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VALUING RISKY PROJECTS WITH CONTINGENT PORTFOLIO PROGRAMMING

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Abstract: This paper examines the valuation of multi-period projects in a setting where (i) the investor maximizes her terminal wealth level, (ii) she can invest in securities and private investment opportunities, and (iii) markets are incomplete, i.e. the cash flows of private investments cannot necessarily be replicated using financial securities. Based on Gustafsson and Salo's (2005) Contingent Portfolio Programming, we develop a multi-period mixed asset portfolio selection model, where project management decisions are captured through project-specific decision trees. This model properly captures the opportunity costs imposed by alternative investment opportunities and determines the appropriate risk-adjustment to the projects based on their effect on the investor's aggregate portfolio risk. The project valuation procedure is based on the concepts of breakeven selling and buying prices, which require the solution of mixed asset portfolio selection models with and without the project being valued. The valuation procedure is demonstrated through numerical experiments.

Keywords: project valuation, mixed asset portfolio selection, contingent portfolio programming

1 Introduction

The valuation of risky multi-period projects is one of the most fundamental topics in corporate finance (Brealey and Myers 2000, Luenberger 1998). Among the numerous methods developed for this purpose, the risk-adjusted net present value (NPV) method (Brealey and Myers 2000, Chapter 2) and the decision tree technique (Raiffa 1968; Brealey and Myers 2000, Chapter 10) are among the most popular ones. These two methods are complementary in the sense that a decision tree can be used to structure the management decisions and uncertainties related to a project, while the risk-adjusted NPV method gives the present value of each risky cash flow stream that can be acquired from the decision tree, provided that the appropriate risk-adjusted discount rate is known.

For projects of market-traded companies, the appropriate discount rate can, in principle, be derived using the Capital Asset Pricing Model (CAPM; Sharpe 1964, Lintner 1965; see Rubinstein 1973). However, there are no direct guidelines for determining the discount rate when the investor is not a public company, e.g., when the investor is an individual, a governmental agency, or a non-listed firm, which do not have a share price to maximize. In a series of recent papers, Gustafsson et al. (2004), De Reyck et al. (2004) and Gustafsson and Salo (2004) have studied the valuation of risky projects in this setting under the assumptions that (i) the investor maximizes her terminal wealth level, (ii) she can invest in securities in financial markets as well as in a portfolio of private investment opportunities, and (iii) markets are incomplete, i.e. the cash flows of private investments cannot necessarily be replicated using financial securities. However, due to the complexity of modeling a portfolio of multi-period projects, the developed models have been limited to two time periods only and hence cannot be applied to the valuation of a multi-period project whose management strategy can be altered over the course of the project's life cycle.

In this paper, we extend the valuation framework presented in Gustafsson et al. (2004) and Gustafsson and Salo (2004) to multiple time periods and intermediate project management decisions. Both the original framework and its multi-period extension are based on a *mixed asset portfolio selection* (MAPS) model – a portfolio model including both projects and securities – and the concepts of *breakeven selling and buying prices*, which compare the values of optimal portfolios including and not including the analyzed project. We employ Gustafsson and Salo's (2005) *Contingent Portfolio Programming* (CPP) to develop a multi-period MAPS model that captures the management decisions and uncertainties related to risky multi-period projects through project-specific decision trees. We extend CPP to include securities and generalize its objective function to several types of preference models, ranging from the expected utility model to various non-expected utility models.

The main contribution of this paper is the development of a generic framework for the valuation of risky multi-period projects in a setting where the investor maximizes her terminal wealth level and where the possible investment assets include both projects and securities. Also, we discuss how the results of the earlier MAPS-based project valuation papers extend to the multi-period setting. In particular, we show that breakeven prices remain consistent with contingent claims analysis when the investor is a non-expected utility maximizer, and that the resulting prices are, under certain conditions, the same as in Hillier's (1963) method. The use of the resulting framework is demonstrated through numerical experiments.

The remainder of this paper is structured as follows. Section 2 reviews earlier approaches to the valuation of multi-period projects and discusses their assumptions and shortcomings. Section 3 gives an overview of the model developed in this paper. Section 4 discusses the modeling of states, securities, and projects in the present approach, and Section 5 gives an explicit formulation of the portfolio selection model. Section 6 describes how the developed portfolio model can be used in project valuation, and Section 7 illustrates the approach with numerical experiments. Finally, Section 8 draws conclusions.

2 Earlier Approaches

The literature on corporate finance contains a large number of apparently rivaling methods for the valuation of risky multi-period projects. In what follows, we briefly review some of the most widely used approaches, namely, (i) decision trees (Hespos and Strassman 1965, Raiffa 1968), (ii) expected utility theory (von Neumann and Morgenstern 1947, Raiffa 1968), (iii) the risk-adjusted NPV method (Brealey and Myers 2000), (iv) real options (Dixit and Pindyck 1994, Trigeorgis 1996), (v) Robichek and Myers' (1966) certainty equivalent method (see also Brealey and Myers 2000, Chapter 9), (vi) Hillier's (1963) method, and (vii) Smith and Nau's (1995) method. These methods are summarized in Table 1.

Table 1. Methods for the valuation of risky multi-period investments.

Method	Use CE	Use PV	Use ST	Formula / explanation
Risk-adjusted NPV		X		$NPV = -I + \sum_{t=1}^{T} \frac{E[c_t]}{(1+r_{adj})^t}$
Decision tree			X	A flow chart with decision and chance nodes
Expected utility theory	X			$CE[X] = u^{-1}\left(E[u(X)]\right)$
Contingent claims analysis		X		$NPV = -I + \sum_{i=0}^{n} S_i x_i^*$
Robichek and Myers (1966)		X		$NPV = -I + \sum_{t=1}^{T} \frac{CE[c_t]}{(1+r_f)^t}$
Hillier (1963)		X		$NPV = -I + CE\left[\sum_{t=1}^{T} \frac{c_t}{(1+r_f)^t}\right]$
Smith and Nau (1995)		X		NPV = breakeven selling or buying price Preference model for cash flow streams: $U[(c_1, c_2, \ldots, c_T)] = E\left[u^*(c_1, c_2, \ldots, c_T)\right]$
Gustafsson et al. (2004) Gustafsson and Salo (2004)		X		NPV = breakeven selling or buying price Preference model for terminal wealth levels

Key: **CE** = Certainty equivalent for a risky alternative, **PV** = Present value of a risky cash flow stream, **ST** = Structuring of decision opportunities and uncertainties, I = investment cost, c_t = risky cash flow at time t, r_{adj} = risk-adjusted discount rate, u = utility function, S_i = price of security i, x_i^* = amount of security i in the replicating portfolio, r_f = risk-free interest rate, u^* = intertemporal (multi-attribute) utility function.

2.1 Decision Trees and Related Approaches

In general terms, a decision tree describes the points at which decisions can be made and the way in which these points are related to unfolding uncertainties. Conventionally, decision trees have been utilized with expected utility theory (von Neumann and Morgenstern 1947) so that each end node of the decision tree is associated with the utility implied by the earlier actions and the uncertainties that have resolved earlier. This decision tree formulation does not explicitly include the time axis or provide guidelines for accounting for the time value of money.

In corporate finance, decision trees are used to describe how project management decisions influence the cash flows of the project (see, e.g., Brealey and Myers 2000, Chapter 10). Here, decision trees are typically applied together with the risk-adjusted NPV method, whereby an explicitly defined time axis is also constructed. However, the selection of an appropriate discount rate for the NPV method is often problematic, mainly because the rate is influenced by three confounding factors, (i) the risk of the project, which depends on the project's correlation with other investments, (ii) the opportunity costs imposed by alternative investment opportunities, and (iii) the investor's risk preferences, which affect the two other factors by determining (a) the degree of risk-adjustment for the project and (b) the optimal alternative investment portfolio when the investor does not invest in the project.

Several methods for selecting the discount rate have been proposed in the literature. However, most of them have problematic limitations. For example, the weighted average cost of capital (WACC) is appropriate only for average-risk investments in a firm, whereas discount rates based on expected utility theory do not account for the opportunity costs imposed by securities in financial markets. The real options literature suggests the use of contingent claims analysis (CCA) to derive the appropriate discount rate by constructing replicating portfolios using market-traded securities. Still, it may be difficult to construct replicating portfolios for private projects in practice. Last, the use of a CAPM discount rate is appropriate only for market-traded companies.

2.2 Robichek and Myers' and Hillier's Methods

Robichek and Myer's (1966) and Hillier's (1963) methods are two alternative ways of determining a risk-adjustment to a discount rate in a multi-period setting. These methods have been widely discussed in the literature on corporate finance (see, e.g., Keeley and Westerfield 1972, Bar-Yosef and Mesznik 1977, Beedles 1978, Fuller and Kim 1980, Chen and Moore 1982, Gallagher and Zumwalt 1991, Ariel 1998, and Brealey and Myers 2000). They both employ expected utility theory or a similar preference model to derive a certainty equivalent (CE) for a risky prospect.

In Robichek and Myers' method, the investor first determines a CE for the cash flow of each period, and then discounts it back to its present value at the risk-free interest rate. Yet, because CEs are taken separately for each cash flow, the method does not account for the effect of cash flows' temporal correlation on the cash flow stream's aggregate risk; hence, it may lead to an unnecessarily large risk-adjustment. For example, consider a cash flow stream that yields a random cash flow X at time 1 and $-X(1+r_f)+a$ at time 2 (a is a constant and r_f is the risk-free interest rate). Assuming that the investor invests the funds obtained at time 1 in the risk-free asset, the investor will acquire cash equal to a for sure at time 2. To avoid arbitrage opportunities, this cash flow stream has to be discounted at the risk-free interest rate. Still, since cash flows at times 1 and 2 are separately risky, Robichek and Myers' method leads to a discount rate that is higher than the risk-free interest rate, which is incorrect. The importance of recognizing intertemporal correlation of cash flows is further discussed in Fuller and Kim (1980).

On the other hand, in Hillier's (1963) method, which is in the context of certainty equivalents also referred to as the single certainty equivalent (SCE) method (Keeley and Westerfield 1972), we first determine the cash flow streams that can be acquired with the project in different scenarios and then calculate the NPVs of these streams using the risk-free interest rate. The result is a probability distribution for risk-free-discounted NPV, for which a CE is then determined. However, the use of the risk-free interest rate essentially means that any money received before the end of the planning horizon is invested in the risk-free asset. Yet, it might be more advantageous to invest the funds in risky securities instead. Therefore, Hillier's method is, strictly speaking, applicable only in settings, where the investor cannot invest in risky securities.

Perhaps the most restrictive assumption used in Hillier's method as well as in Robichek and Myers' method is that the risk-adjustment to the discount rate is determined only using the investor's risk preferences without considering alternative investment opportunities. Yet, in general, the appropriate discount rate depends also on the investment portfolio in which the investor would invest if she did not invest in the project. For example, consider a risk-neutral investor who can invest in one project and in risky securities in financial markets. We know that, in financial markets, such an investor would invest all her funds in the security with the highest expected return. Therefore, the appropriate discount rate for the NPV of the investor's project is equal to the expected rate of return of this security. Yet, both of the methods discussed in this section would give a discount rate equal to the risk-free interest rate for the project. The fact that Robichek and Myers' and Hillier's methods do not adequately consider alternative investment opportunities limits their applicability to a setting where the investor can invest only in the analyzed project and in the risk-free asset.

2.3 Smith and Nau's Method

The idea behind Smith and Nau's (1995) method, which Smith and Nau call "*full decision tree analysis*," is to explicitly account for security trading in each decision node of a decision tree. The main advantage of the approach is that it appropriately accounts for the effect that the possibility to invest in securities has on the discount rate of a risky project. However, the method does not consider alternative projects, which impose an opportunity cost on the project being valued. Also, the method relies on the assumption that the investor maximizes the (intertemporal) utility of the project's cash flows rather than the utility of the investor's terminal wealth level. Therefore, it does not necessarily lead to an investment portfolio with maximal NPV, when NPV is defined as the present equivalent of the investment's future value (Luenberger 1998).

Also, practically appealing forms of the method rely on several restrictive assumptions: in order to develop a useful rollback procedure, Smith and Nau (1995) assume (i) additive independence (Keeney and Raiffa 1976), (ii) constant absolute risk aversion (CARA), and (iii) partial completeness of markets. Yet, as pointed out by Keeney and Raiffa (1976), additive independence entails possibly unrealistic preferential restrictions. The CARA assumption seems also questionable, because it leads to an exponential utility function with utility bounded from above. This is known to result in an unrealistic degree of risk aversion at high levels of outcomes (see, e.g., Rabin 2000). In practice, it may also be difficult to create a replicating portfolio for market-related cash flows of a project, as assumed in partially complete markets.

In view of the limitations of earlier approaches, we develop a valuation method for risky multi-period projects where project management decisions are structured as project-specific decision trees. The method relies on fewer assumptions about the investor's preference structure than Smith and Nau's integrated procedure, allowing the use of a wide range of preference models. We also employ the objective of maximization of the investor's terminal wealth level, which ensures consistency with NPV maximization.

3 Model Overview

In a MAPS model, there are two kinds of assets, projects and securities. Projects produce cash flows according to the chosen project management strategy; cash flows from securities are realized through trading decisions. Similarly to CPP, uncertainties are modeled using a state tree, which depicts the structure of future states of nature (Figure 1). The state tree need not be binomial or symmetric; it may also take the form of a multinomial tree that has different probability distributions in its branches. In each non-terminal state, securities can be bought and sold in any, possibly fractional quantities. As usual in the financial literature, we assume that there are no transaction costs or profits tax.

Using the CPP framework, projects are modeled using decision trees that span over the state tree. Figure 2 describes how project decisions (the figure on the left), when combined with the state tree in Figure 1, lead to project-specific decision trees (the figure on the right). The specific feature of these decision trees is that the chance nodes – since they are generated using the common state tree – are shared by all projects. Security trading is implemented through state-specific trading variables, which are similar to the ones used in financial models of stochastic programming (e.g., Mulvey et al. 2000) and in Smith and Nau's (1995) method.

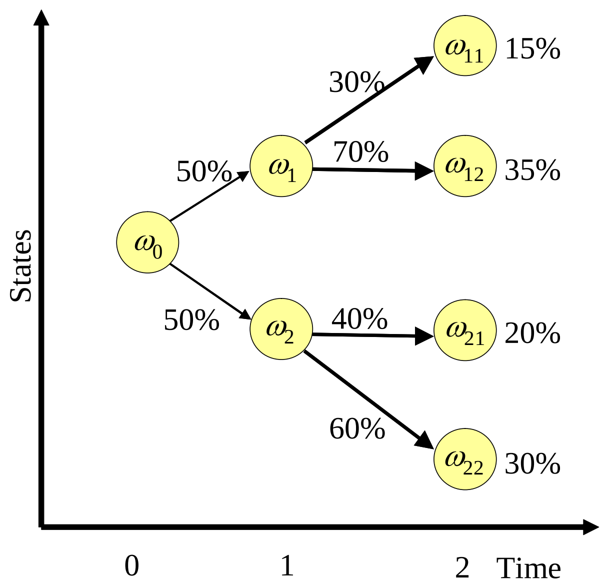

Figure 1. A state tree.

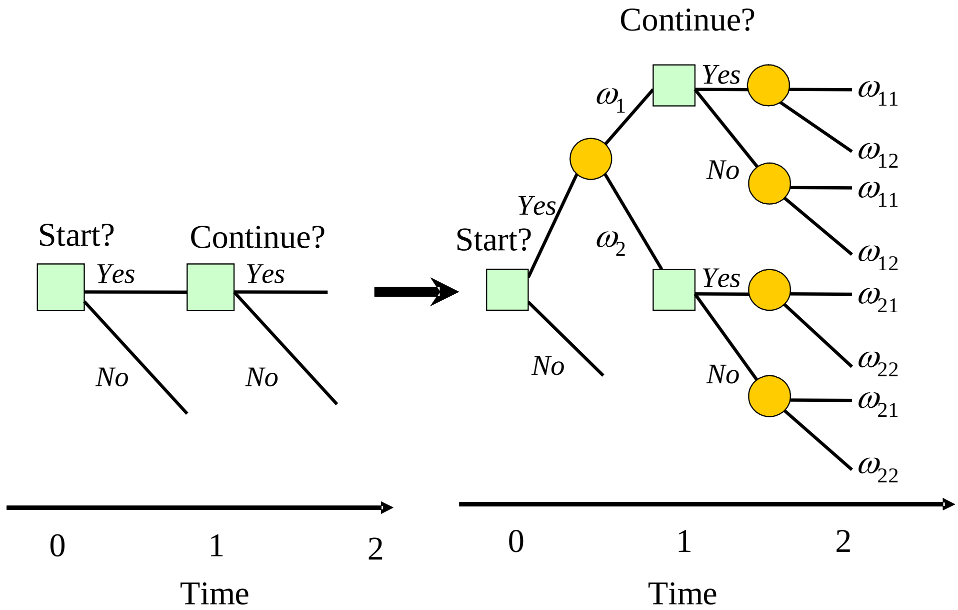

Figure 2. A decision tree for a project.

The investor seeks to maximize the utility of her terminal wealth level. When the investor's preferences are captured by the preference functional U, this objective can be expressed as

$$\max U[X],$$

where the random variable X represents the amount of cash at the end of the planning horizon. When the investor is able to determine a certainty equivalent for any X, U can be expressed as a strictly increasing transformation of the investor's certainty equivalent operator CE. Hence, the objective can be written as

$$\max CE[X],$$

which is the general objective function used in CPP.

Alternatively, it is possible to use a risk-constrained mean-risk model (Gustafsson et al. 2004), where the investor maximizes the mean of X and constrains the risk level for X. Let ρ be the investor's risk measure and R her risk tolerance, i.e. the maximum level for risk. Then, the objective function is

$$\max E[X]$$

and the risk constraint is

$$\rho[X] \leq R.$$

A risk-constrained model is often relevant, because without a risk constraint several preference models may yield unbounded solutions when there is a possibility to purchase securities and borrow without limit (see Gustafsson and Salo 2004).

4 Formulation of States, Assets, and Cash Flows

4.1 States

Let the planning horizon be $\{0,...,T\}$. The set of states in period t is denoted by Ω_t, and the set of all states is $\Omega = \bigcup_{t=0}^{T} \Omega_t$. The state tree starts with base state ω_0 in period 0. Each non-terminal state is followed by at least one state. This relationship is modeled by the function $B : \Omega \to \Omega$ which returns the immediate predecessor of each state, except for the base state, for which the function gives $B(\omega_0) = \omega_0$.

The probability of state ω, when $B(\omega)$ has occurred, is given by $p_{B(\omega)}(\omega)$. Unconditional probabilities for each state, except for the base state, can be computed recursively from the equation $p(\omega) = p_{B(\omega)}(\omega) \cdot p(B(\omega))$. The probability of the base state is $p(\omega_0) = 1$.

4.2 Assets

Let there be n securities available in financial markets. The amount of security i bought in state ω is indicated by trading variable $x_{i,\omega}$, $i = 1,\ldots,n$, $\omega \in \Omega$, and the price of security i in state ω is denoted by $S_i(\omega)$. Under the assumption that all securities are sold back in the next period (they can immediately be re-bought), the cash flows implied by security i in state $\omega \neq \omega_0$ is $S_i(\omega) \cdot \left(x_{i,B(\omega)} - x_{i,\omega}\right)$. In base state ω_0, the cash flow is $-S_i(\omega_0) \cdot x_{i,\omega_0}$.

The investor can invest privately in m projects. The decision opportunities for each project are structured as a decision tree which is formed of *decision points*. For each project $k = 1,\ldots,m$, the decision tree is implemented using a set of decision points D_k and the function $ap(d)$ that gives the action leading to decision point $d \in D_k \setminus \{d_k^0\}$, where d_k^0 is the first decision point of project *k*. Let A_d be the set of actions that can be taken in decision point $d \in D_k$. For each action a in A_d, a binary action variable $z_{k,a}$ indicates whether the action is selected or not. These variables are bound by the restriction that, at each decision point, only one of them can be equal to one at a time. The action in decision point d is chosen in state $\omega(d)$.

For a project *k*, the vector of all action variables $z_{k,a}$ relating to the project, denoted by $\mathbf{z_k}$, is called the management strategy of project *k*. The vector of all action variables of all projects, denoted by $\mathbf{z}$, is termed the project portfolio management strategy. The pair $(\mathbf{x},\mathbf{z})$, composed of all trading and action variables, is the (mixed asset) portfolio management strategy.

4.3 Cash Flows and Cash Surpluses

Let $CF_k^p(\mathbf{z_k},\omega)$ be the cash flow of project k in state ω when the project management strategy is $\mathbf{z_k}$. When $C_{k,a}(\omega)$ is the cash flow in state ω implied by action a, this cash flow is given by

$$CF_k^p(\mathbf{z_k},\omega) = \sum_{\substack{d\in D_k: \\ \omega(d)\in\Omega^B(\omega)}} \sum_{a\in A_d} C_{k,a}(\omega)\cdot z_{k,a} \ ,$$

where the restriction in the summation of decision points guarantees that actions yield cash flows only in the prevailing state and in the future states that can be reached from the prevailing state. The set $\Omega^B(\omega)$ is defined as $\Omega^B(\omega) = \left\{\omega' \in \Omega \mid \exists\, k \geq 0 \text{ such that } B^k(\omega) = \omega'\right\}$, where $B^n(\omega) = B(B^{n-1}(\omega))$ is the n:th predecessor of ω ($B^0(\omega) = \omega$).

The cash flows from security i in state $\omega \in \Omega$ are given by

$$CF_i^s(\mathbf{x_i},\omega) = \begin{cases} -S_i(\omega)\cdot x_{i,\omega} & \text{if } \omega = \omega_0 \\ S_i(\omega)\cdot\left(x_{i,B(\omega)} - x_{i,\omega}\right) & \text{if } \omega \neq \omega_0 \end{cases}$$

Thus, the aggregate cash flow $CF(\mathbf{x},\mathbf{z},\omega)$ in state $\omega \in \Omega$, obtained by summing up the cash flows for all projects and securities, is

$$\begin{aligned} CF(\mathbf{x},\mathbf{z},\omega) &= \sum_{i=1}^{n} CF_i^s(\mathbf{x_i},\omega) + \sum_{k=1}^{m} CF_k^p(\mathbf{z_k},\omega) \\ &= \begin{cases} \displaystyle\sum_{i=1}^{n} -S_i(\omega)\cdot x_{i,\omega} + \sum_{\substack{d\in D_k: \\ \omega(d)\in\Omega^B(\omega)}} \sum_{a\in A_d} C_{z,a}(\omega)\cdot z_{k,a}, & \text{if } \omega = \omega_0 \\ \displaystyle\sum_{i=1}^{n} S_i(\omega)\cdot\left(x_{i,B(\omega)} - x_{i,\omega}\right) + \sum_{\substack{d\in D_k: \\ \omega(d)\in\Omega^B(\omega)}} \sum_{a\in A_d} C_{z,a}(\omega)\cdot z_{k,a}, & \text{if } \omega \neq \omega_0 \end{cases} \end{aligned}$$

Together with the initial budget of each state, cash flows define *cash surpluses* that would remain in state $\omega \in \Omega$ if the investor chose portfolio management strategy $(\mathbf{x},\mathbf{z})$. Assuming that excess cash is invested in the risk-free asset, the cash surplus in state $\omega \in \Omega$ is given by

$$CS_\omega = \begin{cases} b(\omega) + CF(\mathbf{x},\mathbf{z},\omega) & \text{if } \omega = \omega_0 \\ b(\omega) + CF(\mathbf{x},\mathbf{z},\omega) + (1 + r_{B(\omega)\to\omega})\cdot CS_{B(\omega)} & \text{if } \omega \neq \omega_0 \end{cases}, \qquad (1)$$

where $b(\omega)$ is the initial budget in state $\omega \in \Omega$ and $r_{B(\omega)\to\omega}$ is the short rate at which cash accrues interest from state $B(\omega)$ to ω. Cash surplus in a terminal state is the investor's terminal wealth level in that state.

5 Optimization Model

When using the preference functional U, the objective function for the MAPS model is written as a function of cash surplus variables of the last time period, i.e.

$$\max_{\mathbf{x},\mathbf{z},\mathbf{CS}} U\left(\mathbf{CS_T}\right)$$

where $\mathbf{CS_T}$ denotes the vector of cash surplus variables related to period T. Under the risk-constrained mean-risk model, the objective is to maximize the expectation of the investor's terminal wealth level, viz.

$$\max_{\mathbf{x},\mathbf{z},\mathbf{CS}} \sum_{\omega\in\Omega_T} p(\omega)\cdot CS_\omega .$$

Three types of constraints are imposed on the model: (i) *budget constraints*, (ii) *decision consistency constraints*, and (iii) *risk constraints*, which apply to risk-constrained models only. Different versions of the MAPS model are summarized in Table 2.

5.1 Budget Constraints

Budget constraints ensure that there is a nonnegative amount of cash in each state. They can be implemented using continuous *cash surplus variables* CS_ω, which measure the amount of cash in state ω. Using (1), these variables lead to the following budget constraints:

$$CF(\mathbf{x},\mathbf{z},\omega_0) - CS_{\omega_0} = -b(\omega_0)$$
$$CF(\mathbf{x},\mathbf{z},\omega) + (1+r_{B(\omega)\to\omega})\cdot CS_{B(\omega)} - CS_\omega = -b(\omega) \quad \forall\,\omega\in\Omega\backslash\{\omega_0\} .$$

Note that if CS_ω is negative, the investor borrows money at the risk-free interest rate to cover a funding shortage. Thus, CS_ω can also be regarded as a trading variable for the risk-free asset.

5.2 Decision Consistency Constraints

Decision consistency constraints implement the projects' decision trees. They require that (i) at each decision point at which the investor arrives, only one action is selected, and that (ii) at each decision point at which the investor does not arrive, no action is taken, implying that the point does not incur any cash flows. Decision consistency constraints can be written as

$$\sum_{a\in A_{d_k^0}} z_{k,a} = 1 \quad k=1,\dots,m \tag{2}$$

$$\sum_{a\in A_d} z_{k,a} = z_{k,ap(d)} \quad \forall\, d\in D_k\backslash\left\{d_k^0\right\} \quad k=1,\dots,m , \tag{3}$$

where (2) ensures that one action is selected in the first decision point, and (3) implements the above requirements for other decision points.

Table 2. Multi-period MAPS models.

	Preference functional model	General mean-risk model	Deviation-based mean-risk model
Objective function	$\max_{\mathbf{x},\mathbf{z},\mathbf{CS}} U\left(\mathbf{CS_T}\right)$	$\max_{\mathbf{x},\mathbf{y},\mathbf{CS}} \sum_{\omega \in \Omega_T} p(\omega) \cdot CS_\omega$	
Budget constraints	$CF(\mathbf{x},\mathbf{y},\omega_0) - CS_{\omega_0} = -b(\omega_0)$ $CF(\mathbf{x},\mathbf{y},\omega) + (1 + r_{B(\omega)\to\omega}) \cdot CS_{B(\omega)} - CS_\omega = -b(\omega) \quad \forall \omega \in \mathbf{\Omega} \backslash \{\omega_0\}$		
Decision consistency constraints	$\sum_{a \in A_{d_k^0}} z_{k,a} = 1 \quad k = 1,\ldots,m$ $\sum_{a \in A_d} z_{k,a} = z_{k,ap(d)} \quad \forall d \in D_k \backslash \left\{d_k^0\right\} \quad k = 1,\ldots,m$		
Risk constraints	-	$\rho\left(\mathbf{CS_T}\right) \le R$	$\rho\left(\Delta^-,\Delta^+\right) \le R$ $CS_\omega - \tau(\mathbf{CS_T}) - \Delta_\omega^+ + \Delta_\omega^- = 0 \quad \forall \omega \in \mathbf{\Omega}_T$
Variables	$z_{k,a} \in \{0,1\} \quad \forall a \in A_d \ \forall d \in D_k \ k = 1,\ldots,m$ $x_{i,\omega}$ free $\quad \forall \omega \in \mathbf{\Omega} \quad i = 1,\ldots,n$ CS_ω free $\quad \forall \omega \in \mathbf{\Omega}$		$z_{k,a} \in \{0,1\} \quad \forall a \in A_d \ \forall d \in D_k \ k = 1,\ldots,m$ $x_{i,\omega}$ free $\quad \forall \omega \in \mathbf{\Omega} \quad i = 1,\ldots,n$ CS_ω free $\quad \forall \omega \in \mathbf{\Omega}$ $\Delta_\omega^- \ge 0 \quad \forall \omega \in \mathbf{\Omega}_T$ $\Delta_\omega^+ \ge 0 \quad \forall \omega \in \mathbf{\Omega}_T$

5.3 Risk Constraints

A risk-constrained model includes one or more risk constraints. For the sake of analogy with Gustafsson et al. (2004), we focus here on the single constraint case. When ρ denotes the risk constraint and R the risk tolerance, the risk constraint can be expressed as

$$\rho\left(\mathbf{CS_T}\right) \le R.$$

Many common dispersion statistics such as variance (V; Markowitz 1952, 1987), semivariance (SV; Markowitz 1959), absolute deviation (AD; Konno and Yamazaki 1991), lower semi-absolute deviation (LSAD; Gotoh and Konno 2000), and expected downside risk (EDR; Eppen et al. 1989) can be formulated through *deviation constraints* introduced in Gustafsson and Salo (2005) (see also Gustafsson et al. 2004). In general, deviation constraints are expressed as

$$CS_\omega - \tau(\mathbf{CS_T}) - \Delta_\omega^+ + \Delta_\omega^- = 0 \quad \forall \omega \in \mathbf{\Omega}_T,$$

where $\tau(\mathbf{CS_T})$ is a function defining the target value from which the deviations are calculated, and Δ_ω^+ and Δ_ω^- are nonnegative *deviation variables* which measure how much the cash surplus in state $\omega \in \mathbf{\Omega}_T$ differs from the target value. For example, when the target value is the mean of the terminal wealth level, the deviation constraints are written as

$$CS_\omega - \sum_{\omega' \in \Omega_T} p(\omega') CS_{\omega'} - \Delta_\omega^+ + \Delta_\omega^- = 0 \qquad \forall \omega \in \Omega_T,$$

Using these deviation variables, some common dispersion statistics can be written as

AD: $\sum_{\omega \in \Omega_T} p(\omega) \cdot (\Delta_\omega^- + \Delta_\omega^+)$.

LSAD: $\sum_{\omega \in \Omega_T} p(\omega) \cdot \Delta_\omega^-$.

V: $\sum_{\omega \in \Omega_T} p(\omega) \cdot (\Delta_\omega^- + \Delta_\omega^+)^2$

SV: $\sum_{\omega \in \Omega_T} p(\omega) \cdot (\Delta_\omega^-)^2$.

The respective fixed-target value statistics can be obtained with the deviation constraints

$$CS_\omega - \tau - \Delta_\omega^+ + \Delta_\omega^- = 0 \quad \forall \omega \in \Omega_T,$$

where τ is the fixed target level. EDR, for example, can then be obtained from the sum $\sum_{\omega \in \Omega_T} p(\omega) \cdot \Delta_\omega^-$.

6 Project Valuation

The value of a project can be defined as the lowest price at which a rational investor would be willing to sell a project, if she had the project, and as the highest price at which a rational investor would be willing to buy the project, if she did not have it. These prices are referred to as the *breakeven selling price* and *breakeven buying price* of the project, respectively (Gustafsson et al. 2004, Luenberger 1998, Smith and Nau 1995). In this section, we extend the definitions of these prices, as presented in Gustafsson et al. (2004), to a multi-period MAPS setting.

The breakeven prices for a project can be determined from optimization problems where the investor invests and does not invest in the project (see Table 3). Let the project being valued be indexed with j. The action associated with not starting the project is denoted by a^*. When calculating the breakeven selling price, we first determine the optimal investment portfolio with the project. Then, we iteratively solve another optimization problem where the investor does not invest in the project but where an amount v_j^s is added to the budget at time 0 instead. The breakeven selling price is the amount v_j^s that makes the portfolios with and without the project equally preferable. In a preference functional model, this means that the utilities of the two portfolios are identical. In a mean-risk model, two portfolios are equally preferred if they are equal in terms of the expected terminal wealth level and they both satisfy the risk constraint.

The breakeven buying price for a project is determined similarly as its breakeven selling price.

The difference is that, in the first setting, the investor does not invest in the project, and in the second setting, she invests in the project and deducts an amount v_j^b from the budget at time 0. The breakeven selling price is the amount that makes the investor indifferent between the portfolios with and without the project. In general, the breakeven buying price and the breakeven selling price for a project are not equal to each other.

Table 3 describes how breakeven selling and buying prices can be determined based on the optimization problems in Table 2. In each setting, an additional constraint is added to the optimization model. Also, the budget at time 0 is modified in the second setting. The objective function values of the resulting models at optimum are denoted by W^+ and W^- depending on whether the investor invests in the project or not.

Table 3. Definitions of the value of project *j*. Each setting is based on the model in Table 2.

	Breakeven selling price	**Breakeven buying price**
Definition	v_j^s such that $W_s^+ = W_s^-$	v_j^b such that $W_b^+ = W_b^-$
Status quo	Additional constraint: $z_{j,a^*} = 0$, i.e. invest in project Budget at time 0: $b(\omega_0)$ Optimal objective function value: W_s^+	Additional constraint: $z_{j,a^*} = 1$, i.e. do not invest in project Budget at time 0: $b(\omega_0)$ Optimal objective function value: W_b^-
Second setting	Additional constraint: $z_{j,a^*} = 1$, i.e. do not invest in project Budget at time 0: $b(\omega_0) + v_j^s$ Optimal objective function value: W_s^-	Additional constraint: $z_{j,a^*} = 0$, i.e. invest in project Budget at time 0: $b(\omega_0) - v_j^b$ Optimal objective function value: W_b^+

Breakeven selling and buying prices exhibit a notable property: they yield the same result as CCA whenever the method is applicable, i.e. whenever the cash flows of the project can be replicated by trading financial securities. In general, the CCA value of the project is defined as the value of the portfolio that is required to create a trading strategy replicating the future cash flows of the project minus the project's investment cost at time 0 (see Table 1). Smith and Nau

(1995) show that this property holds for investors with an intertemporal utility function over cash flow streams, whereas Gustafsson and Salo (2004) prove the same for non-expected utility investors who maximize their terminal wealth level in a two-period setting. The following proposition generalizes the latter result to a multi-period setting.

PROPOSITION 1. *If there is a replicating trading strategy for a project, the breakeven buying price and breakeven selling price for the project are equal and yield the same value as CCA.*
PROOF: See Appendix.

Gustafsson and Salo (2004) and Gustafsson et al. (2004) also discuss valuation formulas that facilitate the computation of project values for (i) certain types of mean-risk investors, (ii) Choquet-expected utility maximizers exhibiting constant absolute risk aversion, and (iii) investors using Wald's (1950) maximin criterion. These formulas are applicable in the two-period case and are of the form

$$v = \frac{V^{+} - V^{-}}{1 + r_f},$$

where V^{+} and V^{-} are the optimal objective function values with and without the project, respectively, and r_f is the risk-free interest rate. Since the propositions rely on the examination of the last time period with no consideration for the number of intermediate periods, these formulas can also be used in a multi-period setting with the modification that the objective function values are discounted using the time-T spot-rate, s_T, viz.

$$v = \frac{V^{+} - V^{-}}{(1 + s_T)^{T}}.$$

The formal proof of the formulas in a multi-period setting is almost identical to their proof in the two-period setting and is hence omitted.

As shown by Proposition 2, there exists also an analogous formula for the breakeven selling price when the investor can invest only in a single project and the risk-free asset. This proposition holds for any rational preference model accommodated by the portfolio models in Table 2. If we further assume that the investor exhibits constant absolute risk aversion, i.e. the investor's certainty equivalent operator satisfies $CE[X + b] = CE[X] + b$ for all random variables X and constants b, breakeven selling and buying prices are equal to each other; they are also independent of the budget, and they can be computed from the certainty equivalent of the project's future value (Proposition 3). The proofs of these two propositions are in Appendix.

PROPOSITION 2. *When the investor can invest in a single project and in the risk-free asset, the breakeven selling price for the project can be computed using the formula* $v = \frac{V^+ - V^-}{(1+s_T)^T}$ *, where* V^+ *and* V^- *are the investor's certainty equivalents for her terminal wealth level with and without the project, respectively, and* s_T *is the time-*T *spot rate.*

PROPOSITION 3. *When the conditions of Proposition 2 hold and the investor's certainty equivalent operator satisfies* $CE[X+b] = CE[X] + b$ *for all random variables X and constants b, breakeven buying and selling prices are equal and can be computed using the formula* $v = \frac{V^+ - V^-}{(1+s_T)^T} = \frac{CE_T}{(1+s_T)^T}$ *, where* CE_T *is the certainty equivalent for the project's future value and* s_T *is the time-*T *spot rate.*

If all short rates are equal to r_f, the formula in Proposition 3 can be expressed as

$$v = -I + \frac{CE\left[\sum_{t=1}^{T} c_t (1+r_f)^{T-t}\right]}{(1+r_f)^T},$$

where, following the notation in Table 1, I is the investment cost at time 0 and c_t is the random time-t cash flow implied by the project under the optimal project management strategy. Notice that if the investor's certainty equivalent operator further satisfies $CE[aX] = aCE[X]$ for all random variables X and constants a (that is, the investor exhibits also constant relative risk aversion), the formula reduces to Hillier's method (Table 1). This observation is formalized with the following proposition. The proof is obvious from the above and is hence omitted.

PROPOSITION 4. *When the conditions of Proposition 2 hold and the investor's certainty equivalent operator satisfies* $CE[aX+b] = aCE[X] + b$ *for all random variables X and constants a and b, the breakeven buying price and breakeven selling price are equal and yield the same value as Hillier's method, given by the formula* $NPV = -I + CE\left[\sum_{t=1}^{T} \frac{c_t}{(1+s_t)^t}\right]$*, where I is the investment cost of the project,* c_t *is the cash flow at time t, and* s_t *is the time-t spot rate.*

For example, Proposition 4 can be applied with risk-neutral investors, some mean-risk models (see, e.g., Gustafsson and Salo 2005), and preference models based on Yaari's (1987) dual theory. However, no risk-averse expected utility maximizer exhibits both constant absolute and constant relative risk aversion, so that no risk-averse expected utility maximizer is, in general, consistent with Hillier's method.

While Hillier's method can be obtained as a special case of breakeven selling and buying prices, Robichek and Myers' (1966) method does not comply with these prices, unless restrictive assumptions about the analyzed cash flow streams are made. For example, when a project's cash flows are perfectly correlated with each other and the conditions of Proposition 4 hold, Robichek and Myer's method reduces to Hillier's method, and hence gives consistent results with the breakeven prices. On the other hand, Smith and Nau's (1995) method coincides with the present approach when the chosen intertemporal utility function rewards for the maximization of the expected utility of the terminal wealth level, i.e., when the utility functions for periods other than the last are identically zero. Indeed, maximization of intertemporal utility and that of the utility of terminal wealth level are not equivalent objectives: in effect, it can be shown that an intertemporal utility function can lead to an investment portfolio, which is stochastically dominated by another investment portfolio in terms of terminal wealth levels (Proposition 5). Since the terminal wealth level is widely regarded as the future value equivalent of NPV, care should be taken when applying Smith and Nau's method in a setting where the investor aims to maximize the NPV of the investment portfolio. Table 4 summarizes the conditions under which other project valuation methods give the same results as the present approach.

PROPOSITION 5. *Maximization of intertemporal utility may lead to a portfolio that is stochastically dominated by another portfolio in terms of the terminal wealth level.*
PROOF: See Appendix.

Table 4. The conditions when the present approach coincides with other project valuation methods.

Method	**Preferences**	**Cash flows**	**Available investments**
Hillier (1963)	$CE[aX+b]=aCE[X]+b$	–	One project, risk-free asset
Robichek and Myers (1966)	$CE[aX+b]=aCE[X]+b$	Perfectly correlated	One project, risk-free asset
Smith and Nau (1995)	Utility function for each period except the last is identically 0	–	One project, risk-free asset, securities

Finally, it is worth noting that the results based on the CAPM in Gustafsson and Salo (2004) and Gustafsson et al. (2004), such as the convergence of project values to the CAPM market prices, may not hold in a multi-period setting. The reason for this is that the mean-variance model, on which the CAPM is based, is a *dynamically inconsistent* preference model (Machina 1989). Therefore, the optimal portfolio for a mean-variance investor maximizing her terminal wealth level is, in general, different from the portfolio obtained using the two-period CAPM consecutively in a multi-period setting.

7 Numerical Experiments

In the previous sections, we developed a framework for valuing risky multi-period projects in incomplete markets, and showed that the framework gives consistent results with CCA and, under certain conditions, with Hillier's (1963) method. In this section, we demonstrate the approach through a set of numerical experiments and show that most of the phenomena observed in a two-period MAPS setting also occur in the multi-period setting. In particular, we cast light on the following issues:

- Q1. How are the values of multi-period projects influenced by the possibility to invest in risky securities?
- Q2. How are the values of multi-period projects influenced by the possibility to invest in other projects?
- Q3. How do the values of multi-period projects depend on the investor's risk attitude?

7.1 Experimental Setup

The experimental setup involves 3 time periods, 4 projects, and 2 securities that together constitute the market. Uncertainties related to projects and securities are captured through a state tree which consists of 8 equally likely time-1 states and 64 equally likely time-2 states (Figure 3). Each state is formed of two underlying states, (i) the market state (denoted by M), which determines the prevailing security prices, and (ii) the private state (denoted by P) that influences project outcomes. Security prices at time 2 are given in Figure 3. Security prices at time 1 are computed from these prices using the CAPM (Table 5), and the prices at time 0 are obtained similarly from the time-1 security prices (Table 6).

			Security 1	Security 2	Project A	Project B	Project C	Project D
State 0	State M1/P1	State M11/P11	$60	$36	$150	$150	$100	$180
		State M12/P11	$50	$12	$150	$100	$120	$60
		State M13/P11	$40	$36	$150	$50	$90	$180
		State M14/P11	$30	$12	$150	$0	$80	$60
		State M11/P12	$60	$36	$50	$250	$170	$180
		State M12/P12	$50	$12	$50	$200	$200	$60
		State M13/P12	$40	$36	$50	$150	$40	$180
		State M14/P12	$30	$12	$50	$100	$50	$60
	State M2/P1	State M21/P11	$70	$36	$150	$200	$130	$180
		State M22/P11	$60	$12	$150	$150	$70	$60
		State M23/P11	$50	$36	$150	$100	$130	$180
		State M24/P11	$40	$12	$150	$50	$110	$60
		State M21/P12	$70	$36	$50	$300	$100	$180
		State M22/P12	$60	$12	$50	$250	$180	$60
		State M23/P12	$50	$36	$50	$200	$40	$180
		State M24/P12	$40	$12	$50	$150	$50	$60
	State M3/P1	State M31/P11	$80	$36	$150	$250	$190	$180
		State M32/P11	$70	$12	$150	$200	$45	$60
		State M33/P11	$60	$36	$150	$150	$60	$180
		State M34/P11	$50	$12	$150	$100	$100	$60
		State M31/P12	$80	$36	$50	$350	$110	$180
		State M32/P12	$70	$12	$50	$300	$50	$60
		State M33/P12	$60	$36	$50	$250	$140	$180
		State M34/P12	$50	$12	$50	$200	$90	$60
	State M4/P1	State M41/P11	$90	$36	$150	$300	$65	$180
		State M42/P11	$80	$12	$150	$250	$105	$60
		State M43/P11	$70	$36	$150	$200	$100	$180
		State M44/P11	$60	$12	$150	$150	$115	$60
		State M41/P12	$90	$36	$50	$400	$175	$180
		State M42/P12	$80	$12	$50	$350	$90	$60
		State M43/P12	$70	$36	$50	$300	$210	$180
		State M44/P12	$60	$12	$50	$250	$180	$60
	State M1/P2	State M11/P21	$60	$36	$80	$220	$50	$180
		State M12/P21	$50	$12	$80	$170	$40	$60
		State M13/P21	$40	$36	$80	$120	$110	$180
		State M14/P21	$30	$12	$80	$70	$105	$60
		State M11/P22	$60	$36	$40	$260	$125	$180
		State M12/P22	$50	$12	$40	$210	$115	$60
		State M13/P22	$40	$36	$40	$160	$235	$180
		State M14/P22	$30	$12	$40	$110	$170	$60
	State M2/P2	State M21/P21	$70	$36	$80	$270	$250	$180
		State M22/P21	$60	$12	$80	$220	$215	$60
		State M23/P21	$50	$36	$80	$170	$180	$180
		State M24/P21	$40	$12	$80	$120	$190	$60
		State M21/P22	$70	$36	$40	$310	$130	$180
		State M22/P22	$60	$12	$40	$260	$100	$60
		State M23/P22	$50	$36	$40	$210	$70	$180
		State M24/P22	$40	$12	$40	$160	$90	$60
	State M3/P2	State M31/P21	$80	$36	$80	$320	$240	$180
		State M32/P21	$70	$12	$80	$270	$150	$60
		State M33/P21	$60	$36	$80	$220	$120	$180
		State M34/P21	$50	$12	$80	$170	$100	$60
		State M31/P22	$80	$36	$40	$360	$230	$180
		State M32/P22	$70	$12	$40	$310	$100	$60
		State M33/P22	$60	$36	$40	$260	$120	$180
		State M34/P22	$50	$12	$40	$210	$240	$60
	State M4/P2	State M41/P21	$90	$36	$80	$370	$100	$180
		State M42/P21	$80	$12	$80	$320	$160	$60
		State M43/P21	$70	$36	$80	$270	$250	$180
		State M44/P21	$60	$12	$80	$220	$290	$60
		State M41/P22	$90	$36	$40	$410	$200	$180
		State M42/P22	$80	$12	$40	$360	$100	$60
		State M43/P22	$70	$36	$40	$310	$50	$180
		State M44/P22	$60	$12	$40	$260	$45	$60

Figure 3. State tree with security prices and project cash flows at time 2.

Table 5. Securities in states M1–M4.

	M1		M2		M3		M4	
	1	2	1	2	1	2	1	2
Beta	0.821	1.527	0.782	1.795	0.755	2.064	0.735	2.332
Market price	$39.32	$20.00	$48.58	$20.00	$57.83	$20.00	$67.09	$20.00
Capitalization weight	74.68%	25.32%	78.46%	21.54%	81.26%	18.74%	83.42%	16.58%
Expected rate of return	14.46%	20.00%	13.23%	20.00%	12.39%	20.00%	11.78%	20.00%
St. dev. of rate of return	28.44%	60.00%	23.02%	60.00%	19.33%	60.00%	16.66%	60.00%
Market expected return	15.86%		14.68%		13.82%		13.15%	
Market st. dev.	31.15%		26.49%		23.05%		20.39%	

Table 6. Securities in state 0.

	Security	
	1	2
Market price in M1	$39.32	$20.00
Market price in M2	$48.58	$20.00
Market price in M3	$57.83	$20.00
Market price in M4	$67.09	$20.00
Beta	1.264	0.000
Market price in 0	46.85	18.52
Capitalization weight	79.14%	20.86%
Expected rate of return	13.58%	8.00%
St. dev. of rate of return	22.10%	0.00%
Market expected return	12.41%	
Market standard deviation	17.49%	

The CAPM requires us to specify the amounts of outstanding shares and the expected rate of return of the market portfolio in each state. We have assumed here that there are a total of 15,000,000 shares of security 1 and 10,000,000 shares of security 2. In each state, the expected rate of return of the market portfolio is determined so that the *market price of risk* – the excess rate of return of the market portfolio divided by the standard deviation of the market – remains constant at 0.252. This is a reasonable way of selecting the expected rate of return of the market, because hereby it is, in each state, proportional to the standard deviation of the market (see Tables 5 and 6). Also, this implies that security 2 is priced at $20 in each time-1 market state, which is desirable, because the time-2 prices for security 2 are independent of the time-1 market state (Figure 3). The risk-free interest rate is 8%.

Table 7 presents the investments required to start projects A, B, C and D. The investments are made in two rounds; the first financing round takes place at time 0 and the second at time 1. If the investment is made in both rounds, the project produces a cash flow at time 3. Figure 3 gives this cash flow as a function of the time-2 state. The specific characteristics of these cash flows are summarized in Table 7. The cash flows of project A depend only on the private states, implying that the project is uncorrelated with the market. On the other hand, the cash flows of project B can be separated into private and market related components; the private component depends on the private state and the market component on the market state. Furthermore, the markets are *partially complete* (see Smith and Nau 1995) with respect to project B in the sense that the market component of project B can be replicated by buying 5 shares of security 1; the private component is obtained by taking the negative of project A. Similarly, the outcome of project C depends on both private and market states but it cannot be separated into market and private components. In contrast, the cash flows of project D are entirely market-dependent and they can be replicated by purchasing 5 shares of security 2.

Table 7. Projects.

	Project			
	A	**B**	**C**	**D**
Private state dependence	Yes	Yes	Yes	No
Market state dependence	No	Yes	Yes	Yes
Market component replication	No	Yes	No	Yes
Investment cost at time 0	\$10	\$20	\$30	\$40
Investment cost at time 1	\$60	\$150	\$70	\$40
Expected outcome	\$80.00	\$220.00	\$127.42	\$120.00
St. dev. of outcome	\$43.01	\$90.00	\$62.66	\$60.00

We examine two preference models, (i) the risk-constrained mean-variance (MV) model, with which we use the deviation-based mean-risk model in Table 2, and (ii) the expected utility (EU) model with constant absolute risk aversion (CARA). As the base case, the latter model employs the preference functional $U[X] = E[u(X)] = -E\left[e^{-\alpha X}\right]$ with $\alpha = 0.003$ unless otherwise noted, and the former the risk tolerance of \$250 for the standard deviation of the terminal wealth level. For both preference models, breakeven selling and buying prices are equal (see Gustafsson et al. 2004, Gustafsson and Salo 2004) and therefore they are displayed in a single entry. We begin our experiments by analyzing project values when securities are not available and then continue to the setting with securities. The optimizations were carried out using the GAMS software package with SBB and CONOPT algorithms (www.gams.com).

7.2 Single Project without Securities

We first solve project values in a setting where the investor can invest only in a single project and the risk-free asset. Under both preference models, each of the projects is started at time 0. The optimal management strategies for the projects at time 1 are described in the first eight data rows of Table 8. The mark "X" indicates that a further investment is made in the project, while the mark "–" means that the project is terminated. The first mark is the optimal strategy for the MV model; the second one indicates the optimal action for the EU investor. In the MV model, the risk constraint is not binding, wherefore it now corresponds to expected value maximization.

Table 8. Optimal project management strategies and project values.

	Project			
Time-1 State	**A**	**B**	**C**	**D**
M1/P1	X/X	–/–	X/X	X/X
M2/P1	X/X	X/X	X/X	X/X
M3/P1	X/X	X/X	X/X	X/X
M4/P1	X/X	X/X	X/X	X/X
M1/P2	–/–	X/–	X/X	X/X
M2/P2	–/–	X/X	X/X	X/X
M3/P2	–/–	X/X	X/X	X/X
M4/P2	–/–	X/X	X/X	X/X
Project value, MV/EV model	$5.09	$33.69	$14.43	$25.84
Project value, EU model	$3.17	$25.59	$9.58	$21.24

In Table 8, observe that a further investment is made in project A only if private state P1 obtains. This is because the outcomes of project A in the states ensuing P1 are substantially higher than in the states following P2 (Figure 3). Project B is terminated in state M1/P1 under both preference models. In contrast, only the MV investor chooses to continue the project in state M1/P2. The termination decisions are explained by the low outcome with the market component of project B (i.e., security 1); furthermore, project B is more profitable in private state P2 than in P1, explaining the continuation decision with the (risk-neutral) MV investor.

Both MV and EU investors choose to continue projects C and D in all states. With project D this is because the realized time-1 state does not convey any information about the future cash flows of project D (Figure 3); with project C the future outcome distribution in each time-1 state is different but in each case it warrants a continuation decision.

Since we analyze here only a single project with the risk-free asset, we can use the project values in Table 8 to numerically verify Proposition 3. Let us take project A, for example. The certainty equivalent for an EU investor's terminal wealth level with project A is \$586.90, while it is \$583.20 without the project. Discounting the difference of these values back to its present value yields $v_A = (\$586.90 - \$583.20)/1.08^2 = \$3.17$, which coincides with the project's breakeven selling and buying prices given in Table 8.

7.3 Single Project with Securities

We next consider a setting where the investor can invest in a single project, in the risk-free asset, and in securities 1 and 2. We first examine the investor's trading strategy in a case where the investor cannot invest in projects, because we can consequently examine how the starting of a project influences the optimal trading strategy. Table 9 gives the optimal trading strategy for the MV investor. Note that purchase of security 2 at time 0 is equivalent to investment in the risk-free asset; for the sake of clarity, we have assumed here that the investor invests in the risk-free asset and not in security 2. The column CS gives the amount of money lent in each state.

Table 9. Optimal trading strategy for the MV investor without projects.

	Portfolio weights		Securities bought		
State	**1**	**2**	**1**	**2**	**CS**
0	100%	0%	16.557	0	-\$275.602
M1/P1	74.7%	25.3%	14.203	9.468	-\$394.475
M2/P1	78.5%	21.5%	11.809	7.872	-\$224.445
M3/P1	81.3%	18.7%	9.414	6.275	-\$10.034
M4/P1	83.4%	16.6%	7.019	4.679	\$248.724
M1/P2	74.7%	25.3%	14.203	9.468	-\$394.475
M2/P2	78.5%	21.5%	11.809	7.872	-\$224.445
M3/P2	81.3%	18.7%	9.414	6.275	-\$10.034
M4/P2	83.4%	16.6%	7.019	4.679	\$248.724

In comparison, an EU investor would buy 7.634 shares of security 1 at time 0, and 5.169 shares of security 1 and 3.478 shares of security 2 in each time-1 state. Since the price of security 1 changes between the time-1 states, the portfolio weights for an EU investor do not remain constant across time-1 states. An EU investor lends \$142.032 in state 0, implying that the EU model with $\alpha = 0.003$ is a more risk averse preference model than the mean-standard deviation model with $R = \$250$.

When an MV investor invests in project A, the portfolio weights of the optimal security portfolio remain the same as in Table 9, but absolute amounts of securities in the portfolio change. The constant portfolio weights are explained by the fact that project A is uncorrelated with the market; this was proven in a two-period setting by Gustafsson et al. (2004). For the same reason, an EU investor preserves her investment behavior: even when she invests in project A, the optimal trading strategy is, as before, to buy 5.169 shares of security 1 and 3.478 shares of security 2 in each time-1 state.

Recall that the cash flows of project B can be formed by buying 5 shares of security 1 and deducting the cash flows of project A from the resulting portfolio. The weights of the optimal security portfolio for an MV investor now change, because project B counts for 5 shares of security 1; if these 5 shares were added to the security portfolio, the portfolio weights would match with the weights in Table 9. With the EU investor, we clearly observe this behavior: she invests in 0.169 shares of security 1 in each time-1 state where the project is continued (see Table 10) and 5.169 shares otherwise.

In contrast, the cash flows of project C cannot be separated into private and market components, wherefore the weights of the security portfolio change without a clear pattern. Project D can be replicated using 5 shares of security 2. Therefore, in the optimum, an EU investor shorts $5 - 3.478 = 1.522$ shares of security 2 and buys 5.169 shares of security 1. When these 5 shares are added to the security portfolio of the MV investor, the weights of the portfolio correspond to the ones in Table 9.

Table 10 presents the optimal project management strategy and the values of the projects for MV and EU investors. The optimal strategies for project A, C and D are the same as in Table 8, whereas project B is now terminated in states M2/P1 and M1/P2 under both preference models due to opportunity costs imposed by the securities, i.e. it is more profitable to invest in the securities in states M2/P1 and M1/P2 than in project B. In particular, this implies that the opportunity costs imposed by securities can have a major effect on projects' optimal management strategies.

Table 10. Optimal project management strategies and project values.

	Project			
	A	**B**	**C**	**D**
M1/P1	X/X	–/–	X/X	X/X
M2/P1	X/X	–/–	X/X	X/X
M3/P1	X/X	X/X	X/X	X/X
M4/P1	X/X	X/X	X/X	X/X
M1/P2	–/–	–/–	X/X	X/X
M2/P2	–/–	X/X	X/X	X/X
M3/P2	–/–	X/X	X/X	X/X
M4/P2	–/–	X/X	X/X	X/X
Project value, MV model	$4.13	$14.53	$8.54	$15.56
Project value, EU model	$3.17	$14.15	$7.07	$15.56

In each case, except for project A under the EU model, the opportunity costs lower the resulting project values. Notice also that project D is priced at $15.56 due to the existence of a replicating portfolio: the price of 5 shares of security 2 minus the risk-free discounted NPV of the costs is $5 \cdot \$20/1.08 - \$40 - \$40/1.08 = \$15.56$.

In summary, securities typically lead to lower project values by imposing additional opportunity costs. However, as noted by Gustafsson et al. (2004), they may also increase project values by providing more efficient diversification possibilities. Furthermore, if a replicating portfolio exists for a project, the project will be priced at its CCA value. Also, the introduction of securities may, at times, change the optimal management strategies for projects. [Q1]

7.4 Project Portfolio with Securities

In this section, we consider the case where the investor can simultaneously invest in projects A–D as well as in the risk-free asset and in the two securities. Here, the optimal project management strategies are the same as in Table 10, except that both MV and EU investors make here a further investment in project B in states M2/P1; this is largely due to the fact that projects A and B are negatively correlated with each other. Note also that, apart from project D, the values of the projects increase, because with more investment assets available, the risks of each project can be diversified more efficiently than previously.

Table 11 presents the optimal security portfolio of the MV investor when the projects are managed using the optimal project management strategy. This portfolio combines the effects that were described in the previous section: in states where project B is continued (in all states except M1/P1 and M1/P2), the investor buys five shares of security 1 less than she would otherwise; on the other hand, the starting of project D implies that the investor buys five shares of security 2 less in each state. With these modifications, the optimal trading strategy is very close to the optimal trading strategy when only project C was started. The small differences are due to correlations between projects.

Table 11. Optimal trading strategy for the MV investor with projects A–D.

State	Portfolio weights		Securities bought		CS
	1	2	1	2	
0	100.0%	0.0%	11.136	0.000	-$121.69
M1/P1	74.7%	25.3%	9.495	6.319	-$363.27
M2/P1	76.2%	23.8%	5.167	3.912	-$239.70
M3/P1	105.3%	-5.3%	4.731	-0.685	-$67.29
M4/P1	116.5%	-16.6%	4.019	-1.915	$64.41
M1/P2	95.1%	4.9%	17.451	1.769	-$525.09
M2/P2	77.3%	22.7%	4.180	2.989	-$113.30
M3/P2	94.5%	5.5%	2.046	0.343	$127.43
M4/P2	114.8%	-14.8%	2.488	-1.076	$210.33

Tables 12 and 13 present sensitivity analyses for project values with respect to the MV investor's risk tolerance R and the EU investor's risk aversion parameter α, respectively. At low risk tolerance levels (high degree of risk aversion), an MV investor gives high values for projects A and B, whereas the values of the projects decrease as the investor becomes more risk tolerant. In contrast, the price behavior of project C is opposite: project C obtains low values at small risk tolerance levels and the values increase as the investor becomes more risk tolerant. Project D is constantly priced at $15.56 due to the existence of a replicating portfolio. An EU investor prices the projects similarly: values of projects A and B decrease by diminishing risk aversion, while the value of project C increases. Also in this case, the value of project D is constant due to the existence of a replicating portfolio.

In both cases, project values converge to limit values as the investor becomes less risk averse. However, these limit values are different, and in neither case they coincide with a risk neutral investor's project values (−\$4.13, \$9.35, \$1.86, \$12.83), obtained by using a short rate 13.58% in period 0 (the expected rate of return of security 1; Table 6) and 20.00% in period 1 (the expected rate of return of security 2; Table 5). Gustafsson et al. (2004) showed in their numerical experiments that, in a two-period setting, the project values for an MV investor converge towards the CAPM prices of the projects; the experiments in De Reyck et al. (2004) showed a similar behavior for EU investors with CARA. The observed converge behavior for both types of investors shows that a similar phenomenon can happen also in a multi-period setting.

Table 12. Project values for an MV investor.

	Project			
Risk level	**A**	**B**	**C**	**D**
75	\$11.70	\$16.87	\$4.12	\$15.56
100	\$8.49	\$16.21	\$6.20	\$15.56
125	\$7.55	\$15.96	\$7.20	\$15.56
150	\$7.05	\$15.82	\$7.80	\$15.56
175	\$6.73	\$15.73	\$8.21	\$15.56
200	\$6.50	\$15.67	\$8.51	\$15.56
225	\$6.34	\$15.62	\$8.74	\$15.56
250	\$6.21	\$15.58	\$8.92	\$15.56
300	\$6.02	\$15.53	\$9.19	\$15.56
350	\$5.89	\$15.49	\$9.38	\$15.56
400	\$5.79	\$15.46	\$9.52	\$15.56
450	\$5.71	\$15.44	\$9.63	\$15.56
500	\$5.65	\$15.42	\$9.71	\$15.56
1000	\$5.38	\$15.35	\$10.10	\$15.56
1500	\$5.28	\$15.32	\$10.23	\$15.56
2500	\$5.21	\$15.30	\$10.33	\$15.56
5000	\$5.15	\$15.28	\$10.41	\$15.56
50000	\$5.10	\$15.27	\$10.48	\$15.56

Table 13. Project values for an EU investor.

	Project			
α	**A**	**B**	**C**	**D**
0.005	$8.03	$16.67	$5.97	$15.56
0.004	$7.43	$16.41	$6.87	$15.56
0.003	$6.83	$16.15	$7.79	$15.56
0.002	$6.24	$15.90	$8.74	$15.56
0.001	$5.67	$15.66	$9.71	$15.56
0.0001	$5.15	$15.46	$10.59	$15.56
0.00001	$5.10	$15.44	$10.68	$15.56
0.000001	$5.09	$15.44	$10.69	$15.56

In summary, project values when the investor can invest simultaneously in projects A–D can be different from the values obtained with single projects due to correlations between projects and better diversification possibilities implied by a greater number of investment assets in the portfolio. Also, the optimal management strategies for the projects may change. [Q2] Sensitivity analysis reveals that the value of a project may either decrease or increase with increasing risk aversion depending on its correlation with the rest of the investment portfolio. Furthermore, a project with a replicating portfolio is constantly priced at its CCA value. Also, with decreasing level of risk-aversion, project values converge towards prices that are different from the projects' risk-neutral values. These values are analogous to the CAPM prices and EU convergence prices observed in a two-period setting. [Q3]

8 Summary and Conclusions

In this paper, we have presented a model for selecting a multi-period mixed asset portfolio and a framework that uses this model in the valuation of multi-period projects. In the MAPS model, project management decisions are structured as project-specific decision trees. The project valuation framework is based on the concepts of breakeven selling and buying prices (Gustafsson et al. 2004, Luenberger 1998, Smith and Nau 1995); the breakeven selling price is the lowest price at which a rational investor would sell the project if she had it, whereas the breakeven buying price is the highest price at which the investor would buy the project if she did not have it.

The present project valuation framework differs from Smith and Nau's (1995) "full decision tree analysis" in that (i) it maximizes the investor's terminal wealth level rather than the intertemporal utility implied by the project and in that (ii) it takes into account the opportunity costs imposed by alternative projects in the portfolio. The framework also (iii) allows the use of a wide range of preference models, and it leads to an explicit optimization problem, which can readily be solved using standard techniques of nonlinear programming. In comparison with Smith and Nau's (1995) integrated rollback procedure, which is also an explicit project valuation method, the present framework relies on fewer assumptions about the investor's preference structure and the nature of project cash flows.

In addition to formulating the project valuation framework, we showed in this paper that, when the investor uses either a preference functional or a mean-risk model, the framework gives consistent values with CCA, wherefore it can be regarded as a generalization of CCA to incomplete markets. In addition, we showed that the framework leads to Hillier's (1963) method when the investor can invest only in a single project and in the risk-free asset, and when she exhibits both constant relative and constant absolute risk aversion.

In our numerical experiments, we examined the pricing behavior of mean-variance and expected utility maximizers when markets were assumed to abide by the standard CAPM. We observed that many of the phenomena occurring in a two-period setting also took place in the multi-period setting. For example, we observed that, when compared to the situation without securities, the presence of securities typically resulted in lower project values due to imposed opportunity costs. Furthermore, replicating portfolios led to project values that were independent of the investor's risk attitude. We also observed that the value of a project, when it did not have a replicating portfolio, could either increase or decrease with increasing risk aversion, depending on its correlation with the rest of the investment portfolio. When the investor became less risk averse, project values converged towards prices that were different from the prices that a risk-neutral investor would give to the projects.

This work suggests several areas for further research. For example, it would be relevant to examine the effect of alternative security pricing models, such as the multi-period models of Merton (1973), Stapleton and Subrahmanyam (1978), and Levy and Samuelson (1992), on project values and optimal trading strategies. Also, the convergence behavior of project values demands more analysis: the results from Gustafsson et al. (2004) indicate that in a two-period setting project values for a mean-variance investor converge towards the projects' CAPM prices as the investor becomes less risk averse; a similar behavior could potentially be proven in the multi-period case.

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Appendix

Proofs of Propositions 1, 2, 3 and 5

Proof of Proposition 1: Let $\mathbf{x}_{\omega}^{*}$ denote the portfolio of securities in state ω that replicates the cash flow pattern of the next period. Let $P_j(\omega) = \sum_{i=0}^{n} S_i(\omega) x_{\omega,i}^{*}$ be the price of this replicating portfolio. Let $\mathbf{x}_{\omega}^{**}$ be the portfolio whose cash flow is equal to the value of the replicating portfolio in each state of the next period. (For convenience, let us assume that $\mathbf{x}_{\omega}^{**} = \mathbf{0}$ if ω belongs to the last or the second last period and that $\mathbf{x}_{\omega}^{*} = \mathbf{0}$ if ω belongs to the last period.) Then, $\mathbf{x}_{\omega}^{***} = \mathbf{x}_{\omega}^{*} + \mathbf{x}_{\omega}^{**}$ is the portfolio that replicates the cash flows of the next period and which can be used to construct replicating portfolios for the period after the next period. Let us then define $\mathbf{x}_{\omega}^{****}$ so that it is equal to $\mathbf{x}_{\omega}^{***}$ if ω belongs to the last or the second last period, and in other periods it is the portfolio whose cash flow is equal to the value of the replicating portfolio $\mathbf{x}_{\omega'}^{****}$ in each state ω' of the next period. If there is a replicating trading strategy for the project, there exists portfolios $\mathbf{x}_{\omega}^{****}$ for all states ω. Thus, the time-0 portfolio $\mathbf{x}_{\omega_0}^{****}$ can be used to construct a replicating strategy for the project. Its price is $\sum_{i=0}^{n} S_i(\omega_0) x_{\omega_0,i}^{****}$. Since a shorted replicating trading strategy, which can be created using portfolio $-\mathbf{x}_{\omega_0}^{****}$, nullifies the cash flows of the project, the setting where the investor invests in the project and shorts $\mathbf{x}_{\omega_0}^{****}$ is identical with the situation where the investor cannot invest in the project and receives money equal to $-C_j^0 + \sum_{i=0}^{n} S_i(\omega_0) x_{\omega_0,i}^{****}$, where C_j^0 is the investment cost of the project. When the time-0 budget in the setting without the project is $b(\omega_0)$, the respective budget is $b(\omega_0) - C_j^0 + \sum_{i=0}^{n} S_i(\omega_0) x_{\omega_0,i}^{****}$ with the project and portfolio $-\mathbf{x}_{\omega_0}^{****}$. Hence, the breakeven buying price of the project is given by $b(\omega_0) - C_j^0 + \sum_{i=0}^{n} S_i(\omega_0) x_{\omega_0,i}^{****} - v_j^b = b(\omega_0)$, or $v_j^b = -C_j^0 + \sum_{i=0}^{n} S_i(\omega_0) x_{\omega_0,i}^{****}$, which is also the CCA price of the project. Conversely, in the setting with the project and budget $b(\omega_0)$, we can nullify the project cash flows by shorting $\mathbf{x}_{\omega_0}^{****}$ and obtain an effective budget of

$b\left(\omega_0\right) - C_j^0 + \sum_{i=0}^{n} S_i(\omega_0) x_{\omega_0,i}^{****}$. Hence, in the setting without the project, we will obtain an equally desirable portfolio by increasing the budget by $v_j^s = -C_j^0 + \sum_{i=0}^{n} S_i(\omega_0) x_{\omega_0,i}^{****}$, which is both the breakeven selling price and the CCA price of the project. Q.E.D.

Proof of Proposition 2: Since the investor can invest only in a single project and in the risk-free asset, in the setting without the project the investor invests her entire budget in the risk-free asset; consequently, the portfolio is riskless and is its own certainty equivalent, i.e. $V^- = (1+s_T)^T b(\omega_0) + O$, s_T is the time-T spot rate and O is the future value of the budgets at states other than ω_0. Also, changing the time-0 budget to $b(\omega_0) + \delta$ will give a portfolio value of $V^* = (1+s_T)^T \left(b(\omega_0) + \delta\right) + O$ without the project. Setting $V^+ = V^*$ we obtain $V^+ = V^- + (1+s_T)^T \delta$, wherefrom it follows $\frac{V^+ - V^-}{(1+s_T)^T} = \delta$, which is also, by definition, the breakeven selling price of the project. For a mean-risk investor, the proposition follows from the observation that, when the risk-constraint is not binding, the preference model corresponds to a preference functional model with $CE[X] = E[X]$. If the risk constraint prevents the starting of the project, then $V^+ = V^- = (1+s_T)^T b(\omega_0) + O$, implying a zero value for the project. Q.E.D.

Proof of Proposition 3: When the investor can invest only in a single project and in the risk-free asset, we have $V^- = (1+s_T)^T b(\omega_0) + O$, where s_T is the time-T spot rate and O is the future value of the budgets at states other than ω_0. Also, we can separate the optimal terminal wealth level with the project to random variable X, which equals to project's future value (project's cash flows with the accrued interest), and to f that is the future value of the budgets. Since $CE\left[X+f\right] = CE\left[X\right] + f$, and denoting $CE[X]$ by CE_T, we have $V^+ = CE_T + f$ and $V^- = f$. Increasing the budget by δ both with and without the project will now increase the respective certainty equivalent by $(1+s_T)^T \delta$. Let V^* be the certainty equivalent of the optimal portfolio when the investor invests in the project and the time-0 budget is $b(\omega_0) - \delta$. Then, $V^* = V^+ - (1+s_T)^T \delta$. Setting $V^- = V^*$, we have $V^- = V^+ - (1+s_T)^T \delta$, and hence we have $\frac{V^+ - V^-}{(1+s_T)^T} = \delta$, which is also the breakeven buying price of the project. Noticing that $V^+ - V^- = CE_T$, we immediately obtain the desired formula. Using further this observation and Proposition 2, the proposition is also proved for the breakeven selling price. For a mean-risk investor, the proposition follows from the observation that, when the risk-constraint is not binding, the preference model corresponds to a preference functional model with $CE[X] = E[X]$. If the risk constraint prevents the starting of the project, the project will have a zero value. Q.E.D.

Proof of Proposition 5: The proposition is proven with a three-period counterexample. Consider an investor with an intertemporal utility function $u^*(c_0, c_1, c_2) = 5 - e^{-1c_1} - e^{-0.95c_2} - e^{-0.9c_3}$, where we have assumed CARA and additive independence for the sake of analogy with Smith and Nau (1995). Note that the parameters of the utility function reflect the investor's risk aversion and the investor's subjective perception about the relative importance of the cash flows received at different periods, and hence they can be selected independently of the risk-free interest rate or available securities. Consider then a project that yields a cash flow stream (–1, 1.1, 0) with 50% and a cash flow stream $(-1, 0, 1.22)$ with 50%. There is also a riskless zero-coupon bond yielding the cash flow stream $(-1, 0, 1.21)$ for sure. The risk-free interest rate is 10%, the time-0 budget is 1, and no other securities are available. The terminal wealth level with the project is 1.21 with 50% and 1.22 with 50%. The bond yields a terminal wealth level of 1.21 for sure. Note that the terminal wealth level implied by the project stochastically dominates the one implied by the bond. The expected intertemporal utility for the project is 0.939 and for the bond 0.945. Thus, the bond is preferred to the project even though the project's terminal wealth level stochastically dominates that of the bond. Q.E.D.