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PROJECT VALUATION UNDER AMBIGUITY

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Abstract: This paper examines the valuation of risky projects in a setting where the investor's probability estimates for future states of nature are ambiguous and where he or she can invest both in a portfolio of projects and securities in financial markets. This setting is relevant to investors who allocate resources to industrial research and development projects or make venture capital investments whose success probabilities may be hard to estimate. Here, we employ the Choquet-Expected Utility (CEU) model to capture the ambiguity in probability estimates and the investor's attitude towards ambiguity. Projects are valued using breakeven selling and buying prices, which are obtained by solving several mixed asset portfolio selection (MAPS) models. Specifically, we formulate the MAPS model for CEU investors, and show that a project's breakeven prices for (i) investors exhibiting constant absolute risk aversion and for (ii) investors using Wald's maximin criterion can be obtained by solving two MAPS problems. We also show that breakeven prices are consistent with options pricing analysis when the investor is a non-expected utility maximizer. The valuation procedure is demonstrated through numerical experiments.

Keywords: project valuation, capital budgeting, portfolio selection, ambiguity, mathematical programming, decision analysis

1 Introduction

Recently, Gustafsson et al. (2004) proposed a framework for valuing risky projects in a setting where the investor can invest in a portfolio of private projects and market-traded securities. This framework assumes that the investor knows – by subjective estimation, quantitative analysis, or otherwise – the probability distribution of returns for each asset. Yet, in practice, it can be difficult to obtain well-founded probability estimates for real-world events, such as for the success of a research and development (R&D) project or the increase in the stock price of a newly listed company.

Several approaches for dealing with incomplete information about probabilities (or, ambiguity) have been developed over the past few decades. Prominent approaches include (i) Wald's (1950) maximin analysis, which can be applied when probabilities are unknown, (ii) Dempster-Shafer theory of evidence (Dempster 1967, Shafer 1976), where beliefs can be assigned to sets of states, (iii) second-order probabilities (Marschak 1975, Howard 1992), where a (second-order) probability distribution is associated with each probability estimate, and (iv) Choquet-Expected Utility (CEU) theory, where the expectation is taken with respect to a nonadditive probability measure (Schmeidler 1982, 1989; Gilboa 1987). Among these, the CEU approach has attracted substantial attention among researchers and found several applications in economic analysis, not least due to its robust theoretic foundation (see, e.g., Camerer and Weber 1992, Sarin and Wakker 1992, Wakker and Tversky 1993, Salo and Weber 1995).

In this paper, we examine the valuation of risky projects when the investor's probability estimates for future states of nature are ambiguous. This setting is relevant to investors who allocate resources to industrial R&D projects or make venture capital investments in new technologies. We use the CEU approach to capture the ambiguity of probability estimates as well as the investor's attitude towards ambiguity. We also assume that the investor is consistent with first-degree stochastic dominance, which is widely accepted as a plausible requirement of rational decision making. In this case, the general CEU model reduces to a more amenable form, which is effectively identical to the Rank Dependent Expected Utility (RDEU) model (Quiggin 1982, 1993; see Wakker 1990).

The CEU/RDEU model, hereafter referred to as the CEU model for short, has several intuitively appealing implications. For each portfolio, an ambiguity averse investor adjusts the initial probability estimates, if such are given, so that the probabilities of the states in which the portfolio performs poorly are adjusted upwards, while the probabilities of the states with high outcomes are adjusted downwards. Thus, each portfolio is evaluated by using a dedicated set of state probabilities, obtained by adjusting each state's initial probability estimate according to the investment portfolio's performance in that state.

The main contributions of this paper are (i) the explicit formulation of a mixed asset portfolio selection (MAPS) model for CEU investors and (ii) the development of a computational framework for applying this model in project valuation. We consider three special cases of the general CEU model: (i) the quadratic ambiguity aversion model, which leads to linear probability distortion, (ii) the exponential ambiguity aversion model, and (iii) Wald's (1950) maximin model, which can be obtained as a limit case of the exponential ambiguity aversion model. We demonstrate these models through numerical experiments and contrast the results to expected utility models with exponential utility functions.

For a practitioner, this paper offers (a) a model for selecting an investment portfolio that is robust with respect to probability estimation errors and (b) a framework for that uses this model in project valuation. The framework can be particularly beneficial in the valuation of high technology investments in large corporations, because in these firms resources are often diversified across financial instruments and several technology projects, whereby the related probabilities entail ambiguities.

The remainder of this paper is structured as follows. Section 2 reviews CEU theory and its link to RDEU theory. Section 3 presents the MAPS models based on CEU theory. Section 4 describes how projects can be valued in a MAPS setting when the investor is a CEU maximizer. Section 5 uses the framework in a series of numerical experiments. Section 6 summarizes our findings and discusses their managerial implications.

2 Choquet Expected Utility

Since the publication of the *Ellsberg paradox* (Ellsberg 1961), which questioned the empirical validity of Savage's (1954) subjective expected utility (SEU) theory, several ways of dealing with incomplete information about probabilities have been developed. The CEU approach is based on the hypothesis that the investor's response to ambiguity can be captured through nonadditive probabilities. An analogous idea of probability distortion was utilized already in prospect theory, where Kahneman and Tversky (1979) introduced a probability distortion function (called the weighting function) expressing the investor's chance attitude. While prospect theory was motivated by empirical evidence about human behavior under risk, several axiomatic models featuring distorted probabilities, or *decision weights*, were developed later on. These approaches include weighted utility theory by Chew and MacCrimmon (1979), rank dependent utility theory by Quiggin (1982, 1993), subjectively weighted linear utility of Hazen (1987), dual theory of Yaari (1987), and cumulative prospect theory axiomatized by Wakker and Tversky (1993), among others.

The key difference between the early decision weight models and the CEU theory is that most of the earlier models deal with decision making under risk (see Starmer 2000), where probabilities are either subjectively or objectively known. In contrast, the CEU approach is concerned with decision making under uncertainty, where probabilities are not precisely known (see Camerer and Weber 1992). The approach is largely based on the works of Schmeidler (1982, 1989) and Gilboa (1987) who show that a nonadditive capacity measure ("nonadditive probability measure"; Choquet 1955) represents the investor's preferences when certain axioms hold. Further axiomatizations of nonadditive probabilities can be found in Wakker (1989) and Sarin and Wakker (1992).

A capacity measure is a function $c: 2^{\Omega} \to [0,1]$ which satisfies $c(\varnothing) = 0$, $c(\Omega) = 1$, and $A \subset B \Rightarrow c(A) \le c(B)$, where Ω is the set of states of nature and 2^{Ω} is the power set of Ω. If c is also additive, i.e. $c(A \cup B) = c(A) + c(B)$ for all disjoint A and B, then it is also a probability measure. For an act $X: \Omega \to \mathbb{R}$, the Choquet-integral with respect to c, called the Choquet-expectation of X, is

$$E_c[X] = \int_{-\infty}^{0} \left(c(\{X > x\}) - 1 \right) dx + \int_{0}^{\infty} c(\{X > x\}) \, dx. \tag{1}$$

Although a capacity measure is not, in general, linked to a probability measure, it can be shown (Wakker 1990) that if the investor is consistent with first-degree stochastic dominance with respect to probability measure P, capacity is of the form $c(\{X > x\}) = \varphi\left(P(\{X > x\})\right) = \varphi\left(G_X(x)\right)$, where $G_X(x) = 1 - F_X(x)$ is the decumulative distribution function of X and φ is nondecreasing with $\varphi(0) = 0$ and $\varphi(1) = 1$. For example, such a probability measure may reflect the investor's best but ambiguous probability estimate or, in the case of total uncertainty, the maximum entropy (uniform) distribution. In the following, we assume that such a probability measure exists, and that φ is differentiable.

Using the transformation function and integrating by parts, the Choquet-expection in (1) can be expressed as the Lebesgue-Stieltjes integral

$$
\begin{aligned}
E_c[X] &= \int_{-\infty}^{0} \left(\varphi\left(G_X(x)\right) - 1\right) dx + \int_{0}^{\infty} \varphi\left(G_X(x)\right) dx \\
&= \Big|_{-\infty}^{0} x\left(\varphi\left(G_X(x)\right) - 1\right) - \int_{-\infty}^{0} x\varphi'\left(G_X(x)\right) dG_X(x) + \\
&\quad \Big|_{0}^{\infty} x\varphi\left(G_X(x)\right) - \int_{0}^{\infty} x\varphi'\left(G_X(x)\right) dG_X(x) \\
&= -\int_{-\infty}^{\infty} x\varphi'\left(G_X(x)\right) dG_X(x) = \int_{-\infty}^{\infty} x\varphi'(1 - F_X(x)) dF_X(x)
\end{aligned}
$$

This shows that the Choquet-expectation with respect to c is effectively the same as the expectation with regard to P where probabilities are distorted by the factor $\varphi'(1 - F_X(x))$. Note that when $\varphi(x)$ is increasing and convex, $\varphi'(1 - x)$ is nonnegative and decreasing, expressing ambiguity aversion. Thus, the Choquet-expected utility can be written as

$$
CEU[X] = E_c\left[u(X)\right] = \int_{-\infty}^{\infty} s\varphi'(1 - F_{u(X)}(s)) dF_{u(X)}(s).
$$

With a strictly increasing u, we obtain $F_{u(X)}(s) = P\left(\left\{u(X) \le s\right\}\right) = P\left(\left\{X \le u^{-1}(s)\right\}\right)$ $= F_X(u^{-1}(s))$. A change of variables $s \to u(x)$ gives

$$
CEU[X] = E_c\left[u(X)\right] = \int_{-\infty}^{\infty} u(x)\varphi'(1 - F_X(x)) dF_X(x). \tag{2}
$$

This is essentially the same preference model as in the RDEU theory (Quiggin 1982, 1993; see especially Wakker 1990). Note that since RDEU applies to decision making under risk, the transformation function φ can alternatively be interpreted to be an additional component of the investor's risk preferences, rather than to describe aversion to ambiguous probabilities. However, regardless of the interpretation of φ, a convex φ always influences the investor's behavior by increasing the probabilities of states with low utility outcomes and decreasing those with high utility outcomes.

3 Mixed Asset Portfolio Models

Following Gustafsson et al. (2004), we consider a two-period setting with discrete states. Let there be n securities, m projects, and l states of nature at time 1. The price of security $i = 0, \ldots, n$ at time 0 is denoted by S_i^0 and the respective random price at time 1 by $\tilde{S}_i^1$. The outcome of $\tilde{S}_i^1$ in state $\omega = 1, \ldots, l$ is $S_i^1(\omega)$. The 0-th asset is the risk-free asset with $S_0^0 = 1$ and $S_0^1(\omega) \equiv 1 + r_f$, where r_f is the risk-free interest rate. The amounts of securities in the investor's portfolio are indicated by continuous decision variables $x_i, i = 0, \ldots, n$. The investment cost of project $k = 1, \ldots, m$ at time 0 is C_k^0, and the respective random cash flow at time 1 is $\tilde{C}_k^1$. The outcome of $\tilde{C}_k^1$ in state $\omega = 1, \ldots, l$ is $C_k^1(\omega)$. Binary variable z_k indicates whether project k is started or not.

An investor's preferences over risky mixed asset portfolios can be captured with a preference functional U, which gives the utility of each risky portfolio. For example, under expected utility theory, the investor's preference functional is $U[X] = E[u(X)]$, where u is the investor's von Neumann-Morgenstern (1947) utility function. A MAPS model using a preference functional U can thus be formulated as follows:

$$\max_{\mathbf{x},\mathbf{z}} U\left[\sum_{i=0}^{n} \tilde{S}_i^1 x_i + \sum_{k=1}^{m} \tilde{C}_k^1 z_k\right] \tag{3}$$

subject to

$$\sum_{i=0}^{n} S_i^0 x_i + \sum_{k=1}^{m} C_k^0 z_k = B \tag{4}$$

$$z_k \in \{0,1\} \quad k = 1,\ldots,m \tag{5}$$

$$x_i \text{ free} \quad i = 0,\ldots,n\,, \tag{6}$$

where B is the budget. The expression inside U in (3) is the investor's terminal wealth level. The budget constraint (4) is written as equality, because we know that in the presence of a risk-free asset a rational investor always spends her entire budget in the optimum.

3.1 Binary Variable Model

In general, the MAPS model for the maximization of CEU can be expressed by adapting (2) as the preference functional in (3). Let $F_{\mathbf{x},\mathbf{z}}$ be the cumulative distribution function for the terminal wealth level of mixed asset portfolio $(\mathbf{x},\mathbf{z})$, i.e., for the random variable $W_{\mathbf{x},\mathbf{z}} = \sum_{i=0}^{n} \tilde{S}_i^1 x_i + \sum_{k=1}^{m} \tilde{C}_k^1 z_k$. Letting $W_{\mathbf{x},\mathbf{z}}(\omega) = \sum_{i=0}^{n} S_i^1(\omega) x_i + \sum_{k=1}^{m} C_k^1(\omega) z_k$ denote the investor's terminal wealth level in state ω, the objective function maximizing CEU can be written as

$$\max_{\mathbf{x},\mathbf{z}} \sum_{\omega=1}^{l} p_\omega \varphi'\left(1 - F_{\mathbf{x},\mathbf{z}}\left(\sum_{i=0}^{n} S_i^1(\omega) x_i + \sum_{k=1}^{m} C_k^1(\omega) z_k\right)\right) u\left(\sum_{i=0}^{n} S_i^1(\omega) x_i + \sum_{k=1}^{m} C_k^1(\omega) z_k\right). \tag{7}$$

A challenge in this formulation is to calculate the values for the cumulative distribution function, because the function $F_{\mathbf{x},\mathbf{z}}$ depends on the portfolio $(\mathbf{x},\mathbf{z})$. For the given target level t, we can calculate $F_{\mathbf{x},\mathbf{z}}(t) = \sum_{\omega=1}^{l} p_\omega \xi_\omega$ for each portfolio $(\mathbf{x},\mathbf{z})$ using the following linear constraints:

$$t + \varepsilon - \sum_{i=0}^{n} S_i^1(\omega) x_i - \sum_{k=1}^{m} C_k^1(\omega) z_k \le M \xi_\omega \quad \omega = 1,\ldots,l\,, \tag{8}$$

$$\sum_{i=0}^{n} S_i^1(\omega) x_i + \sum_{k=1}^{m} C_k^1(\omega) z_k - t - \varepsilon \le M(1 - \xi_\omega) \quad \omega = 1,\ldots,l\,, \text{ and} \tag{9}$$

$$\xi_\omega \in \{0,1\} \quad \omega = 1,\ldots,l\,, \tag{10}$$

where M is some large number and ε some small positive number. The variable ξ_ω indicates whether the value of the portfolio is less than or equal to t in state ω; (8) ensures that ξ_ω is equal to 1 when the terminal wealth level is less than $t+\varepsilon$; (9) ascertains that ξ_ω equals 0 when the terminal wealth level is greater than $t+\varepsilon$. Note that the use of a non-zero ε is necessary, because (8) and (9) do not define the value of ξ_ω at $t+\varepsilon$.

In the CEU model, we need to calculate the values of $F_{\mathbf{x},\mathbf{y}}$ for each state, using the state-specific portfolio outcome $W_{\mathbf{x},\mathbf{z}}(\omega) = \sum_{i=0}^{n} S_i^1(\omega)x_i + \sum_{k=1}^{m} C_k^1(\omega)z_k$ as the target value. Extending our earlier notation, let $\xi_{\omega'}^{\omega}$, $\omega' = 1,\ldots,l$, be a binary variable indicating whether the portfolio outcome in state ω' is less than or equal to the target level $W_{\mathbf{x},\mathbf{z}}(\omega)$. The associated constraints can now be expressed with (8)–(10) by replacing t with the appropriate target value.

With the help of binary variables $\xi_{\omega'}^{\omega}$, the CEU model can be written as the following mixed integer non-linear programming (MINLP) model:

$$\max_{\mathbf{x},\mathbf{z},\boldsymbol{\xi}} \sum_{\omega=1}^{l} p_\omega \varphi'\left(1 - \sum_{\omega'=1}^{l} p_{\omega'}\xi_{\omega'}^{\omega}\right) u\left(\sum_{i=0}^{n} S_i^1(\omega)x_i + \sum_{k=1}^{m} C_k^1(\omega)z_k\right) \tag{11}$$

subject to

$$\sum_{i=0}^{n} S_i^0 x_i + \sum_{k=1}^{m} C_k^0 z_k = B \tag{12}$$

$$\sum_{i=0}^{n} S_i^1(\omega)x_i + \sum_{k=1}^{m} C_k^1(\omega)z_k - \sum_{i=0}^{n} S_i^1(\omega')x_i - \sum_{k=1}^{m} C_k^1(\omega')z_k + \varepsilon \le M\xi_{\omega'}^{\omega} \qquad \omega = 1,\ldots,l,\ \omega' = 1,\ldots,l \tag{13}$$

$$\sum_{i=0}^{n} S_i^1(\omega')x_i + \sum_{k=1}^{m} C_k^1(\omega')z_k - \sum_{i=0}^{n} S_i^1(\omega)x_i - \sum_{k=1}^{m} C_k^1(\omega)z_k - \varepsilon \le M(1 - \xi_{\omega'}^{\omega}) \qquad \omega = 1,\ldots,l,\ \omega' = 1,\ldots,l \tag{14}$$

$$z_k \in \{0,1\} \quad k = 1,\ldots,m \tag{15}$$

$$\xi_{\omega}^{\omega'} \in \{0,1\} \quad \omega = 1,\ldots,l,\ \omega' = 1,\ldots,l, \tag{16}$$

$$x_i \text{ free} \quad i = 0,\ldots,n \tag{17}$$

Equivalently, the objective function can be expressed in a form similar to the certainty equivalent formula in expected utility theory. This form has some useful properties, which are discussed later in this paper. This “certainty equivalent” form is given by

$$\max_{\mathbf{x},\mathbf{z},\boldsymbol{\xi}} u^{-1}\left(\sum_{\omega=1}^{l} p_\omega \varphi'\left(1-\sum_{\omega'=1}^{l} p_{\omega'}\xi_{\omega'}^{\omega}\right) u\left(\sum_{i=0}^{n} S_i^1(\omega)x_i + \sum_{k=1}^{m} C_k^1(\omega)z_k\right)\right).$$

The model (11)–(17) has n continuous decision variables, $m+l^2$ binary variables, and $1+2l^2$ linear constraints. Such a model can usually be solved in a reasonable time for models with less than 30 states.

3.2 Rank-Constrained Model

Even when the MINLP model (11)–(17) has a reasonable number of binary variables, it can be relatively hard to solve numerically due to the complex relationships defining the variables $\xi_{\omega'}^{\omega}$ and the normalization problems implied by the large values with M. Our experiments with the GAMS software package (see www.gams.com) indicate that standard MINLP algorithms do not necessarily find the optimal solution for models of this type. Therefore, we propose an alternative method for solving the CEU MAPS model. The method relies on solving a separate MINLP model for each possible rank order of states (i.e., ranking is based on the outcome of the portfolio in each state). In each model, portfolios must satisfy the given rank-order. Consequently, distorted probabilities (decision weights) in each model are constant and can be used as parameters in the optimization problem. The decision weight of state ω is $p_\omega^* = p_\omega \varphi'\left(1-F_{\mathbf{x},\mathbf{z}}\left(W_{\mathbf{x},\mathbf{z}}(\omega)\right)\right)$, where $F_{\mathbf{x},\mathbf{z}}\left(W_{\mathbf{x},\mathbf{z}}(\omega)\right)$ is the sum of the probabilities of the states where the portfolio outcome is less than or equal to the portfolio outcome in state ω, $W_{\mathbf{x},\mathbf{z}}(\omega) = \sum_{i=0}^{n} S_i^1(\omega)x_i + \sum_{k=1}^{m} C_k^1(\omega)z_k$.

Assuming that ω_h denotes the state with h:th rank, the rank-constrained model can be formulated as

$$\max_{\mathbf{x},\mathbf{z}} \sum_{\omega=1}^{l} p_\omega^* u\left(\sum_{i=0}^{n} S_i^1(\omega)x_i + \sum_{k=1}^{m} C_k^1(\omega)z_k\right) \tag{18}$$

subject to

$$\sum_{i=0}^{n} S_i^0 x_i + \sum_{k=1}^{m} C_k^0 z_k = B \tag{19}$$

$$\sum_{i=0}^{n} S_i^1(\omega_{h+1})x_i + \sum_{k=1}^{m} C_k^1(\omega_{h+1})z_k - \sum_{i=0}^{n} S_i^1(\omega_h)x_i - \sum_{k=1}^{m} C_k^1(\omega_h)z_k \le 0 \quad h=1,\ldots,l-1 \tag{20}$$

$$z_k \in \{0,1\} \quad k=1,\ldots,m \tag{21}$$

$$x_i \text{ free} \quad i=0,\ldots,n\,, \tag{22}$$

where $p_\omega^* = p_\omega \varphi'\left(1-F_{\mathbf{x},\mathbf{z}}\left(W_{\mathbf{x},\mathbf{z}}(\omega)\right)\right)$. This model is solved for each possible rank-order of states, which leads to a total of $l!$ models. The rank-order with the highest optimum gives the optimal solution to the problem (11)–(17).

An advantage of this approach is that its complexity does not depend on the transformation function φ. However, because the calculation of the optimal solution requires the solution of $l!$ rank-constrained models, this approach is computationally attractive only for problems with some 10 states or less.

3.3 Specific CEU Models

In this section, we consider three interesting special cases of the general CEU model, (i) the quadratic ambiguity aversion model, (ii) the exponential ambiguity aversion model, and (iii) Wald's (1950) maximin model. The pricing behavior of these models is studied in Sections 4 and 5.

3.3.1 Quadratic Ambiguity Aversion

An investor with quadratic ambiguity aversion is characterized by a convex quadratic transformation function $\varphi(x) = ax^2 + bx + c$, $a > 0$. The normalization $\varphi(0) = 0$, $\varphi(1) = 1$ imply that $c = 0$ and $b = 1 - a$, and hence $\varphi(x) = ax^2 + (1-a)x$ and $\varphi'(x) = 2ax + 1 - a$. To ensure that φ is nondecreasing over the interval $[0,1]$, the constant a must be less than or equal to 1. This also ensures that distorted probabilities (decision weights) will be nonnegative.

The appeal of the quadratic transformation function comes from the linearity of the probability distortion, which potentially facilitates the solution of the resulting models. Figure 1 illustrates the decision weights obtained by applying a quadratic transformation function in a setting with six equally likely states of nature. For the sake of comparison, the decision weights are normalized so that they to sum up to 1.

Together with a linear utility function $u(x) = x$, a quadratic transformation function leads to a mixed integer quadratic programming model, where the objective function is

$$\max_{\mathbf{x},\mathbf{z}} \sum_{\omega=1}^{l} p_\omega \left[1 + a\left(1 - 2 \cdot F_{\mathbf{x},\mathbf{y}}\left(\sum_{i=0}^{n} S_i^1(\omega)x_i + \sum_{k=1}^{m} C_k^1(\omega)z_k \right)\right)\right]\left(\sum_{i=0}^{n} S_i^1(\omega)x_i + \sum_{k=1}^{m} C_k^1(\omega)z_k \right).$$

This model is interesting, because (i) it can be computationally appealing and (ii) it falls within the scope of Yaari's (1987) dual theory, expressing the properties observed under this theory, such as linearity in the payments.

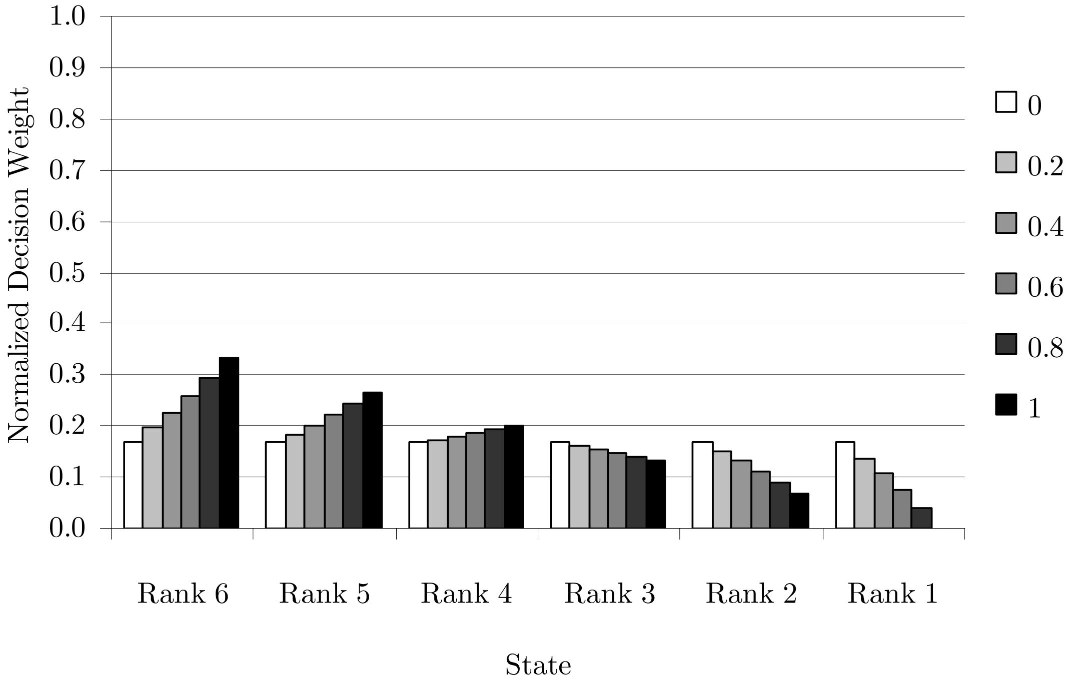

Figure 1. Quadratic ambiguity aversion: Normalized decision weights for different levels of a when there are 6 equally likely states. Rank i indicates the i:th best state.

3.3.2 Exponential Ambiguity Aversion

When the investor's transformation function is $\varphi(x) = a + be^{\gamma x}, \gamma > 0$, the investor is said to express exponential ambiguity aversion. The condition $\varphi(0) = 0$, $\varphi(1) = 1$ imply that $a = -b = 1/(1-e^{\gamma})$ so that $\varphi(x) = (1-e^{\gamma x})/(1-e^{\gamma})$ and $\varphi'(x) = \beta e^{\gamma x}/(e^{\gamma}-1)$. Figure 2 illustrates the decision weights implied by applying exponential ambiguity aversion in a setting with six equally likely states of nature. Again, the decision weights are normalized to sum up to one.

In numerical experiments (Section 5), we also consider investors who exhibit both constant absolute risk aversion and exponential ambiguity aversion. In this case, the investor's preferences are captured by a concave exponential utility function $u(x) = -e^{-\alpha x}$, $\alpha > 0$, and a convex exponential transformation function. This implies a CEU objective function

$$\max_{\mathbf{x},\mathbf{z}} \sum_{\omega=1}^{l} \left(p_\omega \frac{\gamma}{1-e^{-\gamma}} e^{-\gamma F_{\mathbf{x},\mathbf{y}}\left(\sum_{i=0}^{n} S_i^1(\omega)x_i + \sum_{k=1}^{m} C_k^1(\omega)z_k\right)} \right) \left(-e^{-\alpha\left(\sum_{i=0}^{n} S_i^1(\omega)x_i + \sum_{k=1}^{m} C_k^1(\omega)z_k\right)} \right). \tag{23}$$

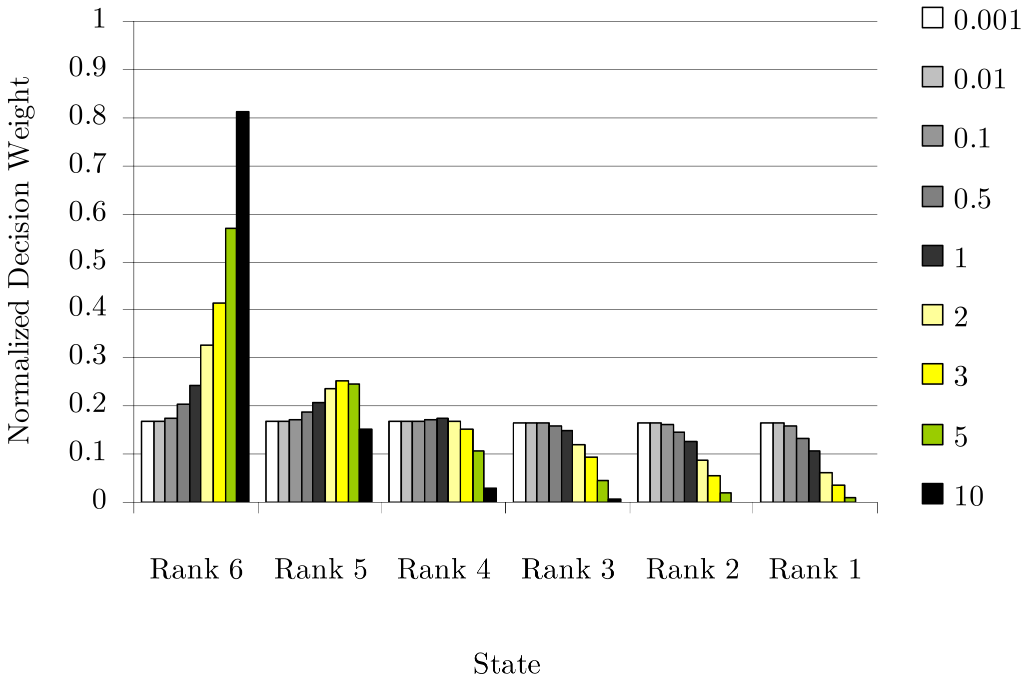

Figure 2. Exponential ambiguity aversion: Normalized decision weights at different γ-levels when there are 6 equally likely states. Rank i indicates the i:th best state.

3.3.3 Wald's Maximin Criterion

With Wald's (1950) maximin criterion, the investor maximizes the utility of the least preferred state. Wald's (1950) model is a special case of the general CEU model, as it is obtained with a capacity measure that gives 0 to all proper subsets of Ω; it is also an extreme case of the exponential CEU model when γ goes to infinity.

A MAPS model using the maximin criterion can be formulated as the following mixed integer linear programming (MILP) problem:

$$\max_{\mathbf{x},\mathbf{z},\pi} \pi \tag{24}$$

subject to

$$\sum_{i=0}^{n} S_i^0 x_i + \sum_{k=1}^{m} C_k^0 z_k = B \tag{25}$$

$$\pi - \sum_{i=0}^{n} S_i^1(\omega) x_i - \sum_{k=1}^{m} C_k^1(\omega) z_k \le 0 \qquad \omega = 1,\ldots,l \tag{26}$$

$$z_k \in \{0,1\} \qquad k = 1,\ldots,m \tag{27}$$

$$x_i \text{ free} \qquad i = 0,\ldots,n \tag{28}$$

$$\pi \text{ free} \tag{29}$$

In contrast to nonlinear CEU models, this model does not require additional binary variables or the solution of several MINLP models. Hence, maximin models can be solved relatively rapidly even with a large number of states.

4 Valuing Projects: CEU Investors

In a MAPS setting, an individual project can be valued by comparing the values of the optimal portfolios when the investor does and does not invest in the project. Formally, the comparison is carried out according to the concepts of *breakeven selling and buying prices* (De Reyck, Degraeve and Gustafsson 2003, Luenberger 1998, Smith and Nau 1995). Specifically, breakeven selling price is the lowest price at which a rational investor would be willing to sell the project if he or she had it, while the breakeven buying price is the highest price at which the investor would buy the project if he or she did not have it. Hence, when determining the breakeven selling price, we assume that the investor invests in the project at the beginning (referred to as the *status quo*) and compare the result to the situation where he or she does not have the project (referred to as the *second setting*). The breakeven selling price is the budget increment which makes the portfolio in the second setting equally preferred to the portfolio in the status quo. Similarly, when calculating the breakeven buying price, we assume that the investor does not have the project in the status quo and that he or she has it in the second setting. The breakeven buying price is the budget reduction that makes the portfolio with the project equally desirable to the portfolio without the project.

Optimization problems for calculating breakeven selling and buying price of project j are given in Table 1. For CEU maximizers, we can use the explicit portfolio optimization model (11)–(17) or (18)–(22), adding the relevant constraint on project j to the model in each setting (i.e., $z_j = 1$ or $z_j = 0$). The breakeven prices are optimized iteratively by changing the parameter v_j^s or v_j^b, as appropriate, until the optimal values of the portfolio selection problems in the status quo and in the second setting are identical. In general, the breakeven selling price and the breakeven buying price are not equal.

When the investor can borrow funds beyond any limit and no risk constraint is imposed on the portfolio, the optimization problems in Table 1 may yield unbounded solutions; if so, the breakeven prices are undefined. For example, this may happen when the utility function of a CEU maximizer is linear. For example, the investors abiding by Yaari's (1987) dual theory are CEU maximizers with a linear utility function. There are also other classes of CEU models which may lead to unbounded solutions, such as models using logarithmic utility functions.

Indeed, if limitless borrowing is possible, it may be necessary to use a risk or borrowing constraint with several types of preference models in order to avoid unbounded solutions.

Table 1. Definitions of the value of project j. For CEU investors U is as in Section 3.

	Breakeven selling price	**Breakeven buying price**
Definition	v_j^s such that $W^- = W^+$	v_j^b such that $W^- = W^+$
Status quo	$W^+ = \max_{\mathbf{x},\mathbf{z}} U\left[\sum_{i=0}^{n} \tilde{S}_i^1 x_i + \sum_{k=1}^{m} \tilde{C}_k^1 z_k\right]$ subject to $\sum_{i=0}^{n} S_i^0 x_i + \sum_{k=1}^{m} C_k^0 z_k = B$; $z_j = 1$; $z_k \in \{0,1\} \quad k = 1,\ldots,m$; x_i free $\quad i = 0,\ldots,n$	$W^- = \max_{\mathbf{x},\mathbf{z}} U\left[\sum_{i=0}^{n} \tilde{S}_i^1 x_i + \sum_{k=1}^{m} \tilde{C}_k^1 z_k\right]$ subject to $\sum_{i=0}^{n} S_i^0 x_i + \sum_{k=1}^{m} C_k^0 z_k = B$; $z_j = 0$; $z_k \in \{0,1\} \quad k = 1,\ldots,m$; x_i free $\quad i = 0,\ldots,n$
Second setting	*Project sold* $W^- = \max_{\mathbf{x},\mathbf{z}} U\left[\sum_{i=0}^{n} \tilde{S}_i^1 x_i + \sum_{k=1}^{m} \tilde{C}_k^1 z_k\right]$ subject to $\sum_{i=0}^{n} S_i^0 x_i + \sum_{k=1}^{m} C_k^0 z_k = B + v_j^s$; $z_j = 0$; $z_k \in \{0,1\} \quad k = 1,\ldots,m$; x_i free $\quad i = 0,\ldots,n$	*Project bought* $W^+ = \max_{\mathbf{x},\mathbf{z}} U\left[\sum_{i=0}^{n} \tilde{S}_i^1 x_i + \sum_{k=1}^{m} \tilde{C}_k^1 z_k\right]$ subject to $\sum_{i=0}^{n} S_i^0 x_i + \sum_{k=1}^{m} C_k^0 z_k = B - v_j^b$; $z_j = 1$; $z_k \in \{0,1\} \quad k = 1,\ldots,m$; x_i free $\quad i = 0,\ldots,n$

On the other hand, portfolio optimization problems typically remain bounded when a CEU investor has a concave exponential utility function. Moreover, for such an investor, there exists a pricing formula analogous to the one introduced in Gustafsson et al. (2004) for mean-risk investors, provided that the preference functional is expressed in expected utility theory's "certainty equivalent" form. The proof of the following proposition is in Appendix.

PROPOSITION 1. *Let the investor be a CEU maximizer with* $u(x) = -e^{-\alpha x}$, $\alpha > 0$, *and the preference functional be* $U[X] = u^{-1}\left(E_c\left[u(X)\right]\right)$. *Then, breakeven selling and buying prices are identical and can be computed using the formula* $v = \dfrac{V^+ - V^-}{1 + r_f}$, *where* V^+ *and* V^- *are the optimal preference functional values with and without the project, respectively.*

It can also be shown that, in arbitrage-free markets, an investor who applies Wald's (1950) maximin criterion never invests in securities, if he or she does not invest in projects (Proposition 2). However, if he or she does invest in projects, it may be optimal to invest in securities as well. The breakeven selling and buying prices for a maximin investor can be computed by calculating the value of the portfolio when the investor does and does not invest in the project and by discounting the difference to its present value at the risk-free interest rate (Proposition 3). The difference from analogous formulas in Gustafsson et al. (2004) is that the value of the portfolio is here defined as the value obtained in the worst possible state, as given by decision variable π in (24). The proofs of the propositions are given in Appendix.

PROPOSITION 2. *In arbitrage-free markets, if a maximin investor does not invest in projects, she does not invest in securities, unless the resulting portfolio replicates the risk-free asset. If a maximin investor invests in projects, she may also invest in a risky security portfolio.*

PROPOSITION 3. *If the investor abides by Wald's maximin criterion, breakeven selling and buying prices for any project are identical and can be computed by the formula* $v = \dfrac{V^+ - V^-}{1 + r_f}$*, where* V^+ *and* V^- *are the values of the portfolio in the worst state (given by decision variable* π*) with and without the project, respectively.*

Breakeven selling and buying prices for a CEU investor (including a maximin investor) exhibit an important consistency property. Namely, a project's breakeven selling and buying price are consistent with options pricing analysis, or contingent claims analysis (CCA), which is used in the theory of finance (see, e.g., Brealey and Myers 2000, Hull 1999, Luenberger 1998). An analogous property was proven by Smith and Nau (1995) for expected utility maximizers; the following proposition generalizes their result to non-expected utility maximizers. The consistency property holds in the sense that CCA and the breakeven prices yield the same result whenever CCA is applicable, i.e. when there is a replicating portfolio for the project. The CCA price of project j is given by

$$v_j^{CCA} = -C_j^0 + \sum_{i=0}^{n} x_i^* S_i^0 ,$$

where x_i^* is the amount of security i in the replicating portfolio. Due to this property, breakeven buying and selling prices can be regarded as generalizations of CCA for incomplete markets. The proof of the proposition is in Appendix.

Proposition 4. *If there is a replicating portfolio for a project, breakeven selling and buying prices for the project are identical and equal to the value given by options pricing analysis.*

5 Numerical Experiments

In this section, we examine CEU-based project valuation through numerical experiments. We first offer comparative results with expected utility and expected value maximizers and then analyze investors with varying degrees of ambiguity aversion. In particular, we consider the following research questions:

Q1. When an expected utility maximizer becomes less risk averse, do the project values converge to the values given by an expected value maximizer?

Q2. To which values do project values converge with increasing risk aversion?

Q3. When a CEU maximizer becomes less ambiguity averse, do the project values converge to the values given by the respective expected utility maximizer?

Q4. To which values do project values converge with increasing ambiguity aversion?

5.1 Experimental Setup

The experimental setup consists of 6 states of nature, 4 projects (Table 3), and 2 securities (Table 2) which constitute the market. Initial state probabilities are generated by assuming maximum entropy, implying a probability of 1/6 for each state. The security prices are obtained from the Capital Asset Pricing Model (CAPM; Sharpe 1964, Lintner 1965; see also Luenberger 1998), where the expected rate of return of the market portfolio has been set to 13.62%, so that the price of security 2 is $20. This price is used by Smith and Nau (1995), Trigeorgis (1996), and Gustafsson et al. (2004) for this security. The standard deviation of the market portfolio is then 19.68%. Betas are calculated with respect to the resulting market portfolio. The market prices in Table 3 are the prices that the CAPM would give to the projects if they were traded in financial markets, assuming infinite divisibility and a negligible market capitalization. Italicized text indicates a computed value. The risk-free interest rate is 8%.

Table 2. Securities.

	Security	
	1	**2**
Shares issued	15,000,000	10,000,000
State 1	$60	$36
State 2	$50	$36
State 3	$40	$36
State 4	$60	$12
State 5	$50	$12
State 6	$40	$12
Beta	*0.66*	*2.13*
Market price	*$44.75*	*$20.00*
Capitalization weight	*77.05%*	*22.95%*
Expected rate of return	*11.72%*	*20.00%*
St. dev. of rate of return	*18.24%*	*60.00%*

Table 3. Projects.

	Project			
	A	**B**	**C**	**D**
Investment cost	$80	$100	$104	$40
State 1	$150	$140	$180	$20
State 2	$50	$140	$180	$40
State 3	$150	$150	$180	$40
State 4	$150	$110	$60	$20
State 5	$50	$170	$60	$60
State 6	$150	$100	$60	$60
Beta	*0.000*	*0.240*	*2.13*	*-1.679*
Market price	*$28.02*	*$23.46*	*-$4.00*	*$0.58*
Expected outcome	*$116.67*	*$135*	*$120*	*$40*
St. dev. of outcome	*$47.14*	*$23.63*	*$60*	*$16.33*

We consider investors who exhibit (i) constant absolute risk aversion and (ii) exponential, quadratic, or maximin ambiguity aversion. By Propositions 2 and 4, breakeven selling and buying prices are identical, and hence both prices are displayed in a single entry. Unless otherwise indicated, we assume that the investor's budget is $500. Optimizations were carried out with the GAMS software package and the rank-constrained formulation in (18)–(22), except for the maximin model, for which the model (24)–(29) was employed.

5.2 Expected Utility, Expected Value, and Maximin Investors

Before considering the impacts of ambiguity aversion, we first examine three cases without ambiguity aversion, namely, (i) expected utility (EU) maximizers, (ii) expected value (EV) maximizers (risk-neutral investors), and (iii) maximin investors. Table 4 gives the project values for an expected utility maximizer as a function of the risk-aversion coefficient α , as well as the optimal mixed asset portfolios at each level of risk aversion. At each level, the investor starts projects A, B, and D. Weights of securities 1 and 2 in the security portfolio are denoted by w_1 and w_2 , respectively. The amount of funds invested in the risk-free asset is given in the column "x_0".

Table 4. Breakeven prices and optimal portfolios for an EU maximizer.

	Project value				Optimal mixed asset portfolio				
α	A	B	C	D	Expectation	St.dev.	w_1	w_2	x_0
0.000001	$28.67	$23.06	-$4.00	$0.64	$83,218.7	$289,177.2	76.99%	23.01%	-$1,468,382.6
0.00001	$28.67	$23.07	-$4.00	$0.64	$8,858.8	$28,919.6	77.01%	22.99%	-$146,659.7
0.0001	$28.68	$23.10	-$4.00	$0.65	$1,422.8	$2,894.1	77.14%	22.86%	-$14,487.5
0.001	$28.72	$23.40	-$4.00	$0.79	$679.3	$293.5	78.38%	21.62%	-$1,271.5
0.005	$28.65	$24.78	-$4.00	$1.44	$613.7	$69.8	82.08%	17.92%	-$102.1
0.010	$27.77	$26.54	-$4.00	$2.28	$606.1	$47.7	83.89%	16.11%	$37.7
0.015	$26.77	$28.26	-$4.00	$3.09	$603.9	$42.5	83.96%	16.04%	$80.0
0.020	$25.74	$29.85	-$4.00	$3.81	$603.1	$40.8	83.18%	16.82%	$98.4
0.030	$23.90	$32.42	-$4.00	$4.96	$602.6	$40.1	80.79%	19.21%	$114.0
0.040	$22.50	$34.02	-$4.00	$5.77	$602.6	$40.4	78.43%	21.57%	$121.1

In Table 4, we observe that when α approaches zero, the values of projects A–D converge to $28.67, $23.06, -$4.00, and $0.64, which are close to the projects' CAPM market prices (Table 3). This is surprising, because one would expect that the project values would converge to the values given by an EV maximizer, i.e., to $17.22, $12.50, –$4.00, and –$6.67. These values are obtained by discounting the project cash flows at a 20% discount rate, the expected rate of return of the security with the highest expected return (security 2). Also, the financial portfolio converges close (but not exactly) to the CAPM market portfolio, containing 76.99% of security 1 and 23.01% of security 2 when $\alpha = 10^{-6}$. A risk-neutral portfolio would contain 100% of security 2. Even though the amount of funds borrowed and the monetary worth of the mixed asset portfolio approach infinity with decreasing risk aversion, project values and the weights of the investor's financial portfolio converge to specific values. Our experiments also show that, when the investor cannot invest in projects, he or she invests in a security portfolio with weights (76.99%, 23.01%) at all levels of risk aversion. Taken together, these observations indicate that the pricing behavior of EU maximizers with constant absolute risk aversion is similar to mean-variance investors whose project values converge to CAPM market prices with increasing risk tolerance, and who always invest in the CAPM market portfolio when there are no projects (De Reyck, Degraeve and Gustafsson 2003).

Because the optimal security portfolio for the EU investor is near to the CAPM market portfolio, the expected rate of return of the mixed asset portfolio converges to a value that is close to the excess rate of return of the CAPM market portfolio. Likewise, the volatility of the mixed asset portfolio approaches a value near to the volatility of the market portfolio. Notice that volatility does not decrease monotonically with increasing risk aversion. This is because standard deviation does not exactly measure the risk perceived by an EU maximizer. Hence, it may happen that, means being equal, a portfolio with more standard deviation is preferred to the portfolio with less standard deviation.

On the other hand, it can be shown that, when the risk aversion parameter α goes to infinity, project values converge towards the values for a maximin investor, who prices the projects at \$17.69, \$25.37, -\$4.00, and \$8.15. In the optimum, a maximin investor invests in a security portfolio with weights (76.32%, 23.68%) and lends \$104.3. Table 4 gives project values and mixed asset portfolio statistics up to the α-level of 0.04. At higher α-levels optimization problems become computationally unstable due to the very large negative values in the exponent. Notice that the expected convergence behavior is not clear from Table 4. This suggests that some solutions at α-levels below 0.04 may be distorted by computational issues or convergence to a local optimum, for example.

Between the two extremes, the value of project A decreases with increasing risk aversion. In contrast, the values of projects B and D grow when risk aversion is increased. The behavior with project D can be explained by the negative correlation of project D with the rest of the portfolio; the negative correlation is partly highlighted by the project's negative beta (Table 3). As indicated by the results from Gustafsson et al. (2004), the value of a nearly-zero-correlation project may decrease by growing risk aversion, explaining the convergence behavior of project A. However, the value of project B increases with α, even though the beta of the project is positive. This may be explained by the project's negative correlation with project A (–0.599). Note that the value must start to decrease at some point, because the maximin value of project B is \$25.37, which is less than the price of project B when $\alpha = 0.04$, \$34.02. Hence, the convergence to maximin values need not be monotonic. Project C is priced at a constant –\$4 level, because there exists a replicating portfolio: it is possible to replicate the cash flows of project C by buying 5 shares of security 2. Since the replicating portfolio costs \$100, it implies a CCA price of –\$4 for project C.

In summary, when the risk aversion parameter α approaches zero, the project values given by an EU maximizer with constant absolute risk aversion converge to values near to CAPM market prices, rather than to the values for an EV maximizer [Q1]. Also the optimal financial portfolio converges close to the CAPM market portfolio. With increasing risk aversion project values approach those of a maximin investor [Q2]. The results also indicate that project values may either decrease or increase with growing risk aversion, and that the convergence behavior is not necessarily monotonic. Finally, a replicating portfolio implies a constant pricing behavior.

5.3 Choquet-Expected Utility Maximizers

In this section, we examine investors with exponential and quadratic ambiguity aversion. The risk aversion parameter α is 0.005 in each experiment. Table 5 describes project values and mixed asset portfolio statistics for exponentially ambiguity averse investors as a function of ambiguity aversion parameter γ. At each value of γ, the investor invests in projects A, B, and D. When γ approaches zero, the values converge, as expected, towards project values given by an expected utility maximizer. The prices for such an investor are given on the row where $\gamma = 0$. At the other extreme, when γ approaches infinity, project values converge towards the values given by a maximin investor. These values are described on the row with $\gamma = \infty$.

Table 5. Breakeven prices and optimal portfolios for a CEU investor with exponential ambiguity aversion.

γ	Project value				Optimal mixed asset portfolio				
	A	**B**	**C**	**D**	**Expectation**	**St.dev.**	w_1	w_2	x_0
0	\$28.65	\$24.78	-\$4.00	\$1.44	\$613.7	\$69.8	82.08%	17.92%	-\$102.1
0.1	\$28.51	\$24.96	-\$4.00	\$1.53	\$612.4	\$65.9	82.75%	17.25%	-\$81.1
0.5	\$27.98	\$25.89	-\$4.00	\$1.84	\$607.7	\$52.1	85.12%	14.88%	\$0.0
1	\$27.35	\$27.41	-\$4.00	\$2.14	\$604.2	\$43.4	81.08%	18.92%	\$84.0
2	\$25.25	\$31.05	-\$4.00	\$3.67	\$603.8	\$43.5	76.32%	23.68%	\$104.1
3	\$22.86	\$34.03	-\$4.00	\$4.96	\$603.8	\$43.5	76.32%	23.68%	\$104.1
5	\$19.78	\$29.90	-\$4.00	\$8.05	\$603.8	\$43.5	76.32%	23.68%	\$104.1
∞	\$17.69	\$25.37	-\$4.00	\$8.15	\$603.8	\$43.5	76.32%	23.68%	\$104.1

Similarly to increasing risk aversion, the value of project A decreases and the value of project D increases with increasing ambiguity aversion. This is because project A has a near-zero correlation with the rest of the portfolio, whereas project D has a negative correlation, which reduces the aggregate risk of the portfolio. Also, because it is possible to construct a replicating portfolio, project C has a constant price which is equal to its CCA value. Note also that that the value of project B increases at low ambiguity aversion levels and decreases at higher levels. This suggests that a similar behavior is likely to happen with growing risk aversion, as conjectured in the previous section.

For values of γ higher than and equal to 2, the optimal portfolio is almost identical with a maximin investor's portfolio, even though project values at these γ-levels are different. The reason for this is that, even though the optimal portfolios without restrictions on what projects can be started are the same at each level, optimal portfolios when the investor must or must not start a project (the settings in Table 1) are different, which implies different project values.

Table 6 presents project values and optimal mixed asset portfolios under quadratic ambiguity aversion. As before, the investor invests in projects A, B, and D at all levels of ambiguity aversion considered. Project values converge towards those for an expected utility maximizer when a goes to 0. At the other extreme, however, project values do not anymore reach maximin values, because a is bounded from above by 1. When $a = 1$, the level of ambiguity aversion roughly corresponds to exponential ambiguity aversion when γ is in the range $[1.5, 4]$. Overall, quadratic ambiguity aversion gives lower values to projects A and B and a higher value to project D than exponential ambiguity aversion.

Table 6. Breakeven prices and optimal portfolios for a CEU investor with quadratic ambiguity aversion.

	Project value				Optimal mixed asset portfolio				
a	A	B	C	D	Expectation	St.dev.	w_1	w_2	x_0
0	\$28.65	\$24.78	-\$4.00	\$1.44	\$613.7	\$69.8	82.08%	17.92%	-\$102.1
0.2	\$27.95	\$25.64	-\$4.00	\$1.79	\$608.8	\$54.4	85.15%	14.85%	-\$16.7
0.4	\$26.97	\$26.70	-\$4.00	\$1.99	\$605.6	\$45.9	83.45%	16.54%	\$53.6
0.6	\$25.84	\$28.11	-\$4.00	\$2.57	\$604.4	\$43.3	80.14%	19.86%	\$88.4
0.8	\$24.19	\$29.05	-\$4.00	\$3.31	\$604.1	\$43.5	76.32%	23.68%	\$104.1
1	\$22.03	\$29.33	-\$4.00	\$4.10	\$603.8	\$43.5	76.32%	23.68%	\$104.1

In summary, when ambiguity aversion approaches zero, the project values and the optimal portfolios are approach those of the expected utility maximizer [Q3]. When exponential ambiguity aversion goes to infinity, the project values approach the values given by a maximin investor [Q4]. Under quadratic ambiguity aversion, however, this occurs only to a certain extent, because the ambiguity aversion parameter a is limited by 1 from above. Also, the convergence to the extreme values is not necessarily monotonic.

6 Summary and Conclusions

In this paper, we have examined project valuation in a setting where the investor's probability estimates are ambiguous and where she can construct an investment portfolio that includes both projects and securities. We have used the CEU theory to capture the investor's preferences for portfolios characterized by ambiguity. In this framework, projects are valued using the concepts of breakeven selling and buying prices, which require the solution of MAPS models with and without the analyzed project.

We formulated two alternative MINLP models for solving MAPS problems for CEU investors, (i) the binary variable model, where the values of the portfolio's cumulative distribution function are captured by dedicated binary variables, and (ii) the rank-constrained model, which is solved separately for each possible rank-order of states. Although the former model is suitable for a larger number of states than the latter, our experience with standard MINLP algorithms suggests that models of the former type may be difficult to solve. The latter model is computationally more tractable in this respect.

We showed that a project's breakeven selling and buying prices for (i) CEU maximizers who exhibit constant absolute risk aversion and for (ii) maximin investors can be computed by solving MAPS models with and without the project and by discounting the difference of the obtained portfolio values back to its present value at the risk-free interest rate. This makes it possible to calculate breakeven selling and buying prices from two optimization problems only. We also showed that breakeven selling and buying prices give consistent results with CCA for non-expected utility maximizers whenever the method is applicable, i.e., when a replicating portfolio exists for the project. Therefore, these pricing concepts can be regarded as generalizations of CCA.

In our numerical experiments, we examined investors who exhibit (i) constant absolute risk aversion and (ii) either quadratic, exponential, or maximin ambiguity aversion. The experiments show that when an expected utility maximizer becomes less risk averse, project values approach values that are close to the projects' CAPM prices, rather than the values given by a risk-neutral investor. Hence, it appears that expected utility maximizers with constant absolute risk aversion behave similarly to risk-constrained mean-risk investors. When the investor becomes increasingly averse to risk or ambiguity, project values approach the values given by a maximin investor. Under quadratic ambiguity aversion, project values do not reach maximin values, because the ambiguity aversion parameter is bounded from above.

Our analysis has several managerial implications. The present framework makes it possible to (i) select an investment portfolio that is relatively insensitive to changes in probability estimates and to (ii) calculate defensible values for risky projects whose success probabilities are ambiguous. Our results also indicate that when ambiguity is taken into account, assets that perform well when the rest of the portfolio performs poorly often gain higher values than when ambiguity is not accounted for. Therefore, hedging through relevant derivative securities is particularly useful under ambiguity, because many hedging instruments are costless zero-value contracts that make the portfolio outcomes more uniformly distributed and thereby lead to higher values. Also, diversification possibilities can be expanded by using – in addition to usual market-traded securities – over-the-counter (OTC) instruments, such as commodity and credit derivatives, that many investment banks provide for their corporate clients.

This work suggests several avenues for further research. In particular, efficient MINLP algorithms for solving the binary variable model (11)–(17) are called for. Computational issues could also be relaxed if the CEU model were changed to the weighted utility model, which may be computationally more amenable than the CEU model. Also, since many real-world project valuation settings involve multiple time periods, the present model should be extended to a multi-period setting, for example, in the spirit of Gustafsson and Salo (2005).

References

Brealey, R., S. Myers. 2000. *Principles of Corporate Finance*. McGraw-Hill, New York, NY.

Camerer, C., M. Weber. 1992. Recent Developments in Modeling Preferences: Uncertainty and Ambiguity. *Journal of Risk and Uncertainty* **5** 325–370.

Chew, S. H., K. R. MacCrimmon. 1979. Alpha-Nu Choice Theory: An Axiomatization of Expected Utility. Working Paper #669. University of British Columbia Faculty of Commerce.

Choquet, G. 1955. Theory of Capacities. *Annales de l'Institut Fourier* **5** 131–295.

Dempster, A. P. 1967. Upper and Lower Probabilities Induced by a Multivalued Mapping. *Annals of Mathematical Statistics* **38** 325–339.

Ellsberg, D. 1961. Risk, Ambiguity, and the Savage Axioms. *Quarterly Journal of Economics* 75 643–669.

Gilboa, I. 1987. Expected Utility without Purely Subjective Non-Additive Probabilities. *Journal of Mathematical Economics* **16** 65–88.

Gustafsson, J. De Reyck, B., Z. Degraeve, A. Salo. 2004. Project Valuation in Mixed Asset Portfolio Selection. Working Paper. Helsinki University of Technology and London Business School. Downloadable at http://www.sal.hut.fi/Publications/pdf-files/mgus04a.pdf.

Gustafsson, J., A. Salo. 2005. Contingent Portfolio Programming for the Management of Risky Projects. *Operations Research* **53**(5), forthcoming.

Hazen, G. B. 1987. Subjectively Weighted Linear Utility. *Theory and Decision* **23** 261–282.

Howard, R. A. 1992. The Cogency of Decision Analysis. In W. Edwards (ed.), *Utility: Theories, Measurement, Applications*. Kluwer Academic Publications, Dordrecht, Holland.

Hull, J. C. 1999. *Options, Futures, and Other Derivatives*. Prentice Hall, New York, NY.

Kahneman, D., A. Tversky. 1979. Prospect Theory: An Analysis of Decision under Risk. *Econometrica* **47**(2) 263–291.

Lintner, J. 1965. The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets. *Review of Economics and Statistics* **47**(1) 13–37.

Luenberger, D. G. 1998. *Investment Science*. Oxford University Press, New York, NY.

Marschak, J. 1975. Personal Probabilities of Probabilities. *Theory and Decision* **6** 121–153.

Quiggin, J. 1982. A Theory of Anticipated Utility. *Journal of Economic Behavior and Organization* **3** 323–343.

——— 1993. *Generalized Expected Utility Theory: The Rank-Dependent Model*. Kluwer Academic Publishers, Dordrecht, Holland.

Salo, A. A., M. Weber. 1995. Ambiguity Aversion in First-Price Sealed-Bid Auctions. *Journal of Risk and Uncertainty* **11** 123–137.

Sarin, R., P. Wakker. 1992. Simple Axiomatization of Nonadditive Expected Utility. *Econometrica* **60** 1255–1272.

Savage, L. J. 1954. *The Foundations of Statistics*. John Wiley & Sons, New York.

Schmeidler, D. 1982. Subjective Probability without Additivity. *Proceedings of the American Mathematical Society* **97** 255-261.

——— 1989. Subjective Probability and Expected Utility without Additivity. *Econometrica* **57** 571–587.

Shafer, G. 1976. *A Mathematical Theory of Evidence*. Princeton University Press, Princeton.

Sharpe, W. F. 1964. Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk. *Journal of Finance* **19**(3) 425–442.

Smith, J. E., R. F. Nau. 1995. Valuing Risky Projects: Option Pricing Theory and Decision Analysis. *Management Science* **41**(5) 795–816.

Starmer, C. 2000. Developments in Non-Expected Utility Theory: The Hunt for a Descriptive Theory of Choice under Risk. *Journal of Economic Literature* **38** 332–382.

Trigeorgis, L. 1996. *Real Options: Managerial Flexibility and Strategy in Resource Allocation*. MIT Press, Massachussets.

von Neumann, J., O. Morgenstern. 1947. *Theory of Games and Economic Behavior*, Second edition. Princeton University Press, Princeton.

Wakker P. 1989. *Additive Representation of Preferences: A New Foundation for Decision Analysis*. Kluwer Academic Publishers, Dordrecht, Holland.

——— 1990. Under Stochastic Dominance Choquet-Expected Utility and Anticipated Utility Are Identical. *Theory and Decision* **29** 119–132.

Wakker, P., A. Tversky. 1993. An Axiomatization of Cumulative Prospect Theory. *Journal of Risk and Uncertainty* **7** 147–176.

Wald, A. 1950. *Statistical Decision Functions*. John Wiley & Sons, New York.

Yaari, M. 1987. The Dual Theory of Choice under Risk. *Econometrica* **55** 95–115.

Appendix

PROOF OF PROPOSITION 1: We know that an expected utility maximizer with an exponential utility function satisfies the delta property, i.e. $u^{-1}\left(E\left[u(X+b)\right]\right)=u^{-1}\left(E\left[u(X)\right]\right)+b$. Since Choquet-expectation satisfies $E_c\left[aX\right]=aE_c\left[X\right]$, we also have $U\left[X+b\right]=u^{-1}\left(E_c\left[u(X+b)\right]\right)=u^{-1}\left(E_c\left[u(X)\right]\right)+b=U\left[X\right]+b$. Let us next consider the breakeven buying price. Let μ_{SQ}^{b} be the optimal value of the objective function in the status quo, and $\mu_{SS}^{b}=\mu_{SQ}^{b}+\Delta^{b}$ be the optimal value in the second setting, when the budget is *B*. To obtain an optimal value in the second setting which is equal to μ_{SQ}^{b}, we can lower the budget by δ^{b}, whereby the amount borrowed effectively increases by the same amount, lowering the optimal value by $\delta^{b}(1+r_f)$. When $U\left[X^*\right]=u^{-1}\left(E_c\left[u\left(X^*\right)\right]\right)$ is the optimum to the portfolio selection problem with the budget *B*, $U\left[X^*-\delta^{b}(1+r_f)\right]=u^{-1}\left(E_c\left[u\left(X^*\right)\right]\right)-\delta^{b}(1+r_f)$ is the optimum to the portfolio selection problem with budget $B-\delta^{b}$. By requiring that $\delta^{b}(1+r)=\Delta^{b}$ we obtain $\delta^{b}=\Delta^{b}/(1+r_f)$. By denoting that $\mu_{SQ}^{b}=V^{-}$ and $\mu_{SS}^{b}=V^{+}$, we immediately obtain the desired formula. With breakeven selling price, we have the optimum value μ_{SQ}^{s} in the status quo and $\mu_{SS}^{s}=\mu_{SQ}^{s}-\Delta^{s}$ in the second setting when the budget is *B*. Again, we can increase the budget by δ^{s} to obtain an optimal solution $U\left[X^*+\delta^{s}(1+r_f)\right]=u^{-1}\left(E_c\left[u\left(X^*\right)\right]\right)+\delta^{s}(1+r_f)$. By requiring that $\delta^{s}(1+r_f)=\Delta^{s}$ we obtain $\delta^{s}=\Delta^{s}/(1+r_f)$. By observing that $\mu_{SQ}^{s}=V^{+}$ and $\mu_{SS}^{s}=V^{-}$, we get again the desired formula. Q.E.D.

PROOF OF PROPOSITION 2: In arbitrage-free markets, each risky portfolio composed of securities must have a state in which it yields less than the risk-free interest rate, as otherwise it would be possible, by buying the portfolio and borrowing at the risk-free interest rate, to create an arbitrage opportunity, i.e. a portfolio with a chance of yielding a positive amount of money without an initial capital outlay. Since a maximin investor values each portfolio by its worst state, such an investor prefers the risk-free asset to all risky security portfolios. However, in a combined project-security portfolio it may occur that the rate of return of the combined portfolio exceeds the risk-free interest rate in all states, because projects are not bound by no-arbitrage conditions. For example, consider a setting with two equally likely states, a project yielding \$100 in state 1 and \$0 otherwise, and a security yielding \$10 in state 2 and \$0 otherwise. The cost of the project is \$45 and the security is priced at \$4.5. The risk-free interest rate is 8% and the budget is \$90. The optimum is to start the project and buy 10 shares of the security. Q.E.D.

PROOF OF PROPOSITION 3: Let us first consider the breakeven buying price. Let π_{SQ}^{b} be the optimal value in the status quo and $\pi^b_{SS} = \pi^b_{SQ} + \Delta^b$ the optimal value in the second setting when the budget is B. Because the investor invests in the optimal mixed asset portfolio regardless of the budget (as long as it yields more than the risk-free interest rate in the worst state), lessening the budget by δ^b will reduce x_0 by the same amount, and consequently decrease the optimal value (the value in the worst state) by $\delta^b(1+r_f)$. By requiring that $\delta^b(1+r_f) = \Delta^b$ we obtain, $\delta^b = \Delta^b/(1+r_f)$. By denoting that $\pi^b_{SQ} = V^-$ and $\pi^b_{SS} = V^+$, we obtain the desired formula. Let us then consider the breakeven selling price. Here, we have the optimal value π^b_{SQ} in the status quo and $\pi^s_{SS} = \pi^s_{SQ} - \Delta^s$ in the second setting. As above, we can increase the optimal value in the second setting by $\delta^s(1+r_f)$ by increasing the budget by δ^s. By setting $\delta^s(1+r_f) = \Delta^s$ and denoting $\pi^s_{SQ} = V^+$ and $\pi^s_{SS} = V^-$ we again obtain the desired formula. Q.E.D.

PROOF OF PROPOSITION 4: Let us first consider the breakeven buying price and let j indicate the project being valued. Let $(\mathbf{x}^{\mathbf{SQ}}, \mathbf{z}^{\mathbf{SQ}})$ be the optimal portfolio in the status quo (with $z_j = 0$). Notice that when there is a replicating portfolio for project j and the investor invests in the project, it is possible to construct a shorted replicating portfolio, which nullifies the cash flows of the project at time 1, and which jointly with project j yields money equal to $-C^0_j + \sum_{i=0}^{n} x^*_i S^0_i$ at time 0 (x^* is the amount of security i in the replicating portfolio). Therefore, in the second setting, a portfolio $(\mathbf{x}^{\mathbf{SS}}, \mathbf{z}^{\mathbf{SS}}) = (\mathbf{x}^{\mathbf{SQ}} - \mathbf{x}^*, \mathbf{z}^{\mathbf{SQ}} + \mathbf{e_j})$, where $\mathbf{e_j}$ is an m-dimensional vector with a zero value for each element except for the j:th element which has the value of 1, will yield the same optimal value with budget $B - (-C^0_j + \sum_{i=0}^{n} x^*_i S^0_i)$ as the portfolio $(\mathbf{x}^{\mathbf{SQ}}, \mathbf{z}^{\mathbf{SQ}})$ yields with budget B in the status quo. The portfolio $(\mathbf{x}^{\mathbf{SS}}, \mathbf{z}^{\mathbf{SS}})$ is the optimum to the problem in the second setting, because otherwise $(\mathbf{x}^{\mathbf{SQ}}, \mathbf{z}^{\mathbf{SQ}})$ would not be the solution to the problem in the status quo. Therefore, the breakeven buying price for project j is, by definition, equal to $-C^0_j + \sum_{i=0}^{n} x^*_i S^0_i$, which is also the CCA price of the project. A similar logic proves the theorem for breakeven selling prices. Q.E.D.