

STABILIZATION OF AN OVERLOADED QUEUEING NETWORK USING MEASUREMENT-BASED ADMISSION CONTROL

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Abstract

Admission control can be employed to avoid congestion in queueing networks subject to overload. In distributed networks the admission decisions are often based on imperfect measurements on the network state. This paper studies how the lack of complete state information affects the system performance by considering a simple network model for distributed admission control. The stability region of the network is characterized and it is shown how feedback signaling makes the system very sensitive to its parameters.

Keywords: queueing network; admission control; stability; overload; perturbed Markov process

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1. Introduction

Consider an overloaded queueing network where the incoming traffic exceeds the service capacity over a long time period. In this case it is often necessary to employ admission control to avoid the network to become fully congested. Many networks of practical interest are composed of subnetworks, not all of which are administered by a single party. In such a network the admission controller seldom has complete up-to-date system information available. Instead, the admission decisions must be based on partial measurements on the network state.

This paper studies the effect of imperfect information to the performance of the admission control scheme. Typical performance measures for well-dimensioned networks in this kind of setting include the average amount of rejected traffic per unit time, and the mean proportion of time the network load is undesirably high. However, assuming the network under study is subjected to long-term overload, there is another performance criterion that must first be analyzed, namely: *If the network is subjected to a stationary load exceeding the service capacity, how strict admission control rules should one set in order to stabilize the system?*

To deal with the question mathematically, it is assumed that the network can be modeled using the simplest nontrivial model for a distributed network, the two-node tandem network with independent and exponential service times and unlimited buffers. The network state is denoted by $X = (X_1, X_2)$ where X_i is the number of jobs in node i . It is assumed that the admission control can be modeled so that the input to the system is a Poisson process with a stochastic time-varying intensity, the intensity $\lambda = \lambda(X)$ being a function of the network state.

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The lack of complete state information is reflected in the model by assuming that the input rate λ is a function of only one of the X_i . If $\lambda(X) = \lambda(X_1)$, then the analysis of the system can be reduced to the study of birth–death processes, which are well understood. This is why in the following it is always assumed that $\lambda(X) = \lambda(X_2)$, so that the admission control introduces a feedback signaling loop to the system. For example, one can model a network where arriving traffic is rejected when the size of the second buffer exceeds a threshold level K by setting $\lambda(X) = 1(X_2 \leq K)$, see Figure 1. In order to also cover more complex admission policies with multiple thresholds and thinning of input traffic, the shape of $\lambda(X_2)$ will not be restricted in any way.

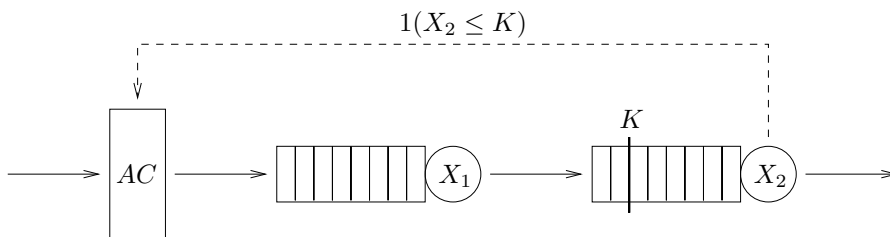


FIGURE 1: Admission control based on the single threshold level K .

More precisely, X is defined as a continuous-time stochastic process as follows. Let λ be a nonnegative function on \mathbb{Z}_+ and $\mu_1, \mu_2 > 0$. Define the transition rates $q(x, y)$ for $x \neq y, x, y \in \mathbb{Z}_+^2$ by

$$q(x, y) = \begin{cases} \lambda(x_2), & y = x + e_1, \\ \mu_1, & y = x - e_1 + e_2 \geq 0, \\ \mu_2, & y = x - e_2 \geq 0, \\ 0, & \text{otherwise,} \end{cases} \quad (1)$$

where e_i denotes the i -th unit vector of \mathbb{Z}_+^2 . As usual, set $q(x, x) = -q(x)$, where the transition rate out of state x is defined by

$$q(x) = \sum_{y \neq x} q(x, y).$$

It is clear that $q(x) < \infty$ for all x , so using the minimal construction [3, 5] the rates $q(x, y)$ define a unique Markov process X on $\mathbb{Z}_+^2 \cup \{\kappa\}$. Here κ denotes an additional state not in \mathbb{Z}_+^2 with $T_\kappa = \inf\{t > 0 : X(t) = \kappa\} \leq \infty$ being the time of explosion of X . The notation $S(\lambda, \mu_1, \mu_2)$ will be used for the set of transition rates corresponding to the triple (λ, μ_1, μ_2) , and the system $S(\lambda, \mu_1, \mu_2)$ is said to be *stable* if the corresponding Markov process is ergodic, that is, irreducible and positive recurrent.

In its most general form, the stability problem may now be stated as

(P1) Characterize the set of all $(\lambda, \mu_1, \mu_2) \in \mathbb{R}_+^{\mathbb{Z}_+} \times \mathbb{R}_+ \times \mathbb{R}_+$ for which $S(\lambda, \mu_1, \mu_2)$ is stable.

Specializing to networks with threshold-based admission control, the offered traffic is assumed to arrive at unit rate, without loss of generality. Denoting the admission threshold by K , (P1) now takes the form

(P2) For each $(\mu_1, \mu_2) \in \mathbb{R}_+^2$, determine for which values of $K \in \mathbb{Z}_+ \cup \{\infty\}$, if any, the system $S(1(\cdot \leq K), \mu_1, \mu_2)$ is stable.

Note that the system corresponding to $K = \infty$ in (P2) is the ordinary tandem queue, for which it is well-known that $\min(\mu_1, \mu_2) > 1$ is sufficient and necessary for stability. On the other hand, assuming overload, answering the question on the existence of a threshold level that can stabilize the system is not as straightforward.

The queueing systems literature includes a vast amount of work on various admission control mechanisms. However, most earlier studies on tandem networks require at least one of the buffers to be finite, so that the two-dimensional nature of the problem can partly be reduced to one-dimensional by applying matrix-geometric methods [9]. For networks with unlimited buffers and state-dependent service times, Bambos and Walrand [4] provide stability results extending to non-Markovian systems, however ruling out networks with the type of feedback signaling loop present here. Concerning the network $S(\lambda, \mu_1, \mu_2)$ defined above, the compensation approach introduced by Adan, Wessels, and Zijm [1] can be used for computing the invariant measure in the special case where λ is constant on $\{n \in \mathbb{Z}_+ : n \geq 1\}$. For more general input rates, Leskelä and Resing [7] have described a numerical method for calculating stationary performance characteristics of the system. Altman, Avrachenkov, and Núñez Queija [2] have recently introduced perturbation techniques that seem appropriate for asymptotically analyzing the behavior of $S(\lambda, \mu_1, \mu_2)$ under suitable parameter scaling.

This paper partially answers (P1) by deriving sufficient and necessary conditions for stability. Furthermore, by showing that in the special case of threshold-based admission control the sufficient and necessary conditions coincide, a complete solution of (P2) is given. In addition, the sensitivity of the system is analyzed with respect to changes in the service rates and it is shown how acceleration of one of the servers may, rather paradoxically, unstabilize the system.

2. A sufficient condition for stability

Let S be a countable set. For a function $V : S \rightarrow \mathbb{R}$, denote

$$\lim_{x \rightarrow \infty} V(x) = \infty$$

if the set $\{x : V(x) \leq M\}$ is finite for all $M \in \mathbb{R}$. Further, the mean drift of V with respect to transition rates $q(x, y)$ is denoted by

$$\Delta V(x) = \sum_{y \neq x} (V(y) - V(x)) q(x, y), \quad (2)$$

assuming the sum on the right-hand side converges.

Definition 1. A map $V : S \rightarrow \mathbb{R}$ is called a *Lyapunov function* for q if it satisfies the following conditions called *Foster's criteria*:

(F1) $\sum_{y \neq x} |V(y) - V(x)| q(x, y) < \infty$ for all x (so that the right-hand side of (2) makes sense).

(F2) $\lim_{x \rightarrow \infty} V(x) = \infty$.

(F3) There is a finite set $S_0 \subset S$ such that $\sup_{x \in S \setminus S_0} \Delta V(x) < 0$.

The following continuous-time analogue of Foster's classical theorem [6] provides a sufficient condition for stability.

Theorem 1. (Tweedie [10].) *Let X be an irreducible Markov process on a countable state space S generated by transition rates $q(x, y)$ so that $q(x) < \infty$ for all x . The existence of a Lyapunov function for q is then sufficient for X to be ergodic.*

Considering the system $S(\lambda, \mu_1, \mu_2)$, let $q(x, y)$ be as defined in (1). Assume V is a function on \mathbb{Z}_+^2 of the form $V(x) = x_1 + v(x_2)$ for some $v : \mathbb{Z}_+ \rightarrow \mathbb{R}$ with $v(0) = 0$. Searching for a Lyapunov function of this type, let us fix a number $r > 0$ and require that the mean drift of V with respect to q satisfies

$$\Delta V(x) = -r \quad \text{for all } x \text{ with } x_1 > 0. \quad (3)$$

It is straightforward to verify that (3) is equivalent to

$$\begin{aligned} v(1) &= 1 - (\lambda(0) + r)/\mu_1, \\ v(n+1) &= 1 - (\lambda(n) + r)/\mu_1 + (1 + \mu_2/\mu_1)v(n) - \mu_2/\mu_1 v(n-1), \quad n \geq 1. \end{aligned}$$

Denoting $\alpha(n) = 1 - (\lambda(n) + r)/\mu_1$ and $w(n) = v(n+1) - v(n)$, the above difference equation can be written as $w(n) = \alpha(n) + \mu_2/\mu_1 w(n-1)$ for $n \geq 1$, with $w(0) = \alpha(0)$. Thus, $w(n) = \sum_{k=0}^{n-1} \alpha(k) (\mu_1/\mu_2)^{k-n}$, so that

$$v(n) = \sum_{j=0}^{n-1} w(j) = \sum_{j=0}^{n-1} \sum_{k=0}^j \alpha(k) (\mu_1/\mu_2)^{k-j},$$

and we conclude that (3) defines for each $r > 0$ the function

$$V_r(x) = x_1 + \sum_{j=0}^{x_2-1} \sum_{k=0}^j (1 - (\lambda(k) + r)/\mu_1) (\mu_1/\mu_2)^{k-j}.$$

Thus we have constructed a family of functions $\mathcal{V} = \{V_r : r > 0\}$ whose elements satisfy $\sup_{x: x_1 > 0} \Delta V_r(x) < 0$, so there are hopes that V_r might satisfy (F3) for a suitably chosen finite subset of \mathbb{Z}_+^2 . In order to investigate whether this is the case, let us study the mean drift of V_r for $x = (0, n)$ with $n \geq 1$,

$$\Delta V_r(0, n) = \lambda(n) - \mu_2(v_r(n) - v_r(n-1)). \quad (4)$$

Definition 2. For $z \geq 0$, denote $Z_n \sim \text{geom}_n(z)$ if Z_n is a random variable on $\mathbb{Z} \cap [0, n]$ with $P(Z_n = j) = cz^j$. For $0 \leq z \leq 1$, denote $Z \sim \text{geom}(z)$ if the random variable Z on \mathbb{Z}_+ satisfies $P(Z = j) = (1-z)z^j$.

In this paper, Z_n and Z will always represent generic random variables with distributions $\text{geom}_n(\mu_1/\mu_2)$ and $\text{geom}(\mu_1/\mu_2)$, respectively. Using this notation, one may verify that (4) can be alternatively written as

$$\Delta V_r(0, n) = \frac{\mathbb{E} \lambda(Z_n) - \mu_2(1 - r/\mu_1) P(Z_n > 0)}{P(Z_n = n)}, \quad Z_n \sim \text{geom}_n(\mu_1/\mu_2). \quad (5)$$

Theorem 2. *The family $\mathcal{V} = \{V_r : r > 0\}$ contains a Lyapunov function for $S(\lambda, \mu_1, \mu_2)$ if and only if*

$$\overline{\lim} \mathbb{E} \lambda(Z_n) < \min(\mu_1, \mu_2), \quad Z_n \sim \text{geom}_n(\mu_1/\mu_2). \quad (6)$$

In particular, if $\lambda(0) > 0$, then (6) is sufficient for the stability of $S(\lambda, \mu_1, \mu_2)$.

The proof of the theorem will utilize the following two lemmas.

Lemma 1. *Condition (6) is equivalent to $\overline{\lim} \Delta V_r(0, n) < 0$ for some $r > 0$.*

Proof. Let $Z_n \sim \text{geom}_n(\mu_1/\mu_2)$ for $n \geq 0$. Observe first that since $\lim \mathbb{P}(Z_n > 0) = \min(1, \mu_1/\mu_2)$,

$$\overline{\lim} \mathbb{E} \lambda(Z_n) - \min(\mu_1, \mu_2) = \overline{\lim} \{\mathbb{E} \lambda(Z_n) - \mu_2 \mathbb{P}(Z_n > 0)\}. \quad (7)$$

Assume now that (6) holds. Then we can choose an $r > 0$ so that $\mathbb{E} \lambda(Z_n) - \mu_2 \mathbb{P}(Z_n > 0) \leq -r$ for n large enough. It follows that $\overline{\lim} \Delta V_r(0, n) < 0$, since using (5) we see that eventually for large n ,

$$\Delta V_r(0, n) \leq \frac{-r + r \mu_2/\mu_1 \mathbb{P}(Z_n > 0)}{\mathbb{P}(Z_n = n)} = -r.$$

For the other direction, assume $\overline{\lim} \Delta V_r(0, n) < 0$ for some $r > 0$. Then there is an $s \in (0, r)$ so that for n large enough, $\Delta V_r(0, n) \leq -s$, and applying (5),

$$\frac{\mathbb{E} \lambda(Z_n) - \mu_2(1 - s/\mu_1) \mathbb{P}(Z_n > 0)}{\mathbb{P}(Z_n = n)} \leq \Delta V_r(0, n) \leq -s.$$

This shows that

$$\mathbb{E} \lambda(Z_n) - \mu_2 \mathbb{P}(Z_n > 0) \leq -s(\mathbb{P}(Z_n = n) + \mu_2/\mu_1 \mathbb{P}(Z_n > 0)) = -s$$

for all n large enough, and in light of (7) it follows that $\overline{\lim} \mathbb{E} \lambda(Z_n) < \min(\mu_1, \mu_2)$.

Lemma 2. *Let f be a function of the form $f(x) = u(x_1) + v(x_2)$ for some $u, v : \mathbb{Z}_+ \rightarrow \mathbb{R}$. Then $\lim_{x \rightarrow \infty} f(x) = \infty$ if and only if $\lim_{x_1 \rightarrow \infty} u(x_1) = \infty$ and $\lim_{x_2 \rightarrow \infty} v(x_2) = \infty$.*

Proof. Assume $\lim u(x_1) = \lim v(x_2) = \infty$, and fix an $M \in \mathbb{R}$. Since $u_0 = \inf u(x_1)$ and $v_0 = \inf v(x_2)$ are finite, we can choose m_1 and m_2 such that $u(x_1) > M - v_0$ for all $x_1 > m_1$, and $v(x_2) > M - u_0$ for all $x_2 > m_2$. Hence, $f(x) > M$ if either $x_1 > m_1$ or $x_2 > m_2$, so that the set $\{x : f(x) \leq M\} \subset [0, m_1] \times [0, m_2]$ is finite. Since M was arbitrary, $\lim_{x \rightarrow \infty} f(x) = \infty$.

Suppose next that $\lim_{x \rightarrow \infty} f(x) = \infty$. Then if $\underline{\lim} u(x_1) < \infty$, there is a $c \in \mathbb{R}$ so that $S = \{x_1 : u(x_1) \leq c\}$ is infinite. This implies that $\{x : f(x) \leq c + v(0)\} \supset S \times \{0\}$ is infinite, contrary to the assumption $\lim_{x \rightarrow \infty} f(x) = \infty$. Thus, $\underline{\lim} u(x_1) = \infty$. Similarly, one proves that $\underline{\lim} v(x_2) = \infty$.

Proof of Theorem 2. Let $r > 0$ and assume $V_r \in \mathcal{V}$ is a Lyapunov function for q . Let S_0 be a finite set so that (F3) holds. Then $\{0\} \times (n_0, \infty) \subset S_0^c$ for some n_0 , which implies

$$\overline{\lim} \Delta V_r(0, n) \leq \sup_{n > n_0} \Delta V_r(0, n) \leq \sup_{x \in S_0^c} \Delta V_r(x) < 0.$$

By Lemma 1, this implies (6).

For the other direction, assume that (6) holds. Applying Lemma 1, we can pick an $r > 0$ so that $\overline{\lim} \Delta V_r(0, n) < 0$. Hence, there is an n_0 and an $\epsilon > 0$ so that $\Delta V_r(0, n) \leq -\epsilon$ for all $n > n_0$. Denoting $S_0 = \{0\} \times [0, n_0]$, it follows that

$$\sup_{x \in S_0^c} \Delta V_r(x) = \max\left\{ \sup_{n > n_0} \Delta V_r(0, n), \sup_{x: x_1 > 0} \Delta V_r(x) \right\} \leq \max\{-\epsilon, -r\} < 0,$$

since by the construction of V_r , $\Delta V_r(x) = -r$ for all x with $x_1 > 0$. Thus, V_r satisfies (F3). Next, observe that using (4),

$$\lambda(n) - \mu_2(v_r(n) - v_r(n-1)) = \Delta V_r(0, n) \leq -\epsilon \quad \text{for } n > n_0.$$

This shows that $v_r(n) - v_r(n-1) \geq \epsilon/\mu_2$ eventually for large n , so that $\lim_{n \rightarrow \infty} v_r(n) = \infty$. By Lemma 2, we conclude that V_r satisfies (F2). Further, (F1) holds trivially since the set $\{x : q(x, y) > 0\}$ is finite for all x . Thus, V_r is a Lyapunov function for q . Finally, note that X is irreducible when $\lambda(0) > 0$. Hence, application of Theorem 1 now completes the proof.

3. Necessary conditions for stability

Assume $\lambda(0) > 0$ so that the system $S(\lambda, \mu_1, \mu_2)$ is irreducible. In the previous section we saw that

$$\overline{\lim} \mathbb{E} \lambda(Z_n) < \min(\mu_1, \mu_2), \quad Z_n \sim \text{geom}_n(\mu_1/\mu_2),$$

is sufficient for the stability of $S(\lambda, \mu_1, \mu_2)$. This section is devoted to studying whether the above condition is also necessary for stability.

3.1. Small perturbations of Markov processes

This section studies how ergodicity is preserved under small perturbations of generators of Markov processes. If $q(x, y)$ and $q'(x, y)$ are generators of Markov processes on a countable state space S , denote

$$D(q, q') = \{x : q(x, y) \neq q'(x, y) \text{ for some } y\}$$

and

$$\overline{D}(q, q') = D(q, q') \cup \{y : q(x, y) > 0 \text{ or } q'(x, y) > 0 \text{ for some } x \in D(q, q')\}.$$

Further, for $F \subset S$ let

$$T_F = \inf\{t > 0 : X(t-) \neq X(t), X(t) \in F\},$$

with the convention $\inf \emptyset = \infty$, and $T_x = T_{\{x\}}$ for $x \in S$.

Lemma 3. *Let X and X' be irreducible Markov processes on a countable state space S generated by $q(x, y)$ and $q'(x, y)$, respectively, with $q(x), q'(x) < \infty$ for all x . Assume that $\overline{D}(q, q')$ is finite. Then X is ergodic if and only if X' is ergodic.*

Proof. By symmetry, it is sufficient to show that the ergodicity of X' implies that of X . So, assume X' is ergodic, and let x be a state in $D = D(q, q')$. Denote the first jump time of X by $\tau = \inf\{t > 0 : X(t-) \neq X(t)\}$. By irreducibility, $\mathbb{E}_x \tau < \infty$, so by the strong Markov property,

$$\mathbb{E}_x T_D = \mathbb{E}_x \tau + \mathbb{E}_x(\mathbb{E}_{X(\tau)} T_D; X(\tau) \notin D).$$

Since $q(x, y)$ and $q'(x, y)$ coincide outside D , and $\mathbb{P}_x(X(\tau) \in \overline{D}) = 1$,

$$\mathbb{E}_x(\mathbb{E}_{X(\tau)} T_D; X(\tau) \notin D) = \mathbb{E}_x(\mathbb{E}_{X(\tau)} T'_D; X(\tau) \in \overline{D} \setminus D) \leq \sup_{y \in \overline{D}} \mathbb{E}_y T'_D.$$

Since X' is ergodic and \overline{D} is finite, so is the right-hand side in the above inequality, and we conclude $\mathbb{E}_x T_D < \infty$. Because X is irreducible, this property implies that X is positive recurrent (Meyn [8], Theorem 4.3:(ii) and Theorem 4.4).

3.2. Bottleneck at node 1

Assume $\mu_1 < \mu_2$. Intuition suggests that in this case the stability of the system depends on whether or not the buffer content at node 1 grows to infinity. Observe that during the periods of time where node 1 remains busy, the input to node 2 is a Poisson process with rate μ_1 . The approach here is to compare the original process to a saturated system where node 2 gets input at rate μ_1 also during the time periods where node 1 is empty, and show that the stability regions for the two systems are close to each other. With this goal in mind, let us introduce another model family denoted by $S^N(\lambda, \mu_1, \mu_2)$. Fix a nonnegative integer N , and define for $x \neq y$,

$$q^N(x, y) = q(x, y) + \mu_1 1(x_1 = 0, x_2 < N, y = x + e_2).$$

It is clear that when $\lambda(0) > 0$, the transition rates $q^N(x, y)$ define using the minimal construction an irreducible Markov process X^N on $\mathbb{Z}_+^2 \cup \{\kappa\}$. By Lemma 3 we know that the stability of $S^N(\lambda, \mu_1, \mu_2)$ is equivalent to that of $S(\lambda, \mu_1, \mu_2)$. Further, by letting N approach infinity, $S^N(\lambda, \mu_1, \mu_2)$ will resemble a network where node 2 receives stationary input at rate μ_1 .

Lemma 4. *Assume that $S^N(\lambda, \mu_1, \mu_2)$ is stable. Then the stationary distribution of X^N satisfies*

$$\mathbb{E} \lambda(X_2^N) = \mu_2 \mathbb{P}(X_2^N > 0) - \mu_1 \mathbb{P}(X_1^N = 0, X_2^N < N), \quad (8)$$

$$\mathbb{P}(X_2^N = n) = (\mu_1/\mu_2)^n \mathbb{P}(X_2^N = 0) - 1(n > N) \sum_{j=N}^{n-1} (\mu_1/\mu_2)^{n-j} \mathbb{P}(X_1^N = 0, X_2^N = j), \quad (9)$$

and for all real-valued f on \mathbb{Z}_+ ,

$$\mathbb{E}(f(X_2^N); X_2^N \leq N) = \mathbb{E} f(Z_N) \mathbb{P}(X_2^N \leq N), \quad Z_N \sim \text{geom}_N(\mu_1/\mu_2). \quad (10)$$

Proof. Starting from the balance equations for X^N , it is not hard to check that $\mathbb{E} \lambda(X_2) = \mu_1 \mathbb{P}(X_1^N > 0)$, and

$$\mu_1 \mathbb{P}(X_1^N > 0) + \mu_1 \mathbb{P}(X_1^N = 0, X_2^N < N) = \mu_2 \mathbb{P}(X_2^N > 0),$$

showing that (8) is true. Further, it is straightforward to verify that for all n ,

$$\mathbb{P}(X_2^N = n + 1) = \mu_1/\mu_2 [\mathbb{P}(X_2^N = n) - 1(n \geq N) \mathbb{P}(X_1^N = 0, X_2^N = n)],$$

from which (9) and (10) follow.

Theorem 3. *Assume $\mu_1 < \mu_2$, and let $Z \sim \text{geom}(\mu_1/\mu_2)$. Then*

$$\begin{aligned} \mathbb{E} \lambda(Z) < \mu_1 &\implies S(\lambda, \mu_1, \mu_2) \text{ is stable,} \\ \mathbb{E} \lambda(Z) > \mu_1 &\implies S(\lambda, \mu_1, \mu_2) \text{ is unstable.} \end{aligned}$$

Proof. Let $Z_N \sim \text{geom}_N(\mu_1/\mu_2)$. Because $\mu_1 < \mu_2$ it follows that $\lim \mathbb{E} \lambda(Z_N) = \mathbb{E} \lambda(Z)$. The first statement now follows from Theorem 2. To prove the second claim, assume that $S(\lambda, \mu_1, \mu_2)$ is stable. Then by Lemma 3, so is $S^N(\lambda, \mu_1, \mu_2)$ for each N . Applying (8) and (10) we see that

$$\mathbb{E} \lambda(Z_N) \mathbb{P}(X_2^N \leq N) = \mathbb{E}(\lambda(X_2^N); X_2^N \leq N) \leq \mu_2 \mathbb{P}(X_2^N > 0). \quad (11)$$

Next, (9) implies

$$\mathbb{P}(X_2^N > N) \leq \sum_{n>N} (\mu_1/\mu_2)^n \quad \text{for all } N,$$

so that $\lim \mathbb{P}(X_2^N \leq N) = 1$. This observation combined with (9) implies that

$$\mathbb{P}(X_2^N = 0) = \frac{\mathbb{P}(X_2^N \leq N)}{\sum_{n=0}^N (\mu_1/\mu_2)^n} \longrightarrow 1 - \mu_1/\mu_2,$$

as $N \rightarrow \infty$. Hence, $\lim \mathbb{P}(X_2^N > 0) = \mu_1/\mu_2$. Letting $N \rightarrow \infty$ on both sides of (11) now shows that $\mathbb{E} \lambda(Z) \leq \mu_1$.

3.3. Bottleneck at node 2

To study necessary stability conditions for the system when $\mu_1 \geq \mu_2$, the following asymptotical property of truncated geometric random variables will be useful.

Lemma 5. *Let $Z_n \sim \text{geom}_n(z)$ with $z \geq 1$. Then for all nonnegative functions f on \mathbb{Z}_+ ,*

$$\underline{\lim} f(n) \leq \underline{\lim} \mathbb{E} f(Z_n) \leq \overline{\lim} \mathbb{E} f(Z_n) \leq \overline{\lim} f(n).$$

Proof. Without loss of generality, assume $\overline{\lim} f(n) < \infty$. Choose a number r so that $\overline{\lim} f(n) < r$. Then there is an n_0 so that $f(n) \leq r$ for all $n > n_0$, and thus

$$\mathbb{E} f(Z_n) \leq r + \frac{\sum_{j=0}^{n_0} (f(n) - r) z^j}{\sum_{j=0}^n z^j} \quad \text{for } n > n_0.$$

This implies that $\overline{\lim} \mathbb{E} f(Z_n) \leq r$, so by letting $r \downarrow \overline{\lim} f(n)$, it follows that $\overline{\lim} \mathbb{E} f(Z_n) \leq \overline{\lim} f(n)$. The proof is completed by applying this inequality to $-f$.

Theorem 4. *Assume $\mu_1 \geq \mu_2$, and let $Z_n \sim \text{geom}_n(\mu_1/\mu_2)$. Then*

$$\begin{aligned} \overline{\lim} \mathbb{E} \lambda(Z_n) < \mu_2 &\implies S(\lambda, \mu_1, \mu_2) \text{ is stable,} \\ \underline{\lim} \lambda(n) > \mu_2 &\implies S(\lambda, \mu_1, \mu_2) \text{ is unstable.} \end{aligned}$$

Especially, if $\lim \lambda(n)$ exists, then

$$\begin{aligned}\lim \lambda(n) < \mu_2 &\implies S(\lambda, \mu_1, \mu_2) \text{ is stable,} \\ \lim \lambda(n) > \mu_2 &\implies S(\lambda, \mu_1, \mu_2) \text{ is unstable.}\end{aligned}$$

Proof. The first statement follows from Theorem 2. To prove the second implication, assume $S(\lambda, \mu_1, \mu_2)$ is stable. Then by Lemma 3, so is $S^N(\lambda, \mu_1, \mu_2)$ for each N . Choose an $r \in \mathbb{R}$ so that $r < \underline{\lim} \lambda(n)$. It follows by Lemma 5 that $\underline{\lim} \lambda(n) \leq \underline{\lim} \mathbb{E} \lambda(Z_n)$. Thus, $\lambda(N) \geq r$ and $\mathbb{E} \lambda(Z_N) \geq r$ for all N large enough. Thus, for all such N ,

$$\begin{aligned}\mathbb{E} \lambda(X_2^N) &= \mathbb{E}(\lambda(X_2^N); X_2^N > N) + \mathbb{E} \lambda(Z_N) \mathbb{P}(X_2^N \leq N) \\ &\geq r \mathbb{P}(X_2^N > N) + r \mathbb{P}(X_2^N \leq N) = r,\end{aligned}$$

so that $\underline{\lim} \mathbb{E} \lambda(X_2^N) \geq r$. Letting r approach $\underline{\lim} \lambda(n)$ we see that $\underline{\lim} \mathbb{E} \lambda(X_2^N) \geq \underline{\lim} \lambda(n)$. Next, $\lim \mathbb{P}(X_2^N > 0) = 1$, because $\mathbb{P}(X_2^N = 0) \leq (\sum_{j=0}^N (\mu_1/\mu_2)^j)^{-1}$ by (9). Moreover, equality (8) shows that $\mathbb{E} \lambda(X_2^N) \leq \mu_2 \mathbb{P}(X_2^N > 0)$ for all N , so that

$$\underline{\lim} \lambda(n) \leq \underline{\lim} \mathbb{E} \lambda(X_2^N) \leq \underline{\lim} \mu_2 \mathbb{P}(X_2^N > 0) = \mu_2,$$

which proves the second claim. In the special case where $\lambda(n)$ has a limit when n tends to infinity, Lemma 5 shows that

$$\underline{\lim} \lambda(n) = \lim \lambda(n) = \overline{\lim} \mathbb{E} \lambda(Z_n),$$

so that the last two implications of the theorem now follow from the first two.

There may exist a substantial gap between the necessary and sufficient stability conditions of Theorem 4 if $\lambda(n)$ is diverging. To gain some insight why characterizing the stability of the system is difficult for such λ , let us consider the behavior of $S(\lambda, \mu_1, \mu_2)$ as μ_1 tends to infinity. Intuition suggests that in this case the system should resemble the single server queue with service rate μ_2 and state-dependent input rate $\lambda(n)$, for which it is known (Asmussen [3], Corollary 2.5) that stability is equivalent to

$$\sum_{n=0}^{\infty} \frac{\lambda(0) \cdots \lambda(n)}{\mu_2^{n+1}} < \infty. \quad (12)$$

Consider for example the input rates $\lambda(n) = a$ for n even, and $\lambda(n) = b$ for n odd, where $0 < a < b$. Then (12) reduces to $\sqrt{ab} < \mu_2$, while with $Z_n \sim \text{geom}_n(\mu_1/\mu_2)$,

$$\underline{\lim} \lambda(n) = a < \frac{\mu_1 b + \mu_2 a}{\mu_1 + \mu_2} = \overline{\lim} \mathbb{E} \lambda(Z_n). \quad (13)$$

Hence, the gap between the necessary and sufficient stability conditions in Theorem 4 grows according to

$$\left[a, \frac{\mu_1 b + \mu_2 a}{\mu_1 + \mu_2} \right] \longrightarrow [a, b], \quad \mu_1 \rightarrow \infty.$$

However, condition (12) may not in general be the correct asymptotical stability characterization of $S(\lambda, \mu_1, \mu_2)$ as $\mu_1 \rightarrow \infty$, due to a fundamental difference between the single-server queue and the tandem network. Namely, if $\lambda(n) = 0$ for some n , then the single-server queue is stable because the queue size cannot exceed n . Obviously, this property is not true for $S(\lambda, \mu_1, \mu_2)$, and this is why the necessary and sufficient stability condition for $S(\lambda, \mu_1, \mu_2)$ must have more complex nature than (12).

3.4. Eventually vanishing input rate function

In most applications it is natural to assume that $\lambda(n)$ becomes eventually zero for large n , so that the admission controller strictly blocks all incoming traffic when the amount of jobs in node 2 becomes too large. In this case $\lim \lambda(n) = 0$, so Theorem 4 shows that for $\mu_1 \geq \mu_2$, $S(\lambda, \mu_1, \mu_2)$ is stable regardless of the shape of the function λ . On the other hand, if node 1 is the bottleneck, then Theorem 3 determines the stability of the system, except in the critical case when $\mathbb{E} \lambda(Z) = \mu_1$. Our intuition about birth–death processes suggests that the system is unstable also in this special case. The validity of this intuition will be proved next. The key to proof is the following lemma which shows that the stability of $S(\lambda, \mu_1, \mu_2)$ implies the stability of the saturated system $S^*(\lambda, \mu_1, \mu_2)$, where node 2 behaves as if node 1 never were empty.

Lemma 6. *Assume $\mu_1 < \mu_2$, and $\lambda(n) = 0$ eventually for large n . If $S(\lambda, \mu_1, \mu_2)$ is stable, then so is the system $S^*(\lambda, \mu_1, \mu_2)$ generated by the transition rates*

$$q^*(x, y) = q(x, y) + \mu_1 1(x_1 = 0, y = x + e_2), \quad x \neq y.$$

Proof. Fix a $K \in \mathbb{Z}_+$ so that $\lambda(n) = 0$ for all $n > K$, and define the transition rates q' by

$$q'(x, y) = q(x, y) + \mu_1 1(x_1 = 0, x_2 > K, y = x + e_2), \quad x \neq y.$$

Because $q'(x) < \infty$ for all x , the rates $q'(x, y)$ define an irreducible Markov process X' on $\mathbb{Z}_+^2 \cup \{\kappa\}$. The first step is to show that X' is ergodic. Note that set of states where q and q' differ is now given by $D(q, q') = \{0\} \times [K + 1, \infty)$. The key to the proof is to observe that the behavior of X' inside $D = D(q, q')$ is similar to a birth–death process with birth rate μ_1 and death rate μ_2 . Denote $x = (0, K + 1)$. Then since $\mu_1 < \mu_2$, it follows that for all $y \in D \setminus \{x\}$,

$$\mathbb{E}_y T'_x = \frac{y_2 - x_2}{\mu_2 - \mu_1}. \quad (14)$$

The ergodicity of X implies $\mathbb{E}_{x-e_2} T'_D = \mathbb{E}_{x-e_2} T_D < \infty$. Next, since $\mathbb{P}_{x-e_2}(T'_D \leq T'_x) = 1$, we can compute using the strong Markov property and (14),

$$\begin{aligned} \mathbb{E}_{x-e_2} T'_x &= \mathbb{E}_{x-e_2} T'_D + \mathbb{E}_{x-e_2}(\mathbb{E}_{X'(T'_D)} T'_x; X'(T'_D) \neq x) \\ &= \mathbb{E}_{x-e_2} T'_D + \mathbb{E}_{x-e_2} \frac{X'_2(T'_D) - x_2}{\mu_2 - \mu_1} \\ &= \mathbb{E}_{x-e_2} T_D + \mathbb{E}_{x-e_2} \frac{X_2(T_D) - x_2}{\mu_2 - \mu_1}. \end{aligned} \quad (15)$$

Since $\mathbb{E}_y T_x = (y_2 - x_2)/\mu_2$ for all $y \in D \setminus \{x\}$, we find in a similar way that

$$\mathbb{E}_{x-e_2} T_x = \mathbb{E}_{x-e_2} T_D + \mathbb{E}_{x-e_2} \frac{X_2(T_D) - x_2}{\mu_2}. \quad (16)$$

Since X is ergodic, comparison of (15) and (16) shows that $\mathbb{E}_{x-e_2} T'_x < \infty$. Conditioning on the first transition of X' now yields

$$\begin{aligned} \mathbb{E}_x T'_x &= \frac{1}{\mu_1 + \mu_2} + \frac{\mu_1}{\mu_1 + \mu_2} \mathbb{E}_{x+e_2} T'_x + \frac{\mu_2}{\mu_1 + \mu_2} \mathbb{E}_{x-e_2} T'_x \\ &= \frac{1}{\mu_1 + \mu_2} + \frac{\mu_1}{\mu_1 + \mu_2} \frac{1}{\mu_2 - \mu_1} + \frac{\mu_2}{\mu_1 + \mu_2} \mathbb{E}_{x-e_2} T'_x, \end{aligned}$$

showing that $E_x T'_x < \infty$. By irreducibility, it now follows that X' is ergodic.

Finally, note that the set $\overline{D}(q', q^*) \subset [0, 1] \times [0, K + 1]$ is finite. Thus, in light of Lemma 3 we may now conclude that the Markov process X^* generated by $q^*(x, y)$ is ergodic.

Theorem 5. *Assume that $\lambda(n) = 0$ eventually for large n .*

(i) *If $\mu_1 < \mu_2$, $S(\lambda, \mu_1, \mu_2)$ is stable if and only if $E \lambda(Z) < \mu_1$ with $Z \sim \text{geom}(\mu_1/\mu_2)$.*

(ii) *If $\mu_1 \geq \mu_2$, $S(\lambda, \mu_1, \mu_2)$ is always stable.*

Proof. In light of Theorems 3 and 4, all we need to show is that the stability of $S(\lambda, \mu_1, \mu_2)$ implies $E \lambda(Z) < \mu_1$ when $\mu_1 < \mu_2$. So, assume $\mu_1 < \mu_2$ and that $S(\lambda, \mu_1, \mu_2)$ is stable. By Lemma 6, so is $S^*(\lambda, \mu_1, \mu_2)$. From the balance equations for X^* it is easy to see that $X_2^* \sim \text{geom}(\mu_1/\mu_2)$. Thus the stationary mean rate of jobs arriving to node 1 equals $E \lambda(Z)$, while the corresponding rate out is equal to $\mu_1 P(X_1^* > 0)$. Because these two quantities must be equal in a stable system, we conclude that

$$E \lambda(Z) = \mu_1 P(X_1^* > 0) < \mu_1,$$

where the last inequality is strict because $P(X_1^* = 0) > 0$ by the ergodicity of X^* .

4. Sensitivity analysis of the stability region

This section focuses on the stability of the system subjected to fluctuations in the system parameters. The treatment here is restricted to the case of eventually vanishing input rates, where Theorem 5 completely characterizes the stable parameter region.

4.1. Sensitivity with respect to varying service rates

The next proposition shows that with nonincreasing input rates, the stability of the system is preserved under speeding up of node 1.

Proposition 1. *Assume λ is nonincreasing and $\lambda(n) = 0$ eventually for large n . Then for all $\mu'_1 \geq \mu_1$,*

$$S(\lambda, \mu_1, \mu_2) \text{ is stable} \implies S(\lambda, \mu'_1, \mu_2) \text{ is stable.}$$

Proof. Assume that $S(\lambda, \mu_1, \mu_2)$ is stable and let $\mu'_1 \geq \mu_1$. If $\mu'_1 \geq \mu_2$, then $S(\lambda, \mu'_1, \mu_2)$ is stable by Theorem 5. On the other hand, if $\mu'_1 < \mu_2$, then also $\mu_1 < \mu_2$, and the necessary condition of Theorem 5 shows that $f(\mu_1/\mu_2) < \mu_1$, where

$$f(x) = (1 - x) \sum_{n=0}^{\infty} \lambda(n) x^n.$$

Because the sequence $\lambda(n)$ is bounded and nonnegative, f is differentiable in $(0, 1)$ with

$$f'(x) = \sum_{n=0}^{\infty} (n+1) (\lambda(n+1) - \lambda(n)) x^n \leq 0,$$

so that $f(\mu'_1/\mu_2) \leq f(\mu_1/\mu_2)$. It follows that $f(\mu'_1/\mu_2) < \mu'_1$, which guarantees the stability of $S(\lambda, \mu'_1, \mu_2)$ by Theorem 5.

To see why it is necessary to require λ to be nonincreasing, consider the following example.

Example 1. Let $\mu_2 = 1$, and assume that $\lambda(n) = 0$ for $n \geq 3$. Then $S(\lambda, \mu_1, \mu_2)$ is stable for all $\mu_1 \geq 1$, and for $\mu_1 \in (0, 1)$, the stability of $S(\lambda, \mu_1, \mu_2)$ is equivalent to

$$\mu_1^{-1}(1 - \mu_1)(\lambda(0) + \lambda(1)\mu_1 + \lambda(2)\mu_1^2) < 1. \quad (17)$$

Figure 2 shows the the left-hand side of (17) as a function of μ_1 , where $\lambda(0) = \lambda(1) = \frac{1}{100}$ and $\lambda(2) = 5$. The plot illustrates that by increasing the service rate μ_1 from $\frac{1}{5}$ to $\frac{1}{2}$ unstabilizes the system.

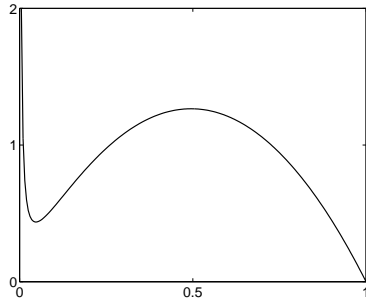


FIGURE 2: The left-hand side of (17) as a function of μ_1 .

Alternatively, we may fix μ_1 and see what happens when μ_2 varies. The following proposition tells a rather surprising result: Even with nonincreasing λ , acceleration of one of the servers may indeed unstabilize the system. The physical intuition behind Proposition 2 is that when μ_2 is very large, the admission controller finds node 2 empty most of the time. This means that the input rate to the system is close to $\lambda(0)$.

Proposition 2. *Assume λ is nonincreasing and $\lambda(n) = 0$ eventually for large n and fix $\mu_1 > 0$. Then*

- for $\lambda(0) \leq \mu_1$, $S(\lambda, \mu_1, \mu_2)$ is stable for all $\mu_2 > 0$,
- for $\lambda(0) > \mu_1$, $S(\lambda, \mu_1, \mu_2)$ becomes eventually unstable for large μ_2 .

Proof. Observe first that by Theorem 5, $S(\lambda, \mu_1, \mu_2)$ is stable for all small $\mu_2 \leq \mu_1$. To study the case with $\mu_2 > \mu_1$, fix a number n_0 so that $\lambda(n) = 0$ for all $n > n_0$. Then with $Z \sim \text{geom}(\mu_1/\mu_2)$,

$$\mathbb{E} \lambda(Z) = (1 - \mu_1/\mu_2) \sum_{n=0}^{n_0} \lambda(n)(\mu_1/\mu_2)^n. \quad (18)$$

If $\lambda(0) \leq \mu_1$, then (18) implies that for all $\mu_2 > \mu_1$,

$$\mathbb{E} \lambda(Z) \leq \lambda(0)(1 - (\mu_1/\mu_2)^{n_0+1}) < \mu_1,$$

which by Theorem 5 is sufficient for stability. Moreover, the right-hand side of (18) converges to $\lambda(0)$ as $\mu_2 \rightarrow \infty$. From this we can conclude that if $\lambda(0) > \mu_1$, then $\mathbb{E} \lambda(Z) > \mu_1$ for large enough values of μ_2 . By Theorem 5, $S(\lambda, \mu_1, \mu_2)$ is unstable for such μ_2 .

4.2. Phase partition for threshold-based admission control

Consider the network with threshold-based admission control, and assume without loss of generality that jobs arrive to the network at unit rate. Denoting the threshold level by K , this system can be modeled as $S(\lambda, \mu_1, \mu_2)$ with $\lambda(n) = 1(n \leq K)$. Theorem 5 now implies that for each $K \in \mathbb{Z}_+ \cup \{\infty\}$, the set of (μ_1, μ_2) for which the system is stable equals

$$R_K = \{(\mu_1, \mu_2) : 1 - (\mu_1/\mu_2)^{K+1} < \min(\mu_1, \mu_2)\}.$$

Since $R_K \supset R_{K+1}$ for all K , the stabilizable region is given by $\cup_{K \leq \infty} R_K = R_0$, while $R_\infty = \{(\mu_1, \mu_2) : \min(\mu_1, \mu_2) > 1\}$ represents the system with no overload. The positive orthant of \mathbb{R}^2 can now be partitioned into four phases as follows:

- $A_1 = R_\infty$ represents the region where the uncontrolled system is stable.
- $A_2 = \cap_{K < \infty} R_K$ is the region where any control stabilizes the overloaded system.
- $A_3 = R_0 \setminus \cap_{K < \infty} R_K$ is the region where the overloaded system is stabilizable using strict enough admission control.
- $A_4 = R_0^c$ is the region where the system cannot be stabilized.

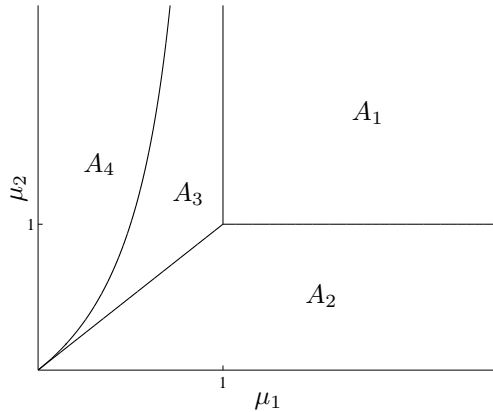


FIGURE 3: Phase diagram of the system with threshold-based admission control.

This partition is depicted in Figure 3. The phase diagram clearly illustrates the content of Propositions 1 and 2, showing that accelerating server 1 drives the system towards more stable regions, while speeding up server 2 may destabilize the network.

5. Conclusion

This paper considered the problem of characterizing the stability region of a two-node queueing network with feedback admission control. For eventually vanishing input rates, the characterization was shown to be complete. It was also illustrated how the presence of feedback signaling breaks down some typical monotonicity properties of queueing networks, by showing that increasing service rates may destabilize the network.

For a diverging input rate function and bottleneck at node 2, the exact characterization of the stability region remains an open problem. Other possible directions for future research include generalizing the results for nonexponential service and inter-arrival times, and considering queueing networks with more than two nodes.

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