Electricity Load Pattern Hedging with Static Forward Strategies

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Abstract

We consider the partial hedging of stochastic electricity load pattern with static forward strategies. We assume that the company under consideration maximizes the risk adjusted expected value of its electricity cash flows. First, we calculate an optimal hedge ratio and after that we use this hedge ratio to solve the optimal hedging time. Our results indicate, for instance that agents with high load volatility hedge later than agents that have low load volatility. Moreover, negative correlation between forwards and electricity load pattern postpones the hedging timing.

Key words: Electricity markets, hedging, derivative instruments

1. Introduction

Electricity spot markets expose market participants to a considerable financial risk. By using electricity derivative instruments the market participants can hedge against electricity price changes and reduce the fluctuations of their cash flows. However, usually these market participants also face load uncertainty that cannot be perfectly hedged by using the derivatives. For example, the amount of hydropower production is a function of reservoir inflow, which, of course, depends on the amount of rainfall. If the rainfall uncertainty does not perfectly correlate with the electricity derivative prices then the load uncertainty cannot be hedged with those derivatives. Therefore, the market is incomplete (for incomplete markets see, e.g., Karatzas et al., 1991, Cuoco, 1997, and Cvitanic et al., 1997).

Transaction costs and illiquidity concerns make the use of dynamic hedging strategies difficult for electricity consumers and producers. When static hedging strategies are used the transaction costs are lower and the implementation of hedging strategies is easier. For instance, Cvitanic et al. (1999) have shown that under proportional transaction costs the minimal super-replicating strategy for European type contingent claims is the least expensive buy and hold strategy. For more about the static hedging of standard options see, e.g., Carr and Wu (2002). Because of the transaction costs and illiquidity concerns, in this paper we consider static hedging of stochastic electricity load pattern with electricity forward contracts. Forward hedging strategies when the underlying asset position is fixed are studied, for example, in Ederington (1979), Frankcle (1980), and Lien and Tse (2002). Hedging of stochastic underlying asset position is studied, for example, in Honda (1983), Paroush and Wolf (1986), and Stulz (1984). These hedging models do not consider electricity markets and cannot be applied directly to electricity load hedging due to the specific features of electricity markets. Fleten and Wallace (1998) and Fleten et al. (1997) study a scenario-based approach for solving the optimal portfolio management problem in electricity markets, and Vehviläinen and Keppo (2003) use Monte Carlo simulation in the managing of electricity market price risk. In the present paper we do not utilize these techniques since we are able to solve analytically the portfolio value distribution. For more about electricity markets, see, e.g., Pilipović (1998).

Modeling of electricity forward curve dynamics (see, e.g., Björk and Landén, 2000, Miltersen and Schwartz, 1998, and Clewlow and Strickland, 1999) has been built on Heath et al. (1992) framework, which was originally used for interest rate markets. Audet et al. (2003) and Koekebakker and Ollmar (2005) expand the study to have time dependent spot volatility. We assume that the company under consideration tries to maximize the risk adjusted expected value of its cash flows. Due to the electricity forward dynamics the utility maximization problem reduces into a minimum variance portfolio selection problem. This kind of hedging is considered, e.g., in Föllmer and Sondermann (1986), Duffie (1991), Lim (2002), Caldentey and Haugh (2003), and Schweizer (1992, 2001). Our model is closest to Caldentey and Haugh (2003) that consider combined production and dynamic hedging decisions. In contrast to their framework we use static hedging strategies and analyze the hedging behavior of different electricity market participants. Combined production and hedging decisions are also analyzed, e.g., in Moschini and Lapan (2002) and Keppo (2002).

When static hedging strategies are used, each moment the agent faces decision whether to hedge with the current load estimate or wait for new information. When we refer to static hedging we mean that once the hedge is created it is not changed after that. First, we compute the optimal hedge ratio as a function of the hedging time. Second, the optimal hedging time is solved. In general this optimal hedging time is a stopping time of stochastic analysis but due to our objective function we are able to solve deterministic hedging time. Because our optimal hedging time is not necessary unique, we also solve load process dynamics under which all hedging times are equally good. This load dynamics is an equilibrium load process in the sense that a representative agent that has this kind of load pattern can trade all the time, which fits the reality since electricity forwards and futures are traded all the time. In other words, if we model the market participants with one representative agent and if we assume that all the time at least one of these market participants executes the static hedging, we can solve for the representative agent's load pattern. We do not argue that representative agents exist in reality, but they can be used to analyze the characteristics of the market participants. Asset prices in a pure exchange economy with one representative agent and one perishable good are studied, for instance, in Lucas (1978), and equilibrium option pricing is studied, e.g., in Naik and Lee (1990).

Our results indicate that over (under) hedge is optimal when the load estimate and the forward prices are positively (negatively) correlated. Moreover, positive correlation between load estimate and forward prices has preemptive effect on the hedging time, whereas increase in the load volatility postpones the hedging decision as it is optimal to wait for better load estimate.

This article is organized as follows: The mathematical model is presented in Section 2. In Section 3 the optimal hedging strategy is derived. A representative agent's load process dynamics are computed in Section 4. In Section 5 we illustrate our model with an example and, finally, Section 6 concludes the article.

2. Mathematical model

We consider an electricity market where spot and derivative instruments are traded continuously in a finite time horizon. In describing the probabilistic structure of the market, we will refer to an underlying probability space (Ω, F, P), where Ω is a set, F is σ-algebra of subsets of Ω, and P is a probability measure on F. The set of market participants is denoted by W. The following assumption characterizes the derivative market.

ASSUMPTION 1. *There exist forward contracts on electricity spot price. The electricity derivative market is complete and there is no arbitrage.*

The no arbitrage assumption states that all portfolios with the same future payoffs have the same current value. The no-arbitrage condition and the completeness of the market ensure the existence of a unique linear pricing function. The linear pricing function can be described by an equivalent martingale measure Q. Under the martingale measure all the expected returns of traded non-dividend paying financial assets are equal to the risk-free interest rate r (see, e.g., Duffie, 2001). Thus, at time t the price of T-maturity derivative on electricity spot price is given as

$$f(t,T) = \exp(-r(T-t))E^Q[\phi(S(T))|F_t] \quad \text{for all} \quad t \in [0,T] \tag{1}$$

where $\phi(.)$ is the payoff function, $S(T)$ is the electricity spot price at time T, E^Q is the expectation operator under the martingale measure Q and the expectation in (1) is taken with respect to the information at time t, F_t. For simplicity, risk-free rate r is assumed to be constant. Note that since electricity is not a financial asset its expected return under Q is not usually equal to the risk-free rate.

We denote the T-maturity forward price at time t by $S(t,T)$. By allowing T to vary from t to $\widetilde{T}$ we get the forward curve $S(t,.):[t,\widetilde{T}] \to \mathbf{R}_+$, where the maximum maturity $\widetilde{T} > t$. There are cycles and peaks in the forward curve due to the seasonality in electricity spot price. The starting point of the forward curve $S(t,t)$ is the electricity spot price at time t. Because the value of a forward contract is by definition zero when initiated, we get from (1)

$$S(t,T) = E^Q[S(T,T)\,|F_t] = E[M(T)S(T,T)\,|F_t], \quad \text{for all} \quad t \in [0,T] \tag{2}$$

where $M(t)$ is Radon-Nikodym derivative dQ/dP on F_t and E is the expectation operator under the objective measure P. In this article we assume that $M(t) = 1$ for all t, i.e., $P = Q$. Thus, forward prices are martingales under P. Koekebakker and Ollmar (2005) report descriptive statistics on the one week-, one year- and two year forward prices in the Nordic electricity market (Nord Pool) during the 1995-2001 period. Their results indicate that forward prices have significant volatility, but the price processes do not have clear trend (see Table 2 in Koekebakker and Ollmar, 2005). Therefore, we make the assumption $M(t) = 1$.

Next we describe our forward price dynamics.

ASSUMPTION 2. *T-maturity forward price follows the following Itô process*

$$dS(t,T) = \exp(-\alpha(T-t))\sigma_S(T)S(t,T)dB_S(t) \quad \text{for all } t \in [0,T] \tag{3}$$

where α is a strictly positive constant $\sigma_S(T) \to \mathbf{R}_+$ is a given bounded deterministic spot volatility, and $B_S(.)$ is a standard Brownian motion on the probability space (Ω, F, P), *along with the standard filtration* $\{F_t : t \in [0,T]\}$.

Assumption 2 implies that the forward volatility is lower than the corresponding spot volatility. Parameter α models the exponential decrease in the forward volatility as a function of maturity. The decrease in the forward volatility can be seen as a consequence of the mean reverting nature of electricity spot price (see, e.g., Clewlow and Strickland, 1999). Assumption 2 is used, e.g., in Audet et al. (2003) and Koekebakker and Ollmar (2005). According to (2) the forward prices are lognormally distributed and, therefore, we have

$$Var(S(t,T)) = S^2(0,T)\left(\exp\left(e^{-2\alpha T}\sigma_S^2(T)\frac{1}{2\alpha}(e^{2\alpha t}-1)\right)-1\right). \tag{4}$$

Let $p^w(t,T)$ denote the T-maturity load estimate at time t for agent $w \in W$, i.e.,

$$p^w(t,T) = E[p^w(T)|F_t], \tag{5}$$

where $p^w(T)$ is the electricity load at time T. We assume that at each time $t \in [0,T]$ each agent has a T-maturity load estimate. The following assumption characterizes the dynamics of the T-maturity load estimate.

ASSUMPTION 3. *T-maturity load estimate of agent $w \in W$ follows the following Itô process*

$$dp^w(t,T) = p^w(t,T)(\sigma_p^w(t,T)dB_p^w(t) + \sigma_{ps}^w(t,T)dB_S(t)) \quad \text{for all } t \in [0,T] \tag{6}$$

where $\sigma_p^w(.,T) : [0,T] \to \mathbf{R}_+$ and $\sigma_{ps}^w(.,T) : [0,T] \to \mathbf{R}$ are bounded deterministic load volatility functions. $B_p^w(.)$ and $B_S(.)$ are independent standard Brownian motions on the probability space (Ω, F, P), *along with the standard filtration* $\{F_t : t \in [0,T]\}$.

Uncertainty in the T-maturity load estimate due to the changes in electricity forward prices is modeled with $B_S(.)$, while $B_p^w(.)$ models the uncertainty that is uncorrelated with the corresponding forward price. Thus, at time t the volatility of T-maturity load estimate for agent $w \in W$ is $\sqrt{(\sigma_p^w(t,T))^2 + (\sigma_{ps}^w(t,T))^2}$ and the correlation with T-maturity forward price is

$$\rho^w(t,T) = \frac{\sigma_{ps}^w(t,T)}{\sqrt{(\sigma_p^w(t,T))^2 + (\sigma_{ps}^w(t,T))^2}}. \tag{7}$$

An agent able to adjust the electricity load due to adverse changes in forward prices has a negative correlation, i.e., $\sigma_{ps}^w(t,T) < 0$. For example, an electricity consumer can decrease its electricity consumption when the forward prices are high. The adjustments can be done, for example, by changing the performance level of an electricity consumption unit. The consumption process parameters can be estimated, e.g., by using Räsänen et al. (1995, 1997). The agents with negative correlation will be called flexible consumers and their set will be denoted by W_- which is a subset of W. An agent able to exploit positive changes in the forward prices has a positive correlation, i.e., $\sigma_{ps}^w(t,T) > 0$. For example, an electricity producer can increase its production by starting up flexible production units that are used only when electricity prices are high. The optimal load pattern can be solved, e.g., by using Gjelsvik et al. (1992), Pereira (1989), and Keppo (2002). The agents with positive correlation will be called flexible producers and their set will be denoted by W_+ and we have $W_+ \subseteq W$. The set of agents with zero correlation will be denoted by W_0 ($W_0 \subseteq W$) and they will be called non-flexible agents. When there is no correlation the total volatility is $\sigma_p^w(t,T)$, i.e., $\sigma_{ps}^w(t,T) = 0$. The load estimates are log-normally distributed and, therefore,

$$Var(p^w(T)) = (p^w(0,T))^2 \left(\exp\left(\int_0^T ((\sigma_p^w(s,T))^2 + (\sigma_{ps}^w(s,T))^2) ds \right) - 1 \right). \quad (8)$$

Note that as we model the conditional expectations there are no restrictions on the load pattern as a function of maturity.

We solve the optimal hedging strategy of T-maturity electricity load for agent $w \in W$. We assume that the optimal hedging strategy is given by optimal hedging amount and time. We solve the hedging amount in terms of hedge ratio that describes the hedging size as a proportion to the load estimate. After the hedging time the position is not readjusted even if there are remarkable changes in the load estimate or in the forward prices. Thus, before the hedging time the agent makes each moment a decision whether to hedge with the current load estimate or wait and make the decision in the future. The electricity cash flows of agent $w \in W$ at time T are given by

$$\pi^w(\tau^w, T, \eta^w(\tau^w)) = p^w(T,T)S(T,T) + \eta^w(\tau^w)p^w(\tau^w,T)(S(\tau^w,T) - S(T,T)), \quad (9)$$

where $\tau^\omega \in: [0,T)$ is the hedging time, $\eta^w(.):[0,T) \to \mathbf{R}$ is the hedge ratio, and therefore, $\eta^w(\tau^w)p^w(\tau^\omega,T)$ is the number of forwards in the portfolio.

The above described hedging strategy is a buy and hold strategy. The strategy is easy to implement, transaction costs are small and they are known in advance. In contrast to these static strategies, the transaction costs of dynamic hedging strategies are often high and they depend on the amount of readjustments done on the portfolio. Moreover, static hedging strategies are more usable in illiquid markets, because continuous trading is not needed (see, e.g., Carr and Wu, 2002). Note that the static forward hedging strategies can remove only linear dependencies on the forward prices and the dynamic strategies hedge perfectly if the load pattern's uncertainty is only from forward prices. Our objective is not to remove all the uncertainty in the portfolio process, but to reduce it considerably. We assume that the agents in the market are risk-averse and the following assumption formalizes the objective of the hedging.

ASSUMPTION 4. *Agent $w \in W$ maximizes the risk adjusted expected value of the electricity portfolio by using buy and hold strategy, i.e., the agent solves*

$$J^{\omega}(t) = \sup_{\tau^{\omega}\in[t,T),\eta^{\omega}(\tau^{\omega})\in\mathbf{R}} E[\pi^{\omega}(\tau^{\omega},T,\eta^{\omega}(\tau^{\omega}))|F_t] - \lambda^{\omega} E[(E[\pi^{\omega}(\tau^{\omega},T,\eta^{\omega}(\tau^{\omega}))| F_t] - \pi^{\omega}(\tau^{\omega},T,\eta^{\omega}(\tau^{\omega})))^2 | F_t] \quad \text{for all } t\in[0,T) \tag{10}$$

where $\pi^{\omega}(\tau^{\omega},T,\eta^{\omega}(\tau^{\omega}))$ *is the T-maturity electricity portfolio defined in (9) and parameter* $\lambda^{\omega} \geq 0$ *is the risk aversion of the agent.*

Note that the optimal hedging strategy is chosen by optimizing the hedging time and hedge ratio, but the load changes due to the spot price changes, i.e. parameter $\sigma^{\omega}_{ps}(t,T)$, is not optimized. Thus, we assume that the hedging strategy is chosen by taking both the spot and load processes as input variables. This assumption is realistic as the companies often optimize their hedging strategies independently from the production processes (see, e.g., Hull, 1997, and Pilipović, 1998). Once the risk aversion parameter λ^{ω} is known the utility of different portfolios can be calculated. Assumption 4 simplifies into the following lemma.

LEMMA 1. *The optimal hedge ratio* $\eta^{\omega}(.):[t,T)\rightarrow\mathbf{R}$ *and the optimal hedging time* $\tau^{\omega}\in[t,T)$ *of agent* $\omega\in W$ *are solved from*

$$\inf_{\tau^{\omega}\in[t,T),\eta^{\omega}(\tau^{\omega})\in\mathbf{R}} Var(\pi^{\omega}(\tau^{\omega},T,\eta^{\omega}(\tau^{\omega}))), \tag{11}$$

where $Var(\pi^{\omega}(\tau^{\omega},T,\eta^{\omega}(\tau^{\omega})))$ *is the variance of T-maturity electricity portfolio.*

Proof: For the expected portfolio value we have

$$E[\pi^{w}(\tau^{w},T,\eta^{w}(\tau^{w}))|F_t] = E[p^{\omega}(T,T)S(T,T) + \eta^{w}(\tau^{w})p^{w}(\tau^{w},T) E[S(\tau^{w},T) - S(T,T)|F_{\tau^{\omega}}]|F_t] \tag{12}$$

Due to the martingale property of the forward prices (Assumption 2)

$$E[S(\tau^{\omega},T) - S(T,T)|F_{\tau^{\omega}}] = 0. \tag{13}$$

Thus, (12) simplifies into

$$E[\pi^{w}(\tau^{w},T,\eta^{w}(\tau^{w}))|F_t] = E[p^{\omega}(T,T)S(T,T)|F_t]. \tag{14}$$

Because the right hand side of (14) is independent of τ^{ω} and $\eta^{\omega}(\tau^{\omega})$, the expected portfolio value does not affect the hedging decision and the problem (10) reduces into

$$\sup_{\tau^{\omega},\eta^{\omega}(\tau^{\omega})} -\lambda^{\omega} E[(E[\pi^{\omega}(\tau^{\omega},T,\eta^{\omega}(\tau^{\omega}))|F_t] - \pi^{\omega}(\tau^{\omega},T,\eta^{\omega}(\tau^{\omega})))^2 | F_t]. \tag{15}$$

This gives (11). Q.E.D

According to Lemma 1 the size and timing of the hedge are calculated by minimizing the variance of the portfolio. This kind of hedging is considered, e.g., in Föllmer and Sondermann (1986) and Schweizer (2001).

3. Optimal hedging strategy

In this section we derive the optimal hedging strategy for the *T*-maturity portfolio of agent $\omega\in W$. First, the optimal hedge ratio is computed and then the optimal hedging time is calculated by using the hedge ratio. The hedging strategy is chosen by minimizing the portfolio variance. To simplify our notation maturity T and agent w are left out whenever there is no possibility for misunderstanding. By using Itô's lemma and by dividing the integrals into two parts (before and after the hedging time) the portfolio hedged at time τ can be written as

$$\pi(\tau) = p(0)S(0)\pi_1(\tau)\pi_2(\tau), \tag{16}$$

where the portfolio process before the hedging time is given by

$$\pi_1(\tau) = \exp\Bigg(-1/2\int_0^\tau (\sigma_p^2(y) + \sigma_{ps}^2(y) + e^{-2a(T-y)}\sigma_s^2)dy + \int_0^\tau \sigma_p(y)dB_p(y) + \int_0^\tau (\sigma_{ps}(y) + e^{-a(T-y)}\sigma_s)dB_s(y)\Bigg) \tag{17}$$

and the portfolio process after the hedging is given by

$$\pi_2(\tau) = \exp\Bigg(-1/2\int_\tau^T (\sigma_p^2(y) + \sigma_{ps}^2(y) + e^{-2a(T-y)}\sigma_s^2)dy + \int_\tau^T \sigma_p(y)dB_p(y) + \int_\tau^T (\sigma_{ps}(y) + e^{-a(T-y)}\sigma_s)dB_s(y)\Bigg) - \eta(\tau)\exp\Bigg(-1/2\int_\tau^T e^{-2a(T-y)}\sigma_s^2 dy + \int_\tau^T e^{-a(T-y)}\sigma_s dB_s(y)\Bigg) + \eta(\tau) \tag{18}$$

The hedge ratio $\eta(\tau)$ is chosen by minimizing the variance of $\pi_2(\tau)$, as the hedge ratio does not affect the portfolio process before hedging, i.e., equation (17) does not contain $\eta(\tau)$. Following proposition states the optimal hedge ratio.

PROPOSITION 1. *The optimal hedge ratio is given by*

$$\eta(\tau) = \frac{\exp(a_2(\tau))(\exp(a_1(\tau) + a_2(\tau)) - 1)}{\exp(a_1(\tau)) - 1} \quad \text{for all } \tau \in [0, \mathrm{T}), \tag{19}$$

where

$$a_1(\tau) = \int_\tau^T \exp(-2\alpha(T-y))\sigma_s^2 dy = \frac{\sigma_s^2}{2\alpha}(1 - \exp(-2\alpha(T-\tau))) \tag{20}$$

$$a_2(\tau) = \sigma_s \int_\tau^T \sigma_{ps}(y)\exp(-\alpha(T-y))dy. \tag{21}$$

Proof: See Appendix A.

According to Proposition 1 when the load estimate and the corresponding forward price are positively correlated, i.e., when $\omega \in W_+$ is optimal to over hedge. In other words, the flexible producer should purchase more forward contracts than indicated by the load estimate. Thus, extra forwards compensate increase in the portfolio variance caused by positive correlation. On the other hand, negative correlation decreases the total variance, i.e., when $\omega \in W_-$, the optimal amount is less than the load estimate. That is, negative correlation can be seen as an additional way to hedge against price changes and, therefore, under hedging is optimal. Naturally, when the load estimate is independent of the forward price, i.e. $\omega \in W_0$, the optimal hedging amount is equal to the load estimate. The expected value and variance of $\pi_2(\tau)$ under the optimal hedge ratio are also calculated in Appendix A and they are given by

$$E[\pi_2(\tau)] = \exp(a_2(\tau)) \tag{22}$$

$$Var(\pi_2(\tau)) = \exp(2a_2(\tau))(\exp(a_1(\tau) + 2a_2(\tau) + a_3(\tau)) - 1) - \frac{(\exp(a_1(\tau) + 2a_2(\tau)) - \exp(a_2(\tau)))^2}{\exp(a_1(\tau)) - 1} \tag{23}$$

where

$$a_3(\tau) = \int_\tau^T (\sigma_p^2(y) + \sigma_{ps}^2(y))dy. \tag{24}$$

The portfolio process before hedging is distributed according to a lognormal distribution, thus we have

$$E[\pi_1(\tau)] = \exp(b_2(\tau)) \tag{25}$$

$$Var(\pi_1(\tau)) = \exp(2b_2(\tau))(\exp(b_1(\tau) + 2b_2(\tau) + b_3(\tau)) - 1), \tag{26}$$

where

$$b_1(\tau) = \int_0^\tau \exp(-2a(T-y))\sigma_s^2 dy = \frac{\sigma_s^2}{2\alpha}(\exp(-2\alpha(T-\tau)) - \exp(2\alpha T)) \tag{27}$$

$$b_2(\tau) = \int_0^\tau \exp(-a(T-y))\sigma_{ps}(y)\sigma_s dy \tag{28}$$

$$b_3(\tau) = \int_0^\tau (\sigma_p^2(y) + \sigma_{ps}^2(y))dy. \tag{29}$$

Following corollary gives the portfolio variance.

COROLLARY 1. *The portfolio variance when optimal hedge ratio is used is given by*

$$Var(\pi(\tau)) = p^2(0)S^2(0)(E[\pi_1(\tau)]^2 Var(\pi_2(\tau)) + E[\pi_2(\tau)]^2 Var(\pi_1(\tau)) + Var(\pi_1(\tau))Var(\pi_2(\tau))), \quad (30)$$

where the expected values and variances of $\pi_1(\tau)$ *and* $\pi_2(\tau)$ *are given in (22), (23), (25), and (26).*

Proof: See Appendix B.

The optimal hedging time is solved by minimizing the total variance. As the total variance in (30) is independent of the portfolio realization the hedging time can be obtained by deterministic minimization of (30). Thus, solving the stopping time of stochastic analysis reduces into deterministic minimization problem. Note, that the variance function (30) is not generally convex with respect to τ, thus it can have inflection points. The global minimum can still be attained with numerical minimization (for optimization methods see, e.g., Fletcher, 1987) as the optimal hedging time is on a finite time interval $[0,T)$. Equation (30) can be constant on a given time interval, and thus there can be multiple global minima. Thus, in general the hedging time is not unique. In the next section we use the non-uniqueness to solve the representative agent's load process dynamics.

4. Representative agent

Electricity consumers and producers hedge their exposures to future uncertainty by buying and selling electricity forward contracts that are traded in over-the-counter market and in organized exchanges. Usually the counterparties of these forwards are electricity consumers that take long positions and electricity producers that take short positions. In the previous section we calculated optimal hedging times for these counterparties when the load dynamics were known. In reality trading in electricity forward markets takes place all the time and not just at one specific time. Thus, each moment is optimal hedging time for some agents. In this section we compute load dynamics under which all hedging times are equally good and in this sense we call this dynamics an equilibrium load process. Further, an agent with such load dynamics will be called a representative agent. In other words, if we model the market participants with one representative agent and if we assume that all the time at least one of the market participants executes the static hedging, we can solve for the representative agent's load pattern. After solving the load pattern the representative agent can be used to analyze the characteristics of the market participants. Note, that we do not argue that representative agents exist in reality.

Load dynamics of a non-flexible agent ($\omega \in W_0$) are described by one parameter $\sigma_p(.)$, whereas the load dynamics of a flexible producer ($\omega \in W_+$) and a flexible consumer ($\omega \in W_-$) described by two parameters $\sigma_p(.)$, and $\sigma_{ps}(.)$. Because of the correlation structure, each of hese groups has their own representative agents with own load volatility structures. It turns out that only the load dynamics of non-flexible representative agent ($r \in W_0$) can be solved analytically. The load processes of flexible producers' ($r^+ \in W_+$) and flexible consumers' ($\bar{r} \in W_-$) representative agents need to be solved numerically. For simplicity, in this paper we study only the non-flexible representative agent. The following proposition gives the representative agent's load process dynamics.

PROPOSITION 2. *The representative agent of non-flexible agents* $r \in W_0$ *has the following load volatility function*

$$\sigma_p^r(\tau) = \sqrt{\frac{\sigma_s^2 \exp(-2\alpha(T-\tau))}{\exp(a_1(\tau))-1}} \quad \text{for all } \tau \in [0,T). \tag{31}$$

Proof: The non-flexible representative agent has $\sigma_{ps}^r(.)=0$ and, therefore, the portfolio variance (30) simplifies into

$$Var(\pi(\tau)) = p^2(0)S^2(0)(\exp(b_3(\tau)+b_1(\tau))(1+\exp(a_3(\tau)+ \\ +a_1(t))-\exp(a_1(\tau)))-1). \tag{32}$$

All hedging times are equally good for the representative agent when the variance (32) is constant for all $\tau \in [0,T)$. Thus, the load dynamics are obtained by solving $\sigma_p^r(\tau)$ from

$$\frac{\partial Var(\pi(\tau))}{\partial \tau} = 0 \quad \text{for all} \quad \tau \in [0,T). \tag{32}$$

This gives (31). *Q.E.D*

Proposition 2 states that the non-flexible representative agent's load volatility is time dependant. Moreover, the volatility increases as the hedging time approaches maturity. This implies that there are different non-flexible agents in the market, i.e., they have different load processes. Non-flexible agents with lower load volatility hedge before non-flexible agents with higher load volatility. Agents with high load volatility postpone their hedging decision to get better load estimates. Note, that already in a market with only non-flexible agents the hedging takes place all the time, and that for all positive and bounded forward parameters $\sigma_S(T)$ and α there exist a non-flexible representative agent.

5. Example

In this section we illustrate our model with an example. The example consists of three parts. First, we give our forward market parameter estimates and compute the representative agent's load dynamics. Secondly, we calculate optimal hedging strategies for typical market participants. Finally, we analyze the sensitivity of the hedging strategies.

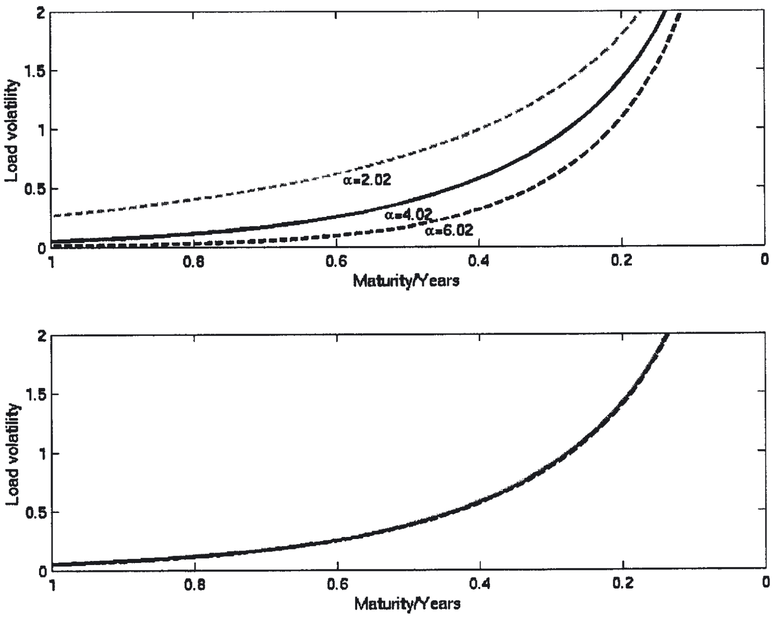
Table 1: Forward market parameters

Parameter	$\sigma_s(1)$	α	T
Value	0.5	4.02	1

Parameters describing the electricity forward dynamics ($\sigma_s(T)$ and α) can be estimated from the electricity forward markets. Audet et al. (2003) estimate these parameters from weekly contracts and we use their parameter estimates. We assume that the maturity of the electricity portfolio is one year. Parameters describing the forward market are summarized in Table 1. These parameters imply that the spot price volatility is 0.5 and the corresponding volatility of a one-month forward is 0.36. The non-flexible representative agent's load volatility dynamics are illustrated in Figure 1. In the upper picture of Figure 1 the volatility is calculated for different values of parameter α. The black solid

Figure 1

Representative agents' load dynamics: upper picture with different volatility discount factors and lower picture with different spot volatilities



line is calculated for α = 4.02, the gray dashed line for α = 2.02, and the black dashed line for α = 6.02. In the upper picture the load volatility curve decreases as the parameter α is increased. Thus, if the mean reversion in the spot price process (i.e. parameter α) increases the load volatility of the representative agent decreases. In the lower picture of Figure 1 the load volatility of the non-flexible representative agent is calculated for different spot volatility values $\sigma_S(1) = \{0.1; 0.5; 0.9\}$. The load volatility curves in the lower picture are almost identical, thus changes in the representative agent's load dynamics due to the spot volatility changes are minor. The form of the curves in Figure 1 illustrates that the non-flexible agents with low uncertainty in the load process tend to hedge their portfolios before the agents with a high volatility.

Next we calculate optimal hedging strategies for typical market participants. We study all three different cases: a flexible producer $\omega_+ \in W_+$, a flexible consumer $\omega_- \in W_-$, and a non flexible agent $\omega_0 \in W_0$. In all these cases the load volatility, $\sqrt{(\sigma_p^\omega(.,1))^2 + (\sigma_{ps}^\omega(.,1))^2}$, is constant and equal to 0.1. Thus, if the load estimate is 1 *GWh* the standard deviation of a one month load estimate is about 29 *MWh* and the standard deviation of a one year load estimate is 100 *MWh.*

Flexible producer $\omega_+ \in W_+$ has a positive correlation between forward prices and load estimate. In this case we assume that the correlation is 0.5. Table 2 summarizes the load process parameters and the optimal hedging strategy. The volatility parameters are calculated from (7) and the optimal hedging time is obtained by minimizing (30). In this case the hedging time is 0, thus it is optimal to hedge the portfolio immediately. As the optimal hedge ratio is 1.21 it is optimal to over hedge the portfolio. The optimal hedge ratio is received from Proposition 1.

Table 2: Flexible producer's load process parameters and the optimal hedging strategy

Parameter	$\rho^{\omega_+}(.,1)$	$\sigma_p^{\omega_+}(.,1)$	$\sigma_{pS}^{\omega_+}(.,1)$	τ^{ω_+}	η^{ω_+}
Value	0.5	0.866	0.05	0	1.21

The consumption of flexible consumer $\omega_- \in W_-$ correlates negatively with electricity prices. In the following example we assume that the correlation is -0.5. The load process parameters and the hedging strategy are summarized in Table 3. The optimal hedging time is 0.57, thus it is optimal to hedge 5 months before the maturity. The optimal hedge ratio is 0.82, i.e., in this case it is optimal to under hedge the portfolio.

Table 3: Flexible consumer's load process parameters and the optimal hedging strategy

Parameter	$\rho^{\omega_-}(.,1)$	$\sigma_p^{\omega_-}(.,1)$	$\sigma_{pS}^{\omega_-}(.,1)$	τ^{ω_-}	η^{ω_-}
Value	-0.5	0.866	-0.05	0.57	0.82

The load estimate of non-flexible agent $\omega_0 \in W_0$ does not correlate with the forward price. In this case the optimal hedging time is 0.17, thus it is optimal to hedge the portfolio 10 months before the maturity. According to Proposition 1 the non-flexible agent's optimal hedge ratio is always one. The parameters and the hedging strategy are summarized in Table 4.

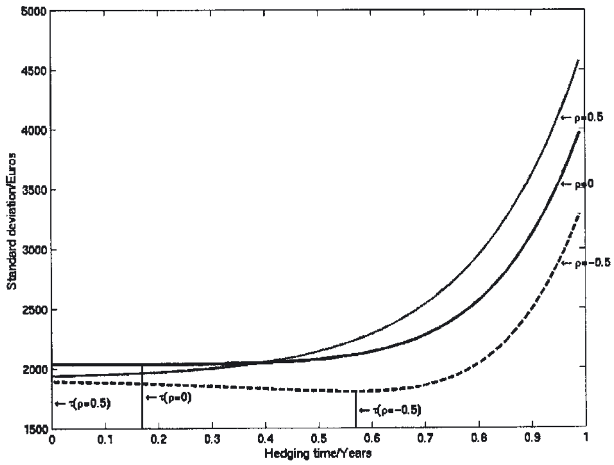
Table 4: Non-flexible agent's load process parameters and the optimal hedging strategy

Parameter	$\rho^{\omega_0}(.,1)$	$\sigma_p^{\omega_0}(.,1)$	$\sigma_{pS}^{\omega_0}(.,1)$	τ^{ω_0}	η^{ω_0}
Value	0	0.1	0	0.17	1

In Figure 2 the standard deviation of the portfolio process is given as a function of the hedging time for all three example cases. We assume that the forward price $S(0,1)$ = 20 €/*MWh* and the load estimate $p^{\omega}(0,1)$ = 1 *GWh*. In Figure 2 the flexible producer is

Figure 2

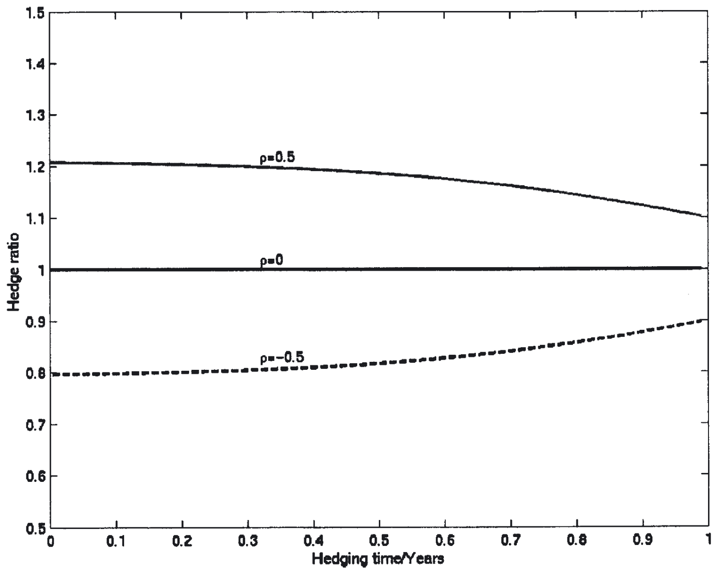
Standard deviations as a function of hedging time for the typical market participants



the gray line, while the dashed black line is the flexible consumer and the black line is the non-flexible agent. The standard deviation curves in Figure 2 have the same load volatility, but the correlations are different. In Figure 2 the smaller the correlation the later the standard deviation attains its minimum. Thus, positive correlation makes early hedging more favorable while negative correlation postpones the hedging decision. Negative correlation can be seen as an additional way to hedge against price changes. Note, that changes in correlation change also the minimum value of the standard deviation. For example, in Figure 2 the portfolio with negative correlation can be hedged most effectively.

In Figure 3 the hedge ratios are presented for all three example cases. Again flexible producer $\omega_+ \in W_+$ is the gray line, the dashed black line is the flexible consumer $\omega_- \in W_-$, and the black line is the non-flexible agent $\omega_0 \in W_0$. Figure 3 indicates that when the correlation is positive it is optimal to over hedge, and when the correlation is negative it is optimal to under hedge. Further, when the correlation is zero the optimal hedging amount is equal to the load estimate. The optimal hedge ratios get closer to one as the maturity decreases.

Figure 3
Optimal hedge ratios for the typical market participants



The three planes in Figure 4 illustrate the optimal hedge ratio as a function of time and correlation for different load volatilities, $\sigma_L = \sqrt{(\sigma_p^\omega(.,1))^2 + (\sigma_{ps}^\omega(.,1))^2} = \{0.05; 0.1; 0.15\}$. In Figure 4 the optimal hedge ratio is equal to one for all volatilities and maturities when the correlation is zero. When the correlation is positive the largest hedge ratio is for the largest load volatility, 0.15. On the other hand, when the correlation is negative the largest hedge ratio is for the smallest load volatility, 0.05. Thus, as the load volatility decreases the optimal hedge ratio gets closer to one for all correlations. In Figure 4 the optimal hedge ratio changes linearly as a function of the correlation, but the rate of change decreases as the maturity decreases.

Figure 4
Hedge ratio as a function of time and correlation

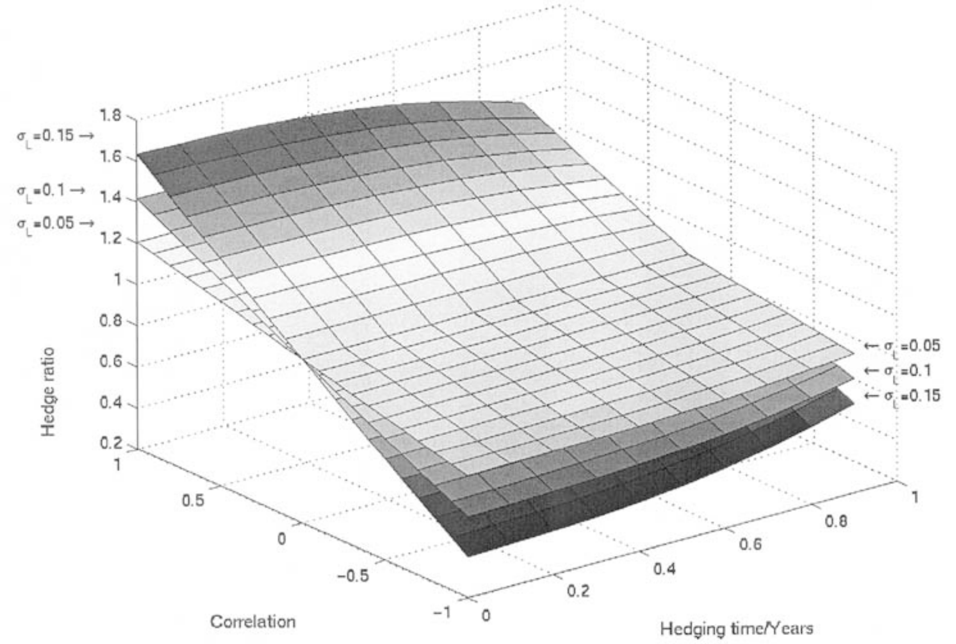
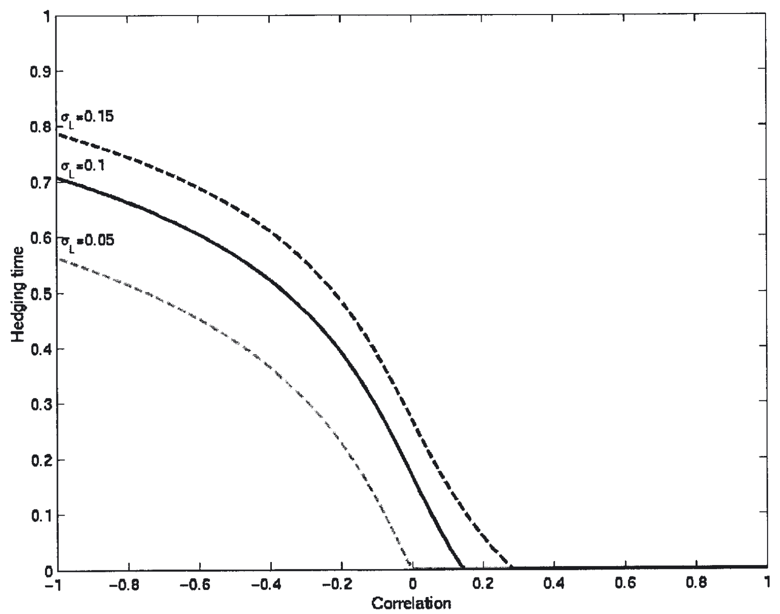


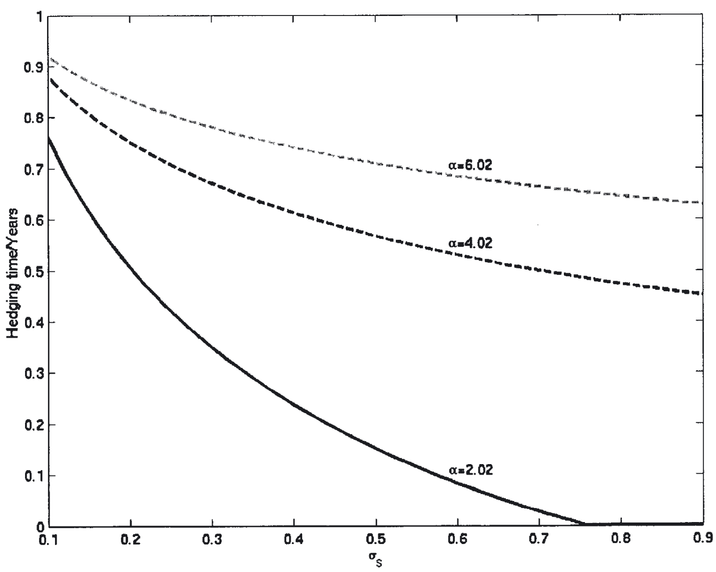
Figure 5
Hedging time as a function of correlation



In Figure 5 the optimal hedging time is presented as a function of correlation for different load volatilities $\sigma_L = \sqrt{(\sigma_p^{\omega}(.,1))^2 + (\sigma_{ps}^{\omega}(.,1))^2}$. The black line is for 0.1, the dashed gray line is for 0.05, and the dashed black line is for 0.15. In Figure 5 when the load volatility is 0.1 it is optimal to hedge immediately if the correlation is over 0.14. As the correlation decreases below 0.14 the hedging is postponed. When the load estimate and the forward price have perfect negative correlation it is optimal to hedge about 3.5 months before the maturity. Correspondingly, if the load estimate and the forward price do not correlate it is optimal to hedge after about 2 months (0.18 years). Also Figure 5 indicates that increase in the load volatility postpones the hedging. In Figure 5 the hedging time is more sensitive to changes in the correlation when the correlation is close to zero.

In Figure 6 the optimal hedging time as a function of spot volatility for the sensitive consumer (described in Table 3) is illustrated. Naturally, similar calculations can be done for agents with different load dynamics. In Figure 6 three different values for the parameter α are used, the black solid line is for α = 2.02, the black dashed line for α = 4.02, and the gray dashed line for α = 6.02. In Figure 6 earlier hedging becomes more favorable as the spot volatility increases, also decrease in the mean reversion of the spot price process (i.e. parameter α) makes earlier hedging more favorable. Thus, increase in the forward variance pre-empts hedging as the uncertainty in the forward price can be eliminated by buying the forward contracts.

Figure 6
Hedging time as a function of spot volatility for a sensitive consumer



Let us summarize our numerical results. It is optimal to over (under) hedge an electricity portfolio when the load estimate and the forward prices are positively (negatively) correlated. Moreover, the optimal hedge ratio gets closer to one as the maturity or load volatility decreases. Positive correlation between load estimate and forward prices has preemptive effect on the hedging time, whereas negative correlation can be seen as an additional way to hedge against price changes. Increase in the load volatility postpones the hedging decision since in this case it is optimal to wait for better load estimate. On the other hand, increase in the forward uncertainty preempts hedging as the forward uncertainty is hedged by using the forward contracts. In Table 5 the sensitivity of the hedge ratio and the hedging time to agent specific parameters are summarized. The first row follows from Figure 4 and the second row from Figure 5.

Table 5: Sensitivity to agent specific parameters

	$\sqrt{(\sigma_p^\omega(\cdot,1))^2+(\sigma_{ps}^\omega(\cdot,1))^2}$ *increases*	$\rho^\omega(t,T)$ *increases*
$\eta^\omega(\tau)$	Increase if $\rho^\omega(t,T)>0$ Decreases if $\rho^\omega(t,T)<0$	Increases
τ^ω	Postpones	Pre-empts

Conclusions

We have studied partial hedging of electricity cash flows with static forward strategies. Our model can be used by electricity consumers and producers. We assumed that the risk adjusted expected value of the portfolio is maximized when the portfolio variance is minimized. Thus, the hedging amount and timing were solved by minimizing the portfolio variance. We also computed the non-flexible representative agent's load dynamics. Under this load dynamics all hedging times are equally good, i.e., all possible hedging times are optimal. The representative agent was used to analyze the behavior of the non-flexible agents in the market. The sensitivity analyses in the example section illustrated our model's sensitivity to parameter changes.

Our results indicate that the agents with high load uncertainty postpone their hedging decision in order to get better load estimates. A load estimate correlating positively with forward prices should be over hedged, while a load estimate with negative correlation should be under hedged. Moreover, positive correlation has a pre-emptive effect on the hedging timing and agents with negative correlation postpone their hedging decision as negative correlation is seen as an additional hedging instrument.

Our hedging strategy is based on the decreasing forward volatility structure, which is widely used in electricity markets. Other commodity markets (e.g. oil market, see e.g. Pilipović, 1998) and the term structure for interest rates (see, e.g., Chapter 21 in Hull, 1997) have similar decreasing volatility characteristics. Thus, our model can possibly be applied also to other commodity and interest rate markets.

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Appendix A

The optimal hedge ratio is computed by minimizing the variance of the portfolio process after the hedging. The portfolio process after the hedging is given in (18). For convenience we restate the equation

$$\pi_2(\tau) = \exp\left(-1/2\int_\tau^T (\sigma_p^2(y) + \sigma_{ps}^2(y) + e^{-2a(T-y)}\sigma_s^2)dy + \int_\tau^T \sigma_p(y)dB_p(y) + \int_\tau^T (\sigma_{ps}(y) + e^{-a(T-y)}\sigma_s)dB_s(y)\right). - \eta(\tau)\exp\left(-1/2\int_\tau^T e^{-2a(T-y)}\sigma_s^2 dy + \int_\tau^T e^{-a(T-y)}\sigma_s dB_s(y)\right) + \eta(\tau) \quad \text{(A1)}$$

The random variable $\pi_2(\tau)$ is a sum of two lognormally distributed variables

$$A = \exp\left(-1/2\int_\tau^T (\sigma_p^2(y) + \sigma_{ps}^2(y) + e^{-2a(T-y)}\sigma_s^2)dy + \int_\tau^T \sigma_p(y)dB_p(y) + \int_\tau^T (\sigma_{ps}(y) + e^{-a(T-y)}\sigma_s)dB_s(y)\right) \quad \text{(A2)}$$

$$B = -\eta(\tau)\exp\left(-1/2\int_\tau^T e^{-2a(T-y)}\sigma_s^2 dy + \int_\tau^T e^{-a(T-y)}\sigma_s dB_s(y)\right), \quad \text{(A3)}$$

and a constant $\eta(\tau)$. The variance of $\pi_2(\tau)$ is

$$Var(\pi_2(\tau)) = Var(A) + Var(B) + 2Cov(A,B). \quad \text{(A4)}$$

Lognormal distribution gives

$$Var(A) = \exp(2a_2(\tau))(\exp(a_3(\tau) + a_1(\tau) + 2a_2(\tau)) - 1) \quad \text{(A5)}$$

$$Var(B) = \eta(\tau)^2(\exp(a_1(\tau)) - 1) \quad \text{(A6)}$$

$$Cov(A,B) = E[AB] - E[A]E[B] = \eta(\tau)(-\exp(a_1(\tau) + 2a_2(\tau)) + \exp(a_2(\tau))), \quad \text{(A7)}$$

where

$$a_1(\tau) = \int_\tau^T \exp(-2\alpha(T-y))\sigma_s^2 dy = \frac{\sigma_s^2}{2\alpha}(1 - \exp(-2\alpha(T-\tau))) \quad \text{(A8)}$$

$$a_2(\tau) = \sigma_s\int_\tau^T \sigma_{ps}(y)\exp(-\alpha(T-y))dy \quad \text{(A9)}$$

$$a_3(\tau) = \int_{\tau}^{T} (\sigma_p^2(y) + \sigma_{ps}^2(y))dy. \quad \text{(A10)}$$

Note that (A4) is a convex function of the hedge ratio and therefore the optimal hedge ratio is obtained from the first order condition as follows

$$\frac{\partial Var(\pi_2(\tau))}{\partial \eta(\tau)} = 0, \quad \text{(A11)}$$

which gives

$$\eta(t) = \frac{\exp(a_2(\tau))(\exp(a_1(\tau) + a_2(\tau)) - 1)}{\exp(a_1(\tau)) - 1} \quad \text{(A12)}$$

When optimal hedge ratio is used the expected value and variance after the hedging are

$$E[\pi_2(\tau)] = \exp(a_2(\tau)) \quad \text{(A13)}$$

$$Var(\pi_2(\tau)) = \exp(2a_2(\tau))(\exp(a_3(\tau) + a_1(\tau) + 2a_2(\tau)) - 1)$$

$$- \frac{(\exp(a_1(\tau) + 2a_2(\tau)) - \exp(a_2(\tau)))^2}{\exp(a_1(\tau)) - 1} \ . \quad \text{(A14)}$$

Appendix B

The portfolio variance is

$$Var(\pi(\tau)) = p^2(0)S^2(0)Var(\pi_1(\tau))\pi_2(\tau)), \tag{B1}$$

where $\pi_1(\tau)$ and $\pi_2(\tau)$ are independent random variables. The well-known equality

$$Var(X) = E[X^2] - E[X]^2 \tag{B2}$$

gives

$$Var(\pi_1(\tau)\pi_2(\tau)) = E[\pi_1^2(\tau)\pi_2^2(\tau)] - E[\pi_1(\tau)\pi_2(\tau)]^2 = E[\pi_1^2(\tau)]E[\pi_2^2(\tau)] \\ -E[\pi_1(\tau)\pi_2(\tau)]^2, \tag{B3}$$

where the second equality follows from the independence of $\pi_1(\tau)$ and $\pi_2(\tau)$. By using (B2) for $\pi_2(\tau)$ we get

$$Var(\pi_1(\tau)\pi_2(\tau)) = E[\pi_1(\tau)]^2(E[\pi_2^2(\tau)] - E[\pi_2(\tau)]^2) + E[\pi_2^2(\tau)]Var(\pi_1(\tau)) \tag{B4}$$

and by using (B2) for π_2 we get

$$Var(\pi_1(\tau)\pi_2(\tau)) = E[\pi_1(\tau)]^2(Var(\pi_2(\tau)) + (Var(\pi_2(\tau)) + E[\pi_2(\tau)]^2) \\ Var(\pi_1(\tau)). \tag{B5}$$

Thus, the total variance is

$$Var(\pi(\tau)) = p^2(0)S^2(0)(E[\pi_1(\tau)]^2 Var(\pi_2(\tau)) + E[\pi_2(\tau)]^2 Var(\pi_1(\tau)) \\ +Var(\pi_1(\tau))Var(\pi_2(\tau))). \tag{B6}$$