varphi_j}{\partial\theta}\frac{\partial\varphi_j}{\partial\theta^T}\right]
\end{aligned} \tag{A12.12}$$

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