GENERALIZED LINEAR-IN-PARAMETER MODELS — THEORY AND AUDIO SIGNAL PROCESSING APPLICATIONS

Tuomas Paatero

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Abstract

This thesis presents a mathematically oriented perspective to some basic concepts of digital signal processing. A general framework for the development of alternative signal and system representations is attained by defining a generalized linear-in-parameter model (GLM) configuration. The GLM provides a direct view into the origins of many familiar methods in signal processing, implying a variety of generalizations, and it serves as a natural introduction to rational orthonormal model structures. In particular, the conventional division between finite impulse response (FIR) and infinite impulse response (IIR) filtering methods is reconsidered. The latter part of the thesis consists of audio oriented case studies, including loudspeaker equalization, musical instrument body modeling, and room response modeling. The proposed collection of IIR filter design techniques is submitted to challenging modeling tasks. The most important practical contribution of this thesis is the introduction of a procedure for the optimization of rational orthonormal filter structures, called the BU-method. More generally, the BU-method and its variants, including the (complex) warped extension, the (C)WBU-method, can be consider as entirely new IIR filter design strategies.

Acknowledgements

Contents

Chapter 1

Introduction

Digital signal processing has become a household word with varying meanings describing alike a field in scientific research as well as the operational principle of a common electric device. Technical solutions from consumer electronics to industrial process control are increasingly relying on digital data management and manipulation. It is difficult to imagine a human activity that is not somehow related to observation, recording and transmission of sampled and quantized representations of the reality - to digital signal processing. Maybe the most apparent and well-recognized example is provided by the technology that enables recording, storage, transmission and reproduction of image and audio information. However, the fast progress in digital signal processing technology is achieved mainly by speeding up the processing; the principled methods that are used may themselves be relatively old and unchanged. It is thus quite common but unfortunate that, for example, such natural objectives as better understanding of the underlying physics of the phenomenon, or utilization of perceptual aspects, may characterize the research as somewhat "organic", with a flavor of being unnecessarily cumbersome or inherently academic. This kind of confrontations, if there are any, are obviously transitory, and it is apparent that the attention is shifting in a natural way towards qualitative and conceptual aspects, and related controllability issues, with a touch of irony, precisely due to the amount of processing power available.

Basic tools in signal processing rely on the possibility of representing signals and systems interchangeably in the time- and frequency-domains using transformations between the descriptions. The reproducing property of these transformations is then in turn a consequence of underlying basis function representations for signals and systems in the corresponding time- or frequency-domains. Basis function representations provide also various parametrizations of signals and systems as well as means for practical reproduction or modeling, utilizing, for example, digital filter implementations. The underlying mathematical concepts that enable these considerations are the notion of *function spaces* for signals and systems, and the concept of related *operators* or *functionals* that describe transformations between function spaces. Further insight and practicality is gained by utilizing such fields in mathematics as *linear algebra* and *complex analysis*.

This thesis was inspired by a renewed interest in the early 1990's towards alternative representations of signals and systems using *rational orthonormal functions* [Heuberger, 1991] [Oliveira e Silva, 1994]. The mathematical foundation of these considerations can be traced back to the 1920's, to the work of Takenaka, Malmquist, and Walsh [Walsh, 1969], where the idea of interpolation and approximation using polynomials [Szegö, 1939] was generalized to rational function expansions. It is obviously allowed to argue that the appearance of rational orthonormal functions is an older and more direct consequence of the *Hilbert space theory* and the *Cauchy integral theorem*.

Apparently unaware of the mathematical background, *rational orthonormal filter structures* were introduced in the engineering literature in the 1950's by Kautz, Huggins and Young [Kautz, 1954] [Huggins, 1956] [Young and Huggins, 1962]. Discrete-time rational orthonormal filter structures can be attributed to Broome [Broome, 1965] as well as the baptizing of the *discrete Kautz functions*, consequently defining the discrete-time *Kautz filter*. Wiener [Wiener, 1949] and Lee [Lee, 1960] had already a decade earlier proposed synthesis networks based on some classical orthonormal polynomial expansions [Szegö, 1939], in particular the *Laguerre functions*. Consequently, the "Laguerre trail" is still far more better known in the signal processing community than its more general counterpart.

The revival of rational orthonormal filter structures is driven by proposed applications to system identification and control engineering [Heuberger et al., 1995] [Van den Hof et al., 1994]   [Oliveira e Silva, 1995]   [Szabó and Bokor, 1997] [Bokor and Schipp, 1998] [Bultheel and De Moor, 2000]. The "Laguerre branch" for using orthogonal filter structures is equally dominated by the system identification perspective [King and Paraskevopoulos, 1979] [Nurges, 1987] [Zervos et al., 1988] [Mäkilä, 1990] [Wahlberg, 1991], although there are also some proposals for adaptive filtering [den Brinker, 1994] [Fejzo and Lev-Ari, 1997] [Merched and Sayed, 2001]. However, publications that would more clearly approach the field of audio signal processing are very rare [Davidson and Falconer, 1991] [den Brinker and Belt, 1997] [Ngia and Gustafsson, 1999] [Campi et al., 1999]. Apparently, a systematic utilization of rational orthonormal filter structures in a genuine filter synthesis sense, with applications to audio signal processing, was initiated in [Paatero et al., 2001].

The author of this thesis has been privileged in being able to utilize his former studies in mathematics in his more recent subject of interest, digital signal processing, acoustics and related audio signal processing in particular. This perspective of a "learning process" has also somewhat unavoidably affected the style and content of this monograph. This is for example reflected in the fact that there are lot of side tracts, or even loose ends, which is however considered as a contribution of the thesis: a potential interested reader is provided with an extensive collection of references. Admittedly, at times there are also lapses towards an educational touch and the form of a textbook; the appearance is at least partly unintentional, but the aim of this thesis is genuinely to build up from the perspective of elementary concepts of signal processing. One of the few clear decisions in the process of writing this thesis is that the mathematics is not pompously displayed and mystified. This is not a thesis in mathematics and it would certainly not qualify as such. Another conscious choice is the substantial utilization of footnotes to maintain readability of

the bread-and-butter text.

More or less basic concepts for representing signals, systems and related transformations are presented in the beginning of Chapter 2. Function space descriptions of signals and systems are introduced in Sections 2.2 and 2.3. In Section 2.4, *conformal mapping techniques* are used to interconnect functions spaces and to deduce basis function representations. Chapter 3 introduces the concept of generalized linear-in-parameter models (GLM) as a general framework for various signal processing tasks. Using this synthetic approach, methods that are conventionally associated to a very limited type of model structure, that is, to an *finite impulse response* (FIR) or *moving-average* filter, are generalized in a natural manner, which also brings out the mathematical origin of some common signal processing routines. In particular, the frequently misplaced barrier between FIR and IIR (*infinite impulse response*) filtering methods is demystified. The GLM concept is also used as an introduction to rational orthonormal filter structures, the topic of Chapter 4. Audio-oriented case studies, including loudspeaker equalization, musical instrument body modeling, and room response modeling, are presented in Section 4.4. The most important practical contribution of this thesis is probably the *BU-method* (dubbed from Brandenstein and Unbehauen) for the optimization of rational orthonormal filter structures. The BU-method and its variants, including the *(complex) warped* extension, the (C)WBU-method, are presented in Section 4.3. These and some other less apparent contributions of the thesis are considered in the concluding Chapter 5.

Chapter 2

Mathematical means in signal processing - from conventional to somewhat advanced

Transformations between time- and frequency-domain representations of signals and systems are essential in any development of signal processing methods. The beginning of this Chapter is more or less "DSP textbook stuff", although some unconventional interconnections are utilized to deduce and relate various transformations. Function space descriptions for signals and systems are introduced in Section 2.2, which gives also further insight into the transformation concept that is in most cases missing from the aforementioned textbook considerations. The Hilbert space framework for representing and modeling signals and systems is presented in Section 2.3. This Chapter concludes by introducing an useful bundle of mathematical topics: particularly, in Section 2.4 conformal mapping techniques are used to interconnect function spaces and to deduce basis function representations into various function spaces.

2.1 Representing signals, systems and transformations

Along with the actual definitions, a number of notational conventions are introduced in this Section. Such concepts as linearity, causality, time-invariance, and stability are included gradually. Some canonical digital filter implementations are also presented as direct consequences of difference equation considerations.

2.1.1 Discrete-time signals and signal transformations

A discrete-time signal is a mapping from some index set representing time to a signal value range. Here the discrete-time signal $x(n)$ is defined as a mapping from a subset

S of the ordered integer axis $\mathbb{Z}$ to the complex plane $\mathbb{C}$, $x : S \to \mathbb{C}$, $x : n \mapsto x(n)$. Sometimes it is convenient to express an ordered collection of component signals as a vector valued mapping, $x : S \to \mathbb{C}^N$, $x : n \mapsto \mathbf{x}(n) = [x_1(n) \ \cdots \ x_N(n)]^T$, where the superscript T denotes matrix transpose and N is the dimensionality of the vector. In both cases, the properties of the complex plane, regarded as a vector space (manifold) and as a multiplicative ring, guarantee well-defined *sample-by-sample* arithmetic operations, such as, signal *scaling*, *additions* and *multiplications*. An important operation on signals is the *delay operator* defined by

$$z^k[x(n)] = x(n+k), \ k \in \mathbb{Z}, \tag{2.1}$$

which is defined to be zero outside the signal support. The most elementary signal is the *unit impulse signal*

$$\delta(n) = \begin{cases} 1, & if \quad n = 0 \\ 0, & if \quad n \neq 0 \ . \end{cases} \tag{2.2}$$

Using the delay operator on the unit impulse, every signal can be *decomposed* as

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k). \tag{2.3}$$

Here too the summation is defined to be void outside the signal support. It should be noted that definition (2.3) does not require any assumptions about the convergence of the summation. It can also be seen as a decomposition of a complex valued function as a collection of assigned discrete values and *characteristic functions* defining the support set[1]. Utilizing this cumbersome interpretation it can be stated that a discrete-time signal has no *physical energy*, defined as the square *Lebesgue integral* over the past (as well as future) times.

The *frequency-domain representation* of a discrete-time signal is based on the *z-transform*, which associates a complex variable, z, to the signal $x(n)$:

$$X(z) \equiv \mathcal{Z}\{x(n)\} := \sum_{n=-\infty}^{\infty} x(n)z^{-n}. \tag{2.4}$$

This *power series expansion* of $x(n)$ is defined in some *annular region* of the complex plane, that is, whenever (2.4) is absolutely summable[2]. If this *region of convergence (ROC)* is not empty, it can be specified by

$$R(r_1, r_2) = \{z = re^{j\theta} \in \mathbb{C} : 0 \leq r_1 < r < r_2 \leq \infty, \ \sum_{n=-\infty}^{\infty} \left| x(n)r^{-n} \right| < \infty\}. \tag{2.5}$$

From the theory of power series it is known that $X(z)$ converges absolutely and *uniformly* in $R(r_1, r_2)$ and that it diverges for all $z \in \mathbb{C} \setminus \overline{R(r_1, r_2)}$, where the bar denotes closure. It is more case-specific what happens on the boundaries, but in the

---

[1]This provides a way to avoid defining the inter-sample signal values to be zero, which would be incorrect in many ways.

[2]Definition (2.4) is meaningful if the summation is convergent, that is, absolutely convergent, for some $z \neq 0$, implying that it is convergent on a circle with radius $|z|$.

upcoming it is mainly the special cases $r = 0$, $r = 1$ and $r = \infty$ that are of interest. Furthermore, the function $X(z)$ is *analytic* in the ROC, that is, differentiable with respect to the complex variable, even up to derivatives of all orders. This and many other properties of the z-transform are consequences of the uniformity, which permits operating inside the summation. One such result is the *inverse z-transform*

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^n \frac{dz}{z}, \tag{2.6}$$

where C is any closed (counter-clockwise) contour in the *ROC*. If the *unit-circle*, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, is within the *ROC*, then the *discrete-time Fourier transform* (DTFT) can be defined by the substitution[3]

$$X(\omega) \equiv X(z)|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}. \tag{2.7}$$

It is easily seen that the Fourier transform is a continuous and periodic function with period 2π.[4] Conversely, every continuous 2π-periodic (square-integrable on $\mathbb{T}$) function has a *trigonometric system* decomposition of the form (2.7). Moreover, $X(\omega)$ has the form of a *Fourier series* with discrete "line-spectra"

$$x(n) = \frac{1}{2\pi} \int_0^{2\pi} X(\omega) e^{j\omega n} d\omega. \tag{2.8}$$

This is the *inverse discrete-time Fourier transform* (IDTFT), and again, it is simply (2.6) evaluated on the unit-circle. The N-point *discrete Fourier transform* (DFT) for a discrete-time signal is defined formally by

$$X(k) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1. \tag{2.9}$$

Once again, the DFT could be defined as a substitution to transforms (2.4) or (2.7), respectively, corresponding to *equidistant sampling* in the angular frequency-domain $0 \leq \omega < 2\pi$.[5] However these definitions would exclude *periodic signals*, that is, signals for which $x(n+P) = x(n)$ for some integer $P > 0$ and all $n \in \mathbb{Z}$, which are by definition infinite in duration and divergent in any measure of summability.

The summation in (2.9) is finite and unconditionally convergent only in the case of a finite-duration signal. However, it is apparent from the formal definition that $X(k)$ has a periodic extension in $\mathbb{Z}$, implying that this infinite but periodic extension has a finite Fourier series expansion

$$X_P(k) = \sum_{n=0}^{N-1} x_P(n) e^{-j2\pi kn/N} \tag{2.10}$$

---

[3]Actually this definition is insufficient in both directions – there are practical signals, with well-defined z-transforms, having no Fourier transforms, and on the other hand, the Fourier transform can be defined for a wider class of square-summable sequences.

[4]In contrast to the continuous-time counterparts where a distinction between periodic and aperiodic signals has to be made.

[5]On the other hand, the definition could be relaxed to any ordered N-point discrete frequency grid in $[0, 2\pi]$, but in the following, unequal frequency sampling is restricted to cases when it can be implemented by *frequency transformations*.

with respect to its *harmonically related frequency components*, and with Fourier series coefficients given by

$$x_P(n) = \frac{1}{N}\sum_{k=0}^{N-1}X(k)e^{j2\pi kn/N}, \quad n = 0, 1, \ldots, N-1. \tag{2.11}$$

This can be seen as a discrete version of (2.8) where the integration is replaced by summation and an appropriate normalization over the measured set. The (periodically extended) discrete-time signal $x_P(n)$ is related to the original signal $x(n)$ through

$$x_P(n) = \sum_{l=\infty}^{\infty}x(n-lN), \tag{2.12}$$

which is seen by forcing (2.9) into the form of (2.11). The signal (2.12) is in general an *aliased* version of $x(n)$, produced by *folding* it over the time-interval $0 \le n < N-1$. The transform pair

$$\tag{2.10}$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1}X(k)e^{j2\pi kn/N} \quad \rightleftharpoons \quad X(k) = \sum_{n=0}^{N-1}x(n)e^{-j2\pi kn/N} \tag{2.13}$$

$$\tag{2.11}$$

produces *perfect reconstruction* (PR) between the signal representations if and only if there is no aliasing in the *time-domain*, that is, if $x_P(n) = x(n)$, $n = 0, \ldots, N-1$, which holds exactly when $x(n)$ itself has a periodic extension of the form (2.12). In other words, PR is achieved for N-periodic or finite signals of duration less or equal to N.[6] Additionally, by substituting the IDFT into (2.4) or (2.7) it is seen that $X(z)$ or $X(\omega)$, respectively, have closed form representations as functions of the spectral samples $X(k)$. [7]

2.1.2 Discrete-time system descriptions

A *discrete-time system* H is defined as an operator or functional that describes an *input-output relationship*

$$\begin{aligned} x(n) &\longrightarrow y(n) \\ &H \\ X(z) &\longrightarrow Y(z). \end{aligned} \tag{2.14}$$

Despite of the seemingly specialized definition for describing one particular relation, $\{x(n), y(n)\}$ or $\{X(z), Y(z)\}$, it is to be interpreted as a mapping from a specified subset of the time- or frequency-domain discrete-time signals with well-defined images $y(n) = H[x(n)]$ or $Y(z) = H[X(z)]$ in the corresponding range of the mapping.

---

[6]In the latter case it is presumed that $x(n)$ is first shifted to the "base interval" $[0, L]$, $L < N$.

[7]This can be interpreted as (Lagrange) polynomial interpolation in terms of the values $\{X(k)\}$ at frequencies $\omega_k = 2\pi k/N$.

It is obviously possible to extend the notion of a discrete-time system to mixed time- and frequency-domain input-output relationships, for example, to include systems that implement a signal transformation as a part of the process.

Another method for characterizing the internal behavior of a discrete-time system is to use some test signals: the system is excited with a well-specified input signal and the corresponding response (or output signal) is gathered and analyzed in some sense. The depth of the analysis may vary from a detailed time- or frequency-domain expression to the measuring of a single quantity that somehow characterizes the system. The sort of "extremes" in the choices of test signals are then the impulse-like or purely sinusoidal signals, respectively. They are complementary in the sense that the former is completely localized in time and evenly distributed in frequency, and vice versa for the latter. The impulse response of an *initially relaxed* [8] system is defined as a collection of responses to shifted unit impulses,

$$h(n, k) = H[\delta(n - k)], \quad n, k \in \mathbb{Z}, \tag{2.15}$$

and in general it is an infinite set of component signals. If the system is excited with a sinusoidal, or rather with a *complex exponential*, $x(n) = e^{j\omega_0 n}$, $\omega_0 \in [0, 2\pi]$, $n \in \mathbb{Z}$, then the following (formal) responses are produced

$$y(n) = H[e^{j\omega_0 n}], \quad n \in \mathbb{Z} \tag{2.16}$$

$$Y(\omega) = H\left[\sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2l\pi)\right], \quad \omega \in \mathbb{R} \tag{2.17}$$

where the summation is by definition the DTFT of the complex exponential[9]. That is, the system is exited with an infinite 2π-periodic *frequency impulse train*. By replacing the summation index l with $(k - m)$ and evaluating the response $Y(\omega)$ to the harmonically related inputs $x(n) = e^{j2\pi nm/N}$, $m = 0, \ldots, N - 1$, at equidistant frequencies $\omega_k = 2\pi k/N$, $k = 0, \ldots, N - 1$, the relation (2.17) results in the general discrete *frequency-domain impulse response* expression

$$H(k, m) \equiv H\left[\sum_{k-m=-\infty}^{\infty} 2\pi\delta\left(\frac{2\pi(k - m)}{N} + 2\pi(k - m)\right)\right] = H[2\pi\delta(k - m)], \tag{2.18}$$

which can be seen as a formal discrete frequency-domain counterpart of (2.15).

---

[8] The term relaxed has to be dragged along for a while. A system is relaxed if it starts at rest. A system is initially relaxed at $n = n_0$, if $y(n) = 0$ for $n < n_0$, and it is used to ensure that the future outputs $y(n)$, $n \geq n_0$, depend only on the inputs $x(n)$, $n \geq n_0$. If the input is spanned over $\mathbb{Z}$, as it is to begin with here, then initially relaxed means relaxed at $n = -\infty$ and the output signal is solely and uniquely determined by the input signal $x(n)$ [Proakis and Manolakis, 1992]. This has not yet anything to do with causality or time-invariancy.

[9] This is an example of a special case where the DTFT can be defined for a sequence that is not absolutely or square-summable. Utilizing the *Dirac delta function* interpretation of $\delta(\cdot)$, the Dirac comb $\tilde{\delta}(\omega) = 2\pi(\sum_{l=-\infty}^{\infty} \delta(\omega + 2\pi l))$ is used to identify the DTFT of $e^{j\omega_0 n}$ as $\tilde{\delta}(\omega - \omega_0)$. The inverse DTFT is then a result of the sampling property of the Dirac delta function: $\frac{1}{2\pi} \int \tilde{\delta}(\omega - \omega_0) e^{j\omega n} d\omega = e^{j\omega_0 n}$.

2.1.3 Linear time-invariant systems

So far, all the means for characterizing the internal behavior of the system are merely theoretical. Definitions (2.14) describe in general an infinite collection of point-wise relations. Equally, the aforementioned "test signals" are in practice unrealizable, in addition to the fact that the produced responses (2.15) and (2.18) would anyhow be impractical. By imposing some qualitative restrictions on the system it is occasionally possible to relate an arbitrary input signal to system descriptions, such as (2.15) or (2.18), to produce a well-defined closed form mathematical expression for the corresponding response.

A system defined by (2.14) is called *linear* if and only if

$$
\begin{aligned}
H[ax(n) + by(n)] &= aH[x(n)] + bH[y(n)] \\
&\text{or} \\
H[aX(z) + bY(z)] &= aH[X(z)] + bH[Y(z)]
\end{aligned}
\tag{2.19}
$$

for any $x(n)$ and $y(n)$, or $X(z)$ and $Y(z)$, respectively, in the domain of the mapping, and for any scalars a and b.[10] Condition (2.19) reflects simultaneously the *scaling and additivity properties*, that is, the *superposition principle*, of a linear system. From (2.19) it follows that if $H[x(n) \equiv 0] = y(n) \neq 0$, then the system is either *nonlinear* or *non-relaxed* [11]. Now, supposing that the input signal has a decomposition $x(n) = \sum_k c_k x_k(n)$ into weighted elementary signal components, the superposition principle may be extended by induction to the form

$$
y(n) = H[x(n)] = H\left[\sum_k c_k x_k(n)\right] = \sum_k c_k H[x_k(n)] = \sum_k c_k y_k(n),
\tag{2.20}
$$

where $y_k(n) = H[x_k(n)]$ and the system is presumed to be relaxed[12]. Previously two such decompositions were formed, that is, the decompositions of a periodic signal into harmonically related components, and the weighted shifted unit delay representation (2.3). The latter is defined for an arbitrary input signal $x(n)$, providing a desired explicit mathematical description,

$$
\begin{aligned}
y(n) &= H[x(n)] = H\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right] && \text{(Eq. (2.3))} \\
&= \sum_{k=-\infty}^{\infty} x(k)H[\delta(n-k)] && \text{(Eq. (2.20))} \\
&= \sum_{k=-\infty}^{\infty} x(k)h(n,k) && \text{(Eq. (2.15))},
\end{aligned}
\tag{2.21}
$$

---

[10] Throughout this work, it is presumed that the domain and range of a signal mapping are at least vector spaces, which ensures that all linear combinations of the elements belong to the same spaces. Additionally, "scalars" means members of the associated *scalar field* of the particular vector space, which here amounts to the set of real or complex numbers, respectively.

[11] A simple example of such a system is $y(n) = ax(n) + b$, $a, b \neq 0$, where the system is described by a linear equation, but the linearity test (2.19) fails. For $b \neq 0$ the system is non-relaxed, or *biased*, but it is a matter of interpretation whether it is linear or not.

[12] This result is very general and the emphasis is clearly on the beginning of the sentence.

for the response $y(n)$ of a relaxed linear system H to an input signal $x(n)$.

Expression (2.21), although explicit, still has the disadvantage of being dependent on a two-dimensional sequence $h(n, k)$, that is, the summation extracts only the other degree of freedom, leaving a *time-dependent* description of the system. If on the other hand it can be presumed that the relaxed linear system is time-independent or *time-invariant*, defined by the implication

$$y(n) = H[x(n)] \qquad \Rightarrow \qquad y(n - k) = H[x(n - k)], \qquad (2.22)$$

for every input signal $x(n)$ and any integer shift k, then $h(n, k) = h(n - k)$, and (2.21) reduces to[13]

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n - k). \qquad (2.23)$$

This is the *convolution sum* representation of a linear time-invariant (LTI) system, which gives the response to any input signal as a function of a single sequence, the unit impulse response $h(m) = H[\delta(m)]$. The shorthand for (2.23) is $y(n) = x(n) * h(n)$ and it is easily seen that convolution is an associative and commutative operation[14]. The input $x(n)$ is said to be convolved, or *filtered*, with the impulse response $h(n)$, that is, at each time-instant n, the product sequence $c(k) = x(k)h(n - k)$ is summed over all indices, where the sequence $h(k)$ is first *folded* over n, or vice versa for $h(k)$ and $x(k)$. In general, direct utilization of (2.23) is possible only in the case of a *finite (duration) impulse response* (FIR), in which case the summation (2.23) is finite for any n and $x(k)$.

The frequency-domain counterpart of the input-output relationship in terms of a *system function* is attained by applying the z-transform or the Fourier transform, respectively, on both sides of (2.23). In the former case, repetitive use of definition (2.4), combined to the LTI-property, provide

$$
\begin{aligned}
Y(z) &= \sum_{n=-\infty}^{+\infty} y(n)z^{-n} = \sum_{n=-\infty}^{+\infty} \left[\sum_{k=-\infty}^{\infty} x(k)h(n - k) \right] z^{-n} \\
&= \sum_{k=-\infty}^{\infty} x(k) \left[\sum_{n=-\infty}^{\infty} h(n - k)z^{-n} \right] \\
&= \sum_{k=-\infty}^{\infty} x(k)z^{-k} \sum_{n=-\infty}^{\infty} h(n)z^{-n} = X(z)H(z), \qquad (2.24)
\end{aligned}
$$

whenever the intermediate definitions for the z-transforms,

$$Y(z) = \sum_{n=-\infty}^{\infty} y(n)z^{-n}, \quad X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad \text{and} \quad H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n},$$

are valid. Additionally, the linearity property and the *convolution-product-identity*

$$y(n) = x(n) * h(n) \qquad \longleftrightarrow \qquad Y(z) = X(z)H(z) \qquad (2.25)$$

---

[13]By definition $h(n, k) = H[\delta(n - k)]$, and the condition (2.22) then particularly applies to $h(n) = H[\delta(n)]$, implying that $h(n, k) = H[\delta(n - k)] = h(n - k)$.

[14]With a change of variable, $m = n - k$, an alternative or dual representation, $y(n) = \sum_{-\infty=k}^{\infty} x(n - k)h(n)$, is attained, where in the last step the dummy variable m is replaced by k.

provide principles for interconnections of systems. For initially relaxed LTI systems H_1 and H_2 that are defined in appropriate domains, relations $H = H_2[H_1]$ and $H = H_1 + H_2$ correspond to *cascade* and *parallel connections* of the systems, respectively. Once again presuming that the following descriptions are meaningful, the response of a parallel system is given by $h(n) = h_1(n) + h_2(n)$ or $H(z) = H_1(z) + H_2(z)$, and by $h(n) = h_1(n) * h_2(n)$ or $H(z) = H_1(z)H_2(z)$ for the cascaded system, respectively[15]. Whenever an *inverse system* H^{-1} can be associated to a system H, it defines an inverted input-output relationship[16].

Another way to characterize the input-output relationship of an LTI system is by directly trying to solve *linear equations* provided by the linear (matrix) operator form $\mathbf{y} = \mathbf{L}\mathbf{x}$ of $y = H[x]$:

$$
\begin{bmatrix} \vdots \\ y(n{+}1) \\ y(n) \\ y(n{-}1) \\ \vdots \end{bmatrix} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \cdot \\ \cdots & h(0) & h(1) & h(2) & \cdots \\ \cdots & h(-1) & h(0) & h(1) & \cdots \\ \cdots & h(-2) & h(-1) & h(0) & \cdots \\ \cdot & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ x(n{+}1) \\ x(n) \\ x(n{-}1) \\ \vdots \end{bmatrix} . \tag{2.26}
$$

In the above, the convolution sum (2.23) is simply augmented into a matrix form. The matrix $\mathbf{L}$ has a special *Toeplitz* form, that is, the elements on the main diagonal and all its sub-diagonals are equal, respectively. The input-output relationship then has a particular shift structure, $y(n - k) = z^{-k}\mathbf{h}^T\mathbf{x}$, $k \in \mathbb{Z}$, where $\mathbf{h}$ is the impulse response sequence. By a somewhat wild association, this can be seen as an infinite collection of difference equations, whose overall solution can in principle be attained from the *constant coefficient difference equations* $\mathbf{a} * \mathbf{y} = \mathbf{b} * \mathbf{x}$, where $\mathbf{a}$ and $\mathbf{b}$ are some (infinite-dimensional) coefficient vectors to be specified[17]. As in (2.24), convolution in the time-domain corresponds to $A(z)Y(z) = B(z)X(z)$ in the z-domain, where $A(z) = \sum_{k=-\infty}^{\infty} a(k)z^{-k}$ and $B(z) = \sum_{k=-\infty}^{\infty} b(k)z^{-k}$ are double-infinite formal polynomials in the variable z. In actually assigning $H(z) = B(z)/A(z)$ a bit more has to be presumed[18].

---

[15]In other words, LTI systems form an associative, commutative and distributive *algebraic ring* $R(h, +, \cdot/*)$ with respect to the sum and product operations defined above. Linear convolution is thus the time-domain product operation. The zero and unit elements of these algebraic structures are $h(n) \equiv 0$ or $H(z) \equiv 0$ and $h(n) = \delta(n)$ or $H(z) \equiv 1$, respectively.

[16]Since the system is defined as a mapping it follows that the invertibility can be characterized as an one-to-one correspondence, or a bijection, in the input-output relationship. In terms of $h(n)$ or $H(z)$, the inverse system $h^{-1}(n)$ or $H^{-1}(z)$, if it exists, is the one for which $h(n) * h^{-1}(n) = \delta(n)$ or $H(z)H^{-1}(z) = 1$, respectively.

[17]The convolution representation is still a point-wise relation, that is, an infinite collection of equations $a(n) * y(n) = b(n) * x(n)$, $n \in \mathbb{Z}$, but due to the shift-invariancy, it suffices to consider $y(n)$ as the solution of the convolution identity $\sum_{k=-\infty}^{\infty} a(k)y(n - k) = \sum_{k=-\infty}^{\infty} b(k)x(n - k)$, $a(0) = 1$. The left hand side of the equation can be associate to the *homogenous* difference equation $\mathbf{a} * \mathbf{y} = 0$ for providing the coefficients of zero-input response using "initial condition" $\mathbf{y}$. The right hand side is then related to the *particular* or *forced-input* solution, where it is presumed that a particular solution can be expressed as a linear combination of shifted input samples. This is motivated by the fact that "solving" means an expression for $y(n)$ in terms of $\mathbf{a}$ and $\mathbf{b}$ for a specific $x(n)$, which is not necessarily a unique description of the system itself, and consequently, due to the LTI property it is presumable that the output takes the same general form as the input. The overall solution as a superposition of the homogenous and particular solutions was already embedded in the expression $\mathbf{a} * \mathbf{y} = \mathbf{b} * \mathbf{x}$.

[18]The corollary is that an LTI system can be associated to a *dynamical process* with time-

An important subclass of the LTI systems are those that possess a *finite difference equation* of the particular form

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k), \qquad (2.27)$$

where for convenience, coefficients are scaled so that $a_0 = 1$. The order of the difference equation is defined to be $\max\{N, M\}$ and it is consistent with the order of the *rational transfer function*[19] composed of the ratio of finite polynomials implied by (2.27) for describing the system function in the z-domain.

An immediate consequence of (2.27) is that for $n > 0$ the response

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k) \qquad (2.28)$$

can not depend on future input values $x(m)$, $m > n$. The general definition for this *causality property* is embedded in the implication

$$x_1(n) = x_2(n), \ for \ n < n_0 \quad \Longrightarrow \quad y_1(n) = y_2(n), \ for \ n < n_0, \qquad (2.29)$$

where $y_1(n)$ and $y_2(n)$ are responses to $x_1(n)$ and $x_2(n)$, respectively. In general this is a point-wise definition, but for LTI systems it reduces to an existential form, that is, an LTI system is causal if (2.29) holds for any fixed n_0. Furthermore, the causality condition for an LTI system impulse response is particularly simple: property (2.29) is equivalent to $h(n) = 0$, for $n < 0$.[20] In the upcoming, a somewhat confusing but frequently used term *causal signal* is to be interpreted as a signal that could be the impulse response of a causal system.

The causality presumption forces the matrix operator (2.26) into a semi-infinite upper-triangular form. In terms of the convolution sum (2.23), the summation becomes onesided:

$$y(n) = \sum_{k=-\infty}^{n} x(k)h(n-k) = \sum_{k=0}^{\infty} h(k)x(n-k). \qquad (2.30)$$

If in addition the input $x(n)$ is restricted to be causal then (2.30) reduces to

$$y(n) = \sum_{k=0}^{n} x(k)h(n-k) = \sum_{k=0}^{n} h(k)x(n-k), \qquad (2.31)$$

---

invariant parameters. In fact, if the delay operator is identified as the discrete-time time-derivative operation, the suggestive term 'difference' can on occasion be replaced by *differential*, in describing the particular form of a dynamical system $y = F[\mathbf{y}, \mathbf{x}]$. It should however be pointed out that this consideration is very speculative and that some of the denotations may not have any true meaning as such. On the other hand, and especially in the case of the double-infinite time-axis $\mathbb{Z}$, it has been demonstrated that the two-operator description $Ay = Bx$ of an input-output system should be considered as the fundamental and more appropriate relation in contrast to the more conventional one-operator form $y = Hx$ [Mäkilä, 2005]. In principle, this observation also question the whole interpretation of the system as a mapping.

[19]The term 'system function' is preserved for a wider meaning.

[20]Proof. Fix n_0, choose $x_1(n) = x_2(n)$, for $n \leq n_0$, and $x_1(n) \neq x_2(n)$, for $n > n_0$. Compose the dual form of (2.23) for $y_1(n_0)$ and $y_2(n_0)$. Separate the summations into two parts, $k \in]-\infty, -1]$ and $k \in [0, \infty[$. Equality $y_1(n_0) = y_2(n_0)$ holds only if $h(k) = 0$ in the former sum.

which implies that the response of a causal LTI system to a causal input is also causal[21]. Summations (2.31) are finite, but in general they require an increasing number of storage locations for the signal values $\{x(k)\}_{k=0}^{n}$ and $\{h(k)\}_{k=0}^{n}$. However, as it was mentioned earlier, in the case of an FIR system, specified by an impulse response $\{h(k)\}_{k=0}^{N}$, the latter form of summation (2.31) provides a direct *non-recursive* implementation

$$y(n) = \sum_{k=0}^{N} h(k)x(n-k) \equiv \mathbf{h}^T \mathbf{x}(n), \ \ \mathbf{h} = \begin{bmatrix} h(0) \\ \vdots \\ h(N) \end{bmatrix}, \ \ \mathbf{x}(n) = \begin{bmatrix} x(n) \\ \vdots \\ x(n-N) \end{bmatrix}, \ \ (2.32)$$

where the input data vector $\mathbf{x}(n)$ is updated at every time-instant n by shifting $\mathbf{x}(n) = z[\mathbf{x}(n-1)]$, that is, the new sample $x(n)$ is added to the front of the previous data vector $\mathbf{x}(n-1)$ where the last element is disregarded. Equations (2.27) or (2.28) provide a straightforward way to generalize the state vector form (2.32) to some systems with *infinite impulse response* (IIR) duration:

$$y(n) = \mathbf{b}^T \mathbf{x}(n) - \mathbf{a}^T \mathbf{y}(n) = \mathbf{c}^T \mathbf{z}(n), \quad \mathbf{c} = \begin{bmatrix} \mathbf{b} \\ -\mathbf{a} \end{bmatrix}, \quad \mathbf{z}(n) = \begin{bmatrix} x(n) \\ \vdots \\ x(n-M) \\ y(n-1) \\ \vdots \\ y(n-N) \end{bmatrix}. \ \ (2.33)$$

Equations (2.33) describe a causal finite dimensional *recursive system*. Because of the perspective of this introduction, systems representable by equations (2.33) form a very limited subset of the more general classes of recursive or IIR systems, respectively, but as *model structures* (2.33) and (2.32) form the basis for *approximating* a causal LTI system.

The LTI assumption guarantees that the difference equation (2.27) has always a solution[22]. This is due to the fact that the constant-coefficients are independent of the input and output sequences. However, it should be noted that although the response in (2.28) is in principle produced by *finite arithmetic operations* this does not mean that the response to a given input is always in practice attainable. Moreover, a system defined by constant-coefficient difference equations (2.27), even if it is causal, is not necessarily linear and time-invariant. The essential difference compared to system descriptions (2.30) or (2.31) is that the response at a given time-instant is separable to a recursive past of the system and a non-recursive contribution of the present and past input values. To which extent these portions are genuinely segregated depends on the aforementioned additional properties of the system. For causal systems described by equations (2.27), the response $y(n)$ for all $n \geq n_0$ is computable knowing $x(n)$, and the initial conditions $\{y(n_0 - 1), \ldots, y(n_0 - N)\}$.

---

[21]By convention, the dimensionality of $x(n)$ and $y(n)$ are the same, if not otherwise stated, which in this case extends the response as $y(n) = 0$, $n < 0$. In the following it is crucial that no ambivalency should appear in identifying operations on one-sided or causal sequences.

[22]The general solution is of the form $y(n) = y_h(n) + y_p(n)$, where $y_h(n)$ is the solution of the *homogeneous* (or zero-input) equation $\mathbf{a}^T \mathbf{y} = 0$ and $y_p(n)$ is any (input-dependent) particular solution of (2.27).
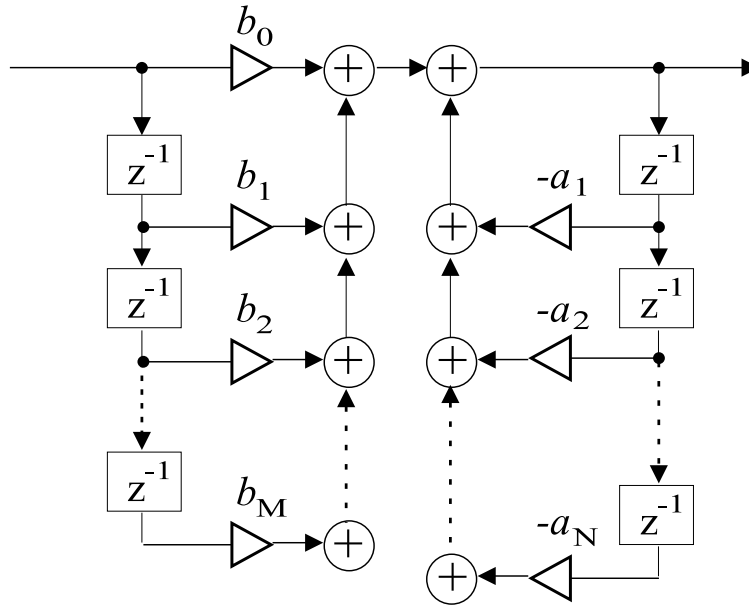
Figure 2.1: A straightforward digital filter implementation of the difference equation (2.27), the direct form I structure of an IIR filter.

Furthermore, the initial conditions reduces to $\{y(-1), \ldots, y(-N)\}$ for causal inputs. Earlier it was stated that the response of a causal LTI system to a causal input is causal, which is thereby equivalent to presuming zero initial conditions. In terms of the general definition (2.14) of a discrete-time system, this is consistent with the assumption of an initially relaxed system at $n = 0$. [23]

2.1.4   Some LTI filter implementations

The more practical implication of the finite difference equation (2.27) is that it has an implementation as a *digital filter*. Using the previous definitions for interconnections of systems, every system possessing the form of equation (2.27) is separable into primitive building blocks, such as, unit delay elements, signal scaling, and branching and summing junctions of the signal flow. The requirement of causality, or from the implementation point of view, the realizability of the system, is reflected in the fact that every feedback loop must contain a delay[24]. A straightforward implementation of equation (2.27) results in the *direct form I structure* of an IIR filter, depicted in Figure 2.1. In addition, the equation (2.33) allows for an immediate *state variable* interpretation of the system: the intermediate signal vector $\mathbf{z}(n)$ is a collection of instantaneous unit delay outputs, that is, the contents of corresponding memory locations.

The filter implementation in Figure 2.1 can be seen as a cascade connection of a non-

---

[23]In fact, a causal system described by constant-coefficient difference equations is strictly linear only in the case of zero initial conditions, although this fact is sometimes pragmatically disregarded in practical LTI filtering.

[24]Actually, arbitrary interconnections of causal LTI systems are allowed since delay free loops are always extractable, for example, using equivalent circuit techniques.
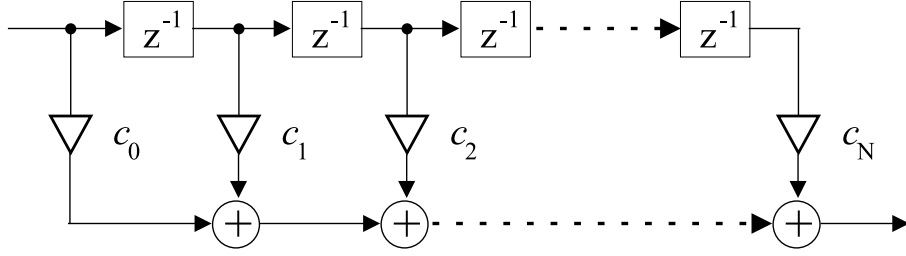
Figure 2.2: A transversal and non-recursive realization of an FIR filter.

recursive FIR filter, depicted as a *tapped transversal delay-line filter* in Figure 2.2, and a purely recursive filter, implemented as a feedback connection of another direct form FIR filter. If the order of these two blocks is interchanged, a *direct form II structure* (Fig. 2.3) is produced, where the common delay-line (of length $\max(M, N)$) introduces a genuine state-variable interpretation of the system. If the inputs and outputs of the delay elements are denoted by $\mathbf{w}(n + 1)$ and $\mathbf{w}(n)$, respectively, the system has a *state-space representation*

$$
\begin{aligned}
\mathbf{w}(n + 1) &= \mathbf{A}\,\mathbf{w}(n) + \mathbf{b}\,x(n) \\
y(n) &= \mathbf{c}\,\mathbf{w}(n) + d\,x(n),
\end{aligned}
\tag{2.34}
$$

where the elements of $(\mathbf{A}, \mathbf{b}, \mathbf{c}, d)$ are attained from the direct form parameters $\{b_k, a_k\}$. In particular, for the direct form II structure with $N = M$, the constants $(\mathbf{A}, \mathbf{b}, \mathbf{c}, d)$ are given by

$$
\mathbf{A} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_N \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},
\tag{2.35}
$$

$$
\mathbf{c} = \begin{bmatrix} b_1 - b_0 a_1 & b_2 - b_0 a_2 & \cdots & b_N - b_0 a_N \end{bmatrix}, \quad d = b_0.
\tag{2.36}
$$

The state-space realization of a system is by no means unique. However, the transfer function of a given realization (2.34) is easily seen to be

$$
H(z) = \mathbf{c}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + d,
\tag{2.37}
$$

where it is presumed that the matrix $(z\mathbf{I} - \mathbf{A})$ is invertible[25]. Figure 2.4 illustrates that the state-space system can be seen as a "vectorized first order resonator". Correspondingly, the impulse response of (2.34) is an exponential sequence, given by

$$
h(n) = \begin{cases} d\,, & n = 0 \\ \mathbf{c}\mathbf{A}^{n-1}\mathbf{b}\,, & n = 1, 2, \ldots \end{cases}
\tag{2.38}
$$

The state-space representation (2.34) will be utilized in Section 3.4.2.

---

[25]The matrix $\mathbf{I}$ is the $N \times N$ unit matrix and the scalar multiplication $z\mathbf{I}$ result in a "z-variable matrix". The matrix function $(z\mathbf{I} - \mathbf{A})$ is the *resolvant* of the matrix $\mathbf{A}$. For causal systems the invertibility is equivalent to the constraint that all eigenvalues of $\mathbf{A}$ should be within the unit circle. These eigenvalues are the solutions of the corresponding characteristic equation $det(z\mathbf{I} - \mathbf{A}) = 0$.
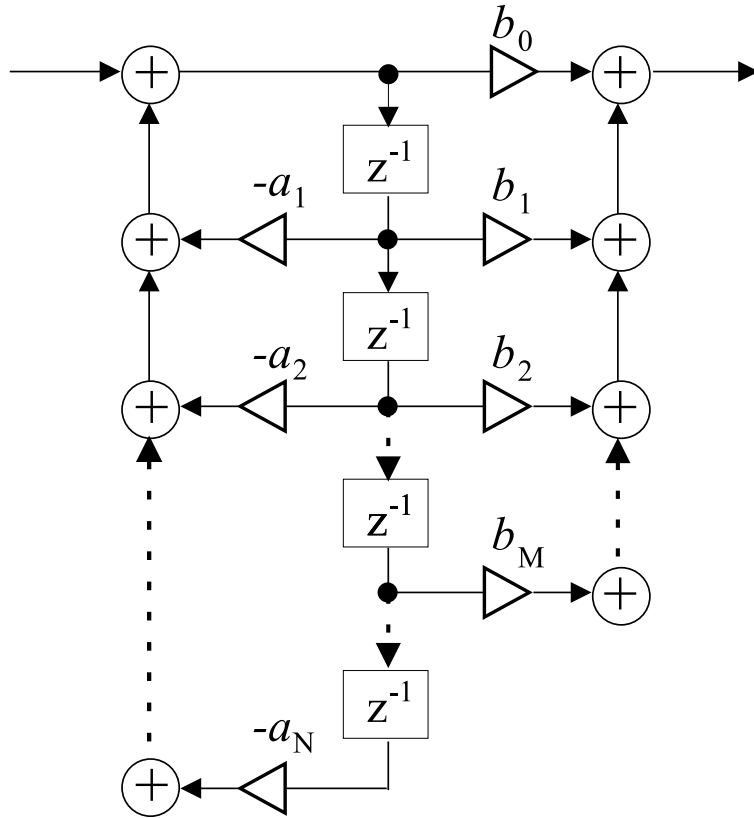
Figure 2.3: The direct form II structure of an IIR filter. The implied choice $M < N$ is a pure technicality.
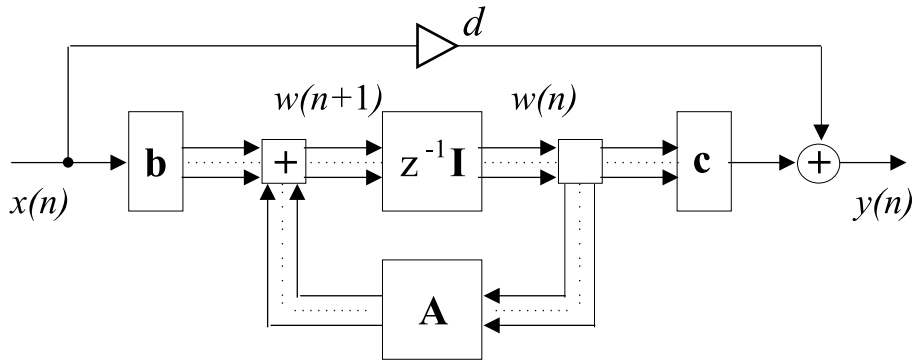
Figure 2.4: The block diagram of a single-input single-output (SISO) LTI system, corresponding to the state-space equations (2.34) and to the transfer function (2.37).

The direct forms are just particular cases of an infinite variety of possible realizations of a given LTI system (2.27).[26] However, the direct form II structure provides a *minimal realization* in the sense that the dimensionality of the state variable is the smallest one possible. A minimal realization is still not unique, but two minimal realizations of the same system are similar in the sense that they can be uniquely recovered from each other using an invertible state transformation. An alternative minimal state-space realization to the direct form structures and their traditional

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[26]Alternative forms are attained, for example, by different decompositions of the rational transfer function, as equivalent circuit manipulations, or as transformed state-space realizations.

variants will be given in Section 3.4.2. Nevertheless, direct forms have the favorable property of being *linear-in-the-parameters*, that is, the output possesses a *linear regressor* form, consisting of a *parameter vector* and a *regressor* (or data vector), as apparent from the expressions (2.32), (2.33) and (2.34), respectively. To what extent this separation into a model and a set of its external parameters is genuine will be the essential question in the forthcoming generalizations.

Some remarks Many of the frequently used concepts and terms are inherently confusing. To begin with, recursive and non-recursive are related to the implementation, and on the other hand, FIR and IIR, when taken literally, are solely properties of the impulse response. Sometimes these terms are intersected, although there exists, for example, recursive FIR filter impltmentations. The other blurred concept is linearity, which was defined by condition(2.19).[27] However, a digital filter is often called linear if it is composed of linear building blocks, although the operation of the system may not be linear. This relaxation allows, for example, forced (non-zero) initial conditions and time-varying parameters[28]. In other words, the model (filter) that produces the regressor of *tap-output* vector is usually presumed to be linear and time-invariant, except for the effect of possible non-zero initial conditions, but the external parametrization of the linear-in-parameter model may be time-varying. The parameters themselves, even in the time-invariant case, are usually obtained by non-linear operations from the data.

2.1.5   Implicit assumptions and alternative interpretations

In the previous section, a variety of definitions was given for representing signals and systems in the time- and frequency-domains, as well as, transforms for transitions between the representations. The contents and depth of reasoning is sufficient for many purposes as such, more or less common text book argumentation, but some of the unstated conditions for the definitions are essential in further generalizations. On a general level, three presumptions were made:

- The signals and systems to be considered have decompositions with respect to some basis functions

- There is a way to determine the components (coefficients, weights, coordinates, parameters, . . .) of these decompositions

- The equations that define the decomposition representations are meaningful

The last item is primarily related to the summability or integrability of the arguments in the expressions. The second article is a combination of the former and properties of the chosen basis functions. For the first, a mild premiss for a set to

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[27]To make things even more complicated, the correct mathematical term would in general actually be *affine*.

[28]Once again, strict linearity would require zero initial conditions, and on the other hand, a linear filter with time-varying coefficients is usually non-linear.

Shifted unit impulses (TD)	$\{\ldots, 0, 0, \delta(n-k), 0, 0, \ldots\}$
Shifted complex variables (TD/FD)	$\{\ldots, z^{\pm 2}, z^{\pm 1}, 1, z^{\mp 1}, z^{\mp 2}, \ldots\}$
Continuous exponentials (TD/FD)	$\{\ldots, e^{\pm j 2\omega}, e^{\pm j\omega}, 1, e^{\mp j\omega}, e^{\mp j 2\omega}, \ldots\}$
Discrete exponentials (TD/FD)	$\{\ldots, z^{\pm j 2\pi/N}, 1, z^{\mp j 2\pi/N}, \ldots\}$
Equidistant sampling (TD/FD)	$\{1, e^{\mp j 2\pi/N}, e^{\mp j 4\pi/N}, \ldots, e^{\mp j(N-1)2\pi/N}\}$
Generic TD basis functions	$\{\ldots, x_{-1}(n), x_0(n), x_1(n), \ldots\}$
Generic FD basis functions	$\{\ldots, X_{-1}(z), X_0(z), X_1(z), \ldots\}$

Table 2.1: Basis functions for representing and transforming discrete-time signals and systems in the time-domain (TD) and in the frequency-domain (FD).

qualify as a base is that it should be capable of representing any member of a somehow specified class of quantities. Table 2.1 presents some of the basis functions used in Section 2.1.1. An additional property of uniqueness of the basis representation is intimately connected to the unambiguity of the corresponding transformation. On the whole, the concept of a *function space* is what is needed for a general and uniform treatment of all three questions. This will be the subject of a few subsequent Sections, starting with the following priming.

The square-summability property was briefly mentioned in connection with the DTFT (2.7): the class of square-summable discrete-time signals is denoted and defined by

$$\ell^2(\mathbb{Z}) = \{x(n) : \sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty\}. \tag{2.39}$$

The corresponding set of causal signals is denoted by $\ell^2(\mathbb{N})$, with an obvious modification in definition (2.39).[29] Actually, a stricter condition of *absolute summability* was used in defining the region of convergence (2.5) of the z-transform (2.4), that is,

$$x(n) \in \ell^1(\mathbb{Z}) = \{x'(n) : \sum_{n=-\infty}^{\infty} |x'(n)| < \infty\} \tag{2.40}$$

implies that $x(n) \in \ell^2(\mathbb{Z})$, and similarly, $\ell^1(\mathbb{N})$ is a subset of $\ell^2(\mathbb{N})$, for the corresponding classes of causal signals[30]. Sequences in $\ell^1(\mathbb{Z})$ or $\ell^1(\mathbb{N})$ are referred to as

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[29]A more engineering oriented term for ℓ^2-signals would be *energy signals*, which distinguish them from *power signals* with infinite energy but with finite average power.

[30]For ℓ^2-signals that are not ℓ^1-signals, the property of uniform convergence of the related transforms is lost and and occasionally replaced by *mean-square convergence*. This would have a substantial effect on the deductions, which is why ℓ^1-convergence was chosen in defining the z-transform (2.4). However, it is precisely the larger classes of ℓ^2-signals that will be associated to the DTFT and the z-transform, respectively, but this generalization requires some additional manoeuvering,

stable, which is again more descriptive for systems than for signals. A necessary condition for an ℓ^1-sequence is that it is decaying, $\lim_{|n|\to\infty} x(n) = 0$. Another immediate consequence, recognized by comparing (2.5) and (2.40), is that the region of convergence must contain the unit circle[31]. Moreover, for causal and stable signals the ROC must be of the form $\{z \in \mathbb{C} : |z| > r, r < 1\}$, that is, the z-transform has to be convergent also for $z = \infty$.[32]

In addition to (2.39) and (2.40), the third physically justifiable categorization of signals is through amplitude variation: the class of *bounded signals* is defined by

$$\ell^\infty(\mathbb{Z}) = \{x(n) : \sup_{n \in \mathbb{Z}} |x(n)| < \infty\}, \qquad (2.41)$$

and similarly, denoted by $\ell^\infty(\mathbb{N})$, in the case of causal sequences. Stable and finite-energy signals are clearly subsets of bounded signals, inducing the inclusion $\ell^1 \subset \ell^2 \subset \ell^\infty$.

A system is said to be *bounded-input-bounded-output stable* (BIBO-stable) if any bounded input produces a bounded output. For an LTI system, the BIBO-stability is equivalent to ℓ^1-stability, which is seen by forming norm inequalities for the convolution sum, $y(n) = h(n) * x(n)$, and conversely by using the "worst case input" $x(n) = \pm M$, with arbitrary alternation of the sign of the bound $M < \infty$. That is, input-output system descriptions using bounded sequences correspond to stable impulse response representations. However, the ℓ^∞-spaces are mathematically poorly equipped for the development of a general framework for signals and systems. Now the point of this all is that a slight modification in the input-system-output setup provides powerful means for the processing of signals and systems. Figure 2.5 displays the change in configuration, where the "input and output spaces" are limited to finite-energy counterparts, and where the system descriptions are extended to the broader class of finite-energy impulse responses.

In the following, the concept of a model for signals and systems is considered as an apparent but profound change in configuration that is essential to further developments. Some models for LTI systems were briefly mentioned in the preceding section. In particular, the FIR model (2.32) was extended to a special class of IIR models (2.33), complemented with the general state-space model description (2.34). Changing the perspective from representation to approximation of a system provides a more general notion of modeling. Moreover, the modified setup allows for a unified approach to the modeling of both signals and systems[33]. In the case of modeling a system, the terms *approximation* and *identification* may be used to reflect the data used in the parameter evaluation, that is, to distinguish between target response approximation and input-output-data identification.

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which will be conducted is Section 2.2.2.

[31]Attributes *exponential* or *majorant* can be added to the term stability, depending on the knowledge on $R(r_1, r_2)$, $0 \le r_1 < 1 < r_2 \le \infty$, and these stricter bounds for stability introduce subsets for $\ell^1(\mathbb{Z})$ or $\ell^1(\mathbb{N})$, respectively.

[32]All the time, it is presumed that one is dealing with the extended complex plane $\mathbb{C} \cup \{\infty\}$.

[33]Any ℓ^2-signal can be interpreted as the (impulse) response of a finite-energy system.
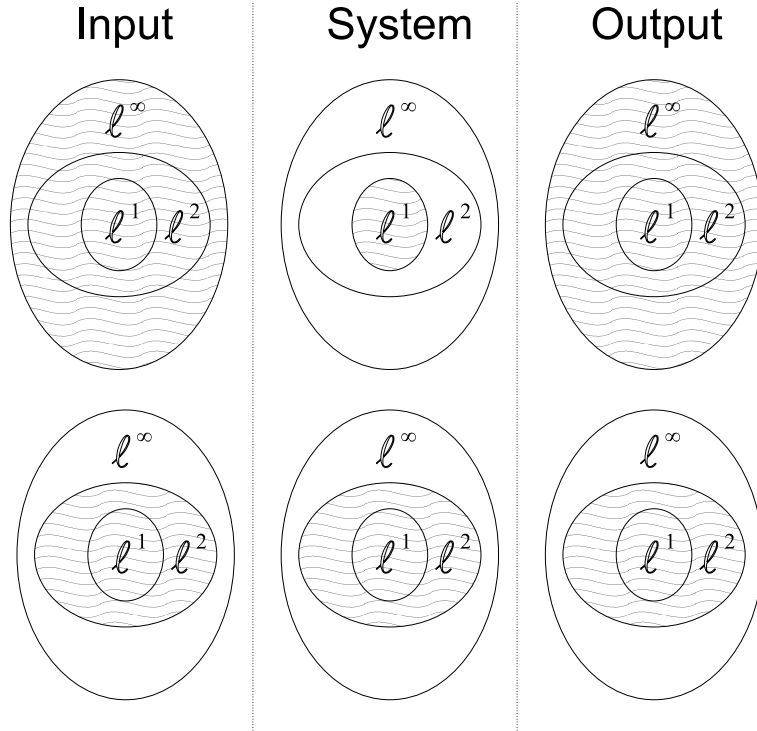
Figure 2.5: The conventional setup for BIBO-stable LTI systems (top) and a modified input-system-output configuration (bottom).

The FIR model provides an asymptotically accurate model for any causal and finite-energy signal or LTI system $h(n)$, that is,

$$h(n) = \lim_{N \to \infty} \sum_{k=0}^{N} h(k)\delta(n - k) \qquad \text{or} \qquad H(z) = \lim_{N \to \infty} \sum_{k=0}^{N} h(k)z^{-k}, \qquad (2.42)$$

where the equality refers to convergence in the ℓ^2-sense. This favorable property of a model is due to the choice of model structure (basis functions) and the chosen parametrization (weighting by the impulse response). However, the FIR model is very restricted since the rate of convergence of the approximation is completely governed by the impulse response sequence. The situation is very different for a recursive model structure of the form (2.33), since in general

- there are no analytic or closed-form solutions for the parameters with respect to a given system and a chosen optimization criterion[34]

- even if a good (locally optimal) solution (in some sense) is attained, it is not necessarily unique (in terms of the parameters), and there are usually very poor means to validate a particular solution with respect to other potential (local or global) solutions

- and consequently, the chain of subsequent solutions as the model order (or

---

[34]The emphasis is on the linking words 'in general': in the following Chapter the modeling setup will be specified as an *output-error formulation*, for example, in contrast to an *equation-error configuration*, which would provide an exception of the statement.

number of parameters) is increased do not constitute a well-behaved evolution for the solution[35]

However, in most practical applications there are strong reasons for presuming that the system to be modeled is inherently recursive, which implies that an essential portion of the system dynamics can be captured with a relatively low number of "recursion parameters". Moreover, since all observable responses are effectively finite in duration, the categorization to FIR or IIR systems is mainly descriptive and rarely the reason for choosing a particular model structure. A clearer view of the modeling characteristics is attained by examining the intermediate signals that are weighted by the parameters, denoted by $\mathbf{x}(n)$ and $\mathbf{z}(n)$ in formulas (2.32) and (2.33), respectively. These *modeling signals* are in both cases produced by causal and linear filtering[36] from the input, but in a very different manner. The modeling or tap-output signals of the FIR filter are simply delayed input samples, independent of the parameters, whereas the tap-outputs of the recursive part of (2.33) are delayed versions of the output sequence, and therefore highly dependent on the parameters $\{a_i\}$. In terms of the tap-output impulse responses, the FIR filter tap-response are completely localized in time, in contrast to the tap-outputs of the recursive part of (2.33), which are distributed over all future time-instants. The objective is to formulate a general framework for linear-in-parameter modeling, where some of the favorable properties of both model types are preserved, namely that the modeling signals are independent of the parameters and yet flexible time-domain signals. In general, the model of a signal or system is supposed to be of the form

$$\hat{y}(n) = \sum_{i=0}^{N} c_i x_i(n), \qquad x_i(n) = g_i(n) * x(n), \qquad (2.43)$$

where $\{x_i(n)\}$ are modeling signals produced by causal LTI filtering of some generating input signal $x(n)$ with the partial model impulse responses $\{g_i(n)\}$. The filtering is presumed to be implemented using realizable digital LTI filtering, causing the tap-outputs to have rational and finite-dimensional transfer functions $G_i(z) = Z\{g_i(n)\}$. The impulse response and transfer function of the model are

$$h(n) = \sum_{i=0}^{N} c_i g_i(n) \quad \text{and} \quad H(z) = \sum_{i=0}^{N} c_i G_i(z), \qquad (2.44)$$

respectively, where the tap-output weights $\{c_i\}$ can always be integrated to the linear filtering operations by denoting $g_i'(n) = c_i g_i(n)$ and $G_i'(z) = c_i G_i(z)$. Equation (2.43) can be interpreted as a generalized or distributed convolution operation, whereas, expressions (2.44) define generalizations to the unit impulse response decomposition and to the z-transform representation, respectively.

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[35]The modeling error, measured in some sense, is not necessarily decreasing for an increase in model order (in contrast to the model (2.42)). In addition, subsequent sets of model parameters are usually completely "disconnected", which complicates the survey.

[36]Assuming time-invariant recursion parameters in the latter.

2.1.6 Deterministic and stochastic signals

The ongoing assumption so far has been that the discrete-time signals and systems are in principle characterized by an explicit, although usually unknown, mapping $n \mapsto x(n)$. However, the classification of signals in (2.39)-(2.41) was based on a much more abstract relation to the individual signal samples, that is, on a single real number reflecting somehow the magnitude of the signal. This observation motivates the following short reference to *random or stochastic signals*, which complements the previous definition of *deterministic* discrete-time signals.

A discrete-time stochastic signal $x(n)$ is defined as a complex *random or stochastic process* defined at discrete and uniformly spaced instants $n \in \mathbb{Z}$ of time, where the stochastic process is characterized by some probabilistic laws. More precisely, the signal $x(n)$ is a realization of a random process related to complex random variables $\{X\}$ with well-defined probability density functions[37]. The signal sample $x(n)$ is the value of a random variable X evaluated at time-instant n, denoted by $X[n]$, and in general a complete description of the signal $x(n)$ would require that the *joint probability density function* of an infinite collection of random variables $\{X[n]\}$ to be known. A stochastic signal is said to be *stationary* if its statistical properties are invariant to shifts in time[38]. That is, a stationary signal $x(n)$ reduces to a sample sequence drawn from the probability distribution of a single complex variable.

Fortunately, in most cases a practical description of stationary stochastic discrete-time systems and signals does not require any reference to the probability theory. This is due to the fact that weaker stationarity conditions are sufficient for the development of stochastic counterparts of the previous concepts. More precisely, stationarity is redefined as the time-invariance of the first and second order stochastic moments, the *mean* and *mean-square value*, $E[x(n)]$ and $E[x^2(n)]$, respectively[39]. These moments have natural physical interpretations and they can be estimated directly from realizations of stochastic processes, that is, without resorting to probability distributions. Once again, the key property is the linearity of the expectation operator and its estimates. According to the linearity, every stationary signal $x(n)$ can be replaced by a more convenient zero mean signal $x(n) - \mu_x$, which will be the additional presumption hereafter. The zero mean assumption provides simplifications in terminology since mean-square and variance type of quantities coincide[40]. An additional consequence of the stationarity is that the *autocorrelation function*

$$r_{xx}(m) = E[x(n+m)x^*(n)], \qquad m \in \mathbb{Z} \tag{2.45}$$

is a function of the time lag m alone, that is, independent of the time variable. The time-invariance of the mean and autocorrelation characterize *weakly stationary*

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[37]Meaning that the stochastic moments, $\mu_r = E[X^r] = \int_{-\infty}^{\infty} \alpha^r p_X(\alpha)d\alpha$, $r = 1, 2, \ldots$, are well-defined with respect to the probability density $p_X(\alpha)$. The expression $E[\cdot]$ is called the mean or expected value of the argument.

[38]Since the random variable is completely characterized by the statistical moments, this is equivalent to presuming that all the moments are time-invariant, that is, constants of the form $\mu_r = \mu_r(n) = E[(X[n])^r]$.

[39]That is, the mean and mean-square value of the process $x(n)$ are defined through the associated random variable X by $E[x(n)] = E[X]$ and $E[x^2(n)] = E[X^2]$, respectively.

[40]Otherwise a distinction between correlation and covariance terms should be made.

signals as the relevant subclass of stationary signals. Furthermore, the set of *wide-sense stationary* (WSS) signals is defined by an additional condition of finite mean-square value, $E[x^2(n)] = r_{xx}(0) < \infty$. The set of WSS signals is denoted suggestively as $\ell^2(\mathbb{Z})$. The *cross-correlation* of two WSS signals, $x(n)$ and $y(n)$, is defined by

$$r_{xy}(m) = E[x(n+m)y^*(n)], \qquad m \in \mathbb{Z}. \tag{2.46}$$

The cross-correlation function resembles the convolution operation, which is even more evident for the deterministic counterpart of (2.46),[41]

$$r_{xy}(m) = \sum_{n=-\infty}^{\infty} x(n+m)y^*(n), \qquad m \in \mathbb{Z}. \tag{2.47}$$

Comparing expressions (2.23) and (2.47), yields $r_{xy}(m) = x(m) * y^*(-m)$. Another property adopted from the deterministic representation is that the correlation sequences of WSS signals are absolutely summable. This observation provides frequency-domain representations for WSS signals and systems as the DTFT or z-transform of the corresponding correlation sequences. Two principal results are needed for the modeling concept of WSS signals. In the first place, the output of a stable LTI system to a WSS input signal is a WSS signal. In addition, the zero mean property of the input signal transforms to the output signal as well. Secondly, any WSS signal can be represented as the output of a causal LTI system[42]. Using definitions and properties of the convolution sum and the correlation terms, the cross-correlation of the input and output signals, $x(n)$ and $y(n)$, related to an LTI system defined by its impulse response $h(n)$, is given by

$$r_{yx}(n) = h(n) * r_{xx}(n). \tag{2.48}$$

The DTDF of (2.48) is simply $S_{yx}(\omega) = H(\omega)S_{xx}(\omega)$, which provides the *cross-spectral density* $S_{yx}(\omega)$ as a product of the frequency response $H(\omega)$ and the power spectral density $S_{xx}(\omega)$.[43] Similarly, the autocorrelation of the output, $r_{yy}(n) = y(n) * y^*(-n)$, can be expressed by

$$r_{yy}(n) = r_{hh}(n) * r_{xx}(n) \tag{2.49}$$

due to the commutativity of the convolution operation. The DTFT of (2.49) is $S_{yy}(\omega) = S_{hh}(\omega)S_{xx}(\omega)$, but the associated filtering operation requires some elaboration. If the transfer function of the causal LTI filter impulse response $h(n)$ is denoted by $H(z)$, then the relation $r_{hh}(n) = h(n) * h^*(-n)$ and some manipulation on the corresponding z-transform provides

$$S_{hh}(\omega) = H(z)H^*(1/z^*)|_{z=e^{j\omega}} = |H(\omega)|^2. \tag{2.50}$$

The transfer function $H^*(1/z^*)$ is associated to the anti-causal part $h^*(-n)$ and the equality $S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega)$ states that when a WSS signal $x(n)$ with power

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[41]The "averaging" of the form $\lim_{N\to\infty}(\frac{1}{2N+1}\sum_{-N}^{N}(\cdot))$ is omitted for notational reasons. It also anticipates that the expectation and summation operators will be related without a normalizing term.

[42]This is a vague version of the *Wold decomposition principle*.

[43]The (DTFT) relation between correlation terms and corresponding spectral densities is known as the *Einstein-Wiener-Khintchine relation*.

spectral density $S_{xx}(\omega)$ is transmitted trough a causal LTI system, the power spectral density of the output equals $S_{xx}(\omega)$ weighted with the square magnitude response of the filter[44]. Above equations, starting from (2.48), provide different input-system-output identification setups depending on the available statistical data. These relations apply also to the partitioned model of the form (2.43) and (2.44), where the system is replaced with the partial model impulse responses, providing the required generalizations to the linear-in-parameter modeling of signals and systems in the stochastic framework. The model is however always considered to be deterministic, although possibly unknown, even if the partial model responses are embedded in a stochastic representation along with the input signal description.

A zero-mean WSS signal with constant power spectral density is referred to as *white noise*. The autocorrelation sequence, attained as the IDTFT, is thus necessarily an impulse sequence, which implies that the white noise signal samples are, by definition, uncorrelated with each other[45]. White noise that is normalized to $S_{xx}(\omega) = 1$ is the stochastic counterpart of the unit impulse sequence[46]. For a white noise input $x(n)$, the power spectral density of the output of a system $H(z)$ is of the form $S_{yy}(\omega) = |H(\omega)|^2\sigma_x^2$, where σ_x^2 is the variance of the white noise input signal. Now presuming that $H(z)$ is a (causal, stable and) finite-dimensional LTI system, representable by a reduced form rational transfer function $H(z) = B(z)/A(z)$, three categorizations of the output can be made depending on the polynomials $B(z)$ and $A(z)$. The output is said to be a *moving average (MA) process* if $A(z)$ is a constant and $B(z)$ a non-constant polynomial, implying that the output is solely dependent on the present and past input values. The process is *autoregressive (AR)* if it is purely a function of the present input and past output samples, corresponding to a constant $B(z)$ and non-constant $A(z)$. The process is said to be *autoregressive moving average (ARMA)* for non-constant $A(z)$ and $B(z)$. The terms *all-pole*, *all-zero* and *pole-zero* are used, respectively, to describe the system that generates the process. More generally, any WSS signal $x(n)$ with a given power spectral density $S_{xx}(\omega) = |H(\omega)|^2\sigma_w^2$ can be produced as the output of a causal LTI filter $H(z)$ to a white noise input signal $w(n)$. Conversely, the WSS signal $x(n)$ with power spectral density $S_{xx}(\omega)$ may be *whitened* by passing $x(n)$ through a causal LTI filter $1/H(z)$. The signal $w(n)$ is called the *innovation process* related to $H(z)$ and $x(n)$.

2.2 Function space descriptions of signals and systems

Function spaces are generalizations of point spaces, such as the set of complex numbers $\mathbb{C}$, where the points of the space are functions defined in a common domain

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[44]Usually it is presumed that the impulse response is real valued, but with these definitions it seems unnecessary.

[45]This reasoning makes the usual attribute 'uncorrelated' redundant in the definition of white noise

[46]In the case of (zero-mean) white noise, the constant value of the power spectral density, $S_{xx}(\omega) = \sigma_x^2$, equals the mean-square value or variance of the signal, that is, the energy of the signal.

and range. All function spaces considered in the following are at the least *vector spaces*; the functions are all complex valued and the point-wise arithmetic operations defining addition and multiplication rely on the vector space properties of $\mathbb{C}$. The construction of a vector space provides means for the study of *linearly independent* members, which is the most important concept in modeling signals and systems. Furthermore, all relevant spaces are *normed* vector spaces, enabling the measurement of distance between terms. An additional property of *completeness* associated to the norm ensures that the inspection of convergence is well-posed.

Function space descriptions enable mappings between classes of functions, functions of functions, or less annoyingly *functionals*. Transformations and operators are special cases of functionals, where the former is usually associated to invertible one-to-one mappings between function spaces and where the latter categorizes mappings within function spaces, such as projection and shift operations. Previous definitions for transformations are sufficient for the purpose of representing signals and systems, but transformations may also be used to transport general properties deduced for a class of functions into another function space, which calls for some specification of the involved classes of functions.

2.2.1 Banach spaces of Lebesgue integrable functions

A *norm* on a real or complex vector space X is a non-negative real-valued function, $\|\cdot\| : X \to \mathbb{R}$, subject to certain conditions. Equipped with a specified norm, X is a *normed linear space*, where the supplement 'linear' is a reflection of the joint properties of X and $\|\cdot\|$. Any such norm induces a natural *metric*, $d(x, y) = \|x - y\|$, a measure of distance of two points $x, y \in X$. A normed linear space X is called a *Banach space*, if X is complete with respect to the natural metric. *Completeness* then means that convergence of a sequence of points $\{x_n\}$ implies convergence to a point $x \in X$, $lim_{n,m \to \infty}\|x_n - x_m\| = 0 \Rightarrow lim_{n \to \infty}\|x_n - x\| = 0$, or for a stricter definition of convergence that any *Cauchy sequence* converges[47].

The previously defined "signal spaces" ℓ^1, ℓ^2 and ℓ^∞ are special cases of the general class of Banach spaces $\ell^p = \{x(n) : \|x(n)\|_p < \infty\}$, when equipped with norms

$$\|x\|_p = \left(\sum_{n \in I} |x(n)|^p\right)^{1/p}, \quad 1 \le p < \infty, \quad \|x\|_\infty = \sup_{n \in I} |x(n)|, \qquad (2.51)$$

where I is the preferred subset of $\mathbb{Z}$. More generally, the p-norm of a complex-valued function, $f : X \to \mathbb{C}$, is based on the integrability of the quantity $|f|^p$ with respect to some *measure* μ on X. In ℓ^p-spaces μ is the counting measure on $\mathbb{Z}$, and fortunately, also the other measure that is needed here, namely the Lebesgue measure on $\mathbb{R}$, has a sufficiently well-known interpretation without further discussion

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[47]A sequence is a Cauchy sequence, if to every $\epsilon > 0$, there is an integer N, such that $\|x_n - x_m\| < \epsilon$ as soon as $n, m > N$. For sequences in $\mathbb{R}$ or $\mathbb{C}$, this Cauchy criterion is usually the definition of convergence, which embeds the fact that $\mathbb{R}$ and $\mathbb{C}$ are complete with respect to the absolute value norm.

on measurability[48]. The general class of Lebesgue integrable functions with respect to some measure μ on X is denoted by $L_X(\mu)$. Here the set over which the integrals are evaluated is the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and the freedom of choice among many different suitable parametrizations of this set is emphasized by the simplified notation $L(\mathbb{T})$. Analogously to 2.51, the class of complex-valued functions f on $\mathbb{T}$ such that $|f|^p \in L(\mathbb{T})$ and the L^p-norm

$$\|f\|_p = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(e^{j\theta})|^p d\theta \right)^{1/p}, \quad 0 < p < \infty, \quad \|f\|_\infty = \sup_{z \in \mathbb{T}} |f(z)| \qquad (2.52)$$

is finite, is denoted by $L^p(\mathbb{T})$. For any fixed p, $0 < p \leq \infty$, $L^p(\mathbb{T})$ is a complex vector space and the integral operator $f \mapsto \int f d\mu$ is a linear functional on $L^p(\mathbb{T})$, but (2.52) as such is not even a norm. The space $L^p(\mathbb{T})$ is enforced to be complete in the natural metric by replacing point-wise equality of members in $L^p(\mathbb{T})$ by equivalence classes of functions that coincide almost everywhere (with respect to the measure). The vectors in $L^p(\mathbb{T})$ are still treated as functions, but the uniqueness is regarded with respect to the metric: $f \sim g$ if and only if $d(f,g) = 0$. Defined in this manner, $L^p(\mathbb{T})$ is a Banach space for any fixed p, $1 \leq p \leq \infty$.

Measurability was the underlying assumption in defining L^p-functions. More familiar regularity properties of functions are given in terms of continuity or differentiability. The class of continuous complex-valued functions on $\mathbb{T}$ is denoted by $C(\mathbb{T})$ and it is a *dense* subset of $L^p(\mathbb{T})$, that is, functions in $L^p(\mathbb{T})$ can be arbitrarily well approximated by functions in $C(\mathbb{T})$ with respect to the L^p-norm. The L^p-norms define metrics in $C(\mathbb{T})$, even in the strict point-wise sense, but actually only the L^∞-norm makes $C(\mathbb{T})$ a Banach space.

The reason why Banach spaces are emphasized here is that the theory of functionals on Banach spaces is the foundation of all transformations in Section 2.1, and for some to come. For an abstract Banach space X, the class of all continuous linear functionals on X is also a Banach space, the *dual* or *conjugate* space X^*, defined by a specific norm. For example, the dual space of $L^p(\mathbb{T})$, $1 \leq p < \infty$, is $L^q(\mathbb{T})$ with $q = p/(p-1)$, and every continuous linear functional on $L^p(\mathbb{T})$ is of the form $f \mapsto \int fg d\mu$, where $g \in L^q(\mathbb{T})$.

2.2.2 Analytic functions and Hardy spaces

In Section 2.1.5, the z-transform (2.4) of a causal and stable signal or system was found to be absolutely convergent in the complement of the open unit disk $\bar{\mathbb{E}} = \{z \in \mathbb{C} : |z| \geq 1\}$. The z-transform of an $\ell^1(\mathbb{N})$-sequence is thus continuous in $\bar{\mathbb{E}}$, but more importantly, it is analytic or differentiable with respect to the complex variable in the region $\mathbb{E} = \{z \in \mathbb{C} : |z| > 1\}$, denoted by $H(\mathbb{E})$.[49] Actually an

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[48]There are several strategies in defining measures on sets and spaces, and corresponding integrals of complex-valued functions [Hoffman, 1962] [Rudin, 1987].

[49]The function class $H(\mathbb{E})$ is clearly a vector space, but it is also an *algebraic ring*, that is, closed with respect to addition and multiplication operations. Moreover, any superposition of functions, $h = g \circ f$, where $f \in H(\mathbb{E})$, $f(\mathbb{E}) \subset \Omega$ and $g \in H(\Omega)$ for some region $\Omega \subset \mathbb{C}$, is again in $H(\mathbb{E})$.

alternative definition for functions in $H(\mathbb{E})$ is that they are all of the form

$$f(z) = \sum_{i=0}^{\infty} c_i z^{-i}, \quad \sum_{i=0}^{\infty} |c_i| < \infty. \tag{2.53}$$

This is the *Laurent series expansion* in the "vicinity of $z = \infty$" and because of the geometry of $\mathbb{E}$, this representation is unique[50]. Conversely, every continuous complex-valued function on $\mathbb{T}$, $f \in C(\mathbb{T})$, has an unique analytic extension $f \in H(\mathbb{E})$, and is therefore representable by (2.53). The z-transform (2.4) and its inverse (2.6) define an one-to-one mapping between $\ell^1(\mathbb{N})$ and $C(\bar{\mathbb{E}}) \cap H(\mathbb{E})$, but the boundary conditions are too restrictive for the definition of a desired normed subspace of functions in $H(\mathbb{E})$.[51]

The boundary function of an $H(\mathbb{E})$-function is defined as the radial limit $f(e^{j\theta}) = \lim_{r \to 1+} f(re^{j\theta(r)})$, where the convergence is with respect to any non-tangential (to $\mathbb{T}$) path $z = re^{i\theta(r)} \to e^{i\theta}$. If the radial limits are bounded in the L^p-norm, $1 \le p \le \infty$, then they exist almost everywhere on $\mathbb{T}$ and define thus a function in $L^p(\mathbb{T})$. The function classes $H^p(\mathbb{E}) = L^p(\mathbb{T}) \cap H(\mathbb{E})$, $1 \le p \le \infty$, are called *Hardy spaces* and they constitute a sequence of nested Banach subspaces for $H(\mathbb{E})$, when equipped with the L^p-norm[52]. Alternatively, the Hardy space $H^p(\mathbb{E})$, $1 \le p \le \infty$, can be seen as a subspace of $L^p(\mathbb{T})$, the space of all $f \in L^p(\mathbb{T})$ with analytic extensions in $\mathbb{E}$.

The main reason why all this effort is made is that finally there is an explicit relation between the Fourier transform and the z-transform. The (discrete-time)Fourier transform pair

$$F(e^{j\omega}) = \sum_{k=-\infty}^{\infty} f(k)e^{-j\omega k} \quad \longleftrightarrow \quad f(k) = \frac{1}{2\pi} \int_0^{2\pi} F(e^{j\omega})e^{j\omega k} d\omega \tag{2.54}$$

defines an one-to-one mapping between $\ell^2(\mathbb{Z})$ and $L^2(\mathbb{T})$, an *isomorphism*, which is also an *isometry*, since $\|f\| = \|F\|$ with respect to the corresponding metrics[53]. Now the formal z-transform pair

$$F(z) = \sum_{k=-\infty}^{\infty} f(k)z^{-k} \quad \longleftrightarrow \quad f(k) = \frac{1}{2\pi j} \oint_{\mathbb{T}} z^k F(z)\frac{dz}{z}, \tag{2.55}$$

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[50]The derivative of $f(z)$ is again of the form (2.53), and thus in $H(\mathbb{E})$, which implies that $f^{(k)}(z) \in H(\mathbb{E})$ for derivatives of all orders $k = 0, 1, \ldots$. By comparing (2.53) and the power series derived for $f^{(k)}$, the coefficients are explicitly given by $c_i = f^{(i)}(\infty)/i!$. The Laurent series can thus be seen as a generalized *Taylor's formula* or as interpolation by polynomials, with respect to $z = \infty$.

[51]Under the sup norm, $C(\bar{\mathbb{E}}) \cap H(\mathbb{E})$ (or using another denotation $C(\mathbb{T}) \cap H(\mathbb{E})$) is actually a Banach space. In all above reasoning, "inverted" domains are used instead of the usual $\mathbb{D}$ and $\bar{\mathbb{D}} = \mathbb{D} \cup \mathbb{T}$, which lean on the assumption that the complex plane is compactly extended. The class of functions $\mathcal{A} = C(\bar{\mathbb{D}}) \cap H(\mathbb{D})$ is called the *disk algebra*.

[52]For $1 < p < s < \infty$, $H^\infty \subset H^s \subset H^p \subset H^1$, and actually the chain can be extended to $N = H^0$ by a specified L^0-norm (N is for Nevanlinna). For $0 \le p < 1$, H^p is a vector space that is however no longer normed.

[53]It is easy to prove that the sequence of Fourier coefficients of any $F \in L^2(\mathbb{T})$ is square-summable. The fact that every sequence in $\ell^2(\mathbb{Z})$ is a sequence of Fourier coefficients for some $F \in L^2(\mathbb{T})$ is known as the *Riesz-Fischer-Theorem*.

can be seen as a generalization of (2.54), although it has very little meaning as such. Earlier, the space $H^p(\mathbb{E})$ was regarded as a subspace of $L^p(\mathbb{T})$ by associating an unique analytic extension to the corresponding $L^p(\mathbb{T})$-functions. For $1 < p \leq \infty$, this subset of $L^p(\mathbb{T})$ is completely characterized by the property that the negative Fourier moments vanish, that is,

$$\int_0^{2\pi} F(e^{j\omega})e^{j\omega k}d\omega = 0, \quad k = -1, -2, \ldots . \tag{2.56}$$

In particular, this applies to $p = 2$ and the Fourier transform relationship (2.54) on the boundary combined to condition (2.56) makes (2.55) a well-defined invertible transformation between $\ell^2(\mathbb{N})$ and $H^2(\mathbb{E}) = L^2(\mathbb{T}) \cap H(\mathbb{E})$.

Some remarks The analysis part of (2.54) that produces the Fourier coefficients is well-defined for any $F \in L^p(\mathbb{T})$, $1 \leq p \leq \infty$, but the validity of the synthesis part, the convergence of Fourier series in the corresponding norm, is guaranteed only in the case $p = 2$.[54] The spaces $\ell^2(\mathbb{N})$, $\ell^2(\mathbb{Z})$, $L^2(\mathbb{T})$ and $H^2(\mathbb{E})$ are all examples of *Hilbert spaces*, which will be the topic of the next Section. The reason why more general cases than just the two-norm spaces are dragged along is that many properties that really "click" only in a Hilbert space, are still useful in more general spaces. For example, the trigonometric system that generates Fourier transforms is complete in many Banach spaces, although it is no longer a (Fourier) base. This result has recently been generalized to the case of rational orthonormal bases in $L^p(\mathbb{T})$ for $1 < p < \infty$ [Akçay, 2000].

2.3   The Hilbert space

The Hilbert space is a construction that generalizes many properties of finite-dimensional Euclidean spaces to infinite-dimensional function spaces. The dimension of a space is the cardinality of a somehow defined basis set of the space. A basis set, a base, should at the least be able to represent any member of the space, that is, to span the space. The properties of a vector space enable the inspection of linearly independent members. A finite set of linearly independent vectors, $\{e_1, \ldots, e_N\}$, span a subspace of a vector space V, $S = span\{e_1, \ldots, e_N\} \subset V$. Every element $x \in S$ has a unique representation $x = \sum_{i=1}^N c_i e_i$. A set of less than N elements can not span S and a set of more than N elements is inevitably linearly dependent. The uniqueness of the coordinate representation, $\{c_1, \ldots, c_N\}$, and the linear independency of the basis vectors, $\{e_1, \ldots, e_N\}$, are equivalent properties. In the following, the potential usefulness of the concept of a base is considered in the case of some infinite-dimensional spaces.

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[54]There are ways to reproduce a function $F \in L^p(T)$, $1 \leq p \leq \infty$, from its Fourier coefficients by replacing the Fourier series with arithmetic means of its partial sums [Hoffman, 1962]

2.3.1 Bases of spaces

In an infinite-dimensional Banach space B, a countably infinite set $\{e_i\}_{i=0}^{\infty}$ is a *Schauder basis*, if for every $x \in B$, there is a unique series $\sum_{i=1}^{\infty} c_i e_i$ such that

$$\lim_{N \to \infty} \left\| x - \sum_{i=0}^{N} c_i x_i \right\| = 0. \tag{2.57}$$

Then the relation is formally written as $x = \sum_{i=1}^{\infty} c_i e_i$, where the equality is to be interpreted in the sense of the Banach space norm. The linear independency of $\{e_i\}_{i=0}^{\infty}$ was not presumed, but it is a necessary condition for the representation to be unique. If a Banach space has a Schauder base, it is necessarily *separable*, that is, contains a countably infinite *dense* set[55]. However, all Banach spaces do not have a Schauder base, and consequently, they may or may not be separable, in contrast to Hilbert spaces that are always (essentially) separable and spanned by an infinitely countable orthonormal set.

Maybe the most familiar Hilbert spaces are the Cartesian product spaces $\mathbb{R}^N$ and $\mathbb{C}^N$, where N is some positive integer. The *inner product* of two vectors $\mathbf{x}$ and $\mathbf{y}$ is defined as the sum of component products, $(\mathbf{x}, \mathbf{y}) = \mathbf{y}^H \mathbf{x}$, where the superscript H denotes complex conjugate transpose. The unit vectors, $\mathbf{e}_k = [0 \; \cdots \; 1 \; \cdots \; 0]$, $k = 1, \ldots, N$, where the kth element is 1, span the spaces $\mathbb{R}^N$ and $\mathbb{C}^N$, respectively. This base is *orthonormal*, since $(\mathbf{e}_i, \mathbf{e}_j) = 0$, if $i \neq j$, and $(\mathbf{e}_i, \mathbf{e}_i) = 1$, which provides a unique representation $\mathbf{x} = \sum_{i=1}^{N} x_i \mathbf{e}_i$, where the projection of $\mathbf{x}$ onto the basis vector $\mathbf{e}_k$ is given by $x_k = (\mathbf{x}, \mathbf{e}_k)$, and it is the kth component or coordinate of $\mathbf{x}$. What actually qualifies $\mathbb{R}^N$ and $\mathbb{C}^N$ as Hilbert spaces is that they are *complete* with respect to a special norm induced by the inner product, a norm given by $\|\mathbf{x}\| = (\mathbf{x}, \mathbf{x})^{1/2}$. The square-norm of $\mathbf{x}$ is then the sum of squares of the components, $\|\mathbf{x}\|^2 = (\mathbf{x}, \mathbf{x}) = \sum_{i=1}^{N} x_i^2$, as is known for real Euclidean spaces.

2.3.2 Basic concepts of the Hilbert space theory

A complex vector space H is an inner product space, if there is a function $(\cdot, \cdot) : H \times H \to \mathbb{C}$ that assigns to each ordered pair of vectors in H a scalar, with certain symmetry and linearity properties[56]. One consequence of these properties is that the relation $\|x\| = (x, x)^{1/2}$ defines a norm in H. A by-product in proving that $\|x\|$ actually is a norm, is the *Schwartz inequality*

$$|(x, y)| \leq \|x\| \|y\| \qquad (\text{or} \quad |(x, y)|^2 \leq (x, x)(y, y)). \tag{2.58}$$

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[55]In terms of the norm, a set A is dense in B, if any point in B is arbitrarily close to a point in A. If B is a topological space (as it here always is), then the closure $\bar{A} = B$. A weaker topological definition for a base $\{e_i\}_{i=0}^{\infty} \in B$ is that the set of all finite linear combinations of the basis vectors, denoted by S, is dense in B, $\bar{S} = B$. With this definition, there is generally not a representation of the form (2.57).

[56]Namely, for $x, y, z \in H$ and $\lambda \in \mathbb{C}$, (i) $(x + y, z) = (x, z) + (y, z)$, (ii) $(\lambda x, y) = \lambda(x, y)$, (iii) $(y, x) = (x, y)^*$, (iv) $(x, x) \geq 0$, and (v) $(x, x) = 0$, only if $x = 0$. Conditions (i) and (ii) say that the mapping $x \mapsto (x, y)$ is a linear functional for any fixed $y \in H$. The exchange rule (iii) provides axioms for the latter component in the inner product.

For real inner product spaces, the Schwartz inequality ensures that the inverse cosine of $(x, y)/(\|x\|\|y\|)$ is a valid measure for the angle between x and y.[57] Inner product spaces are also metric with respect to the natural metric $d(x, y) = \|x - y\|$, the distance between x and y. These properties can be seen to some extent as generalizations of geometrical concepts of Euclidean spaces[58].

The Hilbert space is an inner product space that is complete in the norm $\|x\| = (x, x)^{1/2}$. Thus, the Hilbert space is a Banach space in which the norm is induced by an inner product[59].

Hereafter, the symbol H will denote a generic Hilbert space. *Orthogonality* is the most important concept of a Hilbert space. One implication of the definition of the inner product is that $(0, y) = 0$ for all $y \in H$, that is, the vector $x = 0$ is *orthogonal* to every vector in H.

Orthogonality and orthonormality Two vectors x and y in H are *orthogonal* if and only if $(x, y) = 0$. The orthogonality of x and y is denoted by $x \perp y$. If S is a subset of H, then $S^{\perp} = \{x \in H : (x, y) = 0 \; for \; all \; y \in S\}$ is the *orthogonal complement* of S, the set of all $x \in H$ that are orthonormal to each vector in S. A vector $x \in H$ is *normal*, if $\|x\| = (x, x)^{1/2} = 1$. A subset of H is orthonormal, if all its vectors are normal and mutually orthogonal. An infinitely countably subset $\{x_i\}_{i=0}^{\infty}$ is orthonormal, if $(x_i, x_j) = 0$ whenever $i \neq j$, and $(x_i, x_i) = 1$ for all $i = 0, 1, \ldots$.

An orthonormal set is linearly independent[60]. More importantly, any finite collection of vectors of an orthonormal set in H spans a subspace S that is closed, that is, a Hilbert subspace of H.[61] Any linearly independent spanning set, or base, of S is attained by a linear invertible transformation from the finite orthonormal set. Furthermore, all orthonormal bases of S are related through an unitary (linear) transformation[62]. An ordered infinitely countable set of orthonormal vectors $\{e_i\}_{i=0}^{\infty}$

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[57]The concept of an angle is not very useful in more general inner product spaces, except for the spacial case $(x, y) = 0$.

[58]Additionally, any norm that is induced by an inner product satisfies the parallelogram law

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2), \tag{2.59}$$

and conversely, any such norm originates from an inner product.

[59]Actually, the completeness was defined with respect to the metric, but the condition (v) of the inner product ensures that the norm induces a metric.

[60]A subset of a vector space is linearly independent, if no finite linear combination of its vectors is the null vector, that is, a finite weighted sum vanishes, $\sum_i c_i x_i = 0$, only if $c_i = 0$ for all i.

[61]The linear independency of an orthonormal set is easily established. Let S be the span of any finite collection of orthonormal vectors $\{e_i\}$. Then S is a subspace, and thus contains the null vector, and there is only one linear combination such that $\sum_i c_i e_i = 0$. The coefficients are given by $c_i = (0, e_i)$, which is zero for all i.

[62]Two bases of S, $\{e_i\}$ and $\{f_i\}$, are related by $\mathbf{e} = \mathbf{Lf}$, where $\mathbf{L}$ is an invertible fixed matrix operator. The transformation is unitary, if $\mathbf{L}^H \mathbf{L} = \mathbf{I}$, where $\mathbf{I}$ is the corresponding unit matrix, that is, if $\mathbf{L}^{-1} = \mathbf{L}^H$. The term unitary refers to the property that the operator norm of $\mathbf{L}$ is one.

in H defines a chain of nested subspaces, $H_1 \subset \ldots \subset H_N \subset H_{N+1} \subset \ldots \subset H$. A natural question is then whether or not H is attained as a limiting process for H_N. Before addressing these existence and completeness issues, some general properties of the subspaces are stated, as well as, more practical results concerning the interplay between linear independency and orthonormality.

The Projection Theorem If S is a closed subspace of H, then its orthogonal complement $S^\perp$ is also a closed subspace of H, and every $f \in H$ has a unique decomposition $f = g + h$, where $g \in S$ and $h \in S^\perp$. The space H is a direct sum, $H = S \oplus S^\perp$, of its orthogonal components. The distance between $f \in H$ and the subspace S, $\inf_{g \in S} \|f - g\|$, attains its minimum for some $g \in S$, and $f - g \perp S$ is a sufficient and necessary condition for the determination of the unique element $g \in S$.

In other words, g is the best approximation of $f \in H$ by an element in the subspace S, with respect to the Hilbert space norm. The approximation error $f - g$ is orthogonal to S, an element of $S^\perp$, which can be used to determine g. The name of the Theorem is due to the geometrical interpretation that the decomposition defines a projection operation P, such that $Pf = g$ and $(1 - P)f = f - g \in S^\perp$. The next results shows how the approximation is attained, if the subspace is spanned by a finite set of vectors.

Normal equations Let $\{e_i\}_{i=1}^N$ be any finite linearly independent set of vectors in H. Then $\{e_i\}_{i=1}^N$ spans a (closed) subspace of H, denoted by H_N, and the coefficients of the best approximation $\hat{x} = \sum_{i=1}^N c_i e_i \in H_N$ that minimizes $\|x - \hat{x}\|$ for any $x \in H$ are attained uniquely, since according to the Projection Theorem, $x - \hat{x} \perp H_N$, so that

$$(x - \hat{x}, e_j) = 0, \quad \text{for all} \quad j = 1, \ldots, N, \quad (2.60)$$

and due to the linearity of the inner product, (2.60) is equivalent to

$$\sum_{i=1}^N c_i (e_i, e_j) = (x, e_j), \quad \text{for all} \quad j = 1, \ldots, N. \quad (2.61)$$

The system of equations (2.61) is called the normal equations. If in addition, the set $\{e_i\}_{i=1}^N$ is orthonormal, then (2.61) reduces to $c_j = (x, e_j)$, $j = 1, \ldots, N$.

The term *closed* is actually redundant in the case of finite-dimensional subspaces, since they are always closed. However, in the Projection Theorem finiteness was not required and it is essential that the orthogonal components are closed. In particular, for a countably infinite set of linearly independent vectors $\{e_i\}_{i=0}^\infty$, the term *closed span* should be used to generalize definitions of algebraic finite bases to infinite dimensions. The closed span of $\{e_i\}$, denoted by $\overline{[\{e_i\}]}$, is used to include limiting processes for sequences of vectors[63]. The closed span of any set of linearly

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[63]This can be seen as a way to generalize the algebraic closure with respect to the vector space operations by enlarging the subspace to the smallest closed set that contains the spanning set. The notion of closure is a topological concept induced by the metric that enables the inspection of convergence.

independent vectors is a closed subspace of H, but not even the orthonormality of the set $\{e_i\}_{i=0}^{\infty}$ guarantees that $\overline{[\{e_i\}]}$ is all of H, that is, the set $\{e_i\}_{i=0}^{\infty}$ is not necessarily *complete* in H. The completeness of a spanning set is closely related to the usefulness of generalized Fourier series.

Fourier series Let $\{e_i\}_{i=0}^{\infty}$ be any countably infinite orthonormal set in H. The series $\sum_{i=0}^{\infty} c_i e_i$ converges (in the norm) to an element $x \in H$ if and only if $\sum_{i=0}^{\infty} |c_i|^2 < \infty$ (that is, $(c_i)_{i=0}^{\infty} \in \ell^2(\mathbb{N})$). Then necessarily $c_i = (x, e_i)$ for all $i = 0, 1, \ldots$, and $\|x\| = \|\sum_{i=0}^{\infty} c_i e_i\| = (\sum_{i=0}^{\infty} |c_i|^2)^{1/2}$. The complex numbers $c_i = (x, e_i)$, $i = 0, 1, \ldots$, are the Fourier coefficients and $\sum_{i=0}^{\infty} (x, e_i) e_i$ is the Fourier series of x with respect to the orthonormal set $\{e_i\}_{i=0}^{\infty}$, respectively.

- The Fourier series is always convergent, since for all $x \in H$,

$$\sum_{i=0}^{\infty} |(x, e_i)|^2 \leq \|x\|^2 \qquad (Bessel's\ inequality). \qquad (2.62)$$

- The Fourier coefficients provide the *best fit* with respect to the norm,

$$\left\| x - \sum_{i=0}^{m} (x, e_i) e_i \right\| \leq \left\| x - \sum_{i=0}^{m} a_i e_i \right\|, \qquad (2.63)$$

  for all $x \in H$, any coefficients $a_i \in \mathbb{C}$, and arbitrary choice of approximation order $m > 0$.

The Fourier series for any $x \in H$ is convergent, according to (2.62), but nothing guarantees that it converges to x. The property that it does, that is, $x = \sum_{i=0}^{\infty} (x, e_i) e_i$ for all $x \in H$, is one possible definition for the completeness of $\{e_i\}$. However, a clearer connection to previous results is achieved by considering the closed span $\overline{[\{e_i\}]}$. Now if in the Projection Theorem, the space H has a trivial decomposition $\overline{[\{e_i\}]} = H$ and $\overline{[\{e_i\}]}^{\perp} = 0$, then the property (2.63) ensures that every $x \in \overline{[\{e_i\}]}$ has an unique series representation $x = \sum_{i=0}^{\infty} (x, e_i) e_i$.[64]

Complete Orthonormal Bases An orthonormal countably infinite set $\{e_i\}_{i=0}^{\infty}$ in H is complete, or a base, if and only if any of the following equivalent conditions apply,

(i) $x = \sum_{i=0}^{\infty} (x, e_i) e_i$ for all $x \in H$

(ii) $\overline{[\{e_i\}]}^{\perp} = 0$

(iii) $\overline{[\{e_i\}]} = H$

(iv) $\|x\|^2 = \sum_{i=0}^{\infty} |(x, e_i)|^2$ for all $x \in H$ (Parseval's equation[65])

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[64]By definition, if $x \in \overline{[\{e_i\}]}$, then for every $\epsilon > 0$, there are some $m > 0$, $a_j \in \mathbb{C}$ and $e_j \in \{e_i\}$, such that $\|x - \sum_{i=0}^{m} a_j e_j\| < \epsilon$. Due to (2.63), this is particularly true for the Fourier series, and the limiting process $m \to \infty$ gives the desired convergence result.

[65]Actually, if the Parseval's equation holds, then it has also the generalized form [Akhiezer and Glazman, 1981], $(x, y) = \sum_{i=0}^{\infty} (x, e_i)(e_i, y)$ for all $x, y \in H$

(v) $\lim\limits_{N\to\infty} \xi_N = 0, \quad \xi_N = \|\sum\limits_{i=N+1}^{\infty}(x,e_i)e_i\|^2 = \sum\limits_{i=N+1}^{\infty}|(x,e_i)|^2$, for all $x \in H$

The equivalence of conditions (i) to (iv) is fairly obvious from the above reasoning. Condition (v) is based on the observation that any partition of the Fourier series,

$$x = \sum_{i=0}^{n}(x,e_i)e_i + \sum_{i=n+1}^{\infty}(x,e_i)e_i, \qquad (2.64)$$

is orthogonal in the sense that $\|x\|^2 = \|\sum_{i=0}^{n}(x,e_i)e_i\|^2 + \xi_n$.[66] This will prove to be a very useful result, not only for the verification of completeness, but also for the evaluation of approximation errors.

Fortunately, there are some encouraging results about the existence of Hilbert space bases. Every (separable) Hilbert space has a countably infinite orthonormal base [Hutson and Pym, 1980]. The term *separable* is in parentheses because it is essentially redundant[67]. Moreover, almost all relevant Hilbert spaces are separable and this restriction lightens the survey [Milne, 1980]. Every orthonormal set in a Hilbert space H is contained in a maximal orthonormal set [Rudin, 1987].[68] Here, the term *maximal* is used to generalize completeness to a possibly uncountable set. In a separable Hilbert space, an orthonormal set cannot be uncountable. In particular, any base of a separable Hilbert space is countable[69]. Then in principle, any orthonormal finite set can be extended to a base, or conversely, any countably infinite maximal (or dense) set can be transformed into a base by eliminating redundant members. If the latter procedure is applied on every step of the following Gram-Schmidt orthogonalization process, then it provides a direct proof of the existence of a base in a separable Hilbert space[70].

Gram-Schmidt Orthogonalization process Let $\{y_i\}$ be a linearly independent set of vectors in H such that $\overline{[\{y_i\}]} = H$. To produce a complete orthonormal set $\{x_i\}$, begin with choosing the first basis vector as $x_1 = y_1/\|y_1\|$. Form the vector $z_2 = y_2 - (y_2,x_1)x_1$ and normalize it, $x_2 = z_2/\|z_2\|$. The vectors x_1 and x_2 are then by construction orthogonal,

$$\begin{aligned}(x_2,x_1) &= (((y_2 - (y_2,x_1)x_1)/\|z_2\|,x_1) \\ &= ((y_2,x_1) - (y_2,x_1)(x_1,x_1))/\|z_2\| = 0, \qquad (2.65)\end{aligned}$$

---

[66]The Projection Theorem provides an orthogonal partition $x = x_n + e_n$ and a *triangle equation* for the corresponding norms, $\|x\|^2 = \|x_n\|^2 + \|e_n\|^2$. The rest is a simple consequence of the orthonormality of the basis vectors.

[67]This means that the dimension of a Hilbert spaces, its cardinality, is at most countably infinite, which is in great contrast to, for example, the situation in more general Banach spaces.

[68]This is a consequence of an incredible versatile theorem in mathematics, the *Hausdorff maximality theorem*, or in a slightly different form, the *axiom of choice* or *Zorn's lemma* [Väisälä, 1983] [Rudin, 1987].

[69]Actually, also for an uncountable orthonormal set $\{e_i\}$ in a (non-separable) Hilbert space H, the set $\{e_i : (x,e_i) \neq 0\}$, $x \in H$, is at the most (infinitely) countable [Milne, 1980], a plausible proof of the countably cardinality of any Hilbert space.

[70]If $\{y_i\}$ is separable (but not linearly independent), reject y_n at iteration $n \geq 2$, if y_n and $\{x_1,\ldots,x_{n-1}\}$ are not linearly independent.
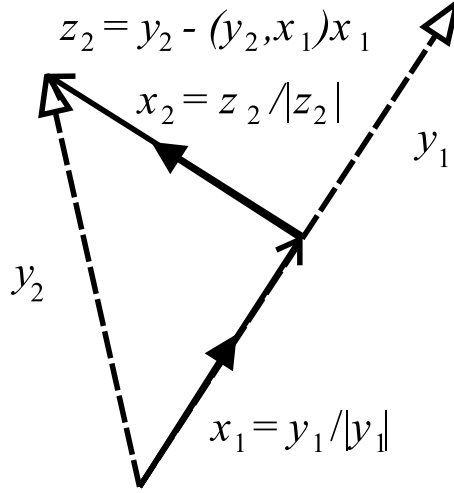
Figure 2.6: A geometrical interpretation of the Gram-Schmidt Orthogonalization process: the Euclidian space $\mathbb{R}^2$ is spanned by any non-parallel (and non-zero) vector pair $\{y_1, y_2\}$ and the GS process provides an orthonormal base $\{x_1, x_2\}$ by utilizing scaling and projections, as depicted.

and they span the same subspace as $\{y_1, y_2\}$. To prove that

$$z_n = y_n - \sum_{i=1}^{n-1}(y_n, x_i)x_i, \qquad x_n = z_n/\|z_n\|, \qquad (2.66)$$

is a valid recursion formula for $n > 2$, presume that the set $\{x_1, \ldots, x_{n-1}\}$ is orthonormal and evaluate the inner products

$$(z_n, x_k) = (y_n, x_k) - \sum_{i=1}^{n-1}(y_n, x_i)(x_i, x_k) = (y_n, x_k) - (y_n, x_k) = 0, \qquad (2.67)$$

for $k \leq n - 1$. Vectors z_n and x_n are by definition parallel and therefore x_n is orthogonal to $\{x_1, \ldots, x_{n-1}\}$. As before, $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ span the same subspace of H, and thus (2.66) will produce an orthonormal set into H that is in addition complete. A geometrical depiction of the first two steps of the Gram-Schmidt Orthogonalization process is given in Figure 2.6.

The following concluding remarks of this Section characterize linear operators on Hilbert spaces. For any fixed $g \in H$, the mapping $g^* : f \mapsto (f, g)$, $H \rightarrow \mathbb{C}$, is a continuous linear functional, denoted as $g^* \in \mathcal{L}(H, \mathbb{C})$.[71] More importantly, every element of the dual space $H^* = \mathcal{L}(H, \mathbb{C})$ has the form of an inner product, that is, there is an unique vector $g \in H$, such that

$$g^*(f) = (f, g), \quad \text{for all} \quad f \in H, \quad \text{and} \quad \|g^*\| = \|g\|. \qquad (2.68)$$

This is the *Riesz Representation Theorem* and it provides an one-to-one isometric mapping between H and H^*, that is, an *isomorphism*. A Hilbert space is thus self-dual, the dual is the space itself, and the isomorphism is actually a *homomorphism*,

---

[71]As it was stated earlier, the set of continuous (or bounded) linear functionals, $\mathcal{L}(B, \mathbb{C})$, on a Banach space B is itself a Banach space, $B^* = \mathcal{L}(B, \mathbb{C})$, the dual of B.

which simplifies the notion of linear operators on a Hilbert space. The set of bounded (or continuous) linear operators, $L : H^* \to H^*$, is denoted plainly by $\mathcal{L}(H)$ and the adjoint operator L^* of L is defined through the inner product relation $(Lf, g) = (f, L^*g)$, $f, g \in H$, and consequently $L^* \in \mathcal{L}(H)$.[72] An operator $L \in \mathcal{L}(H)$ is self-adjoint, if $L^* = L$.

Another useful result on operators is that in every complex Hilbert space H that originates from a measure μ, $H = L^2(d\mu)$, the inner product is of the form

$$(f, g) = \int fg^* d\mu, \quad \text{for all} \quad f, g \in H. \tag{2.69}$$

In the following Section, formula (2.69) is used to produce explicit representations for inner products in various Hilbert spaces by different interpretations of the measure and integration.

2.3.3 Hilbert spaces for signals and systems

According to (2.69) and the corresponding counting measure, the inner product of $\ell^2(\mathbb{N})$, the space of causal and finite-energy signals, is defined by

$$(x, y) = \sum_{n=0}^{\infty} x(n)y^*(n), \qquad x, y \in \ell^2(\mathbb{N}). \tag{2.70}$$

The series on the right side of (2.70) converges, because the product sequence $x(n)y^*(n)$ is in $\ell^1(\mathbb{N})$, which is seen by the Schwartz inequality (2.58). The space $\ell^2(\mathbb{Z})$ is also a Hilbert space, with an obvious extension of (2.70) to the anti-causal part, but the space $\ell^2(\mathbb{N})$ is in a special position since all (separable) Hilbert spaces are isomorphic to $\ell^2(\mathbb{N})$ through some Fourier series representation.

The inner product for the stochastic counterpart of $\ell^2(\mathbb{N})$ is defined by

$$(x, y) = E[x(k)y^*(k)], \quad x, y \in \ell^2(\mathbb{N}), \tag{2.71}$$

where $E[\cdot]$ is the statistical expectation of the stochastic sequence $x(n)y^*(n) \in \ell^1(\mathbb{N})$. The validity of (2.71) is guaranteed by the linearity of $E[\cdot]$, combined to the properties of (zero-mean) wide-sense stationary signals. In particular, comparing to (2.45) and (2.46), the inner product is given by the cross-correlation, $(x, y) = r_{xy}(0)$, and the square-norm of $x \in \ell^2(\mathbb{N})$, the inner product (x, x), is equal to the mean-square value, $\|x\|^2 = E[x^2(n)] = r_{xx}(0)$. Equipped with a proper definition for the expectation, $\ell^2(S)$ is a Hilbert space for any choice of time-axis $S \subset \mathbb{Z}$.

Bearing in mind how functions in $H^2(\mathbb{E})$ were identified in Section 2.2.2, the inner product (F, G), $F, G \in H^2(\mathbb{E})$, is given by

$$(F, G) = \frac{1}{2\pi j} \oint_{\mathbb{T}} F(z)G^*(1/z^*) \frac{dz}{z} = \frac{1}{2\pi} \int_0^{2\pi} F(e^{j\omega})G^*(e^{j\omega})d\omega. \tag{2.72}$$

---

[72]In a Banach space B, the *adjoint* L^* of $L \in \mathcal{L}(B, \mathbb{C})$ is formally defined by an "outer product" relation $< Lf, g > = < f, L^*g >$, $f \in B$ and $g \in B^*$, where $< \cdot, \cdot >$ share at least some of the properties of $(\cdot, \cdot)$.

The same formula applies to the Hilbert space $L^2(\mathbb{T})$ and the validity of (2.72) is ensured in both cases by the fact that the product FG^* is in $L^1(\mathbb{T})$. That (2.72) really is a proper inner product inducing the Hilbert space $H^2(\mathbb{E})$, can be proven directly or by utilizing the isomorphism between $\ell^2(\mathbb{N})$ and $H^2(\mathbb{E})$ given by the z-transform (2.4).[73] For the subclass of rational functions in $H^2(\mathbb{E})$, the integral in (2.72) can always (at least in principle) be evaluated by a formula provided by the *Cauchy's Residue Theorem*.

The stochastic interpretation of (2.72) is easily attained using the isomorphism $(F, G) = (f, g)$, for f and g in $\ell^2(\mathbb{N})$ of WSS signals, and by applying the IDTFT (2.8) on the relation $(f, g) = r_{fg}(0)$,

$$(F, G) = \frac{1}{2\pi j} \oint_{\mathbb{T}} S_{fg}(z)\frac{dz}{z} = \frac{1}{2\pi} \int_0^{2\pi} \tilde{S}_{fg}(\omega)d\omega, \qquad (2.76)$$

where $\tilde{S}_{fg}(\omega) = S_{fg}(e^{j\omega})$ is the cross-spectral density of f and g.

The notion of modeling, introduced in Section 2.1.5, requires some additional definitions for the involved inner products. The modeling signals $x_i(n)$ and $x_j(n)$ are supposed to be the product of causal LTI filtering of some generating input signals $x(n)$ and $y(n)$ with partial model impulse responses $g_i(n)$ and $g_j(n)$, respectively. What is still needed is the formulas for the quantities (x_i, x_j) and (X_i, X_j) in terms of any mixture of time- or frequency-domain representation of the partial models and (two) distinct model excitations. The partial models are always presumed to be deterministic, but the input signals, and consequently the model outputs, are either stochastic or deterministic. The point is that all the ingredients or the inner products are either in $\ell^2(\mathbb{N})$ or in $H^2(\mathbb{E})$. For example the inner product of

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[73]If $F(z) = \sum f(n)z^{-n}$ and $G(z) = \sum g(n)z^{-n}$, then $G^*(1/z^*) = (\sum g(n)(1/z^*)^{-n})^* = \sum g^*(n)z^n$, and a formal calculation provides

$$\begin{aligned} (f, g) &= \sum f(n)g^*(n) = \sum \left(\frac{1}{2\pi j} \oint F(z)z^n \frac{dz}{z} \right) g^*(n) & (2.73) \\ &= \frac{1}{2\pi j} \oint F(z) \left(\sum g^*(n)z^n \right) \frac{dz}{z} = \frac{1}{2\pi j} \oint F(z)G^*(1/z^*)\frac{dz}{z}, & (2.74) \end{aligned}$$

and since the z-transform is an isomorphism, $(F, G) = (f, g)$. Equation (2.72) can also be seen as a consequence of the z-transform property

$$h(n) = f(n)g^*(n) \longleftrightarrow H(z) = \frac{1}{2\pi j} \oint F(w)G^*(z^*/w^*)\frac{dw}{w}, \qquad (2.75)$$

which is attained in a similar way as (2.73), and it is the (conjugate-antisymmetric) counterpart of the fact that convolution in the time-domain, $f(n) * g(n)$, transforms into a product, $F(z)G(z)$, in the z-domain. The inner product (2.70) is the z-transform of $x(n)y^*(n)$ evaluated at $z = 1$, and equally, $(F, G) = H(1)$ by comparing (2.73) and (2.72). The reasoning will become even more confusing, if it is continued utilizing the (deterministic correlation) relation $r_{fg}(m) = f(m) * g^*(-m)$, $(f, g) = r_{fg}(0)$.

$X_i(z) = G_i(z)X(z)$ and $X_j(z) = G_j(z)Y(z)$ is easily deduced from (2.72) as[74]

$$(X_i, X_j) = (G_i X, G_j Y) \;=\; \frac{1}{2\pi j} \oint_{\mathbb{T}} G_i(z)X(z)G_j^*(1/z^*)Y^*(1/z^*)\frac{dz}{z} \quad (2.77)$$

$$= \;\frac{1}{2\pi} \int_0^{2\pi} G_i(e^{j\omega})X(e^{j\omega})G_j^*(e^{j\omega})Y^*(e^{j\omega})d\omega. \quad (2.78)$$

Rearranging and renaming of the arguments on the right hand side gives

$$(X_i, X_j) \;=\; \frac{1}{2\pi j} \oint_{\mathbb{T}} G_i(z)G_j^*(1/z^*)S_{xy}(z)\frac{dz}{z} \quad\quad\quad (2.79)$$

$$= \;\frac{1}{2\pi} \int_0^{2\pi} G_i(e^{j\omega})G_j^*(e^{j\omega})S_{xy}(e^{j\omega})d\omega, \quad\quad (2.80)$$

where the function $S_{xy}(z)$ is alternatively an abbreviation for $X(z)Y^*(1/z^*)$ or the z-transform of a deterministic or stochastic cross-correlation sequence $r_{xy}(m)$, the cross-spectral density[75]. The right hand side of equation (2.79) is also the definition for the time-domain inner product in terms of $x, y, g_i, g_j \in \ell^2(\mathbb{N})$, $(x_i, x_j) = (g_i(n) * x(n), g_j(n) * y(n))$, because in general there seems not to be any meaningful way to expand the relations (2.70) or (2.71) directly in the time-domain[76]. However, utilizing the inversion symmetry of the cross-spectral density, $S_{xy}^*(1/z^*) = S_{yx}(z)$, it is easily deduced from (2.79) that

$$(X_i, X_j) = (G_i S_{xy}, G_j) \qquad \text{or} \qquad (X_i, X_j) = (G_i, G_j S_{yx}), \quad\quad (2.81)$$

which correspond to $(x_i, x_j) = (g_i * r_{xy}, g_j)$ or $(x_i, x_j) = (g_i, g_j * r_{yx})$, respectively, in the time-domain[77]. That is, the "filtered inner product" equals filtering with the cross-correlation sequence of one or the other component in (g_i, g_j).

In the upcoming, there will occasionally be a need to express inner products of vector valued signals and functions as a matrix valued inner product. Basically it is just a matter of partitioning the inner product into element-wise operations, but it still has to be specified how to combine the elements and where to map the product. In some cases the denotation $(\mathbf{x}, \mathbf{y})$ could be used as an abbreviation for $[(x_1, y_1) \;\cdots\; (x_N, y_N)]^T$, that is, to implement a component-by-component inner product, where $\mathbf{x}$ and $\mathbf{y}$ are N-dimensional vectors. Here however, all inner products of the elements of the participating vectors are needed; the inner product of two vectors $\mathbf{x}$ and $\mathbf{y}$, with respective dimensions N and M, is defined by $(\mathbf{x}, \mathbf{y}) = [(x_i, y_j)]_{ij}$, $i = 1, \ldots, N$ and $j = 1, \ldots, M$, and it is a $N \times M$ matrix containing all inner products between the elements of the vectors[78]. This definition is consistent with basic

[74]In this reasoning it is temporarily presumed that also stochastic signals have z-transforms, which is certainly not true, but according to the isomorphisms between all the spaces it is allowed to imagine generic z-transform representations that are however not known, tractable or calculable.

[75]Equation (2.79) is also produced by expanding the convolutions, some rearranging, and by applying the z-transform property (2.75) of product sequences.

[76]Something like $(x_i, x_j) = (g_i, g_j)r_{xy}(0)$ works in special cases, but not in general.

[77]Equations (2.72) and (2.76) can be seen as special cases of (2.79), produced by the choices $S_{xy}(z) = 1$ and $G_i(z) = G_j(z) = 1$, respectively.

[78]This definition can be seen as an "outer product" – for vectors $\mathbf{x}, \mathbf{y} \in \mathbb{C}^N$, the inner product was established as $(\mathbf{x}, \mathbf{y}) = \mathbf{y}^H \mathbf{x}$ and $\mathbf{x}\mathbf{y}^H$ is the corresponding "outer product", and notations like $(\mathbf{x}, \mathbf{y}^T)$ are sometimes used to indicate that the inner product spans a matrix.

vector-matrix calculus rules, for example, in incorporating scalar and (appropriate) matrix multiplications into the expressions. In the case that one of the parties is a scalar, the expressions $(\mathbf{x}, y)$ and $(x, \mathbf{y})$ produce a column and a row vector, respectively, and the diagonal $diag(\mathbf{x}, \mathbf{y})$ is precisely the component-wise inner product of $\mathbf{x}$ and $\mathbf{y}$. Most of the properties of the scalar inner product have natural generalizations, for example the conjugate symmetry, $(\mathbf{y}, \mathbf{x}) = (\mathbf{x}, \mathbf{y})^H$. Given a proper definition for a linear matrix operator $\mathbf{L}$, the above inner products may also be generalized in the form $(\mathbf{x}, \mathbf{y}) = \mathbf{L}[\mathbf{x}\mathbf{y}^H]$, where $\mathbf{L}$ is the addition, expectation or integration operation, respectively, on a matrix valued function $\mathbf{x}(\cdot)\mathbf{y}(\cdot)^H$.

2.4 Useful interconnections of some mathematical topics

There is a certain reason why the theory of complex valued functions of complex variables, the *Function Theory*, has monopolized a name that one would presume to cover a much wider class of function descriptions[79]. Namely, although almost all concepts of real functions can be transformed to the complex domain, this generalization step produces a theory that is much more than a two-dimensional counterpart of elementary real analysis. For example, the complex derivative is defined as an unidirectional total derivative and not as a pair of partial differentials. Thus, functions that have derivatives in every point of some region Ω of the complex plane $\mathbb{C}$, the class of analytic functions denoted by $H(\Omega)$, possess strong "regularity" properties.

Some of the properties of analytic functions have already been mentioned, such as the limits of functions on the boundaries, and the converse, the analytic extensions given a boundary function. These can be seen as generalizations of some particular local representation and uniqueness results. In the following sections an additional regularity property, the *conformability* of analytic functions, is related to other topics such as transformations and operators on function spaces, factorizations of functions and filters, and properties of matrices.

2.4.1 Möbius transformations and conformal mappings

Analytic functions preserve angles in points where the derivative is non-zero. Instead of specifying this conformability property in detail and for more general functions, it is sufficient to study functions $f \in H(\Omega)$ with non-vanishing derivatives in all of the region Ω. Such a function is conformal in the sense that the shape of "small objects" is preserved in the mapping.

The importance of conformal mappings is due to the fact that they can be used

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[79]A more cosmopolite term would be *Complex analysis*: one has to admit that there are strong personal and cultural reasons for taking up this subject, the former referring to authors' lightweight studies in the subject, and the latter to the fact that Function Theory is considerably contributed by Finnish mathematicians. Nevertheless, it is surprising that basic studies in DSP do not usually include an overview of Function Theory or Complex analysis.

to transfer problems stated in one region to another region where the solution is easier to obtain or already deduced. Two regions U and V of the complex field are *conformally equivalent*, if there is a conformal one-to-one mapping $f : U \to V$ that maps U onto V, or in other words, that f is an analytic bijection[80]. For such a function, the inverse $f^{-1}(z)$ always exists and it is a conformal mapping of V onto U. Now if $g \in H(V)$, then the mapping $g \mapsto g \circ f$ is an one-to-one mapping of $H(V)$ onto $H(U)$ which preserves sums and products[81], and problems concerning $H(V)$ can be solved in $H(U)$ and carried back to $H(V)$ using the mapping f. One such equivalence has already been utilized in presenting results for $H(\mathbb{E})$ that were actually deduced in $H(\mathbb{D})$ and then transferred using the inversion $z \mapsto 1/z$, which is a conformal one-to-one mapping from $\mathbb{D}$ onto $\mathbb{E}$.

The *Riemann Mapping Theorem* states that every simply connected region of the complex plane, except the complex plane itself, is conformally equivalent to the unit disk $\mathbb{D}$, and consequently all such regions are conformally equivalent (via the unit disk $\mathbb{D}$)[82]. However, for example $\mathbb{E}$ is not simply connected, but the inversion still provides a conformal equivalence to all simply connected regions other than the complex plane itself.

The above existence result is encouraging, but more importantly, most of the useful conformal mappings can be produced by combinations of simple mappings. The *Möbius transformation* (MT) categorizes four types of such elementary mappings, and it is given by

$$z \mapsto \frac{az + b}{cz + d}, \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0, \qquad (2.82)$$

where a, b, c and d are complex numbers, and the constraint on them ensures that the MT does not reduce to a constant, which would certainly not be an one-to-one mapping. The names *bilinear transformation* or *linear fractional transformation* are also used in relation to (2.82), but they may have corrupted meanings, especially in signal processing. One remarkable property of the MT is that any disk or half-plane of the complex field can be mapped conformally to any disk or half-plane by specifying the form of the function (2.82). Conversely, every conformal mapping between disks and half-planes is a Möbius transformation. Moreover, boundary points are mapped to boundary points and interior points to interior points, and the mapping is attained explicitly and uniquely by fixing it for some few points, usually with respect to three distinct points and their desired images. This can also be seen as a limitation of degrees of freedom, since for example fixing the mapping for three boundary points determines the mapping of the interiors uniquely.

The inverse of a MT is once again a MT and it is given by the substitutions $a' = d$, $b' = -b$, $c' = -c$ and $a' = a$ with respect to the original MT defined in the form

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[80]This sentence contains a tautology, since for one-to-one mappings $f \in H(U)$, $f'(z) \neq 0$ in U, and they are thus conformal.

[81]That is, the mapping is a ring isomorphism of $H(V)$ onto $H(U)$.

[82]A region is by definition (open and) connected, meaning that any two points can be joined with a continuous path within the region. A region is simply connected, if any closed path can be continuously deformed into a single point. Simply speaking, a simply connected region has no holes.

(2.82).[83] Additionally, properly defined compositions of Möbius transformations reduce to the form of a MT, which is apparent also from the above reasoning concerning conformal mappings of disks and half-planes[84].

In addition to the MT, only two other basic types of conformal mappings are required to produce a very general collection of conformally equivalent regions of the complex plane. The power function $z \mapsto z^n$, where $n = 2, 3, \ldots$, is an one-to-one conformal mapping of the sector $\{0 < arg(z) < 2\pi/n\}$ onto the sector $\{0 < arg(z) < 2\pi\}$, where $arg(z) = \theta$ is the angle of the complex number $z = re^{i\theta}$. The corresponding inverse mapping is obviously $z \mapsto z^{1/n}$. Thus every sector in the complex plane is conformally equivalent to any disk or half-plane[85]. The other elementary conformal mapping is the complex exponent function $z \mapsto e^z$, which carries parallel strips $\{\theta_1 < \Im(z) < \theta_2\}$ to sectors $\{\theta_1 < arg(z) < \theta_2\}$. For example the parallel strip $\{0 < \Im(z) < \pi\}$ is conformally equivalent to the upper half-plane $\{\Im(z) > 0\}$ and the corresponding inverse mapping is the complex logarithm function $z \mapsto \ln(z) = \ln|z| + i \arg(z) + in2\pi$, $n \in \mathbb{Z}$, where the branch of the logarithm must be specified (by choosing $n = 0$). For example, sigmoid activation functions in neural networks, such as the logistic function and the hyperbolic tangent, can be seen as compositions of Möbius transformations and the complex exponent function [Mandic, 2000].[86]

2.4.2 Isomorphisms between function spaces

In this section, the conformal equivalency of disks and half-planes is utilized in two special cases. To begin with, z-transform representations of signals and systems are

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[83]If the MT is regarded as a matrix operator, defined by a mutual agreement for something like $M(z) = \mathbf{M}z$ or $M(z) = \mathbf{M}[z\ 1]^T$, where $\mathbf{M}$ is the coefficient matrix $\mathbf{M} = [a\ b; c\ d]$, then the operator product

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} \qquad (2.83)$$

is the unit matrix scaled by the determinant $det(\mathbf{M}) = ad - bc \neq 0$.

[84]Using the *stereographic projection*, every disk and half-plane is transformed to a disk (calotte) on the *Riemann Sphere* (surface of a ball), where MTs form a closed family of all such mappings. Moreover, the subclass of MTs, where $a, b, c, d \in \mathbb{Z}$ and bound by $ad - bc = 1$, forms a *Modular Group* of upper half-plane ($\Im(z) > 0$) mappings, with composition as the group operation [Rudin, 1987]. The stereographic projection maps $z \in \mathbb{C}$ (considered as $\mathbb{R}^2 \subset \mathbb{R}^3$) to the surface of the ball $B(0, 0, 1/2, 1/2)$ (center point $(0, 0, 1/2) \in \mathbb{R}^3$ and radius $1/2$) as the intersection point of the line connecting $(z, 0)$ and $(0, 0, 1)$ (with the surface). This interpretation also "explains" why the special point ∞ in the compactly extended complex plain do not have a direction (angle).

[85]All sectors are included by a combined MT and a properly defined mapping $z \mapsto (z^{1/n})^m = z^{m/n}$ (formally, relying on the fact that rational numbers are dense in $\mathbb{R}$).

[86]The logistic function can be decomposed as

$$f(x) = \frac{1}{1 + e^{-\beta x}} = \mathbf{M}z, \quad \mathbf{M} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad z = e^{-\beta x} \qquad (2.84)$$

and the hyperbolic tangent is of the form

$$f(x) = \frac{1 - e^{-\beta x}}{1 + e^{-\beta x}} = \mathbf{M}z, \quad \mathbf{M} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad z = e^{-\beta x}. \qquad (2.85)$$

related to *Laplace transforms* of continuous-time signals and systems. The latter part categorizes conformal mappings of the unit disk onto itself, which are used to form *homomorphisms* and *isomorphisms* on $H^2(\mathbb{E})$ and its subspaces, respectively. In the next section these mapping are related to the generation and interconnections of basis representations of various spaces.

Möbius transformations that map the unit disk $\mathbb{D}$ conformally to a true half-plane, a half-plane that is defined as either side of a straight line through the origin, are of the form[87]

$$s_a(z) = a\frac{z-1}{z+1}. \tag{2.86}$$

The complex number $a \neq 0$ can be seen as a scaling and rotation factor. For a real and positive scaling factor a, $a > 0$, the unit disk $\mathbb{D}$ is mapped to the left half plane $\{\Re(z) < 0\}$, denoted here as $\mathbb{C}^-$. Correspondingly, the region $\mathbb{E}$ is mapped to the right half-plane $\mathbb{C}^+$. The unit circle is mapped to the imaginary axis, and using a counter clockwise parametrization $e^{j\theta} : [-\pi, \pi] \to \mathbb{T}$, transforms the limits of the boundaries as $s_a(-1) = lim_{\theta \to \pm\pi} s_a(e^{j\theta}) = \pm\infty$. Figure 2.7 characterize how the unit disk is mapped to the left half-plane for $a = 0.4$. The inverse of the mapping (2.86) is easily obtained as

$$z_a(s) = \frac{a+s}{a-s}. \tag{2.87}$$

The reason why mappings (2.86) and (2.87) are introduced here is that they define an isomorphism between discrete- and continuous-time representations of signals and systems[88]. The continuous-time Fourier transform or *Plancherel transform*

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \quad \longleftrightarrow \quad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{i\omega t}d\omega, \tag{2.88}$$

defines an isomorphism in the Hilbert space $L^2(\mathbb{R})$, the space of square-integrable finite-energy functions on the real line $\mathbb{R}$. A purpose-oriented way to distinguish between the original and transform domains is to denote the latter as $L^2(i\mathbb{R})$ and to use rotated functions $F(i\omega) = \hat{f}(\omega)$. Now, similar arguments and deductions as in Section 2.2.2 can be used to relate the Hilbert space pair $L^2(\mathbb{R}) \leftrightarrow L^2(i\mathbb{R})$ to the Hilbert space pair $L^2(\mathbb{R}^+) \leftrightarrow H^2(\mathbb{C}^+)$, but it would be an unnecessary repetition for at least three reasons, which will be addressed below.

The analogy to the z-transform isomorphism $\ell^2(\mathbb{N}) \leftrightarrow H^2(\mathbb{E})$ is the *Laplace transform*

$$F(s) = \int_0^{\infty} f(t)e^{-ist}dt \quad \longleftrightarrow \quad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{ist}ds \tag{2.89}$$

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[87]Actually, from the infinite variety of such mappings, this particular form is attained by setting $s_a(1) = 0$, $s_a(0) = -a$ and $s_a(-1) = \infty$. An interesting consequence of the conformity is that since the segment of a line $[0, 1]$ is perpendicular to the boundary at $z = 1$, so is also $[-a, 0]$ to the straight line defining the boundary of the half-plane at $s = 0$, and thus the *normal* $[-a, 0]$ defines the half-plane in question uniquely.

[88]As stated previously, in order to specify a mapping between $\mathbb{E}$ and $\mathbb{C}^+$, the scaling factor should be real and positive: $a = 1$ is a simple choice, but in fact $a = 1/2$ would be a better one in the sense that then the square matrix operators defined by (2.86) and (2.87) are then readily normalized to produce an unit matrix product.
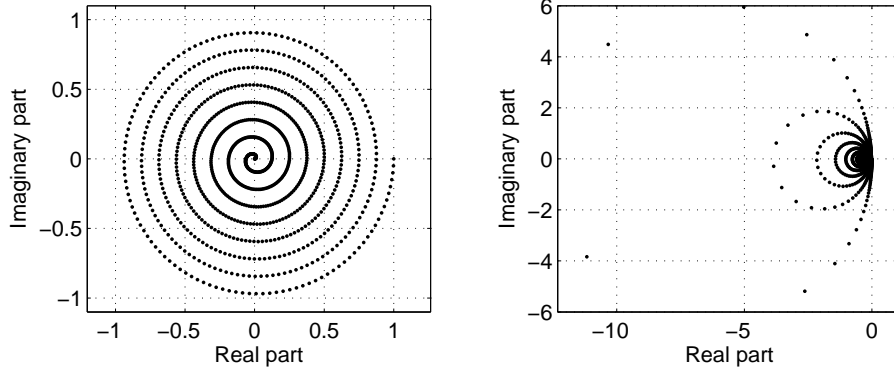
Figure 2.7: En example of the mapping (2.86) for $a = 0.4$. The spiral in the unit disk (on the left hand side) is mapped to the left half-plane in a very non-uniform way (as displayed on the right hand side). The mapping is anchored by $s(1) = 0$ and $s(0) = -0.4$, as expected.

and it defines an one-to-one correspondence between the function classes $L^2(\mathbb{R}^+)$ and $H^2(\mathbb{C}^+)$, where the latter is the space of analytic functions on $\mathbb{C}^+$ that are square-integrable on the imaginary axis[89].

This thesis is confined to discrete-time signal processing and the sole reason why subjects as $L^2(\mathbb{R}^+)$ and $H^2(\mathbb{C}^+)$ are brought in is that the isomorphism between $H^2(\mathbb{E})$ and $H^2(\mathbb{C}^+)$, defined by the conformal mapping (2.86), will be used to generate and interconnect basis functions in $H^2(\mathbb{E})$. Besides the conformal equivalency of the regions, the only additional information that is needed is that $L^2(\mathbb{R}^+)$ and $H^2(\mathbb{C}^+)$ really are Hilbert spaces. The required inner products appear naturally using the generating formula (2.69) and a suitable normalization for the measure. However, the completeness of the spaces and different aspect of the uniqueness and one-to-one correspondence of the elements of these spaces are not discussed further and a reference is made to more detailed deductions [Hoffman, 1962, Rudin, 1987, Oliveira e Silva, 1995]. If $H^2(\mathbb{C}^+)$ is presumed to be a Hilbert space, then the only thing that is still needed to establish the isomorphism between $H^2(\mathbb{E})$ and $H^2(\mathbb{C}^+)$ is the knowledge of how the measure of integration is distorted in the transformation (2.86). Forming the inner products and forcing them to obey an isometric relation (Parseval's equation) shows that the "normalization" of the mapping is given by the complex differentials of (2.86) and (2.87), respectively, which are easily obtained as

$$ds_a = \frac{2a}{(z+1)^2}dz_a \qquad \longleftrightarrow \qquad dz_a = \frac{2a}{(a-s)^2}ds_a. \tag{2.90}$$

The existential reasoning in Section 2.3.2 ensures that all separable Hilbert spaces are isomorphically related, through $\ell^2(\mathbb{N})$ and some Fourier series expansion, but the combination of conformal mappings (2.86) and (2.87), the z-transform and the Laplace transform provide explicit isomorphisms between $\ell^2(\mathbb{N})$, $H^2(\mathbb{E})$, $H^2(\mathbb{C}^+)$, and $H^2(\mathbb{R}^+)$.

Conformal mappings can also be used to produce isomorphisms within the Hilbert

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[89]The fact that there is an one-to-one correspondence between the function classes is known as the *Paley-Wiener Theorem*.

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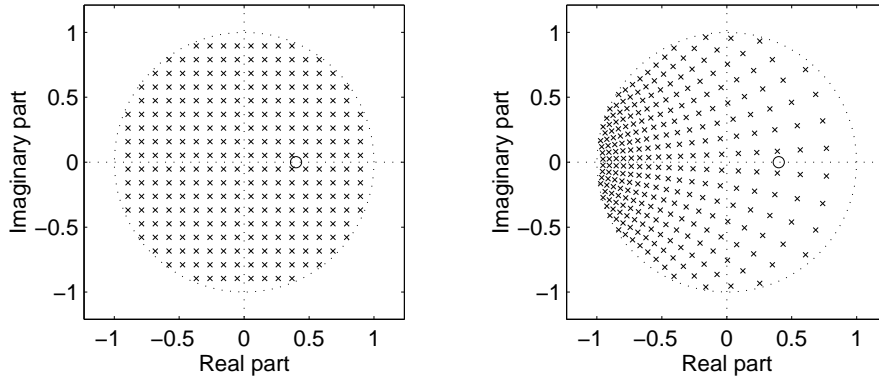
Figure 2.8: A characterization of the mapping (2.91) for $c = 1$ and $a = 0.4$. The uniformly spaced set of points on the left hand side have images that are warped towards the new center of gravity $w(0) = -0.4$, as depicted in the right hand side pane.

space $H^2(\mathbb{E})$. It is not difficult to show that all Möbius transformations that map the unit disk $\mathbb{D}$ onto itself are of the form

$$w_a(z) = c\frac{z - a}{1 - a^*z}, \qquad |c| = 1, \quad a \in \mathbb{D}, \quad z \in \mathbb{D}. \qquad (2.91)$$

The complex number c, $|c| = 1$, is just an unitary rotation and the parameter a, $|a| < 1$, is the fixed single zero of the mapping, $w_a(a) = 0$. An example of the mapping (2.91) is given in Figure 2.8: a uniform distribution of point in the unit disk is mapped to a set of points that is shifted towards $w = -1$, corresponding to the new "center of gravity" $w(0) = -ca = -0.4$, for the choices $c = 1$ and $a = 0.4$. Conversely, all one-to-one $f \in H(\mathbb{D})$, such that $f(\mathbb{D}) = \mathbb{D}$ and $f(a) = 0$, are given by (2.91) [Rudin, 1987]. The function w_a is bounded and continuous in $\bar{\mathbb{D}} = \mathbb{T} \cup \mathbb{D}$, $w_a \in \mathcal{C}(\bar{\mathbb{D}})$, since the single zero of the denominator, the *pole*, is exterior to $\bar{\mathbb{D}}$. As a MT, the unit circle $\mathbb{T}$ and its interior $\mathbb{D}$ are mapped separately to themselves; it may also be more straightforwardly confirmed that $|w_a(z)| = 1$ for all $z \in \mathbb{T}$. This latter property is referred to as *inner*, *allpass* or *lossless*, depending on the context. A similar tautology asserts that $|w_a(z)| < 1$ for all $z \in \mathbb{D}$.

The transformation (2.91) categorizes homomorphisms of $H^2(\mathbb{D})$ and the notation $H^2(\tilde{\mathbb{D}})$ is used to distinguish the transformed space. If the scalar rotation is omitted by choosing $c = 1$, then the inverse mapping in given by $w_a^{-1} = w_{-a}$.[90] Furthermore, if the parameter $a \in \mathbb{D}$ is fixed, then the transformation (2.91) and its inverse defines a change of variable, $w \leftrightarrow z$, into the function descriptions $f(z) \in H^2(\mathbb{D})$ and $\tilde{f}(w) \in H^2(\tilde{\mathbb{D}})$, respectively[91]. The inversion $z \mapsto 1/z$ transforms these results to

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[90]This is seen by a direct calculation or by using the general inversion rule for MTs, given earlier as $\{a, b, c, d\} \leftrightarrow \{a, -b, -c, d\}$.

[91]As in the case of $H^2(\mathbb{E})$ and $H^2(\mathbb{C}^+)$, the actual mapping between elements of $H^2(\mathbb{D})$ and $H^2(\tilde{\mathbb{D}})$ is subject to the distortion of measure on $\mathbb{T}$. The explicit relation is given by

$$\tilde{f}(w) = \frac{\sqrt{1 - a^*a}}{1 - a^*z}f\left(\frac{z - a}{1 - a^*z}\right) \qquad \longleftrightarrow \qquad f(z) = \frac{\sqrt{1 - a^*a}}{1 + a^*w}\tilde{f}\left(\frac{w + a}{1 + a^*w}\right), \qquad (2.92)$$

which is a consequence of the derivative formula $D(w_{\pm a}(z)) = (1 - a^*a)(1 \mp a^*z)^{-2}$.

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$H^2(\mathbb{E})$ and $H^2(\tilde{\mathbb{E}})$. The corresponding change of variable is then given by

$$w^{-1} = \frac{1 - a^*z}{z - a} = \frac{z^{-1} - a^*}{1 - az^{-1}} \quad \leftrightarrow \quad z^{-1} = \frac{1 + a^*w}{w + a} = \frac{w^{-1} + a^*}{1 + aw^{-1}} \qquad (2.93)$$

Purposefully, the inversion is applied as an outer function on (2.91) and its inverse, which genuinely exchanges the roles of $\mathbb{D}$ and $\mathbb{E}$, the complement regions with respect to the unit circle. The explicit mapping between elements $f(z) \in H^2(\mathbb{E})$ and $\tilde{f}(w) \in H^2(\tilde{\mathbb{E}})$ is attained as

$$\tilde{f}(w) = \frac{\sqrt{1 - a^*a}}{1 - az^{-1}} f\left(\frac{z^{-1} - a^*}{1 - az^{-1}}\right) \quad \leftrightarrow \quad f(z) = \frac{\sqrt{1 - a^*a}}{1 + aw^{-1}} \tilde{f}\left(\frac{w^{-1} + a^*}{1 + aw^{-1}}\right), \qquad (2.94)$$

which is seen by forming the inner product (2.72) in either of the spaces $H^2(\mathbb{E})$ or $H^2(\tilde{\mathbb{E}})$, and by compensating for the change of measure in the integration, given by

$$dw = \frac{1 - a^*a}{(1 - a^*z)^2} dz \quad \leftrightarrow \quad dz = \frac{1 - a^*a}{(1 + a^*w)^2} dw \qquad (2.95)$$

The complex power function $z \mapsto z^N$, $N \in \mathbb{N}$, is a conformal mapping of the unit disk $\mathbb{D}$ onto $\mathbb{D}$, which is one-to-many, in fact, every point of the closed unit disk $\bar{\mathbb{D}}$ is mapped precisely n times to itself. It is a product of n Möbius transformations of the form (2.91) having zeros at $z = 0$. Analogously, any finite product of functions (2.91) is a conformal mapping of $\mathbb{D}$ onto $\mathbb{D}$, and conversely, every conformal mapping $A_N : \mathbb{D} \to \mathbb{D}$, $A(\mathbb{D}) = \mathbb{D}$, that has a predefined set of zeros $\{a_1, \ldots, a_N\}$ is given by

$$A_N(z) = c \prod_{i=1}^{N} \frac{z - a_i}{1 - a_i^* z}, \qquad |c| = 1, \quad \{a_1, \ldots, a_N\} \subset \mathbb{D}, \quad z \in \mathbb{D} \qquad (2.96)$$

and this representation is unique up to a possible rotation by its zeros[92]. The function (2.96) is inner, allpass and lossless, that is, $|A_N(z)| = 1$ for $z \in \mathbb{T}$ and $|A_N(z)| < 1$ for $z \in \mathbb{D}$, and it is stable since the poles $\{1/a_1^*, \ldots, 1/a_N^*\}$ are outside the unit circle. These properties suggest once more a change of variable $w^{-1} \leftrightarrow 1/A_N(z)$ in the Hilbert space $H^2(\mathbb{E})$. The mapping is not one-to-one in $\mathbb{E}$, but if $z^{-1} \mapsto 1/A_N(z)$ is regarded as a mapping from $\mathbb{E}$ to $\mathbb{E}^N$, where $\mathbb{E}^N$ is a stacked manifold of $\mathbb{E}$, then the mapping is one-to-one and it defines an isomorphism between $H^2(\mathbb{E})$ and some subspaces of $H^2(\mathbb{E})$. It is probably easier to imagine the actual mapping on the boundary $\mathbb{T}$, $\mathbb{T} \to \mathbb{T}^N$, $e^{j\omega} \mapsto e^{jv(\omega)}$, where the phase function $\omega \mapsto v(\omega)$ is a continuous monotonic function, and thus invertible[93]. Moreover, if

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[92]The points $\{a_1, \ldots, a_N\}$ do not have to be distinct and some of them may even be set to zero. The form of (2.96) that expresses these special cases is given by

$$A_N(z) = z^p \prod_{j=1}^{q} \left(\frac{z - a_j}{1 - a_j^* z}\right)^{p_j}, \qquad (2.97)$$

where p and p_j are the multiplicities of $z_i = 0$ and $z_i = z_j$, respectively. The orders of the terms in (2.97) are bound by $N = p + q + \sum(p_j - 1)$.

[93]Each term in the product (2.96) is of the form $e^{jv_i(\omega)}$ on $\mathbb{T}$, where $v_i : [0, 2\pi] \to [\theta, \theta + 2\pi]$ is a continuous monotonic function for some initial phase $\theta = v_i(0)$. The product of N such terms is then simply $e^{jv(\omega)} = \prod e^{jv_i(\omega)} = e^{j \sum v_i(\omega)}$, and the sum of monotonic functions is monotonic.

the inverse mapping is restricted to one of the layers $\mathbb{E}_i$, $i = 1, \ldots, N$, that is, to one branch of the inverse function, then there is a conformal equivalence between $\mathbb{E}_i$ and a subregion of $\mathbb{E}$ and the union of N such regions is all of $\bar{\mathbb{E}}$.[94] This construction defines a decomposition of $H^2(\mathbb{E})$ into N subspaces $H_i^2(\mathbb{E})$, $i = 1, \ldots, N$, which are mutually orthogonal to each other. In addition, every $H_i^2(\mathbb{E})$ is a Hilbert space and $H^2(\mathbb{E})$ is a direct sum of its components, $H^2(\mathbb{E}) = H_1^2(\mathbb{E}) \oplus \ldots \oplus H_N^2(\mathbb{E})$. It would be unnecessarily difficult to assert this decomposition directly, since later the actual isomorphic relations between $H^2(\mathbb{E})$ and its subspaces $H_i^2(\mathbb{E})$, $i = 1, \ldots, N$, will appear naturally in the context of periodically generated bases of $H^2(\mathbb{E})$.

The function (2.96), characterized by its zeros $\{a_1, \ldots, a_N\}$, is analytic in the unit disk. Interestingly enough, the same applies to an infinite product of Möbius transformations, given by the infinite *Blaschke product*

$$B(z) = \prod_{i=0}^{\infty} \frac{-a_i^*}{|a_i|} \frac{z - a_i}{1 - a_i^* z}, \quad z \in \mathbb{D}. \tag{2.98}$$

The product (2.98) converges (uniformly on compact subsets of $\mathbb{D}$) to an inner function $B \in H(\mathbb{D})$ if and only if $\sum (1 - |a_i|) < \infty$ [Hoffman, 1962]. Moreover, for any bounded function $f \in H(\mathbb{D})$, excluding $f \equiv 0$, the zeros must satisfy the condition

$$\sum_{i=0}^{\infty} (1 - |a_i|) < \infty, \tag{2.99}$$

and conversely, the function f has an unique factorization as $f = Bg$, where B is the Blaschke product formed from the zeros of f and where g is an analytic function without zeros[95]. This factorization property provides an important reversed result that states that if the zeros of $f \in H(\mathbb{D})$ satisfy

$$\sum_{i=0}^{\infty} (1 - |a_i|) = \infty, \tag{2.100}$$

then f is identically zero, $f(z) = 0$, for all $z \in \mathbb{D}$ [Rudin, 1987]. This interplay between the Blaschke product and the conditions (2.99) and (2.100) is used to prove completeness of rational bases for various Hilbert and Banach spaces. For example in the case of $H^2(\mathbb{D})$ or $H^2(\mathbb{E})$, the conjecture is that if the zeros or poles, respectively for $H^2(\mathbb{D})$ or $H^2(\mathbb{E})$, of a rational orthonormal system satisfy (2.100), then the only function that is orthogonal to all the basis functions is $f(z) = 0$, which was condition (ii) for completeness in Section (2.3.2).

The representation of the Blaschke product (2.98) in the domain $\mathbb{E}$ is attained as

$$B_E(z) = 1/B(z) = \prod_{i=0}^{\infty} \frac{|a_i|}{-a_i^*} \frac{1 - a_i^* z}{z - a_i} = \prod_{i=0}^{\infty} \frac{-a_i}{|a_i|} \frac{z^{-1} - a_i^*}{1 - a_i z^{-1}}, \quad z \in \mathbb{E}. \tag{2.101}$$

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[94]In general it is impossible to describe these subregions. In the case of z^N and $z^{1/N}$, the decomposition is achieved by attaching "semi-open segments" $2\pi(k-1)/N \le \arg(z) < 2\pi k/N$, $k = 1, \ldots, N$.

[95]This is a simplified version of a much stronger result for the broad class of Nevanlinna functions, which is the superclass of all H^p spaces [Rudin, 1987]. The attribute bounded is used to avoid specifying the norm. A practical example of bounded $f \in H(\mathbb{D})$ are those that are continuous in $\bar{\mathbb{D}}$.

Now the poles and zeros of (2.101) are a_i and $1/a_i^*$, $i = 0, 1, \ldots$, respectively, and $B_E(z)$ is thus stable and convergent in the region $\mathbb{E}$. If $\sum(1 - |a_i|) < \infty$, then once more, $|B_E(z)| = 1$ for $z \in \mathbb{T}$ and $|B_E(z)| < 1$ for $z \in \mathbb{E}$, and (2.101) could be interpreted as a change of variable in $H^2(\mathbb{E})$. However, in addition to the fact that treating (2.101) as a conformal mapping is a bit far-fetched, the really interesting case is precisely the one when $\sum(1 - |a_i|) = \infty$.

2.4.3 The shift operator as a generator of bases

The Blaschke product is closely related to a class of bounded linear operators, *shift operators* that generate shift-invariant subspaces into various Banach spaces. In particular, the concepts of shift operators and shift-invariant subspaces is useful in producing bases for the Hilbert spaces $H^2(\mathbb{E})$ and $\ell^2(\mathbb{N})$. A closed subspace Y of a Hilbert space H is shift-invariant, if there is a linear transformation $S : H \to H$, such that $S(Y) \subset Y$, that is, $S(f) \in Y$ for all $f \in Y$. The *Beurling-Lax Theorem* characterizes the nature of shift-invariant subspaces: for every inner function A, the subspace of functions denoted and defined by $AH = \{Af : f \in H\}$ is shift-invariant, and conversely, every shift-invariant subspace is of the form AH [Rudin, 1987]. Here the more general term 'inner function' is replaces by rational allpass functions (of the form (2.101)) and the theorem is used to deduce and unify rational orthonormal bases of $H^2(\mathbb{E})$, and their time-domain counterparts in $\ell^2(\mathbb{N})$.

The simplest allpass function, Blaschke product or shift operator on $H^2(\mathbb{E})$ is the unit delay z^{-1}. This right-shift of elements $f(z) \in H^2(\mathbb{E})$ and $x(n) \in \ell^2(\mathbb{N})$ is given by

$$(Sf)(z) = z^{-1}f(z) \quad \text{and} \quad Sx = \{0, x(1), x(2), \ldots\}. \tag{2.102}$$

These shift operators generate the previously defined "standard bases" of $H^2(\mathbb{E})$ and $\ell^2(\mathbb{N})$, that is, the bases $\{z^{-i}\}_{i=0}^{\infty}$, $z^{-i} = S^i(1)$, and $\{\delta(n-i)\}_{i=0}^{\infty}$, $\delta(n-i) = S^i\delta(n)$, respectively. The right-shift operator decomposes $H^2(\mathbb{E})$ and $\ell^2(\mathbb{N})$ into infinite direct sums of subspaces, $H^2(\mathbb{E}) = H_0^2(\mathbb{E}) \oplus H_1^2(\mathbb{E}) \oplus \ldots$ and $\ell^2(\mathbb{N}) = \ell_0^2(\mathbb{N}) \oplus \ell_1^2(\mathbb{N}) \oplus \ldots$. The elements of $H_i^2(\mathbb{E})$ are characterized by the fact that they have a zero of order at least i at $z = \infty$, and correspondingly, sequences in $\ell_i^2(\mathbb{N})$ have at least i leading zeros.

Similarly, the first-order causal allpass operator, $A(z) = (z^{-1} - a^*)/(1 - az^{-1}) \in H^2(\mathbb{E})$, defines a partition of the space $H^2(\mathbb{E})$ that is characterized by multiple poles at $z = a$. The basis functions of the component spaces $H_i^2(\mathbb{E})$ are produced by the change of variable, given in (2.93) and (2.94), applied to the canonical base $\{w^{-i}\}_{i=0}^{\infty}$ of $H^2(\mathbb{E})$,

$$\{w^{-i}\}_{i=0}^{\infty} \qquad \longleftrightarrow \qquad \left\{ \frac{\sqrt{1-a^*a}}{1-az^{-1}} \left(\frac{z^{-1}-a^*}{1-az^{-1}} \right)^i \right\}_{i=0}^{\infty}. \tag{2.103}$$

The set of functions on the right-hand side of (2.103) constitute the *Laguerre base* of $H^2(\mathbb{E})$ and the corresponding basis functions of $\ell^2(\mathbb{N})$, the discrete-time Laguerre functions, are attained as their inverse z-transforms[96]. These Laguerre functions de-

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[96]The Hilbert space $H^2(\mathbb{C}^+)$ (or $L^2(i\mathbb{R})$) do not have a simple unit delay sort of a base, but it

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fine once more decompositions of $H^2(\mathbb{E})$ and $\ell^2(\mathbb{N})$ into infinite sums of subspaces. For $a \neq 0$, the subspaces and basis functions are no longer "trivial", but the conformal equivalency provides simple means to transform and project elements of $H^2(\mathbb{E})$ and $\ell^2(\mathbb{N})$ into different basis representations and partitions of subspaces.

A natural step now is to consider subspace partitions and basis functions induced by the finite Blaschke product (2.96) and its inverse $A(z) = 1/A_N(z)$, which is a causal and stable right shift operator. Two simple choices of poles, namely $a_i = 0$ and $a_i = a \in \mathbb{D}$, and a regrouping of N successive terms in (2.103) into a vector representation, provide readily

$$
\mathbf{f}_j(z) = \begin{bmatrix} 1 \\ \vdots \\ z^{-N+1} \end{bmatrix} (z^{-N})^j, \quad j = 0, 1, \ldots, \tag{2.105}
$$

for the canonical base, and respectively,

$$
\mathbf{l}_j(z) = \begin{bmatrix} \frac{\sqrt{1-a^*a}}{1-az^{-1}} \\ \vdots \\ \frac{\sqrt{1-a^*a}}{1-az^{-1}} \left(\frac{z^{-1}-a^*}{1-az^{-1}} \right)^{N-1} \end{bmatrix} \left(\left(\frac{z^{-1}-a^*}{1-az^{-1}} \right)^N \right)^j, \quad j = 0, 1, \ldots, \tag{2.106}
$$

for the Laguerre base. These vector valued functions provide two different partitions of $H^2(\mathbb{E})$ into a finite direct sum of subspaces, $H^2(\mathbb{E}) = H_1^2(\mathbb{E}) \oplus \ldots \oplus H_N^2(\mathbb{E})$, where subspaces $H_i^2(\mathbb{E})$, $i = 1, \ldots, N$, are characterized by the corresponding components of the vectors (2.105) and (2.106), that is, by basis functions $f_{ij}(z)$ and $l_{ij}(z)$, $i = 1, \ldots, N$, $j = 0, 1, \ldots$, respectively[97]. Each component space $H_i^2(\mathbb{E})$, $i = 1, \ldots, N$, then in turn has a partitioning into an infinite collection of subspaces, $H_i^2(\mathbb{E}) = H_{i1}^2(\mathbb{E}) \oplus H_{i2}^2(\mathbb{E}) \oplus \ldots$, $i = 1, \ldots, N$, with respect to the particular representation of the component basis functions. These factorizations originate from a simple regrouping of the unit-delay or Laguerre shift basis functions (2.103), respectively, but they suggest a natural procedure with respect to a more general choice of the shift operator A.

Once more, the partitioning and the generation of basis functions can be seen to emerge from a change of variable; the constructions (2.105) and (2.106) are of the

---

is interesting to notice that the simplest base is once more attained with the aid of a conformal mapping: the Laguerre base of $H^2(\mathbb{C}^+)$ and the standard base of $H^2(\mathbb{E})$ are connected by

$$
\{z^{-i}\}_{i=0}^\infty \quad \longleftrightarrow \quad \left\{ \frac{\sqrt{2a}}{a+s} \left(\frac{a-s}{a+s} \right)^i \right\}_{i=0}^\infty, \tag{2.104}
$$

where $a > 0$ and $s \in \mathbb{C}^+$, and the relation is due to the change of variable given by (2.87) and (2.90). Functions on the right-hand side are Laplace transforms of continuous-time Laguerre functions. For complex a such that $\Re(a) > 0$, the corresponding modified form of a basis function is $\sqrt{2\Re(a)}(a - s^*)^i/(a + s)^{i+1}$.

[97]The functions $\mathbf{f}_j(z)$ and $\mathbf{l}_j(z)$, $j = 0, 1, \ldots$, constitute bases for the space $\mathbf{H}_N^2(\mathbb{E})$ of vector valued functions with components in $H^2(\mathbb{E})$. Equipped with the matrix valued inner product, $\mathbf{H}_N^2(\mathbb{E})$ is a Hilbert space and the bases $\{\mathbf{f}_j(z)\}_{j=0}^\infty$ and $\{\mathbf{l}_j(z)\}_{j=0}^\infty$ are orthonormal in the sense that $(\mathbf{f}_i, \mathbf{f}_j) = \delta_{ij}\mathbf{I}$ and $(\mathbf{l}_i, \mathbf{l}_j) = \delta_{ij}\mathbf{I}$, respectively.

form $\mathbf{v}(z)(1/A(z))^j$, where the vector valued function $\mathbf{v}(z)$ acts as a normalization[98]. Using the inner product (2.72), it is not difficult to verify that elements in $\mathbf{v}(z)$ are mutually orthonormal, $(\mathbf{v}(z), \mathbf{v}(z)) = \mathbf{I}$, but the more important fact is that the components of $\mathbf{v}(z)$ are orthogonal to $A(z) = 1/A_N(z)$. More generally, since $AH^2(\mathbb{E})$ is a close subspace of $H^2(\mathbb{E})$ for every inner function A, so is also its orthogonal complement $H(A) = H^2(\mathbb{E}) \ominus AH^2(\mathbb{E})$.[99] The elements of $\mathbf{v}(z)$ in (2.105) or (2.106) are examples of bases for the corresponding finite-dimensional subspace $H(A)$, induced by two special cases of the inner function $A(z) = 1/A_N(z)$.

The idea is hopefully becoming clear, the components of the vector valued function

$$\mathbf{g}(z) = \left[\frac{\sqrt{1 - a_1^* a_1}}{1 - a_1 z^{-1}} \quad \cdots \quad \frac{\sqrt{1 - a_N^* a_N}}{1 - a_N z^{-1}} \prod_{i=1}^{N-1} \frac{z^{-1} - a_i^*}{1 - a_i z^{-1}} \right]^T \tag{2.107}$$

are mutually orthonormal, $(\mathbf{g}, \mathbf{g}) = \mathbf{I}$, and they are all orthogonal to the allpass function

$$A(z) = \prod_{i=1}^{N} \frac{z^{-1} - a_i^*}{1 - a_i z^{-1}}. \tag{2.108}$$

The elements of (2.107) form a base of $H(A) = H^2(\mathbb{E}) \ominus AH^2(\mathbb{E})$, where A is the allpass shift given by (2.108), and an orthonormal base for the Hilbert space $\mathbf{H}_N^2(\mathbb{E})$, the space of vector valued functions with components in $H^2(\mathbb{E})$, is obtained as $\{\mathbf{g}_j = A^j \mathbf{g} : j = 0, 1, \ldots\}$. The explicit form of $\mathbf{g}_j(z)$, $j = 0, 1, \ldots$, is then simply a generalization of (2.106), given by

$$\mathbf{g}_j(z) = \left[\begin{array}{c} \frac{\sqrt{1 - a_1^* a_1}}{1 - a_1 z^{-1}} \\ \vdots \\ \frac{\sqrt{1 - a_N^* a_N}}{1 - a_N z^{-1}} \prod_{i=1}^{N-1} \frac{z^{-1} - a_i^*}{1 - a_i z^{-1}} \end{array} \right] \left(\prod_{i=1}^{N} \frac{z^{-1} - a_i^*}{1 - a_i z^{-1}} \right)^j, \quad j = 0, 1, \ldots. \tag{2.109}$$

Correspondingly, the vector signal space $\mathbf{L}_N^2(\mathbb{N})$, whose components are in $\ell^2(\mathbb{N})$, is a Hilbert space and the inverse z-transforms of (2.109) form a base of the space $\mathbf{L}_N^2(\mathbb{N})$. Representations (2.105) and (2.106) are clearly special cases of (2.109), but as such, they induce three genuinely different types of constructions for $\mathbf{H}_N^2(\mathbb{E})$ and $\mathbf{L}_N^2(\mathbb{N})$, respectively. In fact, an infinite variety of partitions is introduced with respect to different choices of $\{a_1, \ldots, a_N\} \subset \mathbb{D}$. The point is that all such constructions produce bases for the spaces $H^2(\mathbb{E})$ and $\ell^2(\mathbb{N})$, respectively. The essential difference is then in the manner the space, for example $H^2(\mathbb{E})$, is interlaced into the space $\mathbf{H}_N^2(\mathbb{E})$, that is, how the components of $\mathbf{H}_N^2(\mathbb{E})$ partition $H^2(\mathbb{E})$, $H^2(\mathbb{E}) = H_1^2(\mathbb{E}) \oplus \ldots \oplus H_N^2(\mathbb{E})$, which can also be expanded and expressed by a

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[98]The change of variable interpretation between (2.105) and (2.106), or directly between $\{z^{-i}\}_{i=0}^{\infty}$ and (2.106), is quite clear, but the "change of measure" introduced by $\mathbf{v}(z)$ is not that obvious. However, for the expanded components $l_{ij}(z)$, it is simply the "Laguerre normalization".

[99]For a shift operator S on a Hilbert space H, the subspace $H(S) = H \ominus SH$ is actually the null space of the adjoint (left shift) operator S^*, $H(S) = ker S^*$. The shift operator is a non-unitary isometry on H, and a lot could be deduced from the related operator theory, but in the underlying case of allpass operators, explicit relations between functions provide sufficient grounds for the reasoning.
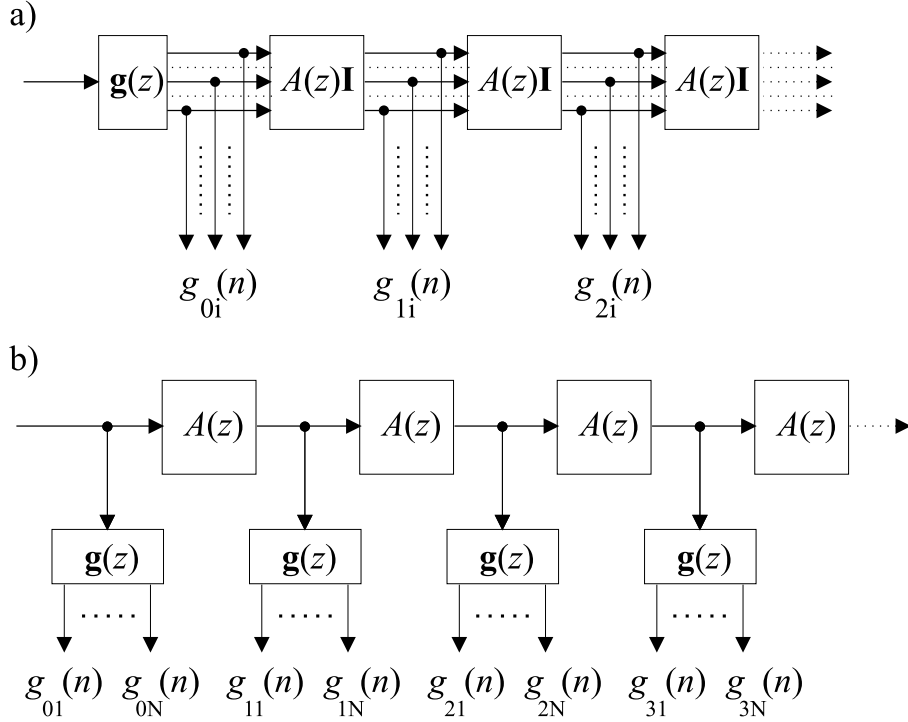
Figure 2.9: Two alternative filter configurations that realize (2.109) as input to tap-output transfer functions.

relation of the form

$$
H^2(\mathbb{E}) \overset{A}{\longleftrightarrow} \mathbf{H}_N^2(\mathbb{E}) = \begin{bmatrix} H_1^2(\mathbb{E}) \\ \vdots \\ H_N^2(\mathbb{E}) \end{bmatrix} = \begin{bmatrix} H_{11}^2(\mathbb{E}) & \oplus & H_{12}^2(\mathbb{E}) & \oplus & \cdots \\ \vdots & & \vdots & & \\ H_{N1}^2(\mathbb{E}) & \oplus & H_{N2}^2(\mathbb{E}) & \oplus & \cdots \end{bmatrix} \quad (2.110)
$$

The change of variable interpretation, $z^{-j} \leftrightarrow \mathbf{g}(z)A(z)^j$, is still plausible for the construction (2.109), although it is not as obvious as in the case of (2.105) and (2.106).[100] This is not particularly disturbing since it is known from Section 2.3.2 that there is an isomorphic relation between $H^2(\mathbb{E})$ and the variously generated subspaces $H_i^2(\mathbb{E})$. At this point it is sufficient to conclude that the component functions of (2.109), the set of functions $\{g_{ij}(z)\}_{j=0}^\infty$, form an orthonormal base in $H_i^2(\mathbb{E})$ and that these functions can be produced by a causal and stable LTI digital filter. Figure 2.9 depicts two alternative configurations for producing all functions $\{g_{ij}(z) : i = 1, \ldots, N, \ j = 0, 1, \ldots\}$ as input to tap-output transfer functions. The corresponding time-domain basis functions $\{g_{ij}(n) : i = 1, \ldots, N, \ j = 0, 1, \ldots\}$ form a base in $\ell^2(\mathbb{N})$, and alternatively to the inverse z-transformation, they may be produced as the tap-output impulse responses of the filters in Figure 2.9.

The following reasoning is not meant to be exact, but hopefully it is instructive and serves as a justification for the introduction of the shift operator. All aforemen-

---

[100]However, equating simply the inner products in the bases $\{w^{-j}\}_{j=0}^\infty$ and $\{g_{ij}(z)\}_{j=0}^\infty$ gives a "change of measure" formula $dw/w = g_i(z)g_i^*(1/z^*)dz/z$, which is easily seen to reduce to the form of (2.95). An alternative deduction of functions of the type (2.109), the *Generalized Orthonormal Basis Functions* (GOBFs) by Heuberger [Heuberger et al., 1995], and a related *Hambo transform theory* of signals and systems [Van den Hof et al., 1994], are considered later.

tioned bases are implicitly presumed to be complete, that is, they provide a basis representation for any member of the corresponding Hilbert space. An old theorem by Walsh dating back to the 1920's implies that a rational orthonormal system is complete if and only if the set of poles satisfy condition (2.100).[101] This condition is very mild, since for example, any infinite sequence of poles such that $|a_i| < r$, for any $0 < r < 1$, will do. Clearly, all bases related to the repeated appearance of a finite set of poles fall into this category, which is particularly the case for the system induced by (2.109) for any choice of N and $\{a_1, \ldots, a_N\} \subset \mathbb{D}$.

Continuing now in terms of the shift operator A, the "total shift", $A^\infty = \prod_{j=0}^\infty A_j$, could be said to be complete if it spans all of $H^2(\mathbb{E})$, that is, if the subspaces it generates form an infinite orthogonal partition of the whole space, $H^2(\mathbb{E}) = \bigoplus_{j=0}^\infty A^j H(A)$, where $H(A) = H^2(\mathbb{E}) \ominus A H^2(\mathbb{E})$ is the generating subspace, as before. On the other hand, if the initial subspace is generated with respect to A^k, $k > 1$, producing $H(A^k) = H^2(\mathbb{E}) \ominus A^k H^2(\mathbb{E})$, then it is easily seen that $H(A^k) = \bigoplus_{j=0}^k A^j H(A)$, since both spaces are spanned by the same orthonormal base. Furthermore, it is known from (2.100) that the total shift A^∞ reduces to the zero function, and by cautiously exploiting the denotation $H(A^\infty) = H^2(\mathbb{E}) \ominus A^\infty H^2(\mathbb{E})$, the conclusion is that the total shift is complete if and only if the poles satisfy condition (2.100). By the same argument, the "single shift", related to the infinite Blaschke product (2.101), generates an orthonormal set of functions

$$\left\{ \frac{\sqrt{1 - a_i^* a_i}}{1 - a_i z^{-1}} \prod_{j=0}^{i-1} \frac{z^{-1} - a_j^*}{1 - a_j z^{-1}} \right\}_{i=0}^\infty \tag{2.111}$$

that is a complete base of $H^2(\mathbb{E})$ if and only if condition (2.100) is satisfied.

---

[101]This result has been the subject of numerous rediscoveries, especially during the last fifteen years. Some of the published results have nonetheless broaden the result to more general Banach spaces and to other rational systems, such as, *rational wavelets* [Akçay, 2000].

Chapter 3

The generalized linear-in-parameter model concept

In this Chapter, some of the concepts and results of the previous Sections are recapitulated and utilized to produce a general framework for the modeling of discrete-time signals and systems. To begin with, a particular modeling setup is established as a blend of system identification and signal processing configurations. Then, the special choice of the model structure introduced in Section 2.1.5 is redefined as an approximation problem in the Hilbert space of causal and finite-energy signals and systems. This concept of a *generalized linear-in-parameter model* (GLM) is then used to deduce many familiar results and methods in signal processing, with an obvious emphasis in mind that there are natural generalizations, or even a more genuine perspective, to the traditional approach.

The GLM concept is not particularly ingenious, or not even entirely novel, but it has not been formulated before as such, that is, from the point of view of linear independency. There is certainly nothing new in constructing a model that is linear in the parameters. For example, the FIR model (2.32), the IIR model (2.33), and the state-space model (2.34), respectively, are linear in the parameters, and in system identification terms, they are special cases of the class of *linear regression models* [Ljung, 1987]. Continuing with pompous interfaces, the GLM concept can also be seen as a generalization of the *Wiener filtering Theory* [Wiener, 1949] [Lee, 1960], or at least, as a generalized framework for some aspects of optimal *least-square* or *minimum mean-square* estimation and filtering. Additionally, all orthogonal series expansions of a given signal or system are, by construction, linear in the parameters; the same applies to a model or an approximation that is based on a finite, sparse or truncated, collection of such expansion terms. The point that the GLM concept tries to make is that the conventional linear regressor approach is perhaps not sufficient enough for a well-posed general modeling setup, but on the other hand, that a categoric restriction to orthogonal model structures is in some cases unnecessary constraining for a practical and flexible generation of the modeling concept.
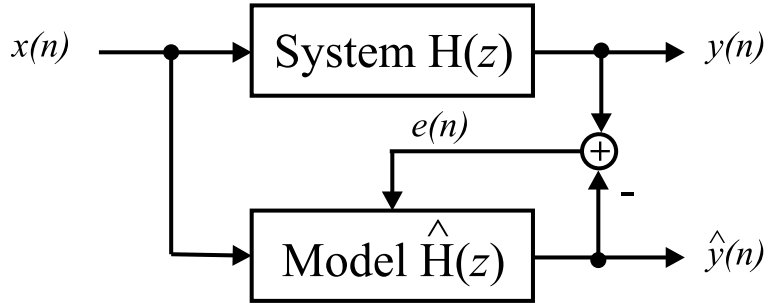
Figure 3.1: The modeling setup - a parallel configuration of an unknown system H and a model $\hat{H}$, with a common input x, and outputs y and $\hat{y}$, respectively. Minimization of the modeling error signal $e = y - \hat{y}$, in some chosen sense, is usually the objective of modeling.

3.1 A general framework for various signal processing tasks

The parallel setup of a possibly unknown system and a model in Figure 3.1 can be used to describe many different assignments in signal processing. Using the terminology of system identification, the setup is by construction in an *output-error* (OE) form [Ljung, 1987]. This does not necessarily exclude other typical identification schemes, since most of the "noise modeling" configurations can be embedded to the modeling setup of Figure 3.1. However, the system identification framework will not be elaborated further since the aim here is somewhat different. In fact, some kind of a reloading of terms is required to avoid misleading interpretations and comparisons. For example, as it can be seen from Figure 3.1, there are no additional noise sources in the system input and output, respectively, corresponding to measurement errors or some other sources of inaccuracy. This is justifiable because the necessary presumptions about the noise signals makes them uninteresting subjects of modeling; in the OE configuration of LTI modeling the noise sources can be commuted to be included into the modeling error, or alternatively, the error terms are regarded as inherent parts of the input and output signals[1].

In the following attempt to categorize different modeling tasks, the symbols $\{x, \delta, H, y, \hat{x}, \hat{H}, \hat{y}\}$ are used generically to denote elements in $\ell^2(\mathbb{N})$ or $H^2(\mathbb{E})$, interchangeably, and both in the deterministic and stochastic sense.

Approximation or modeling by synthesis - the target response is considered as an impulse response of some system H, $y = Hx$, $x = \delta$, and a model $\hat{y} = \hat{H}\delta$ is designed to approximate the system.

Identification or approximate system identification base on input-output-data - the system output $y = Hx$ is approximated with the model response $\hat{y} = \hat{H}x$.

---

[1]The usual presumption about the noise signals is that they are WSS and uncorrelated with the input and output signals. The ongoing assumption here is that WSS signals are zero-mean, which results in a constant term in the error variance.

Filtering The signal y is regarded as a desired response, not necessarily as a response of a particular system $y = Hx$, which is approximated by the response $\hat{y}$ of a digital filter $\hat{H}$ to the input signal x, $\hat{y} = \hat{H}x$.

The above classification is somewhat arbitrary and overlapping. The division between identification and approximation is associated simply with the form of the available data and not as a distinction between "true identification" and approximate identification of a system. Moreover, here modeling means always filtering by digital filters, which makes this all seem apparently stupid. In fact, in most cases the first two items could be replaced with synthesis and analysis by digital filtering, respectively, but these specifications provide flexibility in terminology in such a way that still maintains enough specificity. In addition to analysis, synthesis and modeling by filtering, the subsequent characterizations are more application oriented special cases of filtering.

Prediction by filtering - the desired response $y(n) = x(n + p)$, for some time-lag p, is approximated with the model response $\hat{y} = \hat{H}x$.

Interference cancellation or noise reduction - the interference component in signal y is approximately canceled using a model for the interference, $\hat{y} = \hat{H}x$, based on a reference signal x.

Inverse modeling or transmission channel equalization - the inverse of a system $y = Hx$, or the inverse system response $x = H^{-1}y$, with possible delay, is approximated using the model response $\hat{x} = \hat{H}y$ (where $\hat{H}$ is now an approximation of H^{-1}).

The interference cancellation class includes such typical signal processing tasks as *echo and noise cancellation* [Kuo and Morgan, 1996]. A more natural way to describe inverse modeling and equalization, as well as analysis of a signal, would be to replace the parallel setup with a cascade connection of the system and the model, depicted in Figure 3.2. However, in many practical cases the modeling is based on a somehow attained description of the inverse system, or a desired modification[2] of it, which brings the situation back to the original setup of Figure 3.1, where the system to be modeled is replaced with its inverse.

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[2]One common modification is to make an unstable inverse system stable. Another modification to the pure inverse modeling setup is attained when the desired overall transfer function of the system and the model is chosen to differ from an identity mapping.
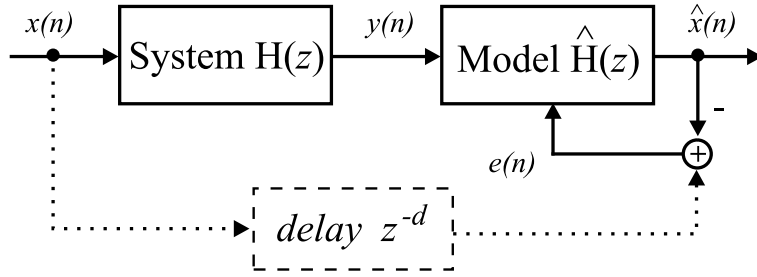
Figure 3.2: Another modeling setup - a cascade configuration of an unknown system H and a model $\hat{H}$. The system output y acts as the input to the model and the desired response is now x, a delayed version of x, or some other target response.

3.1.1 The linear-in-parameter model configuration

As it has been a few times already stated, the model structure that is employed here is of the form

$$\hat{y}(n) = \sum_{i=1}^{N} w_i x_i(n) = \mathbf{w}^T \mathbf{x}(n), \qquad x_i(n) = g_i(n) * x(n) \tag{3.1}$$

$$\hat{Y}(z) = \sum_{i=1}^{N} w_i X_i(z) = \mathbf{w}^T \mathbf{x}(z), \qquad X_i(z) = G_i(z) X(z), \tag{3.2}$$

where w_i, $i = 1, \ldots, N$, are tap-output weights, or model parameters, and where the impulse response of the model is a weighted sum of partial model impulse responses,

$$\hat{h}(n) = \sum_{i=1}^{N} w_i g_i(n) = \mathbf{w}^T \mathbf{g}(n) \quad or \quad \hat{H}(z) = \sum_{i=1}^{N} w_i G_i(z) = \mathbf{w}^T \mathbf{g}(z). \tag{3.3}$$

The responses $G_i(z) = Z\{g_i(n)\}$, $i = 1, \ldots, N$, are supposed to be produced as impulse responses of causal and stable LTI digital filters. However, the weights w_i, $i = 1, \ldots, N$, are not restricted to be time-invariant and thus the model is not necessarily linear or time-invariant[3]. The GLM structure of Fig. 3.3 is a direct illustration of the formulas (3.3), where the partial model filters form a tapped parallel system that is weighted and summed up to produce the model response. This parallel system can sometimes be replaced with a cascaded or *transversal* model structure.

Once more, the modeling concept that is developed here does not fit into the traditional division between *parametric* and *nonparametric* modeling [Ljung, 1987]. The GLM is certainly parametric in the tap-output weights, but choosing of the partial models is not necessarily a conventional data-driven parametrization process, and even when it is, the parameters that define the partial models have a different meaning. In the usual system identification framework, the space of feasible parameters acts like a spanning set for the model structure or the *model class*, from which a

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[3]Time-variant modeling will be discussed in a very limited sense, that is, restricted to the case when the time-variant parameters are local (in time) estimates of the LTI parametrization.
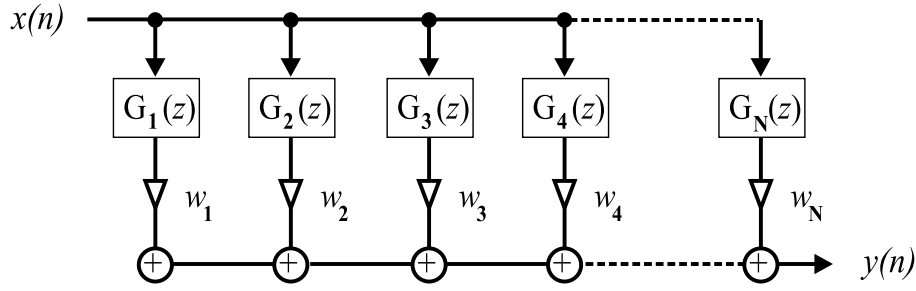
Figure 3.3: The GLM structure as a block diagram of unspecified partial models.
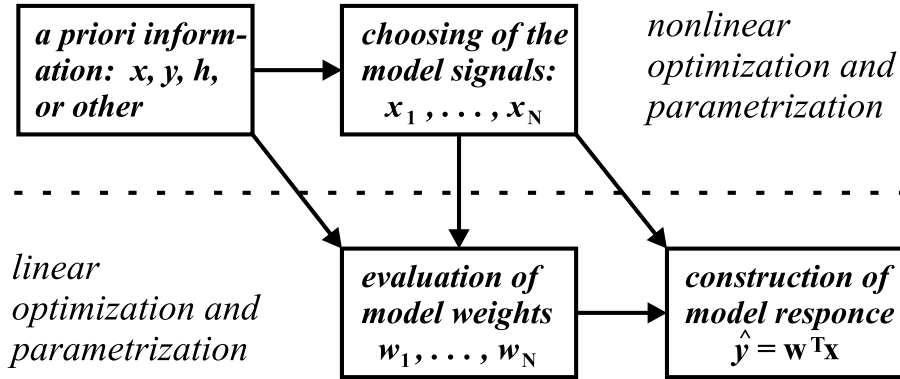
Figure 3.4: The GLM design steps - the division into linear and nonlinear parts in the optimization and parametrization is somewhat case-specific.

particular realization, the model, is attained by fixing a point in this parameter space. In the GLM setup the partial models are fixed, after a possible optimization of parameters, and the model set is subsequently spanned by the variation of the tap-output weights. Due to this incoherence, referring to parameters is avoided in the following by using the terms weights or coefficients[4].

The design steps for the construction of a GLM are illustrated in Fig. 3.4. The partitioning into the linear and nonlinear domains should not be taken too seriously, since exceptions are possible in both directions. Additionally, the terms optimization and parametrization are for the time being somewhat vague expressions. The following is the formal definition of the GLM, which will eventually steer the discussion towards the particular themes of this thesis.

3.1.2 The synthetic definition of the GLM

As it was stated in Section 2.1.5, both the system and the model are supposed to be elements of the space $H^2(\mathbb{E})$ or its time-domain counterpart $\ell^2(\mathbb{N})$. The following definition is based on the concepts and results provided in Section 2.3.

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[4]The abbreviation GLM provides different interpretations for the silent member, such as, the *linear regression* or the *linearly combined or weighted* model, which are all equally ill-matched descriptions.

Definition of the GLM The model given by (3.3) is a Generalized Linear-in-parameter Model (GLM) for the system $y = H[x]$, $x, y \in \ell^2(\mathbb{N})$ and $H \in H^2(\mathbb{E})$, if the partial model responses, $x_i = G_i[x]$, $i = 1, \dots, N$, are mutually linearly independent. The optimal parametrization in the approximation space spanned by the set $\{x_1, \dots, x_N\}$ is attained as the solution of the normal equations

$$\sum_{i=1}^{N} w_i(x_i, x_j) = (y, x_j), \quad j = 1, \dots, N. \tag{3.4}$$

For clarity, the signal and system spaces were polarized as $\ell^2(\mathbb{N})$ and $H^2(\mathbb{E})$, respectively, but this division is completely arbitrary. The notation of a generic Hilbert space $\mathcal{H}$ could have been used to indicate equivalent representations both in the time- and frequency-domains and with respect to the deterministic and stochastic interpretation of the ingredients[5]. The "optimality" of the approximation is with respect to the norm of the Hilbert space, which is an invariant quantity for all realizations of $\mathcal{H}$, and terms such as *least-square* (LS) or *minimum mean-square* (MMS) error may be used to specify between deterministic and stochastic representations. The inner products that define the normal equations (3.4) can be evaluated using various formulas given in Section 2.3.3.

The point in the definition of the GLM is to emphasize the importance of linear independency in well-defined parametric modeling. The formulation may at first seem strange or backward since the linear independency is presumed of signals that are generally unknown. In addition, the restriction to partial model signals appear as an unnecessary limitation compared to a more generic set of "modeling signals" $\{x_1, \dots, x_N\}$ that is not necessarily the product of filtering operations, or at the least, not produced by a common input signal. However, the modeling means here synthesis or filtering by a linear filter which makes the class of modeling signals as general as possible[6]. Moreover, it turns out that the choice of the model structure is the key to guaranteed linear independency, with almost no restrictions on the input signals, which is the main motivation of the GLM concept.

3.1.3 The matrix form of the normal equations

The group of equations given in (3.4), the normal equations, submit to a matrix form

$$\mathbf{Rw} = \mathbf{p} , \tag{3.5}$$

where $\mathbf{R}$ and $\mathbf{p}$ are named the *correlation matrix* and the *correlation vector*, respectively, with a conscious abuse of terminology. The matrix equation (3.5) simply

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[5]This generalization extends to continuous-time signals and systems, as well as, to appropriate vector and matrix valued decompositions of the signal and system spaces.

[6]For example the inclusion of individual partial model excitations, $x_i = G_i[e_i]$, would be redundant in the sense that the different excitations can be integrated to the model, $x_i = G_i E_i[x]$. This is not completely true if and when the model is supposed to be produced by finite, causal, and stable filtering operations.

collects the scalar inner products and the N simultaneous equations into a single equation for the unknown weights, $\mathbf{w} = [w_1 \ \cdots \ w_N]^T$,

$$\mathbf{R} = \begin{bmatrix} (x_1, x_1) & \cdots & (x_N, x_1) \\ \vdots & \ddots & \vdots \\ (x_1, x_N) & \cdots & (x_N, x_N) \end{bmatrix} \quad and \quad \mathbf{p} = \begin{bmatrix} (y, x_1) \\ \vdots \\ (y, x_N) \end{bmatrix}. \tag{3.6}$$

It is easily seen that the correlation matrix is the transpose of the matrix valued inner product of the tap-output vector $\mathbf{x}(n) = [x_1(n) \ \cdots \mathbf{x}_N(n)]^T$ with itself, $\mathbf{R} = (\mathbf{x}, \mathbf{x})^T$, and equally, $\mathbf{p} = (y, \mathbf{x})^T$ for the correlation vector. This is the "unconjugated" form of the normal equations, but the conjugate antisymmetry of the inner products provide many variants[7]. How the inner products are actually evaluated, or approximately estimated, depends on the nature of the available information about the input and output signals and the partial model responses. For example, the expanded partial model response form of the elements of the correlation matrix and vector, $\mathbf{R} = [r_{ij}]$ and $\mathbf{p} = [p_i]$, are simply $r_{ij} = (g_j * x, g_i * x)$ and $p_i = (y, g_i * x)$, which equals $r_{ij} = (g_j, g_i * r_{xx})$ and $p_i = (r_{yx}, g_i)$, respectively, in terms of the (deterministic or stochastic) auto- and cross-correlation sequences[8]. The lower case letters point to the utilization of time-domain formulas (2.70) or (2.71) in the evaluation of the inner products, but the elements of the correlation matrix and vector may also be derived using the frequency-domain representations

$$r_{ij}^* = \frac{1}{2\pi j} \oint_{\mathbb{T}} G_i(z) G_j^*(1/z^*) S_{xx}(z) \frac{dz}{z} \tag{3.7}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} G_i(e^{j\omega}) G_j^*(e^{j\omega}) S_{xx}(e^{j\omega}) d\omega, \qquad i, j = 1, \dots, N, \tag{3.8}$$

and

$$p_i^* = \frac{1}{2\pi j} \oint_{\mathbb{T}} G_i(z) S_{xy}(z) \frac{dz}{z} \tag{3.9}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} G_i(e^{j\omega}) S_{xy}(e^{j\omega}) d\omega, \qquad i = 1, \dots, N, \tag{3.10}$$

where $S_{xx}(z)$ and $S_{xy}(z)$ are z-transforms of the auto- and cross-correlation sequences, respectively[9]. For deterministic signals, the spectral densities are just abbreviations for the function products, $S_{xx}(z) = |X(z)|^2$ and $S_{xy}(z) = X(z)Y^*(1/z^*)$.

---

[7]For example, if the model is defined as $\hat{y} = \mathbf{w}^H \mathbf{x}$, then the correlation matrix and vector are given by $\mathbf{R} = (\mathbf{x}, \mathbf{x})$ and $\mathbf{p} = (\mathbf{x}, y)$. The form of (3.5) is then restored by a complex conjugation operation on all the elements, providing a normal equation for the complex conjugated weights, $\mathbf{R}\mathbf{w}^* = \mathbf{p}$. One of the main properties of any correlation matrix is thus becoming evident, they are *Hermitian symmetric*, that is, complex conjugate symmetric, $\mathbf{R}^H = \mathbf{R}$.

[8]Various formulations are possible (as it was demonstrated in Section 2.3.3) due to the conjugate (anti)symmetry of the inner products and correlation sequences, respectively.

[9]Complex conjugation of the terms is used to make the formulas more transparent – in (3.9) the integrand resulting from $p_i = (y, g_i * x)$ would actually be $S_{yx}(z)G_i^*(1/z^*)$, but $p_i^* = (g_i * x, y)$ yields a neater form.

3.2   General properties of the GLM normal equations

In principle, the solving of the normal equations consists of two steps, 1) evaluation of the inner products, and 2) solving of the matrix equation (3.5) with respect to the unknown variable $\mathbf{w}$. In practice neither of these steps are performed as such. The evaluation of the inner products involves almost always some kind of approximation or estimation of the usually unattainable true values, such as, evaluation based on windowing or other forms of finite data records, both in the time- and frequency-domains. These questions are too wide-ranging for a general discussion and therefore only some case-specific considerations are presented in the forthcoming. Likewise, the discussion on the second step will be restricted to the minimum, since most of the means for solving conventional normal equations [Proakis and Manolakis, 1992] [Haykin, 1996], originating from matrix algebra [Golub and Van Loan, 1989], apply also to the GLM configuration. Thus the aim here is just to show some connections to conventional results and methods through their common origins, providing plausible generalizations, and to point out some dissimilarities. A somewhat more detailed treatment of the properties of the GLM normal equations is given elsewhere [Paatero, 2000].

The solution of the matrix equation (3.5) is simply

$$\mathbf{w} = \mathbf{R}^{-1}\mathbf{p}. \tag{3.11}$$

This solution always exists and it is unique, since the correlation matrix is by definition nonsingular[10]. That is, the correlation matrix is *invertible*, the inverse $\mathbf{R}^{-1}$ exists, and the unique solution of $\mathbf{R}\mathbf{w} = \mathbf{p}$ is $\mathbf{w} = \mathbf{R}^{-1}\mathbf{p}$. However, direct inversion of the correlation matrix is avoided since it is usually a numerically ill-conditioned task[11]. There are many indirect methods to solve matrix equations of the form (3.5) [Golub and Van Loan, 1989]. In general, these methods can be divided into two classes, *triangular matrix factorizations* and various *orthogonal transformations*, where the former represents sophisticated forms of the process of *Gaussian elimination* and where the latter is closely related to the eigenanalysis of the correlation matrix. The conventional autocorrelation matrix is clearly a special case of the GLM correlation matrix for a particular choice of the model. However, many of the traditional means for solving normal equations rely solely on the Hermitian symmetry, providing thus direct generalizations to the more generic GLM configuration. The common grounds of some of these methods are considered in the following. In the next section, some additional symmetry and "regularity" properties of the correlation matrix are related to special cases of the GLM structure.

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[10]The modeling signals, $\{x_1(n), \ldots, x_N(n)\}$, are linearly independent if and only if the correlation matrix, also known as the *Gram matrix*, is nonsingular, that is, its determinant is non-zero, $det(R) \neq 0$ [Akhiezer and Glazman, 1981]. In pure matrix terms, a necessary and sufficient condition is that the correlation matrix should be of full rank.

[11]Directly meaning some known brute force method such as *Gaussian elimination* or forming the inverse as $\mathbf{A}^{-1} = adj(\mathbf{A})/det(\mathbf{A})$, where $adj(\mathbf{A})$ is the *adjugate* of $\mathbf{A}$ formed from all its *cofactors* [Golub and Van Loan, 1989].

3.2.1 The signal matrix representation

As it has been mentioned, the GLM correlation matrix is always Hermitian or Hermitian symmetric, which is a direct consequence of the inner product property $(x, y) = (y, x)^*$. Another and useful way to see this fact is to decompose $\mathbf{R}$ by defining a *modeling signal matrix*,

$$\mathbf{S} = \begin{bmatrix} x_1(0) & x_2(0) & \dots & x_N(0) \\ x_1(1) & x_2(1) & \dots & x_N(1) \\ x_1(2) & x_2(2) & \dots & x_N(2) \\ \vdots & \vdots & \ddots & \vdots \end{bmatrix}, \tag{3.12}$$

where the GLM partial model responses are simply stacked into the columns of $\mathbf{S}$. Using the definition of the time-domain inner products it is easy to verify that

$$\mathbf{R} = \mathbf{S}^H \mathbf{S}, \tag{3.13}$$

and consequently that

$$\mathbf{R}^H = (\mathbf{S}^H \mathbf{S})^H = \mathbf{S}^H \mathbf{S} = \mathbf{R}. \tag{3.14}$$

The correlation vector can also be expressed using (3.12), $\mathbf{p} = \mathbf{S}^H \mathbf{y}$, where $\mathbf{y} = [y(0) \ y(1) \ \dots]^T$ is the system output as a vector[12]. Now the signal matrix form of the normal equations is attained as

$$\mathbf{S}^H \mathbf{S} \mathbf{w} = \mathbf{S}^H \mathbf{y}. \tag{3.15}$$

The solution of the normal equations, in terms of the signal matrix, is then given by

$$\mathbf{w} = (\mathbf{S}^H \mathbf{S})^{-1} \mathbf{S}^H \mathbf{y}, \tag{3.16}$$

which can be seen as a generalization of the conventional *least-square estimate* [Sorenson, 1980].

Representation (3.15) provides also another set of equations for the unknown weights $\mathbf{w}$, in general an infinite dimensional matrix equation[13]

$$\mathbf{S} \mathbf{w} = \mathbf{y} \qquad (\ \mathbf{S} \mathbf{w} \approx \mathbf{y} \ ). \tag{3.17}$$

If there is an appropriate way to make equation (3.17) finite dimensional, for example by truncating the signals into M samples, where $M \geq N$, in such a way that the truncation $\hat{\mathbf{S}}$ is nonsingular[14], then the solution of the equation $\hat{\mathbf{S}} \mathbf{w} = \hat{\mathbf{y}}$ is attained as

$$\mathbf{w} = \mathbf{S}^+ \hat{\mathbf{y}}, \tag{3.18}$$

---

[12]The signal matrix representation (3.12)is inherently explicit or "deterministic", and although it would not be very instructive to form a "stochastic signal matrix", the normal equation has a natural representation in terms of the expected value of the partial model responses; the nth row of the signal matrix, $\mathbf{s}(n) = \mathbf{x}^T(n)$, where $\mathbf{x}(n)$ is the tap-output vector at time n, can be used as a generic variable in defining $\mathbf{R} = E[\mathbf{s}^H(n)\mathbf{s}(n)]$ and $\mathbf{p} = E[\mathbf{s}^H(n)y(n)]$, respectively.

[13]Especially in this case it should be emphasized that the equality is approximative, that is, generally there is not an exact solution and the equality sign means minimization of the equation error $\|\mathbf{S}\mathbf{w} - \mathbf{y}\|$ in the sense of the Hilbert space norm.

[14]Equivalent ways of expressing the same thing would be that the truncated modeling signals should remain linearly independent or that $\hat{\mathbf{S}}$ is required to have *full rank*, $rank(\hat{\mathbf{S}}) = N$.

where $\mathbf{S}^+$ is the *pseudo-inverse* of $\hat{\mathbf{S}}$. In the case of a finite dimensional signal matrix, the solutions (3.16) and (3.18) are in principle identical, but they point to very different computational procedures for evaluating the least-square solution [Haykin, 1996].

Now if the weight vector $\hat{\mathbf{w}}$ is the somehow attained solution of the normal equations, then the response of the GLM model is $\hat{\mathbf{y}} = \mathbf{S}\hat{\mathbf{w}}$, and the substitution (3.16) provides an operator form for the model response,

$$\hat{\mathbf{y}} = \mathbf{S}(\mathbf{S}^H\mathbf{S})^{-1}\mathbf{S}^H\mathbf{y} \equiv \mathbf{P}\mathbf{y}. \tag{3.19}$$

The infinite dimensional matrix operator $\mathbf{P}$ can be seen as a principled but impractical generalization of the *projection operator*; using vector space notations, $\hat{y} = Py$, where P is a projection from the entire signal space, that is, from $\ell^2(\mathbb{N})$ or $H^2(\mathbb{E})$, onto the finite dimensional approximation space, spanned by the GLM partial model responses[15].

3.2.2   Positive definiteness of the GLM correlation matrix

The signal matrix representation provides also a compact way to verify an important property of the GLM correlation matrix, namely that it is always *positive semidefinite*. The GLM response for an arbitrary non-zero weight vector, $\mathbf{w} \in \mathbb{C}^N$, is given by $y = \mathbf{S}\mathbf{w}$, and a simple manipulation shows that

$$0 \leq (y, y) = \mathbf{y}^H\mathbf{y} = (\mathbf{S}\mathbf{w})^H(\mathbf{S}\mathbf{w}) = \mathbf{w}^H\mathbf{S}^H\mathbf{S}\mathbf{w} = \mathbf{w}^H\mathbf{R}\mathbf{w}, \tag{3.20}$$

or $\mathbf{w}^H\mathbf{R}\mathbf{w} \geq 0$ for short, which is the condition for positive semidefiniteness. The expression $\mathbf{w}^H\mathbf{R}\mathbf{w}$ is the quadratic or Hermitian form of a Hermitian matrix $\mathbf{R}$. Strictly speaking, the GLM correlation matrix is actually *positive definite*, $\mathbf{w}^H\mathbf{R}\mathbf{w} > 0$, for any $\mathbf{w} \neq 0$, since positive definiteness of the correlation matrix is equivalent to linear independency of the modeling signals[16]. As in the case of the conventional correlation matrix, the positive definiteness is a combined property of the input signal and the apparatus that perform the correlation operation: it is possible to construct input signals that enforce the correlation matrix to be singular (and not positive definite). However, as a general and not particularly well-founded comment, such anomalies should not in principle have more severe consequences for the GLM concept than they inherently have for the conventional setting. The following characterization is adopted from an analogous result for the correlation matrix of a stationary discrete-time stochastic process [Haykin, 1996]; the GLM correlation matrix is always positive semidefinite and almost always positive definite. The conjecture then is that possible violations of the positive definiteness are due to the properties

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[15]Or in matrix terms, a projection onto the column space of $\mathbf{S}$. Any (Hermitian) symmetric and *idempotent* matrix, $\mathbf{P}^H = \mathbf{P}$ and $\mathbf{P}^2 = \mathbf{P}$, is an orthogonal projection, which is easily seen to be true for $\mathbf{P}$ in (3.19), but this "definition" does not tell anything about what it actually projects. However, the GLM construction in the Hilbert space framework ensures that $Py = \hat{y} \in V$ and $(1 - P)y = y - \hat{y} \in V^\perp$, where the approximation space $V = span\{\mathbf{S}\}$, its complement $V^\perp$, and the Hilbert space partitioning $H = V \oplus V^\perp$, defines the projection in question.

[16]If $\mathbf{w}^H\mathbf{R}\mathbf{w} = 0$, for some $\mathbf{w} \neq 0$, then $\mathbf{w}^H\mathbf{R}\mathbf{w} = \|\sum_{i=1}^N w_i\mathbf{x}_i\|^2 = 0$, that is, the set of modeling signals would be linearly dependent.

of the input signal, and not as such, caused by the choice of the GLM structure[17]. Adopting once more concepts from system identification, the property that ensures positive definiteness is called *persistency of excitation*, and it is strongly linked to identifiability properties of the model [Söderström and Stoica, 1989] [Regalia, 1995]. Loosely speaking, persistency of excitation means that the input signal is "spectrally rich", that is, potentially capable of exciting all modes or frequency components of the model, and long enough in duration. This can be stated in terms of the spectral density function, $S_{xx}(\omega)$, by requiring that it should be nonnegative (which it already is by definition), and in addition, somehow bounded away from zero[18].

3.2.3 Eigenanalysis of the GLM correlation matrix

As in the case of the conventional correlation matrix, many theoretical, methodological as well as numerical properties of solving the normal equations are closely related to the *eigenvalues* of the GLM correlation matrix. The eigenvalues and the associated eigenvectors are solutions of the matrix equation

$$\mathbf{Rq} = \lambda\mathbf{q} \iff (\mathbf{R} - \lambda\mathbf{I})\mathbf{q} = \mathbf{0}, \tag{3.21}$$

where $\mathbf{I}$ and $\mathbf{0}$ are the unit matrix and the zero vector of appropriate sizes, and where the unknown variables, $\mathbf{q} \in \mathbb{C}^N$ and $\lambda \in \mathbb{C}$, represent the eigenvectors and -values, respectively[19]. The solution of the equation (3.21) with respect to λ is attained as the solution of the *characteristic equation*,

$$det(\mathbf{R} - \lambda\mathbf{I}) = 0, \tag{3.22}$$

where the left hand side is a polynomial in λ with, as is well known, precisely N roots that are not necessarily distinct. The eigenvector $\mathbf{q}_i$ associated to the eigenvalue λ_i is then attained as the solution to (3.21), $\mathbf{Rq}_i = \lambda_i\mathbf{q_i}$. Multiplying both sides of this equation from the left by $\mathbf{q}^H$ and resolving for λ_i yields

$$\lambda_i = \frac{\mathbf{q}_i^H\mathbf{Rq}_i}{\mathbf{q}_i^H\mathbf{q}_i}, \quad i = 1, \ldots, N. \tag{3.23}$$

---

[17]The following reasoning is supposed to be plausible but not particularly exact. The GLM is defined for a generic input, actually for an arbitrary input in the appropriate signal spaces, and therefore the occurrence of $\mathbf{w}^H\mathbf{Rw} = 0$ must be an oddity for some special input signals. In particular, the impulse response Gramian, the correlation matrix for an unit impulse excitation, should necessarily always be positive definite, which does not however imply that the correlation matrix is positive definite for an arbitrary input signal: for example, the impulse response Gramian of the FIR model is in addition orthonormal (the unit matrix), but this does not imply that the correlation matrix is always positive definite. A somewhat analogous relaxation of rigors is implied in the definition of the GLM: it would not be very fruitful to enumerate and exclude some signals from the input space, for example the zero signal, although it is obvious that the modeling concept fails in such situations. The interplay between the input signal and the model structure is clearly more involved in the case of the GLM, impossible to characterize in general, but the principled difficulties are essentially the same for any correlation base modeling scheme.

[18]The usual requirements vary from continuity and strict positivity to milder constraints related to the Lebesgue measure, such as, essentially bounded away from zero or non-zero almost everywhere.

[19]The geometrical interpretation of (3.21) is that the direction of the eigenvector is invariant under the linear transformation $\mathbf{R}$, $\mathbf{Rq} = \lambda\mathbf{q}$.

The quotient (3.23) that interconnects eigenvectors and -values is simply the normalized quadratic form for the choice of vector $\mathbf{q}_i$. An immediate consequence of (3.23) is that the eigenvalues are always real and non-negative. In addition, the smallest eigenvalue is clearly a measure of the positive definiteness. The ratio of the largest and smallest eigenvalue, the *condition number* of the GLM correlation matrix $\chi(\mathbf{R}) = \lambda_{max}/\lambda_{min}$, evaluates the numerical sensitivity of solving the normal equations [Golub and Van Loan, 1989].[20]

The above procedure for the determination of the eigenvalues and corresponding eigenvectors is usually not used in practice as such, because solving the characteristic equation is very sensitive to numerical errors. As an alternative, the general representation of the normalized quadratic form, the *Rayleigh quotient*, may be used to evaluate eigenvalues. It is not, for example, difficult to verify that the minimum and maximum of the Rayleigh quotient with respect to the variable $\mathbf{x} \in \mathbb{C}^N$ provides the smallest and largest eigenvalue,

$$\lambda_{min} = \min_{\mathbf{x}\in\mathbb{C}^N, \mathbf{x}\neq\mathbf{0}} \frac{\mathbf{x}^H\mathbf{R}\mathbf{x}}{\mathbf{x}^H\mathbf{x}} \quad and \quad \lambda_{max} = \max_{\mathbf{x}\in\mathbb{C}^N, \mathbf{x}\neq\mathbf{0}} \frac{\mathbf{x}^H\mathbf{R}\mathbf{x}}{\mathbf{x}^H\mathbf{x}}, \qquad (3.24)$$

respectively[21]. According to the relation (3.23), the minimum and maximum of the Rayleigh quotient is attained at the corresponding eigenvectors. This optimization with respect to the Rayleigh quotient provides yet another interesting and facile generalization of concepts, the *GLM eigenfilters*, as a generalization of the FIR eigenfilters [Makhoul, 1981].

The formulas (3.24) provide also a way to evaluate or approximate the condition number of the GLM correlation matrix that does not require an explicit eigenanalysis. This subject will be re-examined in the case of orthogonal GLM structures, in which case it will be proven that the minimum and maximum eigenvalues have bounds that are independent of the particular model structure, that is, they depend solely on the input signal as in the case of the conventional correlation matrix of a stochastic process [Haykin, 1996].

3.3   Breaking the GLM normal equations

Directly or implicitly, the general theory of matrix factorizations provides the basis for linear system analysis, for the solving of normal equations in particular, as well as the foundation for many signal processing routines. The purpose of this Section is not to enumerate possible choices or to present any of the methods in detail, but once more, to link some of the previous results to a few new concepts that are needed in the forthcoming. Moreover, the attention is oriented towards the

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[20]The condition number is defined here in the sense of the matrix 2-norm, $\chi(\mathbf{R}) \equiv \|\mathbf{R}\|_2\|\mathbf{R}^{-1}\|_2 = \lambda_{max}/\lambda_{min}$.

[21]The intermediate eigenvalues can be evaluated recursively starting from either form of (3.24), corresponding to *maximin* or *minimax* constraints with respect to the subspaces spanned by the related subsequence of eigenvectors. The minimax and maximin theorems for an arbitrary eigenvalue, $\lambda_k, \ k = 1, \ldots, N$, may also be formulated without explicitly resorting to the remaining eigenstructure [Haykin, 1996].

solving of the GLM normal equations, where the correlation matrix is, by definition, square, positive definite and Hermitian. This does not, however, mean that less restricted forms of matrix factorizations are not interesting in the case of the GLM. For example, the non-square equation (3.18), the over-determined matrix equation bases on the signal matrix representation, can be solved using various triangular matrix factorizations [Golub and Van Loan, 1989]. Also in this case, the regularity of the underlying GLM setup imposes constraints on the non-square equation (3.18) that limit in a natural way the variety of relevant methods.

3.3.1 Triangular matrix factorizations

The *Cholesky factorization* is a particular triangular matrix factorization enabled by the Hermitian symmetry. Every GLM correlation matrix $\mathbf{R}$ has a decomposition $\mathbf{R} = \mathbf{C}\mathbf{C}^H$, where the matrix $\mathbf{C}$ is either an upper- or lower-triangular *Cholesky factor*. The term "matrix square-root" is also used for the Cholesky factor[22]. Using the Cholesky factorization for the GLM correlation matrix, the matrix normal equation, $\mathbf{C}\mathbf{C}^H\mathbf{w} = \mathbf{p}$, is solved in two steps, utilizing an intermediate variable $\mathbf{z} \in \mathbb{C}^N$ and by solving two triangular equations[23],

$$\mathbf{C}\mathbf{C}^H\mathbf{w} = \mathbf{C}(\mathbf{C}^H\mathbf{w}) \equiv \mathbf{C}\mathbf{z} = \mathbf{p} \qquad \Rightarrow \qquad \begin{cases} \mathbf{C}\mathbf{z} = \mathbf{p} \\ \mathbf{C}^H\mathbf{w} = \mathbf{z} \end{cases} \qquad (3.25)$$

Algorithms for attaining the Cholesky factor of a Hermitian matrix are given in [Golub and Van Loan, 1989]. However, the Cholesky factorization is not proposed as a preferable candidate for the actual solving of the normal equations, neither in the conventional nor in the GLM configuration, but as an intriguing interconnector between the next topic, the QR decomposition, and the *recursive least-square* (RLS) algorithm, which will be derived for the GLM in the end of this Chapter. Intuitively, the idea is that using the square-root of the correlation matrix, a "square-root of the normal equations" could be used to avoid the correlation analysis and to lower the numerical sensitivity of the solution, less surprisingly, proportionally to the square-root of the original condition number of the correlation matrix[24]. The GLM generalizations of *square-root adaptive filters* will not be speculated further in this thesis, although the author is very interested in the related interplay between *Kalman* and *RLS filtering*, which was presented and discussed by Haykin, Sayed and Kailath in an extraordinary and illuminating interaction between two publications [Sayed and Kailath, 1994b] [Haykin, 1996].

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[22]It is not difficult to show that the Cholesky factorization can alternatively be formulated as $\mathbf{R} = \mathbf{C}\mathbf{D}\mathbf{C}^H$, where $\mathbf{D}$ is a real diagonal matrix, and where the Cholesky factor $\mathbf{C}\sqrt{\mathbf{D}}$ is normalized in the sense that the entries on the main diagonal of $\mathbf{C}$ are all ones.

[23]Triangular equations are solved by substitution/back-substitution, depending on the lower/upper-triangularity of $\mathbf{C}$.

[24]Supposing the correlation matrix is a product of its square-root factors, $\mathbf{R} = \mathbf{R}^{1/2}\mathbf{R}^{H/2}$, then the "square-root normal equation" is given by $\mathbf{R}^{H/2}\mathbf{w} = \mathbf{z}$, where $\mathbf{z} = \mathbf{R}^{-1/2}\mathbf{p}$ is the transformed correlation vector. This results in an efficient algorithm (the *QR-RLS adaptive algorithm*), if the ingredients of $\mathbf{R}^{1/2}$ and $\mathbf{z}$ can be updated directly from the input data [Haykin, 1996].

3.3.2 Orthogonal matrix factorizations

Orthogonality is becoming the second most ambiguous term in this thesis, after linearity, and the following will hardly restrain this development. Various orthogonalizations can be used to transform the original problem of solving a matrix equation, whether it be the normal equation (3.5) or the over-determined equation (3.17), into an equivalent but simpler form. For example, if the Gram-Schmidt orthogonalization process (Section 2.3.2) is applied to the signal matrix, and denoted as a matrix transformation, $\mathbf{GS} = \mathbf{S}_\perp$, then the normal equations with respect to the orthonormal modeling signals, $\mathbf{S}_\perp$, reduce to $\mathbf{S}_\perp^H \mathbf{S}_\perp \mathbf{w} = \mathbf{Iw} = \mathbf{S}_\perp^H \mathbf{y}$, that is, the optimal weights, w_i, $i = 1, \ldots, N$, are simply the Fourier coefficients of the model output y, $w_i = (y, x_i^\perp)$, with respect to the orthonormal modeling signals, $x_i^\perp$, $i = 1, \ldots, N$.

The Gram-Schmidt orthogonalization process is an example of a QR factorization of the signal matrix, $\mathbf{S} = \mathbf{Q}\tilde{\mathbf{R}}$, where $\mathbf{Q} \in \mathbb{C}^{M \times M}$ is an *unitary* matrix and where $\tilde{\mathbf{R}} \in \mathbb{C}^{M \times N}$ is upper triangular[25]. The rectangular matrix $\tilde{\mathbf{R}}$ is upper triangular in the sense that it is a row partition of a square (upper) triangular $\tilde{\mathbf{R}}_N \in \mathbb{C}^{N \times N}$ and a lower zero block $\mathbf{0} \in \mathbb{C}^{(M-N) \times N}$, $\tilde{\mathbf{R}} = [\tilde{\mathbf{R}}_N; \mathbf{0}]$. A matrix is unitary or orthogonal, if $\mathbf{Q}\mathbf{Q}^H = \mathbf{I}$, or equivalently $\mathbf{Q}^H\mathbf{Q} = \mathbf{I}$, for an appropriate unit matrix $\mathbf{I}$. The first N columns of the unitary matrix $\mathbf{Q} = [\mathbf{q}_1 \ \cdots \ \mathbf{q}_M]$, $\mathbf{Q}_N = [\mathbf{q}_1 \ \cdots \ \mathbf{q}_N]$, form an orthonormal basis of the approximation space spanned by the signal matrix $\mathbf{S}$. There are many other and usually preferable methods than the Gram-Schmidt orthogonalization process for performing the QR factorization [Golub and Van Loan, 1989].[26] Elementary unitary operations, such as *Householder transformations* and *Givens rotations*, may be used sequentially to modify a matrix into an upper triangular form. The matrix $\mathbf{Q}$ is then the composition of these operations.

The transformation defined by $\mathbf{Q}$, $\mathbf{Qx}$, $\mathbf{x} \in \mathbb{C}^M$, is unitary in the sense that it preserves the 2-norm, $\|\mathbf{Qx}\|_2^2 = (\mathbf{Qx})^H(\mathbf{Qx}) = \mathbf{x}^H\mathbf{Q}^H\mathbf{Qx} = \mathbf{x}^H\mathbf{x} = \|\mathbf{x}\|_2^2$. In particular, applying the unitary operator $\mathbf{Q}^H$ on the equation $\mathbf{Sw} = \mathbf{Q}\tilde{\mathbf{R}}\mathbf{w} = \mathbf{y}$ transforms it into an upper triangular form,

$$\mathbf{Q}^H\mathbf{Q}\tilde{\mathbf{R}}\mathbf{w} = \mathbf{Q}^H\mathbf{y} \qquad \Rightarrow \qquad \begin{bmatrix} \tilde{\mathbf{R}}_N \\ \mathbf{0} \end{bmatrix} \mathbf{w} = \begin{bmatrix} \mathbf{z} \\ \vdots \end{bmatrix}. \tag{3.26}$$

The solving of the over determined matrix equation (3.17) is thus reduced into a lower-order square and triangular equation $\tilde{\mathbf{R}}_N \mathbf{w} = \mathbf{z}$. Moreover, the solution of the normal equation (3.5) is attained in a numerically less sensitive way than the direct (least-square) estimate (3.16). This is beginning to sound like the "square-root approach", and in fact, in the reduced or *thin* version of the QR factorization, $\mathbf{S} = \mathbf{Q}_N\tilde{\mathbf{R}}_N$, the matrix $\mathbf{C} = \tilde{\mathbf{R}}_N^H$ is precisely the unique lower triangular Cholesky factor of the correlation matrix $\mathbf{R}$, $\mathbf{CC}^H = \mathbf{R} = \mathbf{S}^H\mathbf{S}$ [Golub and Van Loan, 1989].[27]

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[25]Here, the dummy notation $\tilde{\mathbf{R}}$ is used to avoid an unfortunate ambiguity that would result from the more natural choice $\mathbf{R}$.

[26]The Gram-Schmidt orthogonalization process can be seen as a linear transformation $G : \mathbf{S} \mapsto \mathbf{Q}_N$, which produces the QR factorization as a by-product.

[27]In general, the square-root of a matrix is by no means unique, but in the case of a Hermitian positive definite matrix the Cholesky factor is unique for a chosen form of the factorization.

3.3.3 Diagonalizations of the correlation matrix

The unitary operator $\mathbf{Q}^H \in \mathbb{C}^{M \times M}$ (multiplication from the left) transforms the signal matrix into an upper triangular form, $\mathbf{Q}^H \mathbf{S} = \tilde{\mathbf{R}}$; another unitary operator $\mathbf{U} \in \mathbb{C}^{N \times N}$ (multiplication from the right) can be used to diagonalize the signal matrix. These two orthogonalizations define the *singular value decomposition* (SVD) of the signal matrix,

$$\mathbf{Q}^H \mathbf{S} \mathbf{U} = \mathbf{D} = \left[\begin{array}{c} \mathbf{D}_N \\ \mathbf{0} \end{array} \right], \quad \mathbf{D}_N = diag\{\sigma_1, \ldots, \sigma_N\}, \tag{3.27}$$

where it is still presumed that $M \geq N$ and that the signal matrix is of full rank[28]. The diagonal elements are called *singular values* and they are real and non-negative. An equivalent form of the SVD of the signal matrix is attained from (3.27) as $\mathbf{S} = \mathbf{Q} \mathbf{D} \mathbf{U}^H$. The SVD provides an interesting chain of transformations on the equation $\mathbf{S}\mathbf{w} = \mathbf{y}$:

$$\mathbf{S}\mathbf{w} = \mathbf{y} \quad \begin{array}{c} \tilde{\mathbf{w}} = \mathbf{U}^H \mathbf{w} \\ \longleftrightarrow \\ \tilde{\mathbf{y}} = \mathbf{Q}^H \mathbf{y} \end{array} \quad \mathbf{D}\tilde{\mathbf{w}} = \tilde{\mathbf{y}} \quad \begin{array}{c} \mathbf{w} = \mathbf{U}\tilde{\mathbf{w}} \\ \longleftrightarrow \\ \mathbf{y} = \mathbf{Q}\tilde{\mathbf{y}} \end{array} \quad \mathbf{w} = \mathbf{U}\mathbf{D}^+ \mathbf{Q}^H \mathbf{y} \tag{3.28}$$

The notation $\mathbf{D}^+$ is used to denote the Pseudo-inverse of $\mathbf{D}$ [Haykin, 1989], although in this case of a diagonal matrix, the generalized inverse is not particularly mysterious, $\mathbf{D}^+ = [\mathbf{D}_N^{-1} \ \mathbf{0}]$, where $\mathbf{D}_N^{-1} = diag\{1/\sigma_1, \ldots, 1/\sigma_N\}$. Moreover, by comparing the rightmost equation in (3.28) to equations (3.16) and (3.18), it is apparent that the matrix operator $\mathbf{U}\mathbf{D}^+\mathbf{Q}^H$ is the pseudo-inverse $\mathbf{S}^+$ of the signal matrix $\mathbf{S}$, $\mathbf{w} = \mathbf{S}^+\mathbf{y} \leftrightarrow \mathbf{S}\mathbf{w} = \mathbf{y}$. The least-square optimal GLM weight vector is thus attained directly from the SVD, but the formula serves also as a reminder of the inevitable numerical sensitivity of solving the normal equations[29].

As implied, the Gram-Schmidt orthogonalization process and the QR factorization can be used in constructing the SVD, but once more the practical methods and aspects, as described for example in [Golub and Van Loan, 1989], are skipped. Analogously to the QR factorization, also the SVD has a reduced or thin version, $\mathbf{S} = \mathbf{Q}_N \mathbf{D}_N \mathbf{U}^H$. By definition, the vectors $\mathbf{q}_i$ and $\mathbf{u}_i$, picked from $\mathbf{Q} = [\mathbf{q}_1 \ \cdots \ \mathbf{q}_M]$ and $\mathbf{U} = [\mathbf{u}_1 \ \cdots \ \mathbf{u}_N]$, are the *left* and *right singular vectors* of the matrix $\mathbf{S}$, respectively. The columns of $\mathbf{Q}_N$ were recognized earlier as the orthonormal base for the signal space $span(\mathbf{S})$, but what is $\mathbf{U}$ besides a base for the *row space* of S (range

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[28]In a more general situation, the SVD may be used to evaluate the rank of the system $\mathbf{A}\mathbf{x} = \mathbf{b}$; the rank of the matrix $\mathbf{A}$ is precisely the number of non-zero singular values.

[29]The matrix formula for the weight vector, $\mathbf{w} = \mathbf{S}^+\mathbf{y} = \mathbf{U}\mathbf{D}^+\mathbf{Q}^H\mathbf{y}$, can be straightforwardly expanded as

$$\mathbf{w} = \mathbf{U}\mathbf{D}^+\mathbf{Q}^H\mathbf{y} = \mathbf{U}[\mathbf{D}_N^{-1}\mathbf{0}] \left[\begin{array}{c} \mathbf{Q}_N^H \\ \vdots \end{array} \right] \mathbf{y} = \sum_{i=1}^N \mathbf{u}_i \frac{\mathbf{q}_i^H \mathbf{y}}{\sigma_i} = \sum_{i=1}^N \mathbf{u}_i \frac{(y, q_i)}{\sigma_i}. \tag{3.29}$$

This expression shows that relatively small changes in $\mathbf{S}$ or $\mathbf{y}$ (or implicitly in the correlation terms $\mathbf{R}$ and $\mathbf{p}$) may be greatly amplified if the singular values are small [Golub and Van Loan, 1989]. It is also noteworthy that the formula (3.29) states clearly that the weight vector is in $span(\mathbf{U})$ and that its coordinates are the scaled Fourier coefficients $(y, q_i)/\sigma_i$.

of S^H). The expansion of the GLM correlation matrix, $\mathbf{R} = \mathbf{S}^H\mathbf{S}$, in terms of the (thin) SVD of the signal matrix $\mathbf{S}$ provides

$$\mathbf{R} = \mathbf{S}^H\mathbf{S} = (\mathbf{QDU}^H)^H(\mathbf{QDU}^H) = \mathbf{UD}^H\mathbf{Q}^H\mathbf{QDU}^H = \mathbf{UD}_N^2\mathbf{U}^H, \qquad (3.30)$$

which is the *eigenvalue decomposition* (EVD) of the correlation matrix. The eigenvalues are squares of the non-zero singular values and with an appropriate ordering, $\lambda_i = \sigma_i^2$, $i = 1, \ldots, N$. The corresponding eigenvectors are the columns of $\mathbf{U}$, $\mathbf{U} = [\mathbf{u}_1 \; \cdots \; \mathbf{u}_N]$, and they are mutually orthogonal and normal. The transposed form of (3.30),

$$\mathbf{U}^H\mathbf{RU} = \mathbf{D}_N^2 = \mathbf{\Lambda}, \qquad (3.31)$$

is the *unitary similarity transformation* of the GLM correlation matrix $\mathbf{R}$. With respect to equations (3.21), the eigenvalues are also referred to as the spectrum of the matrix[30]. The inverse of the GLM correlation matrix is obtained from the EVD as

$$\mathbf{R}^{-1} = \mathbf{U}\mathbf{\Lambda}^{-1}\mathbf{U}^H, \qquad (3.32)$$

where $\mathbf{\Lambda}^{-1} = \mathbf{D}_N^{-2} = diag(\lambda_1^{-1}, \ldots, \lambda_N^{-1})$ for some ordering of the eigenvalues. Inserting (3.32) into the solution of the normal equation (3.11) yields

$$\mathbf{w} = \mathbf{R}^{-1}\mathbf{p} = \sum_{i=1}^{N} \frac{\mathbf{u}_i^H\mathbf{p}}{\lambda_i}\mathbf{u}_i, \quad \mathbf{p} = (y, \mathbf{x})^T = (\mathbf{x}, y)^*. \qquad (3.33)$$

This is probably not in general a practical way to solve the optimal GLM weights, since it is so far computationally the most complex method. Also compared to the SVD form (3.29), the EVD based expansion is more sensitive to the eigenvalue spread than its "square-root" counterparts. However, as in the original formulation of the GLM, the possible (probable) infiniteness considerations are handed back to the evaluation of the correlation terms, which is in general better justified than working with truncations of infinite dimensional matrices, as in the case of all previous factorizations of the signal matrix. Moreover, it is interesting to speculate on a special form of the GLM identification in light of (3.33); if the modeling signals are known and fixed, then the GLM approximation of a desired response y is attained as a function of the correlation term $\mathbf{p} = (\mathbf{x}, y)^*$ and a predefined eigenvalue decomposition. That is, the GLM construction is used to generate an orthonormal base for the parameter space, $\mathbf{w} \in span(\mathbf{U})$. Equation (3.33) is a generalization of a well-known interconnection between the concept of optimal FIR Wiener filtering and the EVD of the autocorrelation matrix [Haykin, 1989].

3.3.4 Orthogonal transformations induced by the GLM

An arbitrary orthonormal set of vectors, $\{\mathbf{u}_1, \ldots, \mathbf{u}_N\}$, $\mathbf{u}_i \in \mathbb{C}^N$, is a base for the (Hilbert) space $\mathbb{C}^N$. Any such base defines an orthogonal transformation, $T : \mathbb{C}^N \rightarrow$

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[30]In Section 3.2.3 the eigenvalues and -vectors were deduced starting from the "eigenequation" (3.21) and arriving at (3.23), which is precisely a partitioned form of (3.31), if the eigenvectors are presumed to be normalized. The normalization is always possible, since a scaled eigenvector is still an eigenvector, showing that eigenvectors are not unique. However, the eigenvalues are unique, but they are not necessarily distinct. A direct derivation of (3.31), on the basis of the eigenequation (3.21), would require the eigenvalues to be distinct [Haykin, 1996].

$\mathbb{C}^N$, $T : \mathbf{x} \mapsto \mathbf{y} = \mathbf{U}^H \mathbf{x}$, and consequently an inverse transform, $T^{-1} : \mathbf{y} \mapsto \mathbf{x} = \mathbf{U}\mathbf{y}$, where the transformations are defined by the unitary matrix $\mathbf{U} = [\mathbf{u}_1 \cdots \mathbf{u}_N]$ for a chosen order of the basis vectors[31]. In particular, the EVD of the GLM correlation matrix defines an orthonormal signal transform for a signal vector $\mathbf{x}(n)$,

$$\mathbf{y}(n) = \mathbf{U}^H \mathbf{x}(n) = \begin{bmatrix} \mathbf{u}_1^H \mathbf{x}(n) \\ \vdots \\ \mathbf{u}_N^H \mathbf{x}(n) \end{bmatrix} \quad \longleftrightarrow \quad \mathbf{x}(n) = \mathbf{U}\mathbf{y}(n) = \sum_{i=1}^{N} y_i(n)\mathbf{u}_i . \quad (3.34)$$

The transform pair (3.34) is simply the Fourier transform with respect to the orthonormal set of basis functions, $\mathbf{U} = [\mathbf{u}_1 \cdots \mathbf{u}_N]$, but it is also a self-contained consequence of the eigenanalysis of the GLM correlation matrix. The left and right hand side of (3.34) are called the analysis and synthesis part, respectively.

Among all unitarily equivalent orthogonal transformations of the form (3.34) there is a special (although not unique) transformation that diagonalizes the correlation matrix of the transformed signal[32],

$$\mathbf{R}_y = (\mathbf{y}, \mathbf{y}) = (\mathbf{U}^H \mathbf{x}, \mathbf{U}^H \mathbf{x}) = \mathbf{U}^H (\mathbf{x}, \mathbf{x})\mathbf{U} = \mathbf{U}^H \mathbf{R}_x \mathbf{U} = \begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_N \end{bmatrix} . \quad (3.35)$$

This is in fact the *unitary similarity transformation* (3.31) of the correlation matrix $\mathbf{R}_x$ and the diagonal elements are thus precisely the corresponding eigenvalues. Such a transformation is called the *discrete Karhunen-Loève transformation* (KLT) and it is the optimal signal transformation in the sense of *energy compaction* and *coding gain* [Vaidyanathan 1993].[33] Conversely, the KLT is sometimes defined as the signal transformation that produces optimal compression or low-rank approximation for a fixed $R \leq N$,

$$\hat{\mathbf{x}}(n) = \sum_{i=1}^{R} c_i(n)\mathbf{u}_i , \qquad \min_{c_i, u_i: i=1,\ldots,R} \|\mathbf{x}(n) - \hat{\mathbf{x}}(n)\|^2, \qquad (3.36)$$

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[31]Square unitary matrices of a fixed dimension $N > 0$, $\mathcal{U}(N)$, form an algebraic group: $\mathcal{U}(N)$ is closed with respect to matrix multiplication, it contains the unit-element $\mathbf{I}$, and every element has an inverse.

[32]For simplicity of notation, the correlation is temporarily defined in its conjugated or outer-product form (see footnote on page 56), $\mathbf{R}_y = (\mathbf{y}, \mathbf{y})$, which is motivated by the observation that "the model", $\mathbf{y} = \mathbf{U}^H \mathbf{x}$, is also in the conjugated form. The calculation rule $(\mathbf{U}^H \mathbf{x}, \mathbf{U}^H \mathbf{x}) = \mathbf{U}^H (\mathbf{x}, \mathbf{x})\mathbf{U}$ can be thought to generalize the scalar inner product axioms, $\lambda(a, b) = (\lambda a, b)$ and $(a, \lambda b) = (a, b)\lambda^*$.

[33]It should however be mentioned that KLT is not the only transformation $\mathbf{y} = \mathbf{T}\mathbf{x}$ with the property that $\mathbf{T}^H \mathbf{R}_x \mathbf{T}$ is diagonal. Phoong and Lin have proposed an interesting non-unitary *Prediction-based Lower triangular Transform* (PLT) which possesses many of the optimality characteristics associated usually solely to the KLT [Phoong and Lin, 2000]. In addition to the decorrelation property, also the optimality in the coding gain is proven for some proposed implementations of the PLT. The name of the transform implies that it is related to the *Linear Prediction Coding* (LPC). In fact, the transform matrix itself is uniquely determined by the coefficient of the optimal *prediction error filters* for successive filter orders, $i = 0, \ldots, N$, with respect to the signal $\mathbf{x}(n)$, and the lower complexity compared to the KLT is due to the efficiency in LPC calculations. It is also interesting to note that as a triangular factorization, the PLT is closely related to the Cholesky factorization of Section (3.3.1).

and the relation to the EVD is then recognized as a sufficient and necessary condition for the optimality; the eigenvectors with respect to the ordered eigenvalues, $\lambda_1 \geq \ldots \geq \lambda_R \geq \ldots \geq \lambda_N$, provide the KLT of rank R. However, as pointed out also in [Ogawa, 1992], the KLT should be defined with respect to the subspace into which it projects and not as an expansion of fixed eigenvectors, since the eigenvectors are not even unique.

Once more, in trying to be general, the above discussion has been blurry about the nature and origin of the signal $\mathbf{x}(n)$, and consequently about the specific meaning of the correlation matrix as well as the measure of optimality. The conventional setup for the KLT is with respect to the transformation of a WSS random vector, for example a sample vector, $\mathbf{x}(n) = [x(n) \;\cdots\; x(n - N + 1)]^T$, drawn from a WSS process $x(n)$ [Haykin, 1996] [Mitra, 2001]. The KLT can then also be seen as a de-correlation of the random variable defined by orthogonal projections onto the basis vectors; for a WSS (zero-mean and finite variance) random variable, the expansion coefficients in (3.34) are zero-mean and uncorrelated (random variables), $E[y_i(n)] = 0$, $E[y_i(n)y_j^*(n)] = 0$, if $i \neq j$, and $E[|y_i(n)|^2] = \lambda_i$. The partial model outputs, $\mathbf{x}(n) = [x_1(n) \;\cdots\; x_N(n)]^T$, of the GLM to a WSS input signal $x(n)$ is clearly a WSS random vector, and thus the KLT or de-correlation of $\mathbf{x}(n)$ in terms of the GLM correlation matrix $\mathbf{R}^* = E[\mathbf{x}(n)\mathbf{x}^T(n)]$ is a straightforward generalization of the conventional KLT. In this form the GLM structure acts as a pre-processing for the KLT; a simple example of such an operation is the blocking or vectorization of the input, $x(n) \rightarrow \mathbf{x}(n) = [x(n) \;\cdots\; x(n - N + 1)]^T$, using an unit delay structure.

More genuine generalizations of the KLT are attained by employing the original setting of transforming an input vector $\mathbf{x}(n)$ using generalized correlations defined by the GLM. The KLT is completely signal dependent, a transformation assigned to a single signal, although this determination is slightly awkward in relation to a random signal. Generalizations of the KLT are thus usually related to an enlargement of the input signal class, or more imaginably, with respect to a set of patterns that characterizes the signal class [Ogawa, 1992].[34] Maybe somewhat surprisingly, also a restriction to a *signal subspace* of $\mathbb{C}^N$, deduced by the rank reduced KLT (3.36), may results in a more general classification of input signals. For example in enhancement of noisy speech signals, the KLT has been used to decompose the signal into *signal* and *noise subspace components* [Ephraim and Van Trees, 1995], which is based on the observation that speech signals, as a class, are inherently rank deficient (redundant). The potential benefits of the GLM-KLT concept are also related to subspace techniques; the generalized KLT generated with respect to GLM correlation analysis of the combined system of the input and the model may be used to efficiently compress the representation of the signal. To be more precise, with a proper choice of the GLM structure with respect to a somehow specified class of signals, the approximate representation of a signal can be made concise, in the sense

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[34]The term "Generalized Karhunen-Loève", transformation or expansion, has been used in various meanings. The generalization in [Nakagawa and Miyahara, 1987] is based on a weighted error criterion. Another type of generalizations can be categorized as (Wiener) filtered KLT – the KLT is constructed with respect to a reference signal (other than $\mathbf{x}$) [Hua and Liu, 1998] [Yamashita and Ogawa, 1996] or with respect to a multiple-signal representation [Goldstein et al., 1999]. As filtering setups, these latter generalizations would have further generalizations in terms of the GLM, but this reasoning is clearly getting off the track.

of $R \ll N$ in (3.36), where the approximation $\hat{\mathbf{x}}(n)$ is the truncated GLM-KLT, that is, the transformation generated by the EVD of the GLM correlation matrix with respect to the R largest eigenvalues[35].

Some remarks There are some existing proposals that can be seen as genuine GLM-KLT constructions. An allpass pre-filtering technique is proposed in [Kuo et al., 1996] for unwrapping linear dependencies in the input signal to improve the performance of the subsequent KLT. The method was entitled *transform encrypted coding* (TEC) to emphasize that as another benefit of the scheme, better security of the coded signal is achieved. The TEC is obviously a special case of the filtered generalizations of the KLT [Yamashita and Ogawa, 1996] [Hua and Liu, 1998] [Goldstein et al., 1999]. Another observation is related to the *transform-domain adaptive filters* (TDAF) [Marshall et al., 1989] [Haykin, 1996]. The KLT can be seen as an adaptive de-correlation that is used to pre-process the input before the actual adaption algorithm. This *self-orthogonalization* improves considerably the performance of the adaptive filter [Gitlin and Magee, 1977] [Lee and Un, 1986].[36] In practice, the KLT is replaced by an approximative self-orthogonalization provided, for example, by a fixed transformation and an adaptive estimate of the eigenvalues [Haykin, 1996]. Alternatively, an adaptive filter may be used to perform the de-correlation. A *gradient-adaptive Laguerre-lattice* (GALL) algorithm [Fejzo and Lev-Ari, 1997] has been proposed as a pre-orthogonalization for an adaptive nonlinear Wiener model [Fejzo and Lev-Ari, 1995]. Various formulations of *recursive-least-squares* (RLS) adaptive Laguerre lattice filters have been deduced [Merched and Sayed, 2001] as generalizations of their conventional RLS lattice counterparts [Haykin, 1996]. These constructions based on the Laguerre filter are genuine examples of non-trivial GLM de-correlators.

3.3.5   A note on fixed orthogonal transformations

The KLT is usually introduced as a mere theoretical reference for more practical orthogonal transforms. Especially in transform coding, it is not just the computational complexity of the KLT but also the coding and transfer of the signal dependent basis that makes the KLT scheme impractical. The DFT based *discrete cosine* and *sine transforms*, DCT and DST, respectively, are examples of widely used fixed and signal independent transforms[37]. The motivation for using suboptimal transforms is both practical in the sense of computational complexity as well as more principled

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[35]Using previous relations between the EVD and the SVD, the GLM-KLT can also be deduced from the SVD of the signal matrix. Nevertheless, here too it is better-justified to integrate potential infiniteness considerations into the evaluation of the correlation terms than to work with in principle infinite dimensional signal matrices. However in practice, and in this case of coding a finite signal block, the dimensions in both interpretations are practically finite: the modeling signals are by definition decaying (for a finite-duration input extended by an infinite zero sequence) so that any given positive threshold for the damping (in some sense) of all the signals is met within a finite data window.

[36]More generally, virtually any kind of subsequent processing benefits from the de-correlation and signal scaling provided by the self-orthogonalization.

[37]Actually, there are four basic forms of both the DCT and the DST [Wang, 1984] [Yip and Rao, 1987]. The third type of sinusoidal transform is the *discrete Hartley transform* (DHT) and there has been propositions for an unified (and at the same time generalized) approach

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with respect to the potentially unknown or time-variant input statistics. Sometimes the choice of a particular fixed transformation can be argued by considering it as an approximation of the KLT for certain input signals. For example in the case of a real autoregressive (AR) process it has been proven that the DCT is a good approximation of the KLT [Huang and Zhao, 2000]. Another common way of saying about the same is that the DCT and the KLT are asymptotically equivalent in the sense of the data extending to infinity and at least for "strongly correlated" or "almost Gaussian" input signals [Nakagawa and Miyahara, 1987].[38] This should not actually be too surprising since in the Hilbert space setup, all orthonormal bases are "asymptotically unitarily equivalent".

The GLM concept can be used to generate orthogonal transforms in several ways. As a signal plus model dependent transformation, the GLM-KLT construction generalizes the signal dependency with respect to an input related set of filtered partial model responses. Alternatively, this filtering can be seen as a means to specialize the transformation for some desired purpose, for example, to reflect an unequal frequency resolution in the transformation, or to anticipate particular types of energy envelopes, such as impulse-like, in the transformed signal. According to the GLM framework, the model input may also be seen as an excitation that is not related to the signal to be modeled or transformed. The GLM-KLT is thus in principle a genuine generalization of both the various fixed orthogonal transformations and the proposed multiple signal (pattern) representation based KL transforms [Goldstein et al., 1999][Cappelli et al., 2001].

Another form of GLM-generated orthogonal transformation is attained directly in terms of the orthogonalized signal matrix, $\mathbf{S}_\perp$, from the introductory of this Section. The GLM weights act now as the "coded signal" and an analogous transform to (3.34) is given by

$$
\mathbf{w}(n) = \mathbf{S}_\perp^H \mathbf{y}(n) = \begin{bmatrix} \mathbf{s}_{1\perp}^H \mathbf{y}(n) \\ \vdots \\ \mathbf{s}_{N\perp}^H \mathbf{y}(n) \end{bmatrix} \quad \longleftrightarrow \quad \mathbf{y}(n) = \mathbf{S}_\perp \mathbf{w}(n) = \sum_{i=1}^{N} w_i(n) \mathbf{s}_{i\perp} \ . \quad (3.37)
$$

The argument n is left to indicate a potential connection to block-by-block transformations and a hat should be attached to $\mathbf{y}(n)$ on the right hand side of (3.37) since the really interesting cases are those when the transformation is approximative. Kind of an implicit assumption is that (3.37) should be a low-rank approximation, the dimensionality of $\mathbf{w}$ should be less than that of $\mathbf{y}$, which does not yet necessarily mean approximation in the reconstruction. In practice, and especially in the case of GLM-generated basis functions, the transform matrix is possibly a truncated version of the truly orthogonal signal matrix, an approximately orthogonal matrix. The corresponding transformation is thus non-unitary, which does not have to be more dramatic than the non-unitarity produced by rank-reduction in the case of a strictly

---

for all these transforms [Wang et al., 1992] [Deng et al., 1994].

[38]The DCT is usually preferred as the candidate for asymptotic equivalence, but this choice is mainly pragmatic. For real signals the KLT is real in contrast to the DFT that is always complex – the DCT is preferred over the DST because it is considered to provides more natural basis functions for most practical cases (caused actually by predetermined choices in indexing, initial phases and block sizes). However in [Jain, 1976], an exact fast KLT was derived in terms of sine functions (although not the DST) for a wider class of signals (first-order Markov processes).

unitary transformation. The transformation (3.37) represents simply "modeling by synthesis" with respect to a chosen set of basis function $\{s_1, \ldots, s_N\}$, or directly $\{s_{1\perp}, \ldots, s_{N\perp}\}$, which is the main approach and theme of the rest of this thesis.

Some remarks Alternatively to the matrix-vector multiplication implementation, the DFT transformation can be performed using a *maximally decimated filter bank* [Vaidyanathan 1993]. The coefficients of the analysis and synthesis (FIR) filters are attained directly from the DFT matrix, and effectively, the combination of filtering and down-sampling is used simply to evaluate inner products, as in (3.34) or (3.37). This identity provides also filter bank implementations for other fixed orthogonal transformations and a genuine interpretation of such transforms in terms of the GLM. However, these constructions rely on the symmetry properties of the implied subband filter coefficients, which are usually lost in a more general case of basis functions. Moreover, in utilizing filtering (convolution) for computing inner products, either the input or the analysis filters must be flipped or time-reversed, and the latter choice is commonly embedded in some form of symmetry of the basis functions. The direct inversion of the filter coefficients is possible only for FIR filters, and thus in general, it is the input that has to be time-reversed[39]. The concept of *lapped* or non-square transforms is also useful in producing more flexible transforms [Malvar, 1990]. These questions are considered in the special case of GLM-oriented transforms provided by various formulations of warped transforms and related warped filter banks [Oppenheim et al., 1971] [Laine, 1992] [Philips, 1994] [Evangelista and Cavalieri, 1998]. The warped discrete Fourier transform (WDFT) was surprisingly rediscovered recently [Makur and Mitra, 2001] and the authors should be acknowledged, if not for the originality of the concept, for the deduction based on the nonuniform DFT [Bagchi and Mitra, 1996] that provide at least in principle a way to generalize the WDFT to include warping with higher order allpass mappings. As a final remark related to fixed transforms, the nature of the GLM suggests that the potentially relevant applications of GLM-related fixed transforms should probably fall into the category of "pulse coding", such as, the coding of electrocardiographic (ECG) signals [Olmos et al., 1999], the compressed storage of seismic data [Spanias et al., 1990], or the modeling of speech waveforms [Ephraim and Van Trees, 1995].

3.4 Two generic GLM structures corresponding to particular types of correlation matrices

The definition of the GLM is based on the assumption of linear independency of the modeling signals, and thus, it can be justly argued that the whole construction is somewhat artificial and hollow as long as there are no guidelines for constructing structures with guaranteed linear independency. However, the definition of linear

---

[39]It is always possible to approximate an IIR subband filter by an FIR filter, and thus to construct an approximative square transform matrix. Direct truncation of the impulse responses is a simple choice, but an alternative re-sampling in the frequency-domain gives probably better results: the transfer function of the subband filters are evaluated at $e^{j2\pi k/N}$, $k = 0, \ldots, N-1$, from which an inverse N-point DFT provides the FIR filter coefficients.

independency is itself arrogantly formulated and not very constructive, a definition based on an implicative relation that is practical mainly in proving linear dependency. This is why the linear independency was related earlier directly to consequential properties such as nonsingularity and positive definiteness of the correlation matrix. Here too, the justifying argument for the general framework is that the contribution of the input signal and the model to the linear independency are essentially separable; the former was previously characterized as the persistency of excitation of the input signal, and in essence, the latter reduces to the requirement of positive definiteness of the model impulse response Gramian, the correlation matrix for unit impulse excitation.

It is possible to formulate a list of principles for constructing GLM structures as parentheses for the ground rule "almost anything will do". For example any irreducible collection of causal LTI partial models is valid, meaning that the partial model transfer functions are mutually irreducible when linearly combined. A much weaker requirement is that none of the partial models should collapse, that is, although some of the partial models may have reducible terms, no pairing of the partial responses should result in a constant scaling relation between the transfer functions. Particularly, for rational partial models it is relatively easy to construct principles in terms of the explicit parametrizations of the partial models and the related model orders. In the following, however, two important conceptual forms of the GLM configuration are induced as consequences of increased regularity or structure of the correlation matrix. In addition, the linear independency presumption is more inherently fulfilled in these constructions, in the sense that the contribution of the model structure is quite explicitly characterized, although the actual building blocks of the models are still relatively generic..

3.4.1 Toeplitz and block-Toeplitz forms

The Hermitian symmetry of the correlation matrix has been the ongoing driving force in deducing GLM counterparts for concepts that are conventional associated only to the FIR model[40]. There is however another orientation of symmetry that can be used to facilitate the solving of linear matrix equations. A square matrix $\mathbf{R}$ is *persymmetric* if it is symmetric with respect to its northeast-southwest diagonal; the persymmetry is characterized by the relation $\mathbf{R} = \mathbf{J}\mathbf{R}^T\mathbf{J}$, where $\mathbf{J}$ is the permutation or counter-identity matrix of appropriate dimensions. More importantly, the inverse of a persymmetric matrix, if it exists, is persymmetric. A *Toeplitz* matrix is a particular form of a persymmetric matrix, where the main diagonal and all its subdiagonals are symmetric in the sense that each of them consists of identical elements. A Hermitian symmetric Toeplitz matrix is thus defined uniquely by $N + 1$

---

[40]In fact, the Hermitian symmetry has implicitly been used in a stronger sense than the sole technical property posed on a matrix: a GLM correlation matrix is also weighty about its main diagonal, measured for example as the distribution of square norms of the elements, which is an important practical observation with respect to the solving of the normal equations, its positive definiteness, and the usability of various matrix factorizations. The relative concentration of energy on the main diagonal, in addition to the Hermitian symmetry, is in fact the property that justifies the use of the term "correlation".

elements,

$$\mathbf{R}_{N+1} = \begin{bmatrix} r_0 & r_1 & \cdots & r_{N-1} & r_N \\ r_1^* & r_0 & & & r_{N-1} \\ \vdots & & \ddots & & \vdots \\ r_{N-1}^* & & & r_0 & r_1 \\ r_N^* & r_{N-1}^* & \cdots & r_1^* & r_0 \end{bmatrix}, \tag{3.38}$$

where the modification in dimension and indexing will be utilized shortly. The elements of (3.38) may in turn be square matrices of a common dimension, in which case the matrix is called block-Toeplitz. It is fairly intuitive that a correlation matrix of the form (3.38) is related to some sort of shift-invariancy of the data that is collected and correlated. In the case of the input signal itself and with respect to an "auto-correlation" of the sequence using the delay operator, the (block-)Toeplitz property is a characterization, or conversely a definition, of the stationarity of the stochastic (vector) signal. This relation between statistical concepts and time-series analysis, glued by the exactitude of the function (Hilbert) space framework, is the very basis of linear filtering and prediction theory[41]. The significance of the (block-)Toeplitz form of the correlation matrix is that it suggest a nested or iterative solution; in terms of the GLM normal equations (3.5), if the solution of the equation $\mathbf{R}_N \mathbf{w}_N = \mathbf{p}_N$ is available, then the order update normal equation $\mathbf{R}_{N+1} \mathbf{w}_{N+1} = \mathbf{p}_{N+1}$ is of the form

$$\begin{bmatrix} \mathbf{R}_N & \mathbf{J}\mathbf{r}_N \\ \mathbf{r}_N^H \mathbf{J} & r_0 \end{bmatrix} \mathbf{w}_{N+1} = \begin{bmatrix} \mathbf{p}_N \\ p_{N+1} \end{bmatrix}, \tag{3.39}$$

where $\mathbf{r}_N = [r_1 \ \cdots \ r_N]^T$ is identified from (3.38) and where p_{N+1} is the new cross-correlation term. It is not difficult to show that the solution of (3.39) is given by[42]

$$\mathbf{w}_{N+1} = \begin{bmatrix} \mathbf{w}_N - \alpha_N \mathbf{J}\mathbf{R}_N^{-1}\mathbf{r}_N \\ \alpha_N \end{bmatrix}, \qquad \alpha_N = \frac{p_{N+1} - \mathbf{r}_N^H \mathbf{J}\mathbf{w}_N}{r_0 - \mathbf{r}_N^H \mathbf{R}_N^{-1}\mathbf{r}_N}, \tag{3.41}$$

where it is essential that the inverse of a persymmetric matrix is also persymmetric [Golub and Van Loan, 1989]. The term $\mathbf{R}_N^{-1}\mathbf{r}_N = \mathbf{w}_N'$ in (3.41) is the solution of the corresponding *Yule-Walker equation*, $\mathbf{R}_N \mathbf{w}_N' = \mathbf{r}_N$, and it is also presumed

---

[41]Kailath provides an interesting historical survey [Kailath, 1974] as well as a collection of "benchmark papers" [Kailath, 1977] on the subject. Particular contributions are associated to Kolmogorow, Wold, Krein [Kailath, 1977] and Wiener [Wiener, 1949]. It is amazing how Kolmogorow states household concepts of digital signal processing, such as *linear (forward and backward) prediction*, *moving-average process*, and *linear convolution*. This is not the only instance when the original formulation is inherently in discrete-time and more general in contrast to the perspective offered by most DSP textbooks. This is at least partly due to the dominance of the so-called *Wiener filtering theory* [Wiener, 1949] and its importance in the first actual signal processing applications using analog network synthesis [Lee, 1960]. It is however paradoxal that Wiener's effort in emphasizing the application and engineering side of his work has resulted in an one-sidedness that does not originate from Wiener. This is however a fortunate historical detorsion for such as the author of this thesis; there is still room for digging between the original ideas and current interpretations and applications, a possible call for yet another sequel in the category "linear filtering - once more with feeling".

[42]Using the partition $\mathbf{w}_{N+1} = [\mathbf{z}_N \ \alpha_N]^T$, the solution is attained by solving the augmented form of (3.39),

$$\begin{cases} \mathbf{R}_N \mathbf{z}_N + \alpha_N \mathbf{J}\mathbf{r}_N &= \mathbf{p}_N \\ \mathbf{r}_N^H \mathbf{J}\mathbf{z}_N + r_0 \alpha_N &= p_{N+1} \end{cases} \tag{3.40}$$

with respect to $\mathbf{z}_N$ and α_N, and utilizing $\mathbf{R}_N^{-1}\mathbf{p}_N = \mathbf{w}_N$ and $\mathbf{R}_N^{-1}\mathbf{J} = \mathbf{J}\mathbf{R}_N^{-1}$.

to be known. Thus the suggested iterative method consists actually of solving two matrix equations in parallel: the weight vector $\mathbf{w}_N$ is first upgraded using $\mathbf{R}_N^{-1}\mathbf{r}_N = \mathbf{w}_N'$ in (3.41) and then the same recursion is utilized directly for the auxiliary parameter $\mathbf{w}_N'$ using r_{N+1} in place of p_{N+1}. The real and scalar version of this procedure is known as the *Levinson algorithm* [Levinson, 1947].[43] In matrix terms it is the solution of a Toeplitz system with a general right hand side (in the normal equation), in contrast to the iterative solution of the Yule-Walker system [Golub and Van Loan, 1989]. The algorithm was much later reinvented in the constrained latter form [Durbin, 1960], which has caused odd logic in terminology. A retrospective renaming has also been used due to a close relationship between the Levinson algorithm and the recurrence relations of the *Szegö orthogonal polynomials* [Szegö, 1939]. These interesting interconnections and some generalizations of the Levinson algorithm have been considered from various aspects, such as, lossless wave scattering [Dewilde et al., 1978], lossless transfer functions [Delsarte et al., 1982] and Hilbert space projections related to the shift operator [Lev-Ari et al., 1984].

The Levinson algorithm is more than just an efficient way to solve Toeplitz systems. In its conventional form for *linear prediction* it provides an one-to-one correspondence between three sets of parameters for characterizing a WSS signal $x(n)$, the (auto)correlation terms $\{r_i\}$, prediction error filter coefficients $\{w_i\}$, and the reflection coefficients $\{k_i\}$ of a corresponding *lattice filter* [Haykin, 1989]. The interest here is not necessarily in the linear prediction setting, but rather in the orthogonalization process implied by the conversion to the lattice form. As an introduction, if the Gram-Schmidt orthogonalization process (Section 2.3.2) is applied to the GLM tap-output vector $\mathbf{x}(n) = [x_1(n) \; \cdots \; x_N(n)]^T$, then the elements of the vector $\mathbf{b}(n) = [b_1(n) \; \cdots \; b_N(n)]^T$, given by

$$
\begin{aligned}
b_i &= x_i - \sum_{j=1}^{i-1}(x_i, b_j)b_j \qquad ; \; b_1 = x_1 \\
&= x_i - \sum_{j=2}^{i-1}(x_i, b_j)b_j - (x_2, x_1)x_1 \quad ; \; b_2 = x_2 - (x_2, x_1)x_1 \qquad (3.42)
\end{aligned}
$$

are mutually orthogonal[44]. The second line in (3.42) is included to demonstrate that the orthogonalization process may also be expressed as a matrix operation, $\mathbf{b} = \mathbf{T}\mathbf{x}$, where the elements of the lower triangular matrix $\mathbf{T}$ are attained by grouping common factors[45]. The signals are here considered as generic entries in $\mathbf{x}$ and $\mathbf{b}$. If instead the explicit (or deterministic) signal vector representation is used, a rearrangement of (3.42) provides the QR decomposition of the signal matrix $\mathbf{S} = [\mathbf{x}_1 \; \cdots \; \mathbf{x}_N]$ (Section 3.3.1).[46] Formula (3.42) has an immediate interpretation

---

[43]Levinson's article was republished as an appendix in Wiener's *Extrapolation, Interpolation, and Smoothing of Stationary Time-Series, with Engineering Applications* [Wiener, 1949], a "trivial procedure ... to facilitate computation", as introduced by the author.

[44]The time-variable and the normalization of the elements $\{b_i\}$ is omitted.

[45]The b's are eliminated from the right hand side by recurrent substitution and linearity of the inner product is used to expand terms – the formula will get really messy after couple of substitutions.

[46]Using column vector denotations, $\mathbf{x}_i$ and $\mathbf{b}_i$, and normalization for the latter terms, $\mathbf{q}_i =$

in light of the Hilbert space Projection Theorem (Section 2.3.2): the entity b_i is the approximation or *prediction* error resulting from approximating x_i in the subspace $V = span\{x_1, \ldots, x_{i-i}\}$ and it is by definition orthogonal to V. Moreover, an orthogonal base $\{b_1, \ldots, b_{i-i}\}$ of V is used and the (LS or MMS) optimal weights are given by the Fourier coefficients (x_i, b_j), $j = 1, \ldots, i-1$. The purpose of this section is to show that the straightforward but increasingly complicated procedure (3.42) for producing the orthogonal signals $\{b_i\}$ from the modeling signals $\{x_i\}$ simplifies greatly in the case of a (block-)Toeplitz correlation matrix by introducing a corresponding *GLM lattice structure*[47].

The GLM structure is by construction parallel, as it was depicted in Figure 3.3. It is not however difficult to deduce that a GLM structure with a Toeplitz correlation matrix has to be equivalent to a tapped transversal structure where the cascaded blocks between the tap-outputs are identical inner, lossless or allpass functions, taken once more as interchangeable terms [Paatero, 2000].[48] Such a configuration is represented in Figure 3.5, where the indexing of the tap-outputs is modified to start from zero. The elements of the correlation matrix may now be expressed using the allpass operator with respect to x_0,

$$r_{ij}^* = (x_i, x_j) = \begin{cases} (A^i x_0, A^j x_0) \\ (A^j x_0, A^i x_0)^* \end{cases} = \begin{cases} (A^{i-j} x_0, x_0) \equiv r_k, & k = i - j \geq 0 \\ (A^{j-i} x_0, x_0)^* = r_{-k}^*, & k = i - j < 0 \end{cases} , \quad (3.43)$$

where conjugate symmetry is used to simplify denotations and to attain one-to-one correspondence to the notation of (3.38).

The shift-invariancy is also reflected to the correlation vector in a certain way. The $N+1$ dimensional GLM tap-output vector $\mathbf{x}_{N+1}$ submit to natural partitions using vectors $\mathbf{x}_0$ and $\mathbf{x}$, respectively, as defined and related by

$$\mathbf{x}_{N+1}(n) = \begin{bmatrix} x_0(n) \\ \vdots \\ x_N(n) \end{bmatrix} = \begin{bmatrix} \mathbf{x}_0(n) \\ x_N(n) \end{bmatrix} = \begin{bmatrix} x_0(n) \\ \mathbf{x}(n) \end{bmatrix} = \begin{bmatrix} x_0(n) \\ A\mathbf{x}_0(n) \end{bmatrix}. \quad (3.44)$$

Now the tap-output vectors $\mathbf{x}_0$ and $\mathbf{x}$ may be used to approximate x_N and x_0, respectively, resulting in a generalization of the conventional forward and backward

---

$\mathbf{b}_i/(b_i, b_i)$, the formula (3.42) transposes into $\mathbf{x}_i = \mathbf{q}_i r_{ii} + \sum_{j=1}^{i-1} r_{ji} \mathbf{q}_j$, or in composite form $\mathbf{S} = \mathbf{QR}$, where $\mathbf{Q} = [\mathbf{q}_1 \cdots \mathbf{q}_N]$ is orthonormal and where the elements of $\mathbf{R}$ now constitute an upper-triangular square matrix.

[47]It would be possible to simply state the lattice structure and then to derive the connection to (3.42) using tedious circuit identification and some additional presumptions, but the following reasoning is more instructive, relatively compact and even fairly original.

The interconnection between the conventional prediction error formulation and the lattice form has been considered in depth in many excellent DSP textbooks [Roberts and Mullis, 1987] [Haykin, 1989] [Proakis and Manolakis, 1992]. More theoretical, general and exhausting deductions where cited in the previous paragraph. In addition to the aforementioned, the following construction is also inspired by other related lattice generalizations [Messerschmitt, 1980] [Oliveira e Silva, 1997], but the setting and the main steps of the deduction are somewhat innovative.

[48]The necessity is plausible at least for rational partial transfer functions, but it is argued here that the result is more general due to uniqueness properties of analytic functions with respect to identities on the boundary: every structure that produces a Toeplitz correlation matrix is in practice equivalent to the described form. The sufficiency of the argument is clear enough and it will be explicitly utilized in the following deduction.

$x(n) \longrightarrow \boxed{G_0(z)} \longrightarrow \boxed{A(z)} \longrightarrow \boxed{A(z)} \cdots\cdots\rightarrow \boxed{A(z)}$

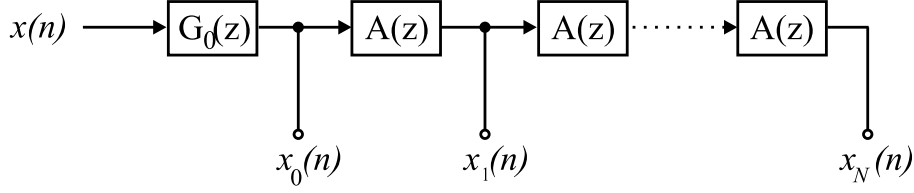$x_0(n) \qquad\qquad x_1(n) \qquad\qquad\qquad x_N(n)$

Figure 3.5: A tapped transversal GLM structure corresponding to a Toeplitz correlation matrix. The common pre-filter (the transfer function from the input to the first tap-output) is arbitrary, but the following blocks are identical allpass functions.

linear prediction error setting [Haykin, 1989],

$$\begin{cases} f_N(n) = \sum_{i=0}^{N} f_{iN}x_i(n) = x_0(n) - (-\mathbf{f}_N)^T\mathbf{x}(n) \\ b_N(n) = \sum_{i=0}^{N} b_{iN}x_i(n) = x_N(n) - (-\mathbf{b}_N)^T\mathbf{x}_0(n) \end{cases}, \qquad (3.45)$$

where the formulas are once more used to define quantities. The double negation is used to account for the approximation error formulation as well as to simplify upcoming expressions. It is easily deduced that the GLM correlation vectors, $\mathbf{p}_N^f = (x_0, \mathbf{x})^T$ and $\mathbf{p}_N^b = (x_N, \mathbf{x}_0)^T$ with respect to the approximation problems (3.45), are related as $\mathbf{p}_N^f = \mathbf{J}(\mathbf{p}_N^b)^*$, that is, they are conjugate persymmetric[49]. In fact, the modeling setup is in the Yule-Walker form, which means that the correlation vectors are also generated by (3.43) and that they have already been denoted as $\mathbf{r}_N^*$ and $\mathbf{Jr}_N$, respectively. The respective normal equations are thus given by $-\mathbf{R}_N\mathbf{f}_N = \mathbf{r}_N^*$ and $-\mathbf{R}_N\mathbf{b}_N = \mathbf{Jr}_N$. By operating either of these equations with $\mathbf{J}$ and using the persymmetry of the correlation matrix, it is clear that also the solutions of the corresponding normal equations are conjugate persymmetric, $\mathbf{b}_N = \mathbf{Jf}_N^*$, or vice versa for $\mathbf{f}_N$ and $\mathbf{b}_N$.[50] This observation has many immediate consequences. For example, the norm (energy) of the prediction error signals (3.45) are necessarily equal. Moreover, the transfer functions of the prediction error filters are "conjugate mirror images" of each other, when expanded as polynomials in the shift operator[51].

---

[49]As stated, $\mathbf{p}_N^f = (x_0, \mathbf{x})^T = (\mathbf{x}, x_0)^*$, and a manipulation of the second term reveals

$$\mathbf{p}_N^b = (x_N, \mathbf{x}_0)^T = \begin{bmatrix} (x_N, x_0) \\ \vdots \\ (x_N, x_{N-1}) \end{bmatrix} = \begin{bmatrix} (x_N, x_0) \\ \vdots \\ (A^N x_0, A^{N-1}x_0) \end{bmatrix} = \begin{bmatrix} (x_N, x_0) \\ \vdots \\ (x_1, x_0) \end{bmatrix} = \mathbf{J}(\mathbf{p}_N^f)^*. \quad (3.46)$$

[50]Notably, also the augmented vectors $[1\ \mathbf{f}_N^T]^T$ and $[\mathbf{b}_N^T\ 1]^T$ are conjugate persymmetric.

[51]These transfer functions are defined as z transforms of (3.45) for $X_0(z) = \mathcal{Z}(x_0) = 1$, that is, as transfer functions from x_0 to x_i, $i = 1, \ldots, N$. Using an apostrophe to indicate the normalization by $X_0(z)$,

$$F_N'(z) = F_N(z)/X_0(z) = \sum_{i=0}^{N} f_{iN}A^i(z)$$

$$B_N'(z) = \sum_{i=0}^{N} b_{iN}A^i(z) = \sum_{i=0}^{N} f_{iN}^* A^{N-i}(z) = A^N(z)\sum_{i=0}^{N} f_{iN}^* A^{-i}(z)$$

$$= A^N(z)\sum_{i=0}^{N} f_{iN}^*(A^i(1/z^*))^* = A^N(z)(F_N'(1/z^*))^*, \qquad (3.47)$$

where the property $A^{-1}(z) = A^*(1/z^*)$ of an allpass function is utilized. Formula (3.47) is a

Now if the Levinson algorithm (3.41) is used to order update $\mathbf{f}_N$, utilizing the relation $-\mathbf{R}_N\mathbf{f}_N = \mathbf{r}_N^* \Leftrightarrow \mathbf{R}_N^{-1}\mathbf{r}_N = \mathbf{f}_N^*$, then the order updated augmented forward prediction error vector is given by

$$
\begin{bmatrix} 1 \\ \mathbf{f}_{N+1} \end{bmatrix} = \begin{bmatrix} 1 \\ \mathbf{f}_N + \alpha_N \mathbf{J}\mathbf{f}_N^* \\ \alpha_N \end{bmatrix} = \begin{bmatrix} 1 \\ \mathbf{f}_N \\ 0 \end{bmatrix} + \alpha_N \begin{bmatrix} 0 \\ \mathbf{J}\mathbf{f}_N^* \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \mathbf{f}_N \\ 0 \end{bmatrix} + \alpha_N \begin{bmatrix} 0 \\ \mathbf{b}_N \\ 1 \end{bmatrix}. \quad (3.48)
$$

The second step is simply a partition, where however knowledge of the particular form of $\mathbf{f}_{N+1}$ is used to force zeros into the partial vectors. The order updated prediction error signal $f_{N+1}(n)$ can then be formed as in (3.45), and subsequently expanded utilizing (3.44) and (3.48), to provide[52]

$$
\begin{aligned}
f_{N+1}(n) &= [1 \ \mathbf{f}_{N+1}^T]\mathbf{x}_{N+1}(n) = [1 \ \mathbf{f}_N^T \ 0]\begin{bmatrix} \mathbf{x}_0(n) \\ x_N(n) \end{bmatrix} + \alpha_N[0 \ \mathbf{b}_N^T \ 1]\begin{bmatrix} x_0(n) \\ A\mathbf{x}_0(n) \end{bmatrix} \\
&= [1 \ \mathbf{f}_N^T]\mathbf{x}_0(n) + \alpha_N[\mathbf{b}_N^T \ 1]A\mathbf{x}_0(n) = f_N(n) + \alpha_N A b_N(n). \quad (3.49)
\end{aligned}
$$

The corresponding formula for the backward prediction error is attained in a similar way, by inserting $\mathbf{J}\mathbf{b}_N = -\mathbf{R}^{-1}\mathbf{r}_N$ into the Levinson algorithm (3.41), or more directly by operating the vector partition (3.48) with $\mathbf{J}(\cdot)^*$, to produce

$$
\begin{aligned}
b_{N+1}(n) &= [\mathbf{b}_{N+1}^T \ 1]\mathbf{x}_{N+1}(n) = [0 \ \mathbf{b}_N^T \ 1]\begin{bmatrix} x_0(n) \\ A\mathbf{x}_0(n) \end{bmatrix} + \alpha_N^*[1 \ \mathbf{f}_N^T \ 0]\begin{bmatrix} \mathbf{x}_0(n) \\ x_N(n) \end{bmatrix} \\
&= [\mathbf{b}_N^T \ 1]A\mathbf{x}_0(n) + \alpha_N^*[1 \ \mathbf{f}_N^T]\mathbf{x}_0(n) = A b_N(n) + \alpha_N^* f_N(n). \quad (3.50)
\end{aligned}
$$

By combining (3.49) and (3.50), the order update recursions can be expressed simultaneously as

$$
\begin{cases} f_{N+1}(n) = f_N(n) + k_N^f A b_N(n) \\ b_{N+1}(n) = A b_N(n) + k_N^b f_N(n) \end{cases} \quad (3.51)
$$

or in the corresponding matrix form

$$
\begin{bmatrix} f_{N+1}(n) \\ b_{N+1}(n) \end{bmatrix} = \begin{bmatrix} 1 & k_N^f \\ k_N^b & 1 \end{bmatrix}\begin{bmatrix} f_N(n) \\ A b_N(n) \end{bmatrix} = \begin{bmatrix} 1 & k_N^f A \\ k_N^b & A \end{bmatrix}\begin{bmatrix} f_N(n) \\ b_N(n) \end{bmatrix}, \quad (3.52)
$$

for $N = 0, 1, \ldots$, and using initializations $f_0(n) = x_0(n)$ and $b_0(n) = x_0(n)$. The "Levinson constant" α_N is temporarily replaced with more generic "coupling coefficients" k_N^f and k_N^b. Formula (3.52) suggests a two-pair processing block defined by the coupling matrix and the shift operator, a generalized feed-forward lattice section, and a cascade connection of such sections forms a generalized feed-forward lattice structure. The parameters of the coupling matrix are called reflection coefficients and another reason for distinguish them from the Levinson constant, besides increased generality, is that they can be evaluated using local correlations obtained directly from the lattice structure signals, that is, without resorting to the transversal-form correlation terms. There are many ways to conclude that minimization of the energy of the prediction error signals (3.51) with respect to the lattice

---

generalization of the relation for real FIR prediction error filters, $B_N(z) = z^{-N}F_N(z^{-1})$, or of the formula for producing a dual set of orthogonal polynomials [Szegö, 1939].

[52]Notably, two different forms of the data vector (3.44) are used to match with the particular form of the parameter vector (3.48).

coefficients results in[53]

$$
\begin{aligned}
k_N^f &= -\frac{(f_N, Ab_N)}{(Ab_N, Ab_N)} = -\frac{(f_N, Ab_N)}{(b_N, b_N)} \\
k_N^b &= -\frac{(Ab_N, f_N)}{(f_N, f_N)} = -\frac{(f_N, Ab_N)^*}{(f_N, f_N)}
\end{aligned}
\qquad (3.54)
$$

In particular, f_{N+1} and b_{N+1} are by construction orthogonal to Ab_N and f_N, respectively, and expanding these orthogonality conditions, $(f_{N+1}, Ab_N) = 0$ and $(b_{N+1}, f_N) = 0$, using (3.51), provides readily (3.54). Notably, the reflection coefficients act as Fourier or projection coefficients with respect to the local approximation spaces spanned by $\{Ab_N\}$ and $\{f_N\}$, respectively, where the denominator terms represent normalization of the corresponding basis functions. Alternatively, the inner product in the numerator can be seen as a correlation term between the lattice stage inputs[54]. In the forward and backward linear prediction setting, the denominator terms are equal, which provides a restatement of the fact that the reflection coefficients are complex conjugates of each other. Another immediate consequence of the symmetry is that the prediction error energies possess simple recurrence relations[55]

$$
\begin{cases}
E_{i+1}^f = (f_{i+1}, f_{i+1}) = (1 - k_i k_i^*)(f_i, f_i) \\
E_{i+1}^b = (b_{i+1}, b_{i+1}) = (1 - k_i k_i^*)(b_i, b_i)
\end{cases}
\Rightarrow \quad E_{i+1} = E_0 \prod_{j=0}^{i}(1 - k_j k_j^*), \quad (3.55)
$$

for $i = 0, 1, \ldots$, whenever $E_{i+1} = E_{i+1}^f = E_{i+1}^b$, and where $E_0 = (f_0, f_0) = (b_0, b_0) = (x_0, x_0)$ is the energy of the input signal $x_0(n)$ to the first lattice section. From the right hand side of (3.55) it can be concluded that the sequence of prediction error energies is strictly decreasing if and only if $|k_i| < 1$ for all $i = 0, 1, \ldots$, which can also be seen as a requirement for (strict) stability of the lattice recursion (3.51).[56]

In practice, the correlation terms in (3.54) have to be estimated from finite data records. For example, if the inner products are interpreted as expectation operations, then some chosen (weighted) time-average may be used to evaluate the lattice

---

[53]For example, the minimum (which is a unique global minimum) of (f_{N+1}, f_{N+1}) with respect to k_N^f is obtained by expanding the inner product as

$$
\begin{aligned}
(f_{N+1}, f_{N+1}) &= (f_N, f_N) + (f_N, k_N^f Ab_N) + (k_N^f Ab_N, f_N) + ((k_N^f Ab_N), k_N^f Ab_N) \\
&= (f_N, f_N) + k_N^{f*}(f_N, Ab_N) + k_N^f(Ab_N, f_N) + k_N^f k_N^{f*}(Ab_N, Ab_N), \quad (3.53)
\end{aligned}
$$

where chosen calculation rules are utilized to extract the reflection coefficient from the inner products. The minimum is then attained directly from the *quadratic form* with respect to k_N^f or by applying suitable conventions for complex derivatives [Brandwood, 1983] of the quantities to locate the minimum. The coefficient k_N^b is attained in a similar manner from minimizing (b_{N+1}, b_{N+1}), $b_{N+1}(n) = Ab_N(n) + k_N^b f_N(n)$.

[54]The term *partial correlation coefficients* (PARCOR or PCC) has been established by various authors [Itakura et al., 1972] [Markel and Gray, 1973] [Makhoul, 1977] [Lee et al., 1981] [Porat and Kailath, 1983].

[55]The recurrence formulas are obtained by forming the prediction error energies with respect to recursions (3.51) and by substituting mixed expressions from (3.54) into the produced equations, for example, by substituting $-k_N^f(Ab_N, Ab_N) = (f_N, Ab_N)$ and $-k_N^{f*}(f_N, f_N) = (Ab_N, f_N)$ into (3.53).

[56]At the present, the term stability is used merely as a condition for the meaningfulness of the prediction error setting; stability issues of the implied inverse model will be considered very briefly in the upcoming. In particular, there are no restrictions on the reflection coefficients when recursions (3.51) are used for general feed-forward (GLM) lattice filter synthesis.

coefficients. Due to the alternative forms of the lattice coefficient formulas, it is apparent that approximation with respect to different quantities in the correlations will most likely have an effect on the attained estimates. In particular, the chosen approximation may produce reflection coefficients with magnitude greater than one. Somewhat better balanced estimates are obtained by using both prediction error signals in the normalization of the correlation terms[57]. The choice of estimated quantities becomes even more crucial when instantaneous or recursively updated estimates are used in a genuinely adaptive lattice formulation. GLM counterparts of the prototype adaptive algorithms will be considered in Section 3.5, along with some indications on their lattice form counterparts.

The GLM lattice structure corresponding to the recursion (3.51) with lattice coefficients $k_N^f = k_N^*$ and $k_N^b = k_N$ is depicted in Figure 3.6. The first filter block $A_0(z)$ is inherited from the original transversal construction and it acts as a pre-filter for the subsequent lattice structure. The choice of $G_0(z)$ is arbitrary in the sense that it does not affect the previous deduction of the GLM lattice structure[58]. In fact, even the (GLM) linear independency presumption is relaxed, since the modeling signals $\mathbf{x}_{N+1}(n) = [x_0(n) \ \cdots \ x_N(n)]^T$ of the transversal structure in Figure 3.5, or the corresponding modeling signals $\mathbf{b}_{N+1}(n) = [b_0(n) \ \cdots \ b_N(n)]^T$ of the lattice structure in Figure 3.6, are inherently linearly independent for any stable and causal $G_0(z)$ and $A(z)$, where the latter is, in addition, still presumed to be an allpass function[59]. However, the point of this whole consideration is that the lattice structure provides an efficient implementation of the orthogonalization operation: the tap-output signals $\mathbf{b}_{N+1}(n) = [b_0(n) \ \cdots \ b_N(n)]^T$ are mutually orthogonal in the ideal GLM setting with respect to the exact reflection coefficients (3.54).[60]

In the hope of getting this consideration to converge, only a few aspects of the lattice orthogonalization are considered in conclusion. The explicit relation between the GLM tap-output vector $\mathbf{x}(n)$ and the orthogonalized tap-output vector $\mathbf{b}(n)$ is

---

[57]Utilization of the geometric mean of the denominators in (3.54), $(E_N^f E_N^b)^{1/2}$, as a common normalizing term is well-justified [Itakura et al., 1972], although it does not result from an actual error criterion [Makhoul, 1977]. In contrast, normalization with respect the arithmetic mean, $(E_N^f + E_N^b)/2$, is directly related to minimizing the sum of prediction errors and this strategy is usually (at)tributed to Burg [Makhoul, 1977] [Haykin, 1989]. It is easy to verify that the corresponding reflection coefficients, k_N^I and k_N^B, are related to the ones in (3.54) through $k_N^I = \pm(k_N^f k_N^b)^{1/2}$ and $k_N^B = 2k_N^f k_N^b/(k_N^f + k_N^b)$, respectively. The Burg-method can thus be seen as the more conservative choice, $|k_N^B| \leq |k_N^I|$, since equality is attained only in the case when the estimates of k_N^f and k_N^b coincide.

[58]The lattice structure was deduced with respect to the reference signal $x_0(n)$, and thus, choices of the form $G_0(z) = A^i(z)$, $i = 0, 1, \ldots$, would release the generalized (auto)correlations (3.43) in terms of the actual input signal $x(n)$. Moreover, some choice relate the subsequent GLM lattice to the conventional feed-forward lattice through a change of variable, $z^{-1} \leftrightarrow A(z)$; the trivial candidate for such an interpretation is $G_0(z) = 1$, but is some cases, there are also better-justified "balancing functions" $G_0(z)$ induced by a particular $A(z)$ (Section 2.4.3).

[59]The linear independency is asserted, for example, by the considerations related to the shift operator in Section 2.4.3.

[60]The orthogonality, $(b_i, b_j) = 0$, $i \neq j$, is a direct consequence of the definition of the backward prediction error (3.45) and the Hilbert space projection theorem (Section 2.3.2): for optimal prediction error coefficients $\mathbf{c}_i$, the error $b_i = x_i - \mathbf{c}_i^T \mathbf{x}_{i-1}$ is orthogonal to the tap-output vector $\mathbf{x}_{i-1} = [x_0 \ \cdots \ x_{i-1}]^T$, denoted as $(b_i, \mathbf{x}_{i-1}) = \mathbf{0}^T$, which implies that $(b_j, b_i) = (b_j, x_i - \mathbf{c}_i^T \mathbf{x}_{i-1}) = (b_j, x_i) - \mathbf{c}_i^H(b_j, \mathbf{x}_{i-1}) = 0$ for $j > i$, and consequently also for $j < i$.
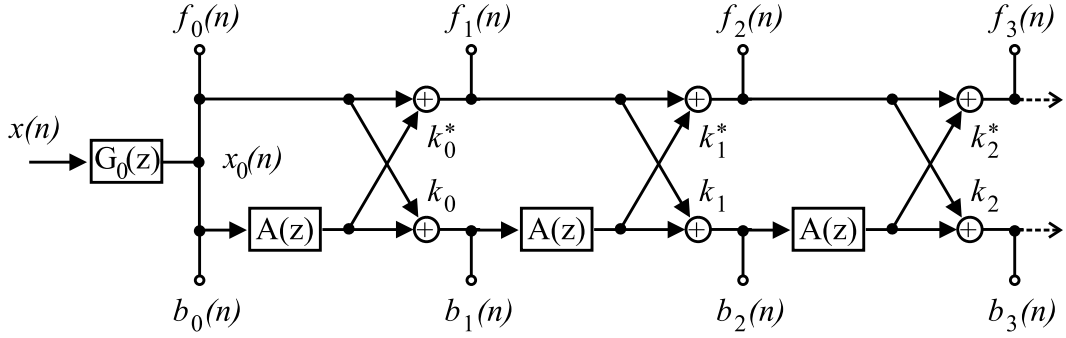
Figure 3.6: The GLM lattice structure deduced from a (block-) Toeplitz correlation matrix and a related transversal structure (Fig. 3.5) with identical allpass blocks. The block-Toeplitz case implies a modified structure with vector valued internal signals $\{\mathbf{x}_i, \mathbf{f}_i, \mathbf{b}_i\}$, matrix delay blocks $A(z)\mathbf{I}$, and reflection coefficient matrices $\{\mathbf{K}_i\}$, which are equally obtained from (3.54) with respect to matrix valued inner products. Notably, the forward and backward coefficient matrices are no longer Hermitian transposes of each other since the denominator terms $(\mathbf{b}_i, \mathbf{b}_i)$ and $(\mathbf{f}_i, \mathbf{f}_i)$ in (3.54) are not necessarily equal, that is, their traces that still represent prediction error energies are equal, but the matrices are not diagonal, which in turn implies that the component signals are not usually mutually orthogonal. The input signal is then alternatively itself a signal vector or then the pre-filter may act as a multiplexer, $\mathbf{x}_0 = \mathbf{G}_0(z)x$; the components of $\mathbf{x}_0$ are presumed to be linearly independent in order to ensure invertibility of the aforementioned matrices.

given by

$$
\begin{bmatrix} b_0(n) \\ b_1(n) \\ b_2(n) \\ \vdots \\ b_N(n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ b_{01} & 1 & 0 & \cdots & 0 \\ b_{02} & b_{12} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{0N} & b_{1N} & b_{2N} & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_0(n) \\ x_1(n) \\ x_2(n) \\ \vdots \\ x_N(n \end{bmatrix}, \tag{3.56}
$$

which is simply a collection of GLM backward prediction error models (or filters) of the form (3.45) for orders $0, \ldots, N$. It is obvious but noteworthy that the transformation (3.56), denoted as $\mathbf{b} = \mathbf{L}\mathbf{x}$, is invertible; the inverse $\mathbf{L}^{-1}$ exists since $\det(\mathbf{L}) = 1 \neq 0$, and in addition, the inverse is also known to be unit lower triangular [Golub and Van Loan, 1989]. Now as it was preluded, the transformation (3.56) can be recognized as the matrix representation of the Gram-Schmidt (GS) orthogonalization process (3.42). This is seen by identifying terms in (3.56) and the augmented form of (3.42), which is another proof of the orthogonality of the backward prediction error signals. In practice, however, it is not meaningful to form mappings between the elements of $\mathbf{L}$ and the inner product terms by trying to replace the GS recursion with a block-operation using back-substitution, as it was boldly implied in (3.42), although the correlation shift-invariancy (3.43) would simplify the expression substantially compared to the general case. The point is that the Yule-Walker form of the Levinson algorithm, or the implied lattice recursion, provide efficient ways to generate all the elements in $\mathbf{L}$; the prediction error coefficients for all orders up to N are determined order-recursively from the correlation terms $\{r_0, \ldots, r_N\}$, or more directly from the estimated lattice parameters

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$\{E_0, k_0, \ldots, k_{N-1}\}$, respectively[61]. More precisely, successive rows of the matrix $\mathbf{L}$ are generated as

$$[\mathbf{b}_{i+1}^T \; 1] = [k_i \; \mathbf{b}_i^T + k_i \mathbf{J} \mathbf{b}_i^H \; 1], \quad \mathbf{b}_i^T = [b_{0i} \; \cdots \; b_{(i-1)i}], \qquad (3.57)$$

which is simply the conjugate persymmetric counterpart of the update formula (3.48) for $k_i = \alpha_i^*$. This expression manifests the decoupling or orthogonality property of the lattice parametrization; the lattice parameters are independent of the approximation order, or conversely, the order update of the whole set of forward and backward prediction error coefficients is determined by a single lattice parameter. The recursion (3.57) for $i = 0, \ldots, N-1$ can be considered as a mapping $\mathcal{L} : \mathbf{k} \mapsto \mathbf{b}$ from the lattice parameter vector to the prediction error coefficient vector of the corresponding (final) prediction order. Moreover, the mapping is invertible and the inverse $\mathcal{L}^{-1}$ is attained using a step-down recursion[62]. This one-to-one correspondence between model descriptions provides useful means to such operations as model reduction, parameter interpolation and stability control, regardless of the actual model implementation. However, the emphasis is still on the more general modeling setup than the prediction error configuration; the parameter mapping $\mathcal{L}$ is just an intermediate operation for deducing the lattice structure or for constructing the signal transformation $\mathbf{L}$, that is, a way to implement favorable pre-processing prior to forming the actual GLM. The following two paragraphs aim to summarize the preceding more clearly in terms of the original modeling task. In particular,

---

[61]The term E_0 is the energy of the lattice input signal, $E_0 = r_0$, and it acts as a signal scaling information in addition to the N-dimensional signal model parametrization. In the linear prediction setting it is common to replace E_0 with E_N, the final (forward) prediction error energy, because it is the required information for the synthesis operation; E_N is given in (3.55) as a function of the parameters $\{E_0, k_0, \ldots, k_{N-1}\}$.

[62]The middle term in the recursion (3.57) is of the form $\tilde{\mathbf{b}}_{N+1} = \mathbf{b}_N^T + k_N \mathbf{J} \mathbf{b}_N^H$, and together with its conjugate persymmetric counterpart, $\mathbf{J} \tilde{\mathbf{b}}_{N+1}^* = \mathbf{J} \mathbf{b}_N^H + k_N^* \mathbf{b}_N^T$, they form an invertible transformation pair

$$\begin{bmatrix} \mathbf{J} \tilde{\mathbf{b}}_{N+1}^* \\ \tilde{\mathbf{b}}_{N+1} \end{bmatrix} = \begin{bmatrix} 1 & k_N^* \\ k_N & 1 \end{bmatrix} \begin{bmatrix} \mathbf{b}_N^T \\ \mathbf{J} \mathbf{b}_N^H \end{bmatrix} \quad \leftrightarrow \quad \begin{bmatrix} \mathbf{b}_N^T \\ \mathbf{J} \mathbf{b}_N^H \end{bmatrix} = \frac{1}{1 - k_N^* k_N} \begin{bmatrix} 1 & -k_N^* \\ -k_N & 1 \end{bmatrix} \begin{bmatrix} \mathbf{J} \tilde{\mathbf{b}}_{N+1}^* \\ \tilde{\mathbf{b}}_{N+1} \end{bmatrix}. \quad (3.58)$$

The inverse operation is in principle defined whenever the determinant of the coupling matrix $\mathbf{T}$ is non-zero, $\det(\mathbf{T}) = 1 - k_N^* k_N \neq 0$. The strict condition $|k_i| < 1$ for all k_i guarantees thus an one-to-one mapping between two parameter sets, but as in the conventional linear prediction setting, this restriction excludes sone interesting and even common practical situations, that is, the mapping could and should be extended with the expense of a non-unique correspondence of the parametrizations [Picinbono and Benidir, 1986] [Benidir and Picinbono, 1987] [Mazel and Hayes, 1988].

It is interesting to notice that the matrix operator $\mathbf{T}$ and its simple inverse $\mathbf{T}^{-1}$ are special cases of the Möbius transformation (Section 2.4.1). In addition, there is a clear similarity between the left hand side of (3.58) and the lattice recursion (3.52), which in turn implies an inverted lattice structure. If fact, the GLM forward and backward prediction error polynomials in the shift operator are generated by including the allpass term A in the coupling matrix, as it appears in (3.52), and therefore the corresponding step-down formulas for the prediction error signals and polynomials would provide a straightforward and natural parallelism to the GLM lattice synthesis structure. However, for many reasons the author is pleased not to be obliged to continue this consideration.

Returning back to the mapping $\mathcal{L}$, the inverse mapping $\mathcal{L}^{-1} : \mathbf{b} \mapsto \mathbf{k}$ is attained by first recognizing that $[k_N \; \tilde{\mathbf{b}}_{N+1}] = \mathbf{b}_{N+1}^T$ and then using $\mathbf{b}_i^T = (\mathbf{J} \tilde{\mathbf{b}}_{i+1} - k_i^* \tilde{\mathbf{b}}_{i+1})/(1 - k_i^* k_i)$ from the right hand side of (3.58) to extract the remaining reflection coefficients, $[k_{i-1} \; \tilde{\mathbf{b}}_i] = \mathbf{b}_i^T$, order-recursively for $i = N, N-1, \ldots, 1$.

$$
\begin{array}{ccccc}
& \mathbf{x}_i & \longrightarrow & \mathbf{r}_i & \longrightarrow & \alpha_i \\
& \nearrow & & & & \searrow \\
x & & \text{filter/correlate/project} & & & \mathbf{L} \\
& \searrow & & & & \nearrow \\
& \left\{ \begin{array}{l} f_i \\ b_i \end{array} \right. & & \longleftrightarrow & & k_i
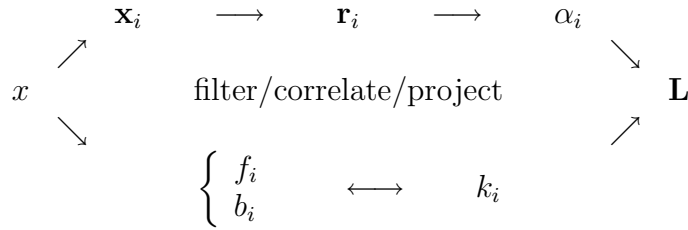\end{array}
$$

Figure 3.7: The upper branch is the direct Levinson (or Yule-Walker) method in contrast to the lower path that stands for evaluation with respect to pure lattice quantities. The two kinds of arrows try to express the fundamental difference in the nature of the order recursions: in the lattice case the filtering and correlation operations are genuinely coupled and modular whereas the "direct form" quantities $\mathbf{x}_i$ and $\mathbf{r}_i$ can be composed independently and then used to feed the actual order update.

a distinction is made between direct utilization of the Levinson recursion and the actual lattice structure implementation.

The Levinson recursion and the implied lattice structure may be viewed as an efficient way to implement matrix factorizations of the correlation or signal matrix, respectively. In fact, the lower-triangular square matrix $\mathbf{L}$ in (3.56) is precisely the inverse of the normalized Cholesky factor of the GLM correlation matrix that is needed both in the QR decomposition and the Cholesky factorization (Section 3.3.1).[63] From this point of view, the underlying orthogonalization or projection operation may be seen as a computational maneuver involving some auxiliary variables without actually constructing the lattice structure. Nevertheless, the two alternative ways to determine the reflection coefficients, and consequently the matrix $\mathbf{L}$, differ essentially in the order and manner that filtering and correlation operations enter into the order update recursion. A simplified diagram of the situation is sketched in Figure 3.7. The cross-correlation part of the GLM normal equation is then solved with respect to the chosen factorization of the correlation matrix. Notably, the Levinson algorithm (3.41) for the general right hand side incorporates these two operations into a simultaneous recursion.

The other possibility is to use the lattice structure to transform the original GLM modeling signals $\{x_i\}$ into their uncorrelated or orthogonal counterparts $\{b_i\}$. The

---

[63]The correlation matrix of the tap-response vector $\mathbf{b}(n)$, $\mathbf{R}_b^* = \mathbf{b}\mathbf{b}^H = (\mathbf{L}\mathbf{x})(\mathbf{L}\mathbf{x})^H = \mathbf{L}\mathbf{x}\mathbf{x}^H\mathbf{L}^H = \mathbf{L}\mathbf{R}_x^*\mathbf{L}^H = \mathbf{D}$, is real and diagonal, $\mathbf{D} = diag((b_0, b_0), \cdots, (b_N, b_N))$, which can be seen as a diagonalization of the GLM correlation matrix $\mathbf{R}_x$. By identifying $\mathbf{C}^* = \mathbf{L}^{-1}\sqrt{\mathbf{D}}$, $\mathbf{R}_x = \mathbf{C}\mathbf{C}^H$ is the Cholesky factorization of the matrix $\mathbf{R}_x$, which has an useful inverse $\mathbf{R}_x^{-1} = \mathbf{C}^{-H}\mathbf{C}^{-1} = \mathbf{L}^T\mathbf{D}^{-1}\mathbf{L}^* = (\mathbf{D}^{-1/2}\mathbf{L})^T(\mathbf{D}^{-1/2}\mathbf{L}^*)$, where the corresponding Cholesky factors are attained directly from $\mathbf{L}$ and multiplication by a simple matrix inverse $\mathbf{D}^{-1/2} = diag((b_0, b_0)^{-1/2}, \cdots, (b_N, b_N)^{-1/2})$.

The linkage to the QR decomposition is seen by transposition and expansion of $\mathbf{b} = \mathbf{L}\mathbf{x}$ into the (deterministic) signal matrix form $\mathbf{B} = \mathbf{S}\mathbf{L}^T$: normalizing $\mathbf{B}$ by multiplying with $\mathbf{D}^{-1/2}$ from the right, denoting the result by $\mathbf{Q} = \mathbf{B}\mathbf{D}^{-1/2} = \mathbf{S}\mathbf{L}^T\mathbf{D}^{-1/2}$, renders $\mathbf{Q}\tilde{\mathbf{R}} = \mathbf{S}$ with $\tilde{\mathbf{R}} = (\mathbf{L}^T\mathbf{D}^{-1/2})^{-1} = \sqrt{\mathbf{D}}\mathbf{L}^{-T}$, which is now in turn upper-triangular.

GLM response with respect to a given target response $y(n)$ is then constructed as

$$\hat{y}_N(n) = \sum_{i=0}^{N} c_i b_i(n) = \mathbf{c}^T \mathbf{b}(n), \qquad c_i = (y, b_i)/(b_i, b_i), \tag{3.59}$$

where the modified indexing of this Section is maintained. The model response (3.59) is the truncated Fourier series expansion of $y(n)$ with respect to the basis functions $b_i(n)$; the normalizing term (b_i, b_i) in the coefficient expression may alternatively be considered as a scaling operation for the signals $b_i(n)$. The modeling signals $\{x_i\}$ and $\{b_i\}$ span by construction the same approximation space, which implies that the approximation and approximation error provided by (3.59) are identical to those resulting from the original GLM configuration[64]. However, the fundamental difference between the model parametrizations is that due to the orthogonality the partial model weights, or coefficients, of (3.59) are independent of each other, that is, independent of the model order and ordering of the modeling signals, which has many desirable consequence concerning the choosing of a particular model. The model structure corresponding to (3.59) and the GLM lattice for producing the partial model responses $\{b_i\}$ is depicted in Figure 3.8. It is called a *tapped lattice* or a *joint section estimator* structure, depending on the point of view. An additional prediction error configuration is included into Figure 3.8 to emphasize the latter perspective: the lattice recursion (3.51) complemented with

$$e_i(n) = e_{i-1} - c_i b_i(n), \qquad e_{-1}(n) = y(n), \ i = 0, 1, \ldots, \tag{3.60}$$

define a joint estimation or prediction of the processes $x(n)$ and $y(n)$. The lattice structure can be seen as a *whitening filter* (with respect to the output signal $f_i(n)$) for the input signal $x(n)$, whereas the joint section defines a whitening process for the output signal $y(n)$ (that is induced by $x(n)$ and de-correlated as $\{b_i(n)\}$).[65] The prediction error configuration for the tapped sections is also useful from a practical point of view. Namely, the block representation for the evaluation of the tap-output weights, suggested by (3.59), may be replaced by local correlation terms

$$c_i = (e_{i-1}, b_i)/(b_i, b_i), \quad i = 0, 1, \ldots, \tag{3.61}$$

which is once more a direct consequence of the orthogonality[66]. This means that using (3.60) and (3.61) also the tapped sections are genuinely order recursive or modular, which is particularly appealing for such applications as adaptive filtering.

Some remarks Potential orthogonality is always embedded into the linear independency assumption through the (GS) orthogonalization process. This general property was demonstrated in terms of the GLM in Section 3.3. The purpose

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[64]The weights of the original GLM response $\hat{y} = \mathbf{w}^T \mathbf{x}$ and the orthonormal expansion coefficients $\mathbf{c}$ are thus related simply as $\mathbf{w} = \mathbf{L}^T \mathbf{c} \leftrightarrow \mathbf{c} = \mathbf{L}^{-T} \mathbf{w}$.

[65]Without further explanation, the "whitening" can be considered as a limiting process for the inverse modeling or system identification setup, $x \mapsto \delta$ or $y \mapsto \delta$, respectively, where δ is alternatively an impulse or a white noise signal (see Sections 2.1.6 and 3.1).

[66]The inner products in the numerators of the coefficient expressions are seen to be equal, $(y, b_i) = (\hat{y}_{i-1} + e_{i-1}, b_i) = (\hat{y}_{i-1}, b_i) + (e_{i-1}, b_i) = (e_{i-1}, b_i)$, since all terms in the approximation $\hat{y}_{i-1}(n) = \sum_{j=0}^{i-1} c_j b_j(n)$ are orthogonal to $b_i(n)$.
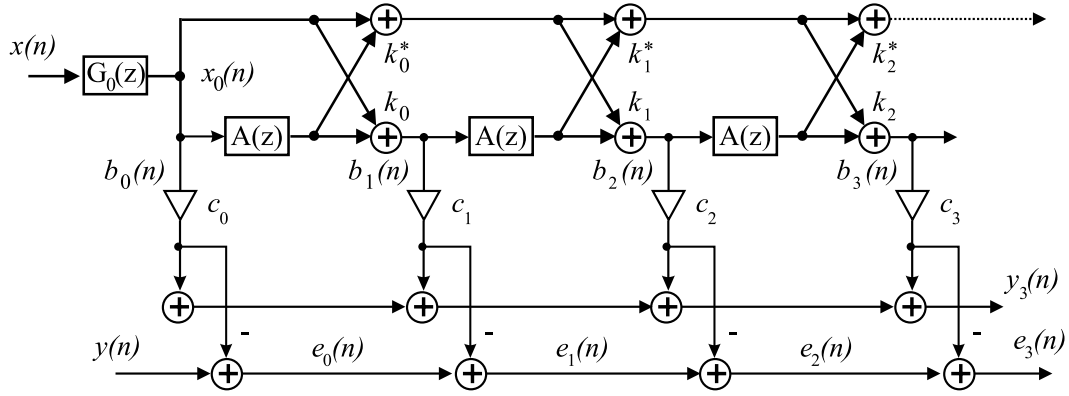
Figure 3.8: The tapped GLM lattice structure.

of this Section was to show that the presumption of a (block-)Toeplitz correlation matrix implies a generalized transversal structure for producing the correlation terms (Figure 3.5), and furthermore, a generalized lattice structure to implement the orthogonalization of the modeling signals (Figure 3.6). This does not however exclude generalizations of the shift structure and the lattice configuration with respect to *non-Toeplitz* correlation matrices. For example, the "different from, but close to Toeplitz" characteristics arising from non-stationarity [Lev-Ari and Kailath, 1984] or varied correlation methods [Morf, 1977] may be tackled using the concept of *displacement-rank* [Kailath, 1973], which (at least in principle) generalizes the Levinson algorithm to any symmetric positive-definite correlation matrix [Friedlander et al., 1978] [Delsarte et al., 1982] [Lev-Ari et al., 1984]. Perhaps even more interesting from the GLM point of view is that there are also generalizations of the Levinson algorithm in the case when the generating model is not restricted to structures with identical allpass blocks [Merched, 2003] [Bultheel and De Moor, 2000].

Another remark is in order to clarify the relation to other lattice configurations, in particular the ARMA or pole-zero lattice structure [Gray and Markel, 1973]. As stated, the GLM lattice structure is a generalization of the feed-forward lattice, in particular, the tapped lattice form of an FIR, MA, or all-zero filter [Griffiths, 1978] [Makhoul, 1978], in contrast to some modified pure lattice structures for producing an arbitrary FIR transfer function [Vaidyanathan, 1986] [Tummala and Parker, 1987]. From the perspective of linear prediction of (speech) signals [Markel and Gray, 1973] [Makhoul, 1975], the forward prediction error part of the lattice structure constitute an analysis apparatus for the corresponding AR or all-pole lattice synthesis filter; typically it is just the synthesis part that is of interest since the analysis operation is usually embedded into direct correlation estimates, with the exception of adaptive linear prediction [Makhoul and Viswanathan, 1978].

The conventional ARMA or pole-zero lattice is then in turn built on the synthesis all-pole lattice; the denominator part of a (presumed to be) rational transfer function is implemented as a lattice structure, whereas the numerator is constructed by tapping, weighting, and summing in the backward path of the lattice [Gray and Markel, 1973]. This would then, using consistent terms, actually be a tapped AR lattice that realizes a pole-zero transfer function. In fact, the numerator

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can be implemented by numerous combinations of output tapping or input injection from or into various nodes of the AR lattice [Lim, 1984]. The original tapped AR lattice is, however, special in the sense that it originates from a related orthogonalization process: the (tap-output numerator) polynomials resulting from the inverse lattice recursion (right hand side of (3.58)) are now orthonormal with respect to normalization by the denominator term. They are precisely the Szegö orthogonal polynomials on the unit circle [Szegö, 1939] [Delsarte et al., 1978] with respect to a weighting function or modified measure in the associated inner product that is induced by the spectral density of the (AR) input signal model. As claimed, all this is at least in principle applicable to the (allpass shift operator driven) GLM framework, including generalizations of vectorized or multidimensional lattice configurations [Lim and Parker, 1984].

In contrast to the forward lattice, the conventional ARMA lattice is not genuinely modular or order recursive since it is generated with respect to a fixed (presumption of the) denominator term. In particular, the overall model does not include the lower order models as substructures and it does not generalize naturally to the unequal order $ARMA(N, M)$ case. There are, however, "true ARMA lattice" configurations based on elementary two-channel or -dimensional MA lattice sections [Morf, 1977] [Friedlander and Maitra, 1981] [Lee et al., 1982] that can be further organized into complex arrays of processing blocks that constitute chains of successive $ARMA(i, j)$ systems, $i \leq N$ and $j \leq M$, for the general (N, M)th order ARMA lattice model [Benveniste and Chaure, 1981] [Karlsson and Hayes, 1987], including exhausting extensions to the multichannel case [Ling and Proakis,1984] [Lev-Ari, 1987] [Chakraborty and Prasad, 1991] [Pan and Levine, 1994]. Here, the feed-forward tapped GLM lattice is proposed as a simple alternative for an ARMA lattice, where the AR part is embedded into the allpass operator, yet in a modular and order recursive manner, with a straightforward extension to the multidimensional case.

3.4.2   Orthonormal GLM structures

The (block-)Toeplitz property of the correlation matrix and the related transversal allpass structure was previously seen as a source of orthogonality and guaranteed linear independency for the GLM. As a somewhat limping converse, structural orthogonality, or mutual orthogonality of the partial model impulse responses, is in the following related to the allpass property and to the appropriateness of the GLM construction, with particular implications to the asymptotic behavior when the number of partial models is increased towards infinity. That is, the orthogonality is introduces as a sufficient condition for potential completeness of the GLM.

The GLM structure is orthogonal if the partial model impulse responses are mutually orthogonal in the Hilbert space $\ell^2(\mathbb{N})$ or $H^2(\mathbb{E})$ sense, respectively. The structure is orthonormal, if in addition, the partial model impulse responses are normalized to unit energy. For example, with respect to the inner product (2.79), the structure is

orthonormal if the correlation matrix[67]

$$\mathbf{R} = [r_{ij}]_{i,j=1}^{N} = \left[\frac{1}{2\pi j} \oint_{\mathbb{T}} G_j(z) G_i^*(1/z^*) S_{xx}(z) \frac{dz}{z} \right]_{i,j=1}^{N} \tag{3.62}$$

becomes the unit matrix $\mathbf{I}$ for the power spectral density $S_{xx}(z) = 1$, corresponding to the unit impulse or white noise input signal, respectively. For such a GLM configuration the normal equations reduce into the trivial form, $\mathbf{Iw} = \mathbf{p}$, or consequently $\mathbf{w} = \mathbf{p} = (y, \mathbf{g})^T$, where $\mathbf{w}$ are now Fourier coefficients of the truncated Fourier series $\mathbf{w}^T \mathbf{g}$ for approximating y with respect to the orthonormal functions $\mathbf{g}$, the composite description of the components $g_i(n) \in \ell^2(\mathbb{N})$ or $G_i(z) \in H^2(\mathbb{E})$, respectively. In addition, the general expression for the (LS or MMS) approximation error simplifies as $\|e\|^2 = (y, y) - \mathbf{w}^H \mathbf{p} = (y, y) - \|\mathbf{w}\|^2$, where $(y, y) = \|y\|^2$ is the energy of the target response y.

The correlation matrix formula (3.62), which is in its general form, suggests some implications for both orthonormal and non-orthogonal GLM structures:

- For an orthonormal structure, the properties of $\mathbf{R}$ (in terms of the eigenvalue spread) are essentially determined by the properties of $S_{xx}(z)$

- In particular, orthogonality guarantees linear independency, characterized as positive definiteness of $\mathbf{R}$, presuming that $S_{xx}(z)$ is positive and somehow bounded away from zero on $\mathbb{T}$

- Moreover, the aforementioned properties can be quantized in a way that is independent of the particular choice of orthonormal functions, that is, as a general consequence of the orthonormality

- The completeness of the model construction, unbiased convergence for increased number of partial models, is analogous to the concept of completeness in the Hilbert space. This completeness transfers as well to the identification setup in terms of the GLM

- For $S_{xx}(z) \neq 1$, the formula (3.62) suggests functions that are orthogonal with respect to $S_{xx}(z)$

- Consequently, it can be argued that for a somehow specified subclass of input signals, there are probably non-orthogonal GLM structures that are preferable in the sense of "approximative orthogonalization" of the overall system

The last two items will not be discussed further, although such observations are obviously essential from the motivation point of view of the GLM concept. The other statements will be tackled in a way that is original mainly in its generality, including though some genuine stretching of the convention [Haykin, 1989] concerning consequences of the GLM configuration. A similar generalization or lifting of the basic concepts has been given in [Oliveira e Silva, 1995] for the more explicit case of sequentially generated rational orthonormal functions.

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[67]Here, the choice of the matrix inner product definition, $\mathbf{R} = (\mathbf{X}, \mathbf{X})^T$, or alternatively $\mathbf{R}^* = (\mathbf{X}, \mathbf{X})$ for $\mathbf{R}^H = \mathbf{R}$, is reflected as $r_{ij} = (X_i, X_j)^* = (X_j, X_i)$.

The key observation behind the previous statements is that for an orthonormal set of functions and an arbitrary constant function $S_{xx}(z) = c$, the formula (3.62) produces a diagonal matrix $c\mathbf{I}$, which is positive definite if and only if c is positive-real. In particular, any bounds for the (presumed to be real on $\mathbb{T}$) function $S_{xx}(z)$, $S_{min} \leq S_{xx}(e^{j\omega}) \leq S_{max}$, result in an inequality $S_{min}\mathbf{I} \leq \mathbf{R} \leq S_{max}\mathbf{I}$, with an appropriate order relation for matrices[68]. In Sections 3.2.2 and 3.2.3 the properties of the GLM correlation matrix in terms of its eigenvalues where characterized using the quadratic or Hermitian form and its normalized counterpart, the Rayleigh quotient. The orthogonality of the GLM structure provides a step further where the eigenvalue spread is bounded directly by the minimum and maximum of the power spectral density of the input signal[69]. Utilizing a compact (conjugated) denotation for (3.62),

$$\mathbf{R}^* = \frac{1}{2\pi j} \oint_{\mathbb{T}} \mathbf{g}(z)\mathbf{g}^H(1/z^*)S_{xx}(z)\frac{dz}{z}, \quad \mathbf{g}(z) = [G_1(z) \ \cdots \ G_N(z)]^T, \qquad (3.63)$$

the quadratic form of the correlation matrix, $\mathbf{w}^H\mathbf{R}\mathbf{w}$, for an arbitrary non-zero vector $\mathbf{w} \in \mathbb{C}^N \setminus \{\mathbf{0}\}$, can be expanded as[70]

$$\mathbf{w}^H\mathbf{R}\mathbf{w} = \mathbf{w}^T\mathbf{R}^*\mathbf{w}^* = \mathbf{w}^T \left[\frac{1}{2\pi j} \oint_{\mathbb{T}} \mathbf{g}(z)\mathbf{g}^H(1/z^*)S_{xx}(z)\frac{dz}{z} \right] \mathbf{w}^* \qquad (3.64)$$

$$= \frac{1}{2\pi j} \oint_{\mathbb{T}} \mathbf{w}^T\mathbf{g}(z)\mathbf{g}^H(1/z^*)\mathbf{w}^* S_{xx}(z)\frac{dz}{z}$$

$$= \frac{1}{2\pi j} \oint_{\mathbb{T}} \|\mathbf{w}^T\mathbf{g}(z)\|^2 S_{xx}(z)\frac{dz}{z}. \qquad (3.65)$$

This expression is still completely general and not particularly revealing as such. However, the norm term in (3.65) is always strictly positive, which passes the non-negativity requirement explicitly to the function $S_{xx}(z)$. Furthermore, either of the integrals (3.64) or (3.65) emphasize that $\mathbf{w}^H\mathbf{R}\mathbf{w} = \mathbf{w}^H\mathbf{w}$ for $S_{xx}(z) = 1$, directly in terms of the orthonormality of the functions $G_i(z)$, $i = 1, \ldots, N$, that is, without resorting to the presumption $\mathbf{R} = \mathbf{I}$. Now supposing that $S_{min} > 0$ is a lower bound for the function $S_{xx}(z)$ on $\mathbb{T}$, possibly the tight one $S_{min} = \min_{\omega \in [0 \ 2\pi]} S_{xx}(e^{j\omega})$, then the quadratic form is bounded below as

$$\mathbf{w}^H\mathbf{R}\mathbf{w} = \mathbf{w}^T \left[\frac{1}{2\pi j} \oint_{\mathbb{T}} \mathbf{g}(z)\mathbf{g}^H(1/z^*)S_{xx}(z)\frac{dz}{z} \right] \mathbf{w}^*$$

$$\geq \mathbf{w}^T \left[\frac{1}{2\pi j} \oint_{\mathbb{T}} \mathbf{g}(z)\mathbf{g}^H(1/z^*)S_{min}\frac{dz}{z} \right] \mathbf{w}^* = S_{min}\mathbf{w}^H\mathbf{w}. \qquad (3.66)$$

The inequality (3.66) states explicitly that the correlation matrix is positive definite. In particular, the GLM modeling signals are guaranteed to be linearly independent.

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[68]For two square matrices $\mathbf{A}$ and $\mathbf{B}$ of equal dimensions, the order relation $\mathbf{A} \leq \mathbf{B}$ is defined as the positive semidefiniteness of $\mathbf{B} - \mathbf{A}$.

[69]For simplicity, the function $S_{xx}(e^{j\omega})$ is considered to be continuous, and thus bounded, on $\mathbb{T}$. As long as the point is to utilize the fact that a function is *essentially* bounded, it would somehow seem unnecessary and unessential to complicate the notation at every step to include oddities in the input signal space. After all, the attribute 'essentially' amounts to "almost everywhere" in the Lebesgue integral (measure) sense, and thus, the power spectral density function can already be considered to be (essentially) represented by its smoothed counterpart.

[70]The fact that $\mathbf{w}^H\mathbf{R}\mathbf{w}$ is real is used to simplify notations through conjugation. Equally obviously it holds that $\mathbf{w}^T\mathbf{w}^* = \mathbf{w}^H\mathbf{w}$.

Similarly, for an upper bound, $S_{max} \geq \max_{\omega \in [0\ 2\pi]} S_{xx}(e^{j\omega})$, the quadratic form is bounded above as

$$\mathbf{w}^H \mathbf{R} \mathbf{w} \quad \leq \quad \mathbf{w}^T \left[\frac{1}{2\pi j} \oint_{\mathbb{T}} \mathbf{g}(z) \mathbf{g}^H(1/z^*) S_{max} \frac{dz}{z} \right] \mathbf{w}^* = S_{max} \mathbf{w}^H \mathbf{w}. \quad (3.67)$$

These two inequalities provide bounds for the normalized quadratic form,

$$S_{min} \leq \frac{\mathbf{w}^H \mathbf{R} \mathbf{w}}{\mathbf{w}^H \mathbf{w}} \leq S_{max}, \quad (3.68)$$

for any $\mathbf{w} \neq \mathbf{0}$ of appropriate dimension. In particular, the bound remain valid for the vector $\mathbf{w}$ that minimize and maximize the Rayleigh quotient, respectively, which according to (3.24), provide the minimum and maximum eigenvalues λ_{min} and λ_{max} of $\mathbf{R}$, and consequently the chain of inequalities

$$S_{min} \leq \lambda_{min} \leq \lambda_{max} \leq S_{max}. \quad (3.69)$$

Finally, the condition number of the correlation matrix, as introduced in Section 3.2.3, has an upper bound[71]

$$\chi(\mathbf{R}) = \frac{\lambda_{max}}{\lambda_{min}} \leq \frac{S_{max}}{S_{min}} \quad (3.70)$$

This upper bound is finite for $S_{min} > 0$ and $S_{max} < \infty$. The remarkable thing about the bounds (3.69) and (3.70) is that they are completely independent of the choice of orthonormal functions. Maybe even more surprisingly, the bounds are insensitive to an arbitrary increase in the number of orthonormal functions, although it is known that the eigenvalue spread and consequently the condition number are increasing functions of the dimension N of the correlation matrix[72]. This is reassuring from the point of view of numerical properties of solving the GLM normal equations for increasing correlation matrix dimensions, but it does not yet imply that the resulting model is unbiased, or asymptotically accurate.

By returning to the original GLM setup, the generic notion of the system to be modeled, $y = Hx$, and its GLM model, $\hat{y} = \hat{H}x$, $\hat{H} = \mathbf{w}^T \mathbf{g}$, provide an operator form for the modeling error $e = y - \hat{y} = (H - \hat{H})x$. The resulting error energy, $\|e\|^2 = (e, e)$, can then be expanded using any of the signal and inner product

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[71]In general, the condition number is defined as the product $\chi(\mathbf{R}) = \|\mathbf{R}\| \|\mathbf{R}^{-1}\|$, where $\|\cdot\|$ is some appropriate matrix norm. The particular expression (3.70) can be seen to result from two alternative choices: i) the spectral norm $\|\mathbf{R}\|_S$, which is the square-root of the largest eigenvalue of $\mathbf{R}^H \mathbf{R}$, and consequently $\|\mathbf{R}^{-1}\|_S = \lambda_{min}^{-1}$, or ii) the (matrix) operator 2-norm $\|\mathbf{R}\|_2 = \sup \|\mathbf{R}\mathbf{w}\|_2 / \|\mathbf{w}\|_2$, where the operand is recognized as the square-root of the Rayleigh's quotient for $\mathbf{R}^H \mathbf{R}$. A simplified interpretation of $\chi(\mathbf{R})$ is that it measures how perturbations in $\mathbf{R}$ and $\mathbf{p}$ are reflected to the evaluation of the weight vector $\mathbf{w} = \mathbf{R}^{-1}\mathbf{p}$. From a more practical point of view, actual algorithms for solving the normal equations "operate on the diagonal", representable by the eigenvalues, and thus the numerical sensitivity is (more of less) directly related to multiplication and division operations on the eigenvalues.

[72]This is due to the *interlacing property* of the eigenvalues of any (Hermitian) symmetric matrix and its sub-matrices [Golub and Van Loan, 1989]: for any consecutive nested sub-matrices, $\mathbf{R}_{N+1} = [r_{ij}]_{i,j=1}^{N+1}$ and $\mathbf{R}_N = [r_{ij}]_{i,j=1}^N$, with respect to an imposed ordering, the eigenvalues are interlaced as $\lambda_1(\mathbf{R}_{N+1}) \leq \lambda_1(\mathbf{R}_N) \leq \lambda_2(\mathbf{R}_{N+1}) \leq \cdots \leq \lambda_N(\mathbf{R}_{N+1}) \leq \lambda_N(\mathbf{R}_N) \leq \lambda_{N+1}(\mathbf{R}_{N+1})$.

interpretations, for example using (2.79) as above, providing[73]

$$\|E\|^2 = (E, E) = \frac{1}{2\pi j} \oint_{\mathbb{T}} |E(z)|^2 \frac{dz}{z} = \frac{1}{2\pi j} \oint_{\mathbb{T}} \left| H(z) - \hat{H}(z) \right|^2 S_{xx}(z) \frac{dz}{z}. \quad (3.72)$$

This expression is valid for an arbitrary model $\hat{H} = \mathbf{w}^T \mathbf{g}$, that is, without presuming the orthogonality of $\mathbf{g}$ or the optimality of $\mathbf{w}$. It is however known that (3.72) is minimized for the solution of the GLM normal equation[74]. The quadratic nature of this formula suggest once more that possible bounds for the spectral density can be used to form bounds for the quantity (3.72),

$$S_{min}\|H - \mathbf{w}^T \mathbf{g}\|^2 \leq \|E\|^2 \leq S_{max}\|H - \mathbf{w}^T \mathbf{g}\|^2, \quad (3.75)$$

The bounding norm term is the energy of the error signal $e_h = h - \hat{h}$, that is, the modeling error energy (3.72) for $S_{xx}(z) = 1$, denoted in the following as $\|E_h\|^2$. Here too, the bounds have real meaning only if $S_{min} > 0$ and $S_{max} < \infty$. The same argument applies to the transposed counterpart of the inequality (3.75) given by

$$\|E\|^2/S_{max} \leq \|H - \mathbf{w}^T \mathbf{g}\|^2 \leq \|E\|^2/S_{min}. \quad (3.76)$$

This latter inequality is interesting from the point of view of potential identifiability of a genuinely unknown system H directly from the input-output data, which also implies that the quantity $\|E\|^2$ is evaluated or approximated with some other means than (3.72), that is, without factoring out the unknown[75].

The formula (3.72) as well as the inequalities (3.75) and (3.76) are still completely general. Actual substance is brought into the consideration by first restricting to the optimal model parametrizations and then by exploiting the consequences of choosing an orthonormal model. The minimization of $\|E\|^2$ and $\|E_h\|^2$, as given above, correspond to two different but well-defined GLM configurations, with unique solutions with respect to the weights, denoted as $\mathbf{w}$ and $\mathbf{c}$, respectively. In particular,

$$S_{min}\|H - \mathbf{c}^T \mathbf{g}\|^2 \leq \|y - \mathbf{w}^T \mathbf{x}\|^2 \leq S_{max}\|H - \mathbf{c}^T \mathbf{g}\|^2, \quad (3.77)$$

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[73]To avoid presuming something like $|(H-\hat{H})X|^2 = |(H-\hat{H})|^2|X|^2$ in the integrand, the generic time-domain inner product may first be expanded as

$$\|e\|^2 = (y - \hat{y}, y - \hat{y}) = ((h - \hat{h}) * x, (h - \hat{h}) * x) = ((h - \hat{h}) * r_{xx}, (h - \hat{h})). \quad (3.71)$$

Either of the last two expressions then result in (3.72) with respect to the inner product (2.79). Alternatively, the inner product $(Y(z) - \hat{Y}(z), Y(z) - \hat{Y}(z))$ can also be expanded directly, producing the integrand $(H(z) - \hat{H}(z))X(z)(H(1/z^*) - \hat{H}(1/z^*))^* X^*(1/z^*)$, which reduces to the desired form.

[74]In the case of optimal model weights, $(e, \hat{y}) = 0$, which simplifies the energy formula,

$$\|e\|^2 = (e, y - \hat{y}) = (e, y) = (y - \hat{y}, y) = (y, y) - (\hat{y}, y) = (y, y) - (\mathbf{w}^T \mathbf{x}, y) = (y, y) - \mathbf{w}^T (\mathbf{x}, y). \quad (3.73)$$

The last inner product is the (conjugated) correlation vector $\mathbf{p}^* = (\mathbf{x}, y) = (\mathbf{g} * x, y) = (\mathbf{g}, r_{yx})$, which provide a companion for (3.72) in terms of the cross-spectral density $S_{xy}(z) = S_{yx}^*(1/z^*)$,

$$\|e\|^2 = (y, y) - \mathbf{w}^T \mathbf{p}^* == (y, y) - \mathbf{w}^T \left[\frac{1}{2\pi j} \oint_{\mathbb{T}} \mathbf{g}(z) S_{xy}(z) \frac{dz}{z} \right]. \quad (3.74)$$

[75]As a reminder, the "genuine system identification setup" in GLM means that in addition to some description of the input and output signals, the partial model responses are also somehow attainable. This does not however necessarily require the explicit knowledge of the the impulse responses $\mathbf{g}$.

where $\mathbf{x}$ constitute of the partial model responses $\mathbf{x} = \mathbf{g}x$ (or $\mathbf{x} = \mathbf{g} * x$).[76] Using the chosen distinguishing terminology, the inequality (3.77) states that the identification or modeling error energy is bounded by the corresponding approximation error energy, within the frame defined by the spectral density of the input signal[77]. This ensures also smoothness in the limiting case $S_{max}/S_{min} \to 1$ in the sense that the error norms as well as the weights must eventually coincide.

The inequality (3.77) and its transposed counterpart of the form (3.76) guarantee that if one of the parties, $\|E\|^2$ or $\|E_h\|^2$, converges to zero, then so must also the other one, provided that $1 \leq S_{max}/S_{min} < \infty$. Moreover, in the case of convergence, the asymptotic rate of decay will coincide, although the relative values of $\|E\|^2$ and $\|E_h\|^2$, as well as the two sets of weights, will not in general ever be the same. The notion of convergence is obviously with respect to the generic dimension of the approximation space spanned by the GLM modeling signals[78]. The point here is that the concept of completeness of a basis system in a Hilbert space (Section 2.3.2) transforms naturally into a condition for universal convergence of the modeling configuration. For example, the approximation error $\|E_h\|^2$ will tend to zero, unconditionally for any H, if and only if the system $\{g_i\}_{i=0}^{\infty}$ is complete, which also forces the corresponding identification setup to converge[79]. In the case of a complete orthonormal system, the condition is somewhat more explicit since the Fourier coefficients are independent of approximation order: an orthonormal system $\{g_i\}_{i=0}^{\infty}$ is complete if and only if

$$\lim_{N \to \infty} \left\| H - \sum_{i=0}^{N} (H, g_i)g_i \right\|^2 = \lim_{N \to \infty} \left| \sum_{i=N+1}^{\infty} (H, g_i) \right|^2 = 0 \qquad (3.78)$$

for all H in the associated Hilbert space. As before, for an input signal such that $1 \leq S_{max}/S_{min} < \infty$, the condition (3.78) ensures that also the corresponding identified configuration is asymptotically accurate or unbiased. That is, a complete orthonormal GLM structure induces a complete set of modeling signals for the system space. The subsequent GLM problems are still in general non-trivial and disconnected, dependent on the choice of model order, but the underlying generating orthonormal structure ensures that the progression is well-posed and easily quantizes in terms of (3.77) and (3.78). In addition, the earlier result (3.70) characterize the numerical sensitivity of solving the GLM normal equations.

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[76]The fact that most likely $\mathbf{w} \neq \mathbf{c}$ was the reason why $\mathbf{w}$ was not allowed to appear simultaneously on different sides of inequalities.

[77]A practical application of (3.77) could be that the solution of the approximation problem (with respect to a known H and a chosen $\mathbf{g}$) can be used to quantize $\|E\|^2$ in terms of the input signal statistics, that is, to anticipate the behavior of an identification configuration. The evaluation of the approximation error energy is particularly simple in the orthonormal case, $\|E_h\|^2 = (H, H) - \|\mathbf{c}\|^2$, where $\mathbf{c} = (H, \mathbf{g})^T$ are Fourier coefficients of H with respect to $\mathbf{g}$.

[78]Any appropriate set of GLM modeling signals, for example $\{g_i\}_{i=0}^{\infty}$ and $\{x*g_i\}_{i=0}^{\infty}$, respectively, define a sequence of "nested" approximation spaces, with corresponding optimal GLM weights, and consequently an unique limit for the approximation error norm. This limit is not necessarily zero and there is not in general "any common structure" in the subsequent solutions.

[79]The same can be stated in terms of $\|E\|^2$ and $\{x_i\}_{i=0}^{\infty}$. In both cases, the linear independency presumption provides an unique decomposition (model parametrization) for all orders, and the property that such a representation converges (in the norm) to an arbitrary element in the space is characterized as completeness of the spanning system. This is one of the occasions when it is essential that the "signal and system spaces" are interchangeable.
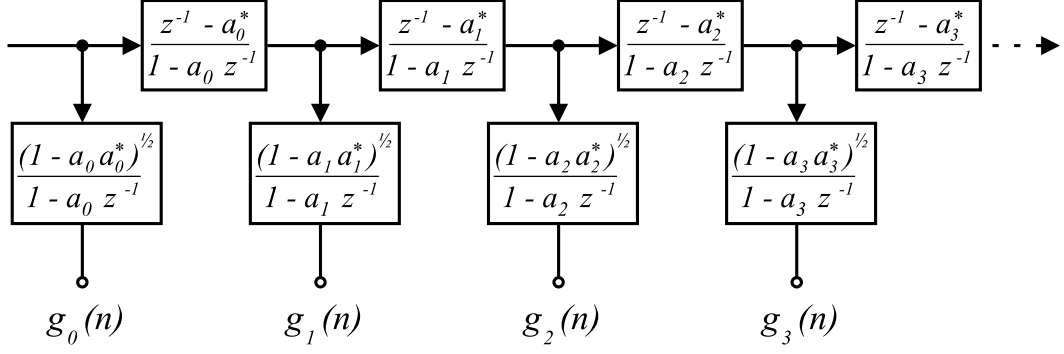
Figure 3.9: A block diagram for implementing rational orthonormal (transfer) functions of the form (3.79). The corresponding time-domain functions, $g_i(n)$, $i = 0, 1, \ldots$, are produced as tap-output impulse responses, and the orthonormality is manifested by the property that $(g_i, g_j) = \sum_{n=0}^{\infty} g_i(n) g_j^*(n) = 0$, $i \neq j$, and $(g_i, g_i) = 1$. The backbone of the filter is a cascade of first-order allpass filters and the tap-output blocks are corresponding first-order all-pole filters with a normalizing gain term.

The existence of complete orthonormal bases for the Hilbert spaces $H^2(\mathbb{E})$ and $\ell^2(\mathbb{N})$ was confirmed in Section 2.4.3 by deducing a base: the z-domain functions

$$G_i(z) = \frac{\sqrt{1 - a_i^* a_i}}{1 - a_i z^{-1}} \prod_{j=0}^{i-1} \frac{z^{-1} - a_j^*}{1 - a_j z^{-1}}, \quad i = 0, 1, \ldots, \quad \{a_j\}_{j=0}^{\infty} \subset \mathbb{D}, \qquad (3.79)$$

are orthonormal in $H^2(\mathbb{E})$ for any choice of poles in $\mathbb{D}$, and furthermore, the set $\{G_i(z)\}_{i=0}^{\infty}$ is complete in $H^2(\mathbb{E})$ if and only if the poles fulfil the additional condition

$$\sum_{j=0}^{\infty} (1 - |a_j|) = \infty. \qquad (3.80)$$

The corresponding complete orthonormal base $\{g_i(n)\}_{i=0}^{\infty}$ of $\ell^2(\mathbb{N})$ is attained as inverse z-transforms of functions (3.79), or equivalently, as impulse responses of synthesis filters with transfer functions (3.79). The latter alternative is emphasized in Figure 3.9, which also explicates that functions (3.79) have a nested or transversal structure.

The general construction of rational orthonormal functions given by (3.79) is precisely of the form that will be used in the next Chapter. However, this sudden burst of explicitness in the choice of GLM structure is somewhat premature in the sense that it is still used to furnish a more general consideration, in particular, to characterize the relation between the orthogonality of a structure and the allpass property.

An immediate and profound implication of the construction (3.79) is that a system of rational functions can be dense in $H^2(\mathbb{E})$; the uncountably infinite space $H^2(\mathbb{E})$ is proven to be separable simply by deducing a countable base. The conclusion is in fact more general, any element in $H^2(\mathbb{E})$ can be approximated arbitrarily well by a (finite) rational system, not necessarily orthogonal, which is very encouraging from the point of view of the GLM concept. The deduction of functions (3.79) in Section 2.4.3 was

admittedly fragmental but it related the derived orthonormal functions in an explicit way to the generating allpass operator[80]. It is not difficult to show that an arbitrary cascade connection of allpass functions can be used to construct (or more precisely to argue the existence of) generic orthonormal structures as a formal generalization of the tapped transversal structure of Figure 3.9 [Paatero, 2000]. However, it is not the case that all orthonormal structures would somehow be related to allpass functions. As a brute example, the sequence of rational functions[81]

$$\tilde{G}_j(z) = C_j \frac{\prod_{i=0}^{j-1}(\prod_{k=0}^{2^i-1}(z^{-1} - a_{kj}^*))}{\prod_{k=0}^{2^j-1}(1 - a_{ki}z^{-1})}, \quad j = 0, 1, \ldots \quad \{a_{kj}\} \subset \mathbb{D} \tag{3.81}$$

is orthogonal, and orthonormal with normalizing coefficients $C_j \in \mathbb{C}$ such that $(\tilde{G}_j, \tilde{G}_j) = 1$, for any choice of pole sequences in $\mathbb{D}$.[82] These functions constitute also a nested or transversal synthesis structure but the intermediate blocks are no longer allpass filters. The whole construction is obviously based on excessive pole-zero cancelations: all poles of $\tilde{G}_l(z)$, $l < j$, are canceled out by zeros of $\tilde{G}_j^*(1/z^*)$ to ensure orthogonality with respect to the z-domain inner product. This suggests that the connection between orthogonal and allpass structures is somehow related through an irreducibility requirement of the system of functions (when linearly combined). From another point of view, the interconnection can be seen to be guaranteed if the overall system admits to a minimal realization. In state-space terms (Section 2.1.4), the minimality of the state-space description[83] is then equivalent to the controllability and observability of the state-variable, and consequently, to the possibility of choosing an orthogonal (or input balanced) realization [Mullis and Roberts, 1976] [Moore, 1981].

The relation between structural orthogonality and the allpass (or loss-less) property has been considered from various points of view under the title *orthogonal filters* [Deprettere and Dewilde, 1980] [Dewilde and Dym, 1981] [Vaidyanathan, 1985]. However, the interest is usually not directed to tapped orthogonal structures, as it is here, and even if it is, it is more commonly the ARMA lattice structure [Gray and Markel, 1975] that emerge as a representative for orthogonal state-space realizations [Vaidyanathan, 1985b] [Regalia et al., 1988]. The lattice counterpart of orthogonal structures [Regalia, 1995] can be seen to result from a feedback connection of (balanced) allpass realizations whereas the state-space realization for

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[80]Another immediate consequence of the construction is that the subspace that a particular allpass operator induces is independent of the ordering of the first order blocks in the Blaschke product (2.101); the basis functions with respect to permutations of the pole set are obviously different but all these bases are related through an unitary transformation.

[81]This is a complex and corrected version of the "general orthonormal construction" proposed by Broome [Broome, 1965] as an intermediate phase in producing the real rational counterpart of functions (3.79). As stated, Broome's construction is incorrect because it misses some of the essential pole-zero cancelations for $j > 2$, which cumulates fast into a completely erratic construction. However, the idea is clear and it was adopted to this anecdotal use. Nevertheless, it is somewhat surprising that this formula or its variants do not, to the authors knowledge, appear anywhere in the literature.

[82]The completeness of the generation is not needed here – at least for rational approximations with respect to disjoint sequences of poles the completeness constraint seem to be of the form (3.80) [Walsh, 1969].

[83]An overview and some historic notes on minimal state-space realizations is given in [De Schutter, 2000]. An early reference of the state-space description itself is provided by [Kalman, 1960].

rational orthonormal functions of the form (3.79) are related to the cascade or forward connection[84]. This latter observation together with state-space balancing techniques [Moore, 1981] has been the basis for deducing rational orthonormal basis functions in state-space form [Heuberger, 1991] [Bodin and Wahlberg, 1994] [Heuberger et al., 1995] [Oliveira e Silva, 1995]. These constructions are usually directed to provide a system identification framework (including multiple-input multiple-output systems) with respect to the generalized basis and a related transform theory [Van den Hof et al., 1994] [Heuberger et al., 2003] [Wahlberg, 2003].[85] In addition to the ease of handling vector and matrix valued signals and functions, the state-space description is particularly justifiable when the whole realization problem is reassigned with respect to the generalized basis representation [Szabó et al., 2000] [de Hoog, 2001].

The aim here is, however, just to lay out an orthogonal state-space structure that generates functions (3.79), which can be done in a pragmatic way. Furthermore, the consideration will be "input oriented", because it is precisely the input-to-state part that is identified as the generating structure[86]. The state-equation (2.34) is first rewritten in the form[87]

$$\mathbf{x}(n) = \mathbf{A}\mathbf{x}(n-1) + \mathbf{b}x(n), \tag{3.82}$$

where the dimension of the square matrix $\mathbf{A}$ and the column vector $\mathbf{b}$ is considered to be indefinite in the sense that it is just supposed to correspond to the number of functions concatenated from the sequence $\{G_i(z)\}_{i=0}^{\infty}$. The linear combination of any such collection of functions then has a minimal state-space realization (for example

---

[84]The family of balanced realizations of allpass filters is closed with respect to appropriate interconnections (as well as substitutions) of elements [Roberts and Mullis, 1987].

[85]An interesting extension of this *Hambo transform theory* relates a basis construction that is induced by a variable length block partition of the generating allpass chain to a linear time-varying operator in the transform domain [de Hoog et al., 2000].

[86]This excludes some of the conventional steps in a more general deduction. Usually the starting point is a stable and minimal state-space description $\mathbf{R} = [\mathbf{A}\ \mathbf{B}; \mathbf{C}\ \mathbf{D}]$ with controllability and observability Gramians $\mathbf{P}$ and $\mathbf{O}$ that are (positive definite) solutions of the *Lyapunov equations* $\mathbf{A}\mathbf{P}\mathbf{A}^H + \mathbf{B}\mathbf{B}^H = \mathbf{P}$ and $\mathbf{A}^H\mathbf{O}\mathbf{A} + \mathbf{C}^H\mathbf{C} = \mathbf{O}$. (Conversely, it is easy to see that the explicit definitions of the Gramians, $\mathbf{P} = \sum_{i=0}^{\infty} \mathbf{A}^i\mathbf{B}\mathbf{B}^H(\mathbf{A}^H)^i$ and $\mathbf{O} = \sum_{i=0}^{\infty}(\mathbf{A}^H)^i\mathbf{C}^H\mathbf{C}\mathbf{A}^i$, fulfill the Lyapunov equations.) This realization can always be made input balanced using a state transformation [Moore, 1981], that is, $\mathbf{P} = \mathbf{I} = \mathbf{A}\mathbf{A}^H + \mathbf{B}\mathbf{B}^H$, which is the orthogonality condition for the structure in [Roberts and Mullis, 1987]. It is then straightforward but somewhat cumbersome to show that the allpass presumption implies that $\mathbf{R}\mathbf{R}^H = \mathbf{R}^H\mathbf{R} = \mathbf{I}$ [Roberts and Mullis, 1987] [Oliveira e Silva, 1995], that is, orthogonality (or unitarity) of the structure in a stricter sense (characterized as orthogonal allpass [Roberts and Mullis, 1987] or lossless bounded(-real)[Vaidyanathan, 1985b] realizations). The latter form of the matrix product states that $\mathbf{O} = \mathbf{I}$ is also a solution of the second Lyapunov equation. The uniqueness of the solution is then either presumed, or alternatively, ensured through the invariability of the Hankel singular values (square-roots of the eigenvalues of $\mathbf{PO}$), which in this case are all equal to unity [Glover, 1984]. It is also possible to conclude more directly that $\mathbf{O} = \mathbf{P} = \mathbf{I}$ (with simultaneous input and output balancing), which guarantee an unique (and equal) solution for the Lyapunov equations [Pernebo and Silverman, 1982].

[87]This modified description is used to avoid a delay in the state-equation. The corresponding output-equation is $y(n) = \mathbf{c}^T\mathbf{x}(n-1) + dx(n)$, where the feed-through term is gathered in the scalar d. The z-transforms of the state-space equations are then $\mathbf{x}(z) = z(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}X(z)$ and $Y(z) = z^{-1}\mathbf{c}^T\mathbf{x}(z) + dX(z)$, which result in an external description $Y(z) = [\mathbf{c}^T(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + d]X(z)$ that is equivalent to the original form (2.37).

given by the "canonical realization" (2.35)). The minimality is reflected in the fact that the input-to-state impulse responses $\mathbf{g}(n) = \mathbf{A}^n\mathbf{b}$, $n = 0, 1, \ldots$, constitute a *controllability matrix* $\mathcal{C} = [\mathbf{b}\ \mathbf{Ab}\ \mathbf{A}^2\mathbf{b}\ \cdots]$ with full row rank. That is, the state-space signals $\mathcal{C}^T$ are linearly independent and the corresponding correlation matrix $\mathbf{P} = \mathcal{C}\mathcal{C}^H$, the *controllability Gramian*, is positive definite. The following relation will also be needed in the upcoming:

$$
\begin{aligned}
\mathbf{P} &= \sum_{n=0}^{\infty}(\mathbf{A}^n\mathbf{b})(\mathbf{A}^n\mathbf{b})^H = \sum_{n=1}^{\infty}\mathbf{A}^n\mathbf{b}\mathbf{b}^H(\mathbf{A}^H)^n + \mathbf{b}\mathbf{b}^H \\
&= \mathbf{A}(\sum_{n=1}^{\infty}\mathbf{A}^{n-1}\mathbf{b}\mathbf{b}^H(\mathbf{A}^H)^{n-1})\mathbf{A}^H + \mathbf{b}\mathbf{b}^H = \mathbf{A}\mathbf{P}\mathbf{A}^H + \mathbf{b}\mathbf{b}^H. \quad (3.83)
\end{aligned}
$$

The transfer function of the state-equation (3.82) is given by

$$
\mathbf{g}(z) = \sum_{n=0}^{\infty}\mathbf{A}^n\mathbf{b}z^{-n} = z(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}, \quad (3.84)
$$

where it is presumed that the eigenvalues of $\mathbf{A}$ (the poles of the system) are strictly within the unit circle to ensure stability. The realization can further be chosen to be input balanced, that is, with a controllability Gramian $\mathbf{P} = \mathbf{I}$, which is precisely the condition for an orthonormal GLM structure,

$$
\begin{aligned}
\mathbf{P}^* &= (\mathbf{g}(z), \mathbf{g}(z)) = (z(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}, z(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}) \\
&= [\mathbf{A}\ \mathbf{b}][\mathbf{A}\ \mathbf{b}]^H = \mathbf{A}\mathbf{A}^H + \mathbf{b}\mathbf{b}^H = \mathbf{I}, \quad (3.85)
\end{aligned}
$$

where the second line is due to (3.83) and it is the orthogonality condition for the state-equation coefficients [Roberts and Mullis, 1987]. The first component of the input-to-state transfer function is now identified as

$$
G_0(z) = \frac{\sqrt{1 - a_0 a_0^*}}{1 - a_0 z^{-1}} = \frac{z\sqrt{1 - a_0 a_0^*}}{z - a_0} = z(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}, \quad (3.86)
$$

where $\mathbf{A} = a_0$ and $\mathbf{b} = \sqrt{1 - a_0 a_0^*} \equiv n_O$. The input pair $(\mathbf{A}, \mathbf{b}) = (a_0, n_0)$ is balanced, $[\mathbf{A}\ \mathbf{b}][\mathbf{A}\ \mathbf{b}]^H = a_0 a_0^* + n_0^2 = 1$, and it implies an unique orthogonal allpass completion $(\mathbf{A}, \mathbf{b}, \mathbf{c}^T, d) = (a_0, n_0, n_0, -a_0^*)$. Every allpass block in the cascaded structure (3.9) has a similar balanced realization, $(a_i, n_i, n_i, -a_i^*)$, with $n_i = \sqrt{1 - a_i a_i^*}$. The augmented state-space realization of an overall cascaded system of $N + 1$ such blocks is then given by[88]

$$
\left[
\begin{array}{ccccc|c}
a_0 & 0 & \cdots & 0 & 0 & n_0 \\
n_1 n_0 & a_1 & \cdots & 0 & 0 & n_1(-a_0^*) \\
n_2(-a_1^*)n_0 & n_2 n_1 & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \ddots & a_{N-1} & 0 & \vdots \\
n_N \prod_{i=1}^{N-1}(-a_i^*)n_0 & \cdots & \cdots & n_N n_{N-1} & a_N & n_N \prod_{i=0}^{N-1}(-a_i^*) \\
\hline
\prod_{i=1}^{N}(-a_i^*)n_0 & \cdots & \cdots & -a_N^* n_{N-1} & n_N & \prod_{i=0}^{N}(-a_i^*)
\end{array}
\right]
=
\left[
\begin{array}{cc}
\mathbf{A} & \mathbf{b} \\
\mathbf{c}^T & d
\end{array}
\right]
$$

$$(3.87)$$

---

[88]This is a scalar allpass version of more general block matrix formulas, given for example in [Oliveira e Silva, 1995] and [de Hoog, 2001], which however share a common origin with this consideration [Roberts and Mullis, 1987]. The point is that the cascaded realization can be generated recursively as $\tilde{\mathbf{A}}_i = [\tilde{\mathbf{A}}_{i-1}\ \mathbf{0}; \mathbf{B}_i\tilde{\mathbf{C}}_{i-1}\ \mathbf{A}_i]$, $\tilde{\mathbf{B}}_i = [\tilde{\mathbf{B}}_{i-1}; \mathbf{B}_i\tilde{\mathbf{D}}_{i-1}]$, $\tilde{\mathbf{C}}_i = [\mathbf{D}_i\tilde{\mathbf{C}}_{i-1}\ \mathbf{C}_i]$, and $\tilde{\mathbf{D}}_i = \mathbf{D}_i\tilde{\mathbf{D}}_{i-1}$ from a sequence of appropriate balanced realizations $(\mathbf{A}_i, \mathbf{B}_i, \mathbf{C}_i, \mathbf{D}_i)$.

This realization is by construction orthogonal. It is in particular input balanced, $\mathbf{AA}^H + \mathbf{bb}^H = \mathbf{I}$, and the elements of the input-to-state transfer function vector $\mathbf{g}(z)$ are identified with the orthonormal functions (3.79):

$$G_{i-1}(z) = g_i(z), \quad \mathbf{g}(z) = z(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} = [g_1(z) \ g_2(z) \ \cdots]^T. \tag{3.88}$$

The construction (3.87) is still "generic" with respect to the order of the generating allpass function in the sense that additional blocks (poles) can be appended (or appropriate sub-systems can be isolated) without affecting the form of the generating structure. It is noteworthy that the eigenvalues of $\mathbf{A}$, the poles of the generating allpass function, appear explicitly on the diagonal, which is a consequence of the lower-triangular form of $\mathbf{A}$. The vector $\mathbf{b}$ contains the feed-through terms of transfer functions (3.79), d is the corresponding zero-lag term of the impulse response of the allpass operator $A(z)$, and the coefficients $\mathbf{c}$ perform the allpass completion in terms of the functions $z^{-1}\mathbf{g}(z)$, $A(z) = z^{-1}\mathbf{c}^T\mathbf{g}(z) + d$.

An excuse This was certainly not the most "pragmatic" way to deduce the relation. It would have been more straightforward just to state the orthogonality property (3.85) of a balanced realization of an allpass operator and then to use its known partition into primitive blocks (3.87) to recognize that the functions (3.79) are indeed produced by the state vector as in (3.88). It is also possible to isolate a state-variable description directly from the difference equations for the development (3.79), and consequently, to construct the orthogonal state-space structure without any reference to balanced realizations. (This has been done in the constrained Laguerre case [King and Paraskevopoulos, 1979] [Nurges and Jaaksoo, 1981] but also for the general (real rational) structure [Wahlberg, 1994].) However, the presented approach emphasizes a more general realization problem: it is straightforward to pass from the state-space description to the transfer function form, but the converse is not as simple. In the spirit of the GLM concept, the state-space of the minimal realization was first recognized to coincide with the subspace spanned by the corresponding functions (3.79). The coordinate change introduced by the input balancing then provided an orthonormal base for the subspace in question. All such bases were known to be unitarily equivalent and the particular structure was formed inductively by identifying the basic building blocks and by utilizing their combining rules. The original idea was also to deduce something illuminating using a more abstract consideration that would have related a (non-minimal) realization of the GLM partial model transfer functions to a reduced description in a proper (controllable) subspace using such concepts as polynomial co-prime factorization and subspace identification. The digest is very intriguing, but the author dare not cite anything aggravating amidst his confusion.

3.5   Prototype adaptive algorithms and the GLM

There is an apparent conceptual conflict between the GLM as a genuine filtering operation and the GLM as the solution of an approximation problem in the Hilbert spaces of signals and systems. Namely, the former is in principle a well-defined, although unspecified, causal signal processing task, but the latter is based on gener-

alized correlations with respect to infinite data records or global and time-invariant descriptions of the ingredients. However, the essential difference is not necessarily related to causality or finiteness requirements, but rather, calls for a correct interpretation of the available information. For example, the input and output signals, and the partial model responses may well have representations that enable the evaluation of the "global" correlation terms, although the actual model is running in "real-time" for some finite epoch starting from a reference time $n = 0$. Such a system is LTI but not causal in the sense that presumptions of the future inputs and outputs are used in the determination of the model parameters[89].

From another point of view, the identification of a system is commonly based on a finite collection of input and output data samples. In the time-domain this may mean that some windowing of the input and output signals is applied prior to the evaluation of the correlation terms[90]. Such a local description of the system results usually in a time-variant overall model,

$$\hat{y}(n) = \mathbf{w}^T(n)\mathbf{x}(n), \tag{3.89}$$

corresponding to a time-varying or *adaptive* normal equation,

$$\mathbf{R}(n)\mathbf{w}(n) = \mathbf{p}(n). \tag{3.90}$$

Equation (3.90) aims to indicate that input and output data up to the time $n > 0$ is somehow used to form the normal equations and to provide the weights for the model (3.89). This does not however necessarily mean that the evaluation of the correlation terms should terminate at time-index n. In fact, a correct interpretation of the GLM in a Hilbert space would require that the evaluation of the correlations should run until the end[91]. In practice this could mean that the "memory of the model" is drained into the evaluation of the correlations by feeding a suitable number of zeros into the model. The term *autocorrelation method* is conventionally associated to such an interpretation, which is however even more confusing in the case of the GLM than it already is for the FIR model [Haykin, 1996]. As a more practical alternative, the correlation terms in (3.90) can be approximated in the time-domain by cumulative sums of the form

$$\mathbf{R}(n) = \sum_{i=0}^{n} \mathbf{s}^H(i)\mathbf{s}(i) \qquad and \qquad \mathbf{p}(n) = \sum_{i=0}^{n} \mathbf{s}^H(i)y(i), \tag{3.91}$$

where $\mathbf{s}(n) = \mathbf{x}^T(n)$, the transpose of the tap-output vector. Continuing with vague analogies, this would then correspond to the *covariance* or *pre-windowing method*

---

[89]Regardless of the interpretation of the signals as deterministic or stochastic, the quantities $X(z)$ and $Y(z)$, or $S_{xx}(z)$ and $S_{xy}(z)$, respectively, determine the weight vector $\mathbf{w}$ with respect to a chosen GLM structure, although the model operate causally according to $\hat{y}(n) = \mathbf{w}^T\mathbf{x}(n)$, $\mathbf{x}(n) = \mathbf{g}(n) * x(n)$, apparently unaware of $x(k)$ and $y(k)$ for $k > n$.

[90]Strictly speaking, the windowing imposes also violations on causality: even in the case when an FIR filter is chosen as the GLM, there are four basic presumptions of the data outside the observation window [Haykin, 1996].

[91]The finite input and output signals $x(n)$ and $y(n)$ are extended to $\ell^2(\mathbb{N})$ by appending zeros, $\{x(0), \ldots, x(n), 0, \ldots\}$ and $\{y(0), \ldots, y(n), 0, \ldots\}$. This implies that the elements of the correlation matrix should be evaluated with respect to the effective length of the partial model responses (in the time-domain), which can be characterized by the essential length of the partial model impulse responses. By the same reasoning, however, the evaluation of the correlation vector terminates at time-index n.

[Haykin, 1996], depending on how the generalization is interpreted. As in the case of the autocorrelation method, an actual block-by-block identification of the model with respect a segmentation of the input and output data would also require a strategy for the insertion of the initial conditions of the model. These and many other aspects of an actual implementation are too wide subjects to be considered in general. However, the following representations of a running or sequential adaptation of the time-varying weights in (3.89) are applicable without specifying a particular GLM.

The form of the cumulative sums (3.91) reveal directly how the contribution of the incoming data can be used to update the correlation terms. By using local estimates of the correlations, the recent past of the data is weighted over the distant past in the evaluation of the model weights. In particular, the aim is to form estimates directly for the ingredients of the solution of the normal equation,

$$\hat{\mathbf{w}}(n) = \hat{\mathbf{R}}^{-1}(n)\hat{\mathbf{p}}(n) \quad \longleftarrow \quad \{\hat{\mathbf{R}}^{-1}(n-1), \hat{\mathbf{p}}(n-1), x(n), y(n)\}. \quad (3.92)$$

It is not a coincident that the appearance of (3.92) reminds of some former topics: matrix factorizations and diagonalizations provide various modifications of the following prototype adaptive algorithms. In addition, the numerical stability and convergence properties of the adaptation schemes are closely related to the general properties of the correlation matrix, in particular to its eigenvalues.

3.5.1 The generalized recursive least-square algorithm

The subsequent derivation of the recursive least-square (RLS) algorithm for the GLM is an almost straightforward generalization of the FIR filter RLS algorithm [Haykin, 1996]. One way to approach the subject is to consider an update formula for the correlation matrix,

$$\mathbf{R}(n) \approx \lambda \mathbf{R}(n-1) + \triangle\hat{\mathbf{R}}(n), \quad n = 1, 2, \dots, \quad (3.93)$$

where $0 < \lambda \leq 1$ is a "forgetting factor" that weights the contribution of the previous correlation matrix $\mathbf{R}(n-1)$. The matrix $\triangle\hat{\mathbf{R}}(n)$ is an estimate of the error $\mathbf{R}(n) - \mathbf{R}(n-1)$ and it is supposed to be a function of the incoming (filtered) data. By construction of the correlation matrix (3.91), the cumulative sum representation itself allow for the recursion formula $\mathbf{R}(n) = \mathbf{R}(n-1) + \mathbf{s}^H(n)\mathbf{s}(n)$, which is a special case of the recursion (3.93) for $\lambda = 1$. The recursion (3.93) introduces an exponential weighting on the evaluation of the the correlation matrix,

$$\mathbf{R}(n) = \sum_{i=0}^{n} \lambda^{n-i}\mathbf{s}^H(i)\mathbf{s}(i). \quad (3.94)$$

Due to the linearity of the model, this weighting can also be seen to result from an exponential windowing (with respect to $\lambda^{1/2}$) of the input signal $x(n)$ or the partial model responses $x_j(n)$, respectively. Moreover, if the model weights are temporarily considered to be fixed, then the operation of weighting or windowing can be passed

through to the model output, and consequently, the square of the corresponding modeling error signal may be expressed as

$$J(n) = \sum_{i=0}^{n} \lambda^{n-i} |e(i)|^2, \quad e(i) = y(i) - \mathbf{w}^T \mathbf{x}(i). \qquad (3.95)$$

Minimization of (3.95) with respect to the model weights is called the *exponentially weighted least-squares method*. The optimum corresponds to the solution of the normal equation (3.90), with $\mathbf{R}(n)$ defined by (3.94) and a similar formula for the correlation vector,

$$\mathbf{p}(n) = \sum_{i=0}^{n} \lambda^{n-i} \mathbf{s}^H(i) y(i). \qquad (3.96)$$

The recursion formulas for the correlation matrix and vector are thus given by

$$\begin{aligned}
\mathbf{R}(n) &= \lambda \mathbf{R}(n-1) + \mathbf{s}^H(n)\mathbf{s}(n), \quad n = 1, 2, \dots, \\
\mathbf{p}(n) &= \lambda \mathbf{p}(n-1) + \mathbf{s}^H(n) y(n), \quad n = 1, 2, \dots.
\end{aligned} \qquad (3.97)$$

The other objective was to form the recursion directly with respect to the solution of the normal equation, as in (3.92). By applying the *matrix inversion lemma* [Golub and Van Loan, 1989], the recursion formula for the inverse of the correlation matrix is attained as

$$\mathbf{R}^{-1}(n) = \lambda^{-1} \mathbf{R}^{-1}(n-1) - \frac{\lambda^{-2} \mathbf{R}^{-1}(n-1) \mathbf{s}^H(n) \mathbf{s}(n) \mathbf{R}^{-1}(n-1)}{1 + \lambda^{-1} \mathbf{s}(n) \mathbf{R}^{-1}(n-1) \mathbf{s}^H(n)}, \qquad (3.98)$$

$n = 1, 2, \dots$. The denotations $\mathbf{P}(n) = \mathbf{R}^{-1}(n)$ and

$$\mathbf{k}(n) = \frac{\lambda^{-1} \mathbf{P}(n-1) \mathbf{s}^H(n)}{1 + \lambda^{-1} \mathbf{s}(n) \mathbf{P}(n-1) \mathbf{s}^H(n)} \qquad (3.99)$$

provide a neater form for the equation (3.98),

$$\mathbf{P}(n) = \lambda^{-1} \mathbf{P}(n-1) - \lambda^{-1} \mathbf{k}(n) \mathbf{s}(n) \mathbf{P}(n-1), \quad n = 1, 2, \dots. \qquad (3.100)$$

A comparison and manipulation of (3.99) and (3.100) result in the relation $\mathbf{k}(n) = \mathbf{P}(n)\mathbf{s}^H(n)$, which states that the *gain vector* $\mathbf{k}(n)$ in the recursion (3.100) is related to the tap-output vector $\mathbf{x}^*(n) = \mathbf{s}^H(n)$ through the transformation defined by the correlation matrix. Finally, by collecting all these relations, a recursion formula for the solution of the normal equation, $\hat{\mathbf{w}}(n) = \mathbf{P}(n)\mathbf{p}(n)$, is attained:

$$\begin{aligned}
\hat{\mathbf{w}}(n) &= \mathbf{P}(n)\mathbf{p}(n) \\
&= \lambda \mathbf{P}(n)\mathbf{p}(n-1) + \mathbf{P}(n)\mathbf{s}^H y(n) \\
&= \lambda \mathbf{P}(n)\mathbf{p}(n-1) + \mathbf{k}(n) y(n) \\
&= \mathbf{P}(n-1)\mathbf{p}(n-1) - \mathbf{k}(n)\mathbf{s}(n)\mathbf{P}(n-1)\mathbf{p}(n-1) + \mathbf{k}(n) y(n) \\
&= \hat{\mathbf{w}}(n-1) - \mathbf{k}(n)\mathbf{s}(n)\hat{\mathbf{w}}(n-1) + \mathbf{k}(n) y(n) \\
&= \hat{\mathbf{w}}(n-1) + \mathbf{k}(n)[y(n) - \mathbf{s}(n)\hat{\mathbf{w}}(n-1)] \\
&= \hat{\mathbf{w}}(n-1) + \mathbf{k}(n)[y(n) - \hat{\mathbf{w}}^T(n-1)\mathbf{x}(n)].
\end{aligned} \qquad (3.101)$$

The term in the brackets is the (*a priori*) estimation error based on the previous weight vector, in contrast to the actual instantaneous (*a posteriori*) estimation error

$$e(n) = y(n) - \hat{\mathbf{w}}^T(n)\mathbf{x}(n). \tag{3.102}$$

In conclusion, the GLM-RLS algorithm is given by equations (3.99), (3.100) and (3.101) with respect to the partial model response $\mathbf{s}(n) = \mathbf{x}^T(n)$ and the estimation error $e(n) = y(n) - \hat{\mathbf{w}}^T(n-1)\mathbf{x}(n)$ for the incoming data $\{x(n), y(n)\}$. The algorithm is initialized by setting $\mathbf{P}(0)$ and $\mathbf{w}(0)$ to some reasonable values. The scaled unit matrix, $\mathbf{P}(0) = \alpha\mathbf{I}$, $\alpha > 1$, and the zero-vector, $\mathbf{w}(0) = \mathbf{0}$, are common choices, but especially in the case of the GLM it could be appropriate to use better balanced or model-specific initial estimates.

The essential differences between the general case and the orthonormal GLM structures, such as the FIR filter, are related to the signal scaling (or gain) introduced by the partial model responses. This implies also that the forgetting factor could or should actually be a matrix, $\mathbf{\Lambda} = diag(\lambda_1, \ldots, \lambda_N)$, where the distributed factors, $0 < \lambda_j \leq 1$, are somehow associated to the GLM tap-output responses. An actual convergence analysis would not be reasonably in the general case, but the guidelines seem to be very similar to that of the FIR model [Haykin, 1996]. The most important observation is that the RLS algorithm can be seen as a genuine limiting process for the method of least-squares, that is, the solution of the GLM normal equations. The convergence properties are once more related to the general properties of the correlation matrix, in particular, the eigenvalue distribution. The main difference between the general case and the orthonormal GLM structure is thus in the way the contribution of the model may be segregated from the properties of the input signal, as it was demonstrated in Section 3.4.2. However, the *independence assumptions* of the input statistics [Haykin, 1996] are by definition equally valid (and unrealistic) for the GLM filtered data vector $\mathbf{s}(n) = \mathbf{x}^T(n)$.[92] In addition, the positive definiteness of the correlation matrix estimates is essential in deducing the RLS algorithm, particularly in the use of the matrix inversion lemma, and it can be argued that constraints in terms of the conventional tap-input vector, $\mathbf{x}(n) = [x(n) \ x(n-1) \ \cdots \ x(n-N+1)]^T$, have natural counterparts in terms of the GLM partial model tap-output vector $\mathbf{x}(n) = [x_1(n) \ x_2(n) \ \cdots \ x_N(n)]^T$. These considerations are closely related to the persistency of excitation of the input signal, which was in Section 3.2.2 alleged to transfer to the data structure defined by the GLM. This opinion is supported by a consideration between persistency of excitation and general adaptive schemes [Bitmead, 1984].

Some remarks The main reason for bringing in the concept of adaptive filtering is to provide an option for reducing the gap between adaptive FIR and IIR filtering methods. The above RLS-GLM concept provides a general framework for some existing proposals for combining the RLS algorithm and special forms of IIR filters, such as, the Laguerre-domain adaptive filter (LDAF) [den Brinker, 1994], the RLS *Gamma filter* [Palkar and Principe, 1994] and the RLS fixed pole adaptive filter (FPAF) [Williamson and Zimmermann, 1996]. It should however be noted that

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[92]The independence assumption (for the FIR least-mean-square algorithm) has recently been removed [Butterweck, 1997], which was acknowledged in the later (fourth) edition of Haykin's book. This update was pointed out to the author by Dr. Bert den Brinker. This does not however affect the argument itself.

a direct utilization of the RLS algorithm on the data structure of the IIR filter (such as the state-variables $\mathbf{z}(n)$ and $\mathbf{w}(n)$ in (2.33) and (2.34), respectively,) do not fall into the category of adaptive GLM. This is obviously due to the fact that the induced partial models are not independent of the "parameters", but also because the standard linear regression formulation corresponds to an equation-error (EE) method [Ljung and Söderström, 1983], in contrast to the GLM that is by construction an output-error (OE) configuration[93]. In practice, and at least from the point of adaptive IIR filtering, it is mainly the OE configuration that has real physical meaning as a system model [Steiglitz and McBride, 1965]. The GLM concept provides an adaptive IIR filter setup that is genuinely in an OE form, and yet, corresponds to an unimodal or quadratic optimization problem, unlike OE IIR filter optimization in general. The distinction between the possible adaptivity or the recursive and non-recursive part of the general adaptive IIR filtering schemes also suggest a construction based on adaptive GLM with slowly varying partial models. There has been a few proposals in this direction [Salama and Cousseau, 1998] [Cousseau et al., 1998] [Ngia and Gustafsson, 1999].

Another remark is related to the algorithm variants of the RLS algorithm and their possible generalizations to other GLM structures than the FIR filter. As it was mentioned in Section 3.3.1, the Kalman filter framework has been used to deduce different QR decomposition-based and lattice-form RLS algorithms [Sayed and Kailath, 1994b].[94] The former is thus in principle based on a general property of the GLM and the latter was earlier related to (block-) Toeplitz forms and to structures with cascaded identical allpass blocks. However, the essential ingredient in both cases, as well as in their combinations, is the utilization of the simple shift structure of the data, which is usually lost in the implied generalizations. Interestingly enough, it has been shown that extended shift structures, in particular those that are related to the allpass shift operator, provide generalizations: an *extended Chandrasekhar recursion* [Sayed and Kailath, 1994] for fast fixer-order implementations and a *generalized Szegö-Levinson realization algorithm* [Dewilde et al., 1978] in the order-recursive case. Both of these ideas have been concretized in the special case of Laguerre filters [Merched and Sayed, 2001a] [Merched and Sayed, 2000]. Merched and Sayed have also proposed extensions to the general case of *transversal orthonormal IIR filters* [Merched and Sayed, 2001b] [Merched, 2003], that is, to the

---

[93]There are undoubtedly linear regression models that can be put into output-error form using filtered data [Ljung, 1987] [Söderström and Stoica, 1989], but the point is that the division between OE and EE methods that is natural in the system identification configuration is somewhat confusing from the signal processing point of view [Johnson Jr., 1984]. Anyone who has experimented with adaptive IIR filters knows that direct utilization of adaptive algorithms do not work: there is a categoric difference in controllability and potential "adaptivity" of the recursive and non-recursive parts, respectively, that has to be taken into account. Typically, slower or conservative adaption steps or various "off-line" techniques are used for the recursive part of the model. Such a difference in the underlying dynamics makes the direct EE configuration slightly awkward. The EE method is usually attributed with 'simple', 'linear','unimodal' but 'biased', whereas the OE method is 'more complicated', 'nonlinear', 'multimodal' but the 'correct choice' [Shynk, 1989b] [Regalia, 1995]. From another point of view, some algorithms, such as the Steiglitz-McBride method [Steiglitz and McBride, 1965] and its generalizations [Regalia, 1992], utilize an interplay between the EE and OE configuration to solve an OE identification problem using "linearizations" provided by an intermediate EE formulation.

[94]The matrix decomposition-based techniques are known as fast or square-root fixed-order algorithms where as the lattice-forms are related to order-recursive realizations.

filter structure of Figure 3.9, but this remark is merely a reminder for the author in the hope of a better understanding of the publications at a later time. In any case, and at least for the transversal GLM structure with identical allpass blocks, there is a clear connection between the generalized lattice structure and the block-Toeplitz form of the correlation matrix, which may in some cases be used to deduce "fast", "square-root", or "order-recursive" counterparts of the GLM-RLS algorithm.

3.5.2 The generalized least-mean-square algorithm

The main motivation for the whole construction of the GLM is that it provides an unimodal or *convex* approximation problem with respect to the chosen approximation subspace and the choice of measure. The uniqueness of the solution of the normal equation has also been utilized indirectly, for example, to provide developments for the correlation matrix, or as in the previous deduction, to validate the RLS algorithm. As an alternative to approximative or iterative solving of the normal equations, the unimodality of the approximation error energy surface may also be used directly to solve for the optimal model weights. The cost function for the approximation error energy, the square or mean-square error of the signal $e = y - \mathbf{w}^T\mathbf{x}$, with respect to the model weights is given by

$$
\begin{aligned}
J(\mathbf{w}) = (e, e) &= (y - \mathbf{w}^T\mathbf{x}, y - \mathbf{w}^T\mathbf{x}) \\
&= (y, y) - \mathbf{w}^H\mathbf{p} - \mathbf{p}^H\mathbf{w} + \mathbf{w}^H\mathbf{R}\mathbf{w} \ , \quad\quad (3.103)
\end{aligned}
$$

where alternative definitions for the correlation terms, $\mathbf{p}$ and $\mathbf{R}$, may be used to expand the inner product. The cost function (3.103) is in *quadratic form* with respect to the variable $\mathbf{w}$, which is just a restatement of the unimodal nature of the error energy surface[95]. By definition, the minimum of the quadratic form (3.103) occur at $\mathbf{w} = \mathbf{R}^{-1}\mathbf{p}$, which is consistent with the result provided by a rewritten form of (3.103), known in elementary calculus as the action of "completing the square".[96] The *method of steepest descent* is an old and widely used method of optimization,

---

[95]The inner product representation for the error energy (3.103) is neat and compact, and possibly by itself sufficient proof for the quadratic nature, but the actual expansions impose feasible tests for the declared conventions and definitions of this thesis. For example, according to the chosen rules for vector-valued inner products,

$$
\begin{aligned}
J(\mathbf{w}) = (y - \mathbf{w}^T\mathbf{x}, y - \mathbf{w}^T\mathbf{x}) &= (y, y) - (y, \mathbf{w}^T\mathbf{x}) - (\mathbf{w}^T\mathbf{x}, y) + (\mathbf{w}^T\mathbf{x}, \mathbf{w}^T\mathbf{x}) \\
&= (y, y) - (\mathbf{w}^T\mathbf{x}, y)^* - \mathbf{w}^T(\mathbf{x}, y) + \mathbf{w}^H(\mathbf{x}, \mathbf{x})^*\mathbf{w} \\
&= (y, y) - \mathbf{w}^H\mathbf{p} - \mathbf{p}^H\mathbf{w} + \mathbf{w}^H\mathbf{R}\mathbf{w} \ ,
\end{aligned}
$$

where in the last step, $\mathbf{p} = (\mathbf{x}, y)^* = (y, \mathbf{x})^T$, $\mathbf{w}^T\mathbf{p}^* = \mathbf{p}^H\mathbf{w}$, and $\mathbf{R} = (\mathbf{x}, \mathbf{x})^* = (\mathbf{x}, \mathbf{x})^T$. On the other hand, by utilizing the vector notation,

$$
\begin{aligned}
J(\mathbf{w}) = \mathbf{e}^H\mathbf{e} &= (\mathbf{y} - \mathbf{S}\mathbf{w})^H(\mathbf{y} - \mathbf{S}\mathbf{w}) = (\mathbf{y}^H - \mathbf{w}^H\mathbf{S}^H)(\mathbf{y} - \mathbf{S}\mathbf{w}) \\
&= \mathbf{y}^H\mathbf{y} - \mathbf{w}^H\mathbf{S}^H\mathbf{y} - \mathbf{y}^H\mathbf{S}\mathbf{w} + \mathbf{w}^H\mathbf{S}^H\mathbf{S}\mathbf{W} \\
&= \mathbf{y}^H\mathbf{y} - \mathbf{w}^H\mathbf{p} - \mathbf{p}^H\mathbf{w} + \mathbf{w}^H\mathbf{R}\mathbf{w} \ ,
\end{aligned}
$$

where the correlation terms are expressed using the signal matrix representations, $\mathbf{p} = \mathbf{S}^H\mathbf{y}$, $\mathbf{p}^H = \mathbf{y}^H\mathbf{S}$, and $\mathbf{R} = \mathbf{S}^H\mathbf{S}$.

[96]For a real polynomial $p(x) = C + Bx + Ax^2$, the extreme point $x = -B/(2A)$ is seen directly from the rewritten form $p(x)/A = (x + B/(2A))^2 - B^2/(2A)^2 + C/A$. It is a minimum point if $A > 0$,

where the minimization of a differentiable function over a vector field is conducted using local linearizations and increments proportional to the negative of the gradient vector. In terms of the function (3.103), the steepest descent algorithm for the weight update is given by

$$\mathbf{w}^{p+1} = \mathbf{w}^p - \mu \nabla J(\mathbf{w}^p), \tag{3.104}$$

where μ is the increment step-size parameter that is possibly time-variant and matrix valued. The iteration index p is denoted as a superscript to indicate that it may be an indirect function of the actual time-variable. The form of (3.104) differ from the usual convention [Widrow and Stearns, 1985] [Haykin, 1996] by omitting the factor $1/2$ in the increment, which is not by itself very interesting, but it is an indication of a more profound difference in the concept of differentiation with respect to a complex (vector) variable. The function (3.103) is by definition real-valued, whether or not the variable is real or complex, $\mathbf{w} \in \mathbb{R}^N$ or $\mathbf{w} \in \mathbb{C}^N$, but the difficulties arise from the fact that complex conjugation, as a composite action, is not an analytic or differentiable function. At least partly due to the same reason also the definition of the gradient itself is usually constructed by partitioning into real and imaginary parts [Huang and Chen, 1989] [Morgan, 1991] [Picinbono, 1991] [Haykin, 1996], which result in the aforementioned difference in the scaling of the gradient vector. In the following, a direct evaluation of the complex gradient is performed using chosen conventions of this thesis and an agreement on complex derivatives [Brandwood, 1983],

$$
\begin{aligned}
\nabla_w J(\mathbf{w}) &= \nabla_w((y,y) - \mathbf{w}^H \mathbf{p} - \mathbf{p}^H \mathbf{w} + \mathbf{w}^H \mathbf{R} \mathbf{w}) \\
&= \nabla_w(-\mathbf{w}^H \mathbf{p}) + \nabla_w(-\mathbf{p}^H \mathbf{w}) + \nabla_w(\mathbf{w}^H \mathbf{R} \mathbf{w}) \\
&= \nabla_w(-\mathbf{p}^H \mathbf{w}) + \nabla_w(\mathbf{w}^H \mathbf{R} \mathbf{w}) \\
&= -\mathbf{p}^* + \mathbf{R}^* \mathbf{w}^*,
\end{aligned}
\tag{3.105}
$$

where the intermediate steps implicate the calculation rules[97]. In addition to the scaling, the correlation terms are conjugated in contrast to the conventional appearance, which is a consequence of the chosen "unconjugated" form of the model. As a formal check, the gradient formula (3.105) ensures that $\nabla J(\mathbf{w}_0) = 0 \Leftrightarrow \mathbf{R}\mathbf{w}_0 = \mathbf{p}$, where $\mathbf{w}_0$ is the unique solution of the normal equation. In this sense the gradient can be seen as a perturbation term in the normal equation. Direct utilization of the gradient (3.105) in the weight update formula (3.104) would result in an algorithm where the correlation terms have to be somehow estimated, for example using recursions (3.97). This formulation, however, has the advantage that it avoids the inverting of the correlation matrix. The actual efficiency in computational complexity is accomplished by replacing the various estimates of the correlation terms by the instantaneous estimates

$$
\begin{aligned}
\hat{\mathbf{R}}(n) &= \mathbf{s}^H(n)\mathbf{s}(n) \tag{3.106} \\
\hat{\mathbf{p}}(n) &= \mathbf{s}^H(n)y(n). \tag{3.107}
\end{aligned}
$$

which is analogous to the assertion that $\mathbf{R}$ is positive definite. The corresponding factorization of (3.103) is given by $J(\mathbf{w}) = (y,y) - \mathbf{p}^H \mathbf{R}^{-1} \mathbf{p} + (\mathbf{w} - \mathbf{R}^{-1}\mathbf{p})^H \mathbf{R}(\mathbf{w} - \mathbf{R}^{-1}\mathbf{p})$, which is seen by expanding the last term. It is also noteworthy that all terms in (3.103) are real if they are grouped with respect to their degree, also the term $-\mathbf{w}^H\mathbf{p} - \mathbf{p}^H\mathbf{w} = -2\Re(\mathbf{p}^H\mathbf{w})$, which results in something close to $x = -B/(2A)$.

[97]Using the definitions proposed by Brandwood, $\nabla(\mathbf{w}^H\mathbf{p}) = 0$, $\nabla(\mathbf{p}^H\mathbf{w}) = \mathbf{p}^*$, and $\nabla(\mathbf{w}^H\mathbf{R}\mathbf{w}) = \mathbf{R}^*\mathbf{w}^*$ [Brandwood, 1983].

These instantaneous estimates are clearly special cases of the weighted correlations (3.94) and (3.96) for $\lambda = 0$, or alternatively, a degenerations of the recursive formulas (3.97) corresponding to memoryless updates[98]. From the point of view of the stochastic correlation terms, $\mathbf{R} = E[\mathbf{s}^H(n)\mathbf{s}(n)]$ and $\mathbf{p} = E[\mathbf{s}^H(n)y(n)]$, the instantaneous estimates are produced simply by omitting the expectation operator. A direct substitution of these estimates to the gradient formula (3.105) transforms the weight update (3.104) into

$$
\begin{aligned}
\mathbf{w}(n+1) &= \mathbf{w}(n) + \mu(\hat{\mathbf{p}}^* - \hat{\mathbf{R}}^*\mathbf{w}^*(n)) \\
&= \mathbf{w}(n) + \mu(\mathbf{s}^H(n)y(n) - \mathbf{s}^H(n)\mathbf{s}(n)\mathbf{w}(n))^* \\
&= \mathbf{w}(n) + \mu\mathbf{s}^T(n)(y(n) - \mathbf{s}(n)\mathbf{w}(n))^* \\
&= \mathbf{w}(n) + \mu\mathbf{x}(n)e^*(n), \qquad n = 0, 1, \ldots,
\end{aligned}
\tag{3.108}
$$

where $e(n)$ is the approximation error signal and where the iteration index and the time-variable are now identified[99]. The weight update algorithm (3.108) is called the *least-mean-square* (LMS) algorithm or the *stochastic gradient* algorithm, where the latter name is descriptive in the sense that it is actually a steepest descent algorithm with respect to the expected value of the weight vector [Widrow and Stearns, 1985].

Externally the GLM-LMS algorithm differs from its FIR filter counterpart mainly in the interpretation of the the data vector $\mathbf{x}(n)$. The algorithm is initialized by setting $\mathbf{w}(0) = \mathbf{0}$ or by a more sophisticated guess of the weight vector. As in the case of the FIR-LMS algorithm, the stability and convergence of the GLM-LMS algorithm is governed by the choice of the step-size parameter μ. Conventional constraints for the step-size parameter are given by

$$
0 < \mu < \frac{2}{\lambda_{max}(\mathbf{R})} \qquad \text{and} \qquad \sum_{i=l}^{N} \frac{\mu\lambda_i}{2 - \mu\lambda_i} < 1 ,
\tag{3.111}
$$

where λ_i, $i = 1, \ldots, N$, are eigenvalues of the corresponding correlation matrix [Widrow and Stearns, 1985] [Haykin, 1996]. The former constraint is related to the stability and the latter essentially to the final mismatch or bias with respect to the optimal solution. The potential convergence rate is obviously an increasing function of the step-size parameter, although conservative choices of μ are usually preferred

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[98]Which does not necessarily mean that the estimate is memoryless. One of the motivations for considering adaptive algorithms is that in the GLM framework the model itself can be seen as a potential storage for past estimates of the correlation terms.

[99]As a mere curiosity, there is also another and more generic way to deduce the algorithm (3.108) directly in the Hilbert space framework, that is, without resorting to the nature of the signals. Namely, the gradient can be approximated as

$$
\begin{aligned}
\nabla_w J(\mathbf{w}) &= \nabla_w(y - \mathbf{w}^T\mathbf{x}, e) = \nabla_w((y, e) - (\mathbf{w}^T\mathbf{x}, e)) \\
&\approx \nabla_w(-\mathbf{w}^T(\mathbf{x}, e)) = -(\mathbf{x}, e),
\end{aligned}
\tag{3.109}
$$

which utilize the fact that due to the orthogonality principle, $(y, e_{opt}) = 0$, also the gradient of (y, e) is negligible, at least in the vicinity of the minimum error e_{opt}. Now the evaluation of the inner product may be analogously estimated by an instantaneous signal product,

$$
\widehat{(\mathbf{x}, e)} \equiv \mathbf{x}(n)e^*(n),
\tag{3.110}
$$

which consequently results in the weight update algorithm (3.108).

since the constraints do not actually guarantee convergence in the deterministic sense and because the final misadjustment (bias in terms of the normalized excess mean-square error) is directly proportional to the product μN [Widrow and Stearns, 1985].

As in the case of the conventional LMS algorithm, the dilemma between convergence rate and steady-state mismatch suggests the use of an adaptive step-size parameter. A simple choice for the adaptive step-size parameter is given by

$$\mu(n) = \frac{\mu_0}{\epsilon + \mathbf{x}^H(n)\mathbf{x}(n)}, \quad 0 << \mu_0 < 2, \tag{3.112}$$

where ϵ is a small positive constant that ensures that the denominator of the expression is always strictly positive. The application of the step-size parameter (3.112) in the weight update (3.108) provides a straightforward generalization of the *normalized LMS algorithm* (NLMS) [Haykin, 1996]. The "normalization" is with respect to an estimate of the tap-output energy, given by the instantaneous tap-output power $\mathbf{x}^H(n)\mathbf{x}(n)$. The formula (3.112) is motivated by the observation that the largest eigenvalue in (3.111) has an upper bound, $\lambda_{max}(\mathbf{R}) \leq \sum_{i=0}^{N} \lambda_i = tr(\mathbf{R})$, where the *trace* of the correlation matrix, $tr(\mathbf{R})$, is much easier to evaluate or approximate than the actual eigenvalues[100]. In addition, an appropriate estimate of the trace of the correlation matrix will provide a bound for the step-size parameter that satisfy also the latter one of the conditions (3.111) [Haykin, 1996].

As the expression (3.112) shows, the trace $tr(\mathbf{R})$ is then approximated by the quantity $\mathbf{x}^H(n)\mathbf{x}(n)$. This simplification can once more be interpreted as an instantaneous estimate: for example using the definition $\mathbf{R} = E[\mathbf{s}^H(n)\mathbf{s}(n)]$, the estimation of the correlation matrix is attained by omitting the expectation operator, $\hat{\mathbf{R}} = \mathbf{s}^H(n)\mathbf{s}(n)$, and the trace of the matrix $\hat{\mathbf{R}}$ is consequently equal to $\mathbf{x}^H(n)\mathbf{x}(n)$. In the case of an orthonormal GLM structure it is also an estimation of the expected sum of squares of the input samples, the "tap-input power" [Haykin, 1996], although the relation is not as trivial as in the case of the FIR model. Moreover, the actual normalization introduced by $\mathbf{x}^H(n)\mathbf{x}(n)$ is even more complicated when the model structure is neither orthonormal nor Toeplitz; the estimate is in principle as valid as the conventional instantaneous tap-input power, and also from the practical point of view, the normalization performs desired averaging over the partial model responses, but this estimation is not as balanced as in the case of the FIR-NLMS algorithm. These observations however suggest that in the general case, the scalar function (3.112) should be replaced by a matrix valued adaptive step-size parameter $\mu(n) = diag\{\mu_1(n), \ldots, \mu_N(n)\}$, where the component functions are attributed separately to the partial model responses. The specialities that should be taken into account are primarily related to the scaling and inertia introduced by a particular

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[100]The trace of a (correlation) matrix is the sum of the main diagonal elements, which according to the unitary similarity transformation (3.31), equals the sum of the eigenvalues. For a Toeplitz matrix, such as the FIR correlation matrix, the diagonal elements are identical, which provides $tr(\mathbf{R}) = N r_{00}$, where r_{00} is one of the main diagonal elements. Now, for an FIR model, or any other orthonormal GLM structure with a Toeplitz correlation matrix, the correlation term r_{00} is also the mean-square value or energy of the input signal $x(n)$, so that the trace can be approximated by a truncation of the inner product (x, x), $tr(\mathbf{R}) \approx \mathbf{x}^H(n)\mathbf{x}(n)$. It should at this point also be noted that in some of the definitions of the GLM correlation terms, the "mean-scaling operation" $1/N(\cdot)$ is omitted since it is cancelled out in the normal equation. This is in fact the only instance in this thesis when this chosen facilitation causes confusion.

GLM. As in the case of the conventional NLMS algorithm, the step-size functions could also be modified subject to various criteria, such as, smoothing, freezing, thresholding, and dependence on the achieved modeling error [Greenberg, 1998].

Some additional remarks The LMS algorithm is usually associated with tracking time-varying systems in various applications of adaptive filtering. However, the fact that it is computationally a "lightweight method" may also be used as an alternative for solving the normal equations, for example, in LTI system identification or to replace block-by-block parametric representations in signal coding. There is also an interesting connection between orthogonal transform coding and the LMS algorithm, in particular, the block-coding based on the DFT may be implemented exactly using the LMS algorithm [Widrow et al., 1987]. In fact, any discrete orthogonal transform may be implemented using the LMS algorithm [Wang, 1991], which could be utilized in Section 3.3.5. The other remark is related to the lattice form of the LMS algorithm, the *gradient adaptive lattice* (GAL) algorithm, although such an introduction is not entirely accurate; the actual GAL update takes the form of a normalized LMS algorithm, but relative to its operation it could also be attributed to the lattice forms of the RLS family of algorithms. At least in principle, all GLM structures that produce a block-Toeplitz correlation matrix should have a GAL implementation, in particular, all GLM structures based on identical allpass blocks. The gradient-adaptive Laguerre-lattice (GALL) algorithm [Fejzo and Lev-Ari, 1997] is an example of such a non-trivial GLM representation.

Chapter 4

Rational orthonormal GLM and some audio oriented applications

In the context of signal processing, *rational orthonormal filter structures* were first introduced in the 1950's by Kautz, Huggins and Young [Kautz, 1954] [Huggins, 1956] [Young and Huggins, 1962]. Kautz showed that an orthogonalization process applied to a set of continuous-time exponential components produces orthonormal basis functions having particular frequency-domain expressions. Much earlier, Wiener and Lee [Lee, 1960] proposed synthesis networks based on some classical orthonormal polynomial expansions [Szegö, 1939]. The idea of representing functions in orthonormal components is elementary, but the essential observation in the aforementioned cases was that some time-domain basis functions have rational Laplace transforms with a recurrent structure, defining an efficient transversal synthesis filter structure.

Discrete-time rational orthonormal filter structures can be attributed to Broome [Broome, 1965] as well as the baptizing of the *discrete Kautz functions*, consequently defining the discrete-time *Kautz filter*. The point of reference in the mathematical literature for the utilization of rational orthonormal function expansions is somewhat arbitrary, but a reasonable choice are the deductions made in the 1920's to prove interconnections between rational approximations and interpolations, and the least-square problem, which were assembled and further developed by Walsh [Walsh, 1969].

There has been a renewed interest towards rational orthonormal filters and model structures over the last ten-fifteen years, mainly from the system identification point of view [Heuberger, 1991] [Van den Hof et al., 1994] [Oliveira e Silva, 1995] [Szabó and Bokor, 1997] [Bokor and Schipp, 1998] [Bultheel and De Moor, 2000]. The perspective has usually been to form generalizations to the well-established *Laguerre models* in system identification [King and Paraskevopoulos, 1979] [Nurges, 1987] [Mäkilä, 1990] [Wahlberg, 1991] and control [Zervos et al., 1988]. In this context, the *Kautz filter or model* has often the meaning of a two-pole generalization of the Laguerre structure [Wahlberg, 1994], whereupon further generalizations restrict, as well, to structures with identical blocks [Heuberger et al., 1995]. Another, and almost unrecognized, connection to recently active topics in digi-

tal signal processing is provided by the orthonormal state-space models for adaptive IIR filtering [Regalia, 1992] with some existing implications to Kautz filters [Cousseau et al., 1998] [Salama and Cousseau, 1998].

Kautz filters have found very little use in audio related signal processing: to cite rarities [Davidson and Falconer, 1991] [Ngia and Gustafsson, 1999] [Campi et al., 1999]. One of the reasons is certainly that results from such fields as system identification and control engineering do not seems to find their way very easily to the convention of audio engineering. The more inherent reason is that there is an independent but related tradition of *frequency warped* structures, which is already well-grounded and sufficient for many tasks in audio signal processing. Frequency warping provides an approximation of the *constant-Q resolution* of modeling [Oppenheim et al., 1971] as well as a good match with the *Bark scale* that is used to describe the psychoacoustical frequency scale of human hearing [Smith and Abel, 1999]. A recent overview of frequency warped filters is given in [Härmä et al., 2000].

The motivation for the following utilization of Kautz filters is based on the observation and opinion that there is a certain void in generality and perspective, both in the proposed utilizations of Kautz filters as well as in the warping-based view to the allocation of frequency resolution into the modeling. In this thesis the use of Kautz filters is demonstrated in the spirit of "pure filter synthesis", that is, given target responses are approximatively modeled as impulse responses of Kautz filters, which was actually their original usage. Quite surprisingly, to the author's knowledge there has not been many (or hardly any) proposals in this direction on the level of modern computational means in design and implementation. The aim is thus to genuinely challenge the conventional FIR and IIR filter design approaches, as well as, their warped counterparts. This mission has been initiated by a series of publications demonstrating potential audio related applications [Paatero et al., 2001] [Härmä and Paatero, 2001] [Penttinen et al., 2001] [Karjalainen and Paatero, 2001] [Paatero and Karjalainen, 2002] and by a few somewhat more methodological considerations [Paatero, 2002] [Paatero, 2003] [Paatero, 2004]. An overview of Kautz filters and their audio applications is given in [Paatero and Karjalainen, 2003].

The rest of this thesis is organized as follows. In Section 4.1 the rational orthonormal functions that have already popped up at several occasions (for example (2.111) and (3.79)) will be restated and placed into the linear-in-parameter model or filter framework. The particular form in which these Kautz filters will be used is pronounced, along with some general properties resulting from this choice. The basic tool for the most important part, the choosing of the Kautz filter poles, is presented in Section 4.3. Audio oriented case studies, including loudspeaker equalization, room response modeling, and modeling of an acoustic guitar body, are demonstrated and compared with more traditional approaches in Section 4.4.

4.1 A particular rational orthonormal GLM: The Kautz filter

As a short farewell for the GLM concept, the underlying model has all along been considered to be implementable as a causal stable finite-dimensional linear filter structure, corresponding to a rational transfer function of the form

$$\hat{H}(z) = \frac{P(z)}{Q(z)} = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_L z^{-L}}, \tag{4.1}$$

where time-invariancy of (all) the parameters $\{b_i, a_i\}$ is not necessarily required. The transfer function (4.1) has an infinite number of different implementations as a digital filter. For example a simple regrouping, $\hat{H}(z) = P(z)/(1 - Q'(z))$, suggests one direction of interpretations. On the other hand, any function of the form (4.1) has a "root-factored" representation

$$\hat{H}(z) = g_0 \frac{(z^{-1} - \beta_1)(z^{-1} - \beta_2) \ldots (z^{-1} - \beta_M)}{(z^{-1} - \alpha_1)(z^{-1} - \alpha_2) \ldots (z^{-1} - \alpha_L)}, \tag{4.2}$$

which in turn implies an infinite variety of cascade and parallel realizations with respect to different partial fraction expansions. The perspective of the GLM is in a sense the opposite: the partial models introduce a model of the form

$$G(z) = \sum_{i=0}^{N} w_i G_i(z) = \sum_{i=0}^{N} w_i \frac{P_i(z)}{Q_i(z)}, \tag{4.3}$$

where $\{G_i(z)\}$ are rational functions with fixed numerator and denominator polynomial factors $P_i(z)$ and $Q_i(z)$, respectively. The GLM is a fixed denominator (or pole) model in the sense that the parameters $\{w_i\}$ interfere only with the numerator terms, which is seen by expanding the expression into the common denominator form[1]. The model (4.3) is by construction parallel, but as it has already been observed, it may also describe a cascaded system, or even a lattice configuration.

4.1.1 Kautz functions and filters

A particularly efficient but at the same time general choice of partial models are the *Kautz functions*

$$G_i(z) = \frac{\sqrt{1 - z_i z_i^*}}{z^{-1} - z_i^*} \prod_{j=0}^{i} \frac{z^{-1} - z_j^*}{1 - z_j z^{-1}}, \quad i = 0, 1, \ldots. \tag{4.4}$$

The efficiency is related to the transversal nature of the functions (Figure 3.9) as well as to the favorable properties resulting from the mutual orthonormality of the functions. The choice is general in the sense that it is capable of representing any

---

[1]Actually some pole-zero cancelations may occur, but the parameters do not introduce new poles (at least other than $z = 0$).
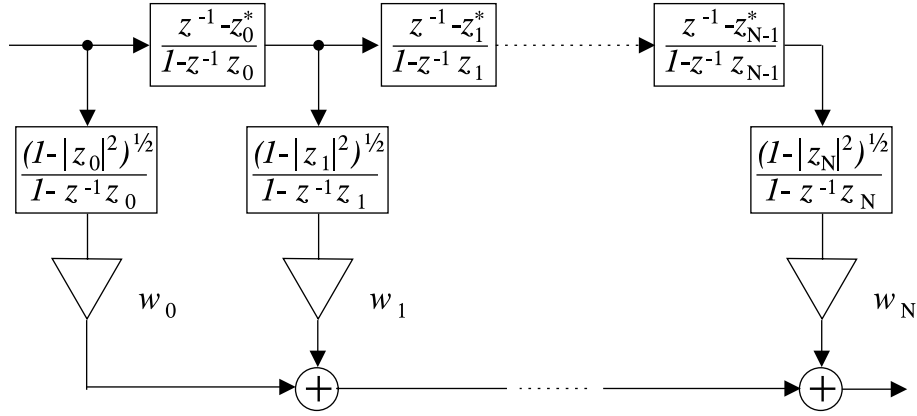
Figure 4.1: The Kautz filter. For $z_i = 0$ in (4.4) it degenerates to an FIR filter and for $z_i = a, -1 < a < 1$, it is a Laguerre filter where the tap filters can be replaced by a common pre-filter.

causal and stable (finite-dimensional) rational system with a proper choice of poles. The model structure induced by this choice of partial models is also efficient among other choices of the form (4.3) with respect to the same generating poles because it is known to posses a minimal realization (Section 3.4.2). The generic form of a Kautz filter is given by the transfer function representation

$$
\begin{aligned}
\hat{H}(z) &= \sum_{i=0}^{N} w_i G_i(z) \\
&= \sum_{i=0}^{N} w_i \left(\frac{\sqrt{1 - z_i z_i^*}}{1 - z_i z^{-1}} \prod_{j=0}^{i-1} \frac{z^{-1} - z_j^*}{1 - z_j z^{-1}} \right),
\end{aligned}
\tag{4.5}
$$

where w_i, $i = 0, \ldots, N$, are somehow assigned tap-output weights. The Kautz filter is thus a truncated weighted sum of Kautz functions as depicted in Figure 4.1. The filter structure (as well as the filter or model order) is uniquely defined by the choice of poles, whereas the tap-output weights are considered as "free parameters" to be specified. Defined in this manner, Kautz filters are merely a class of fixed-pole IIR filters, forced to produce orthonormal tap-output impulse responses. In the GLM framework, the weight determination process is inherently related to the least-square or minimum-mean square criterion, but many other error norms as well as error weighting schemes are possible. In the following however, the consideration is constrained to the LS approximation problem, that is, to the approximate modeling of a given system. It is apparent that the most important task is then how to choose the poles, which embody the essential practical contribution of this thesis.

4.1.2 Least-square approximation by Kautz filters

Clearly the most obvious and simple choice of model parametrization is to use the orthogonal expansion (or projection) coefficients with respect to the chosen orthonormal basis functions. The approximation of a given target response $h(n)$ (or $H(z)$) is obtained as its truncated Fourier series expansion with respect to the time- or

frequency-domain Kautz functions,

$$\hat{h}(n) = \sum_{i=0}^{N} c_i g_i(n), \quad c_i = (h, g_i), \tag{4.6}$$

or as (4.5), with $w_i = c_i = (H, G_i)$. Evaluation of the Fourier coefficients c_i can be implemented in the time-domain by feeding the time-reversed signal $h(-n)$ into the Kautz filter and by subsequently reading the tap outputs $x_i(n) = G_i[h(-n)]$ at $n = 0$: $c_i = x_i(0)$. That is, all inner products in (4.6) are implemented simultaneously using filtering. In the case of an FIR filter this would equal design by truncation. This choice of model or filter parametrization is in accordance with the Hilbert space approximation problem, but also in more engineering-oriented terms, it is the least-square approach, although it should be noted that here the LS criterion is applied on the infinite time horizon and not for example in the time window defined by the target response $h(n)$.

In addition to the ease of evaluating the weights, the Fourier expansion representation (4.6) also implies that it accomplishes simultaneous time- and frequency-domain optimization. Another duality is reflected by the fact that this approximation setting produces equally phase as well as magnitude modeling of the target response. Moreover, in an orthogonal configuration the contribution of each pole to the approximation is explicitly at hand, the approximation is independent of the ordering of the corresponding pole set, and the coefficients are independent of the approximation order, which makes choosing the poles, approximation error evaluation, and model reduction efficient[2]. This choice of LS approximation is also essential to the proposed pole generating process (Section 4.3), which in turn provides the basis for utilizing aforementioned properties due to orthogonality, that is, pruning, tuning and appending an initial set of poles.

4.1.3 Real-valued Kautz functions for complex conjugate poles

A Kautz filter produces real tap output signals only in the case of real poles. In principle this does not in any way limit its potential capabilities of approximating a

---

[2]According to (4.6), the coefficient c_i depends only on the corresponding basis function, $c_i = (h, g_i)$. In other words, the basis function g_i is not affected by any reordering of the preceding pole set, and in addition, it is completely insensitive to any further developments of the sequence of functions with respect to appended poles. Moreover, the approximation error energy for a set of poles $\{z_0, \ldots, z_n\}$ is given "orthogonally" as $E = (h, h) - \sum_{i=0}^{N} |c_i|^2$, which makes model reduction efficient. In practice, the contribution of a particular pole z_i to the approximation is quantized quite well directly by $|c_i|^2$ regardless of its position in the pole sequence. An exact model reduction scheme (with respect to a given pole set) is attained by moving each pole one at a time to the end of the pole sequence; an alternative ordering procedure starting from the most prominent pole was proposed in [den Brinker and Belt, 1997]. It should however be emphasized that for various permutations of the pole set, the basis functions are in general different, but the approximation $\hat{h}(n) = \sum_{i=0}^{N} c_i g_i(n)$ itself is independent of ordering. To be even more specific, actually the magnitude responses of $\{g_i\}$ are indeed invariant to permutations of the "tap-order", but the phase responses depend on the preceding (allpass) block as a whole (but once more, not on the ordering of the individual allpass blocks).
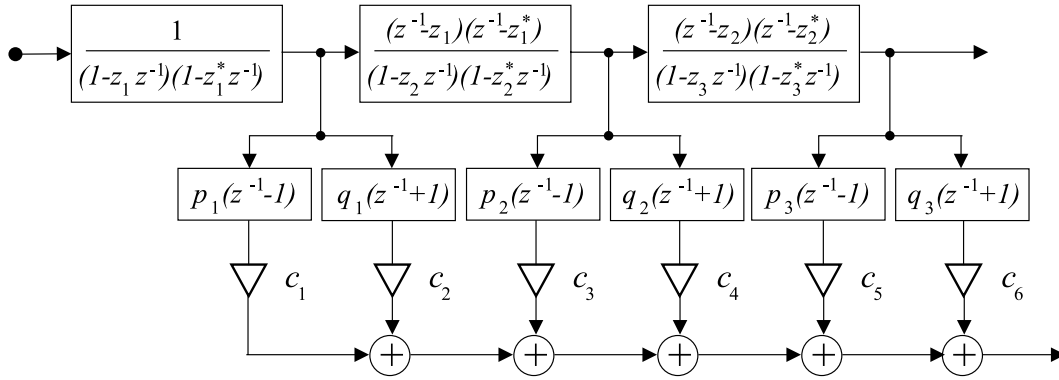
Figure 4.2: One realization for producing real Kautz filter tap-output responses defined by a sequence of complex conjugate pole pairs. The transversal allpass backbone of Fig. 4.1 is restored by moving the denominator terms one step to the right and by compensating for the change in the tap-output filters.

real signal or system. However, it may be preferable that the processing is restricted to real quantities, that is, involving only internal signals, coefficients and arithmetic operations that are real-valued. A restriction to real linear-in-parameter models can also be seen as a categoric step in the optimization of the structure[3].

It is always possible to form a real orthonormal structure from a sequence of real or complex conjugate poles[4]. This can be seen to result from the observation that the corresponding subspace is then generated by a real rational allpass operator (Section 2.4.3) providing a real (valued or rational) orthonormal basis function representation that is produced by an unitary transformation applied to one of its bases. In practice this is done by applying an unitary transformation to the tap-output vector, consisting of 1's corresponding to real poles and 2×2 unitary rotation elements corresponding to a complex conjugate pole pair. There is obviously an infinite variety of (unitarily equivalent) possible solutions. To the practical purposes of this thesis it is sufficient to just choose one particular real rational orthonormal structure: the filter structure in Figure 4.2, proposed originally by Broome [Broome, 1965], is maybe the most "symmetric" and intuitively simple structure.

The second-order section outputs of the transversal structure in Fig. 4.2 are mutually orthogonal, from which each tap output signal is further split into orthogonal components of difference and sum, $x(n) - x(n-1)$ and $x(n) + x(n-1)$, respectively. The tap-output filters are then given by

$$p_i(z^{-1} - 1) \quad \text{and} \quad q_i(z^{-1} + 1), \tag{4.7}$$

where the normalizing coefficients $\{p_i, q_i\}$ are determined by the corresponding pole

---

[3]Assuming that a real-valued response is approximated, it is known that the "true" poles of the system are real or occur in complex conjugate pairs, which makes all other choices sub-optimal.

[4]Some of the Kautz filter deductions are made directly on the assumption of real rational functions [Broome, 1965]. Moreover, the state-space approach to orthonormal structures with identical blocks, the *Generalized orthonormal basis functions* of Heuberger, is based on balanced realizations of real rational allpass functions [Heuberger, 1991].

pair $\{z_i, z_i^*\}$,

$$
\begin{aligned}
p_i &= \sqrt{(1-\rho_i)(1+\rho_i-\gamma_i)/2} \\
q_i &= \sqrt{(1-\rho_i)(1+\rho_i+\gamma_i)/2} \\
\gamma_i &= -2\Re\{z_i\} \\
\rho_i &= |z_i|^2
\end{aligned}
\tag{4.8}
$$

where γ_i and ρ_i can be recognized as corresponding second-order polynomial coefficients. The construction works also for real poles, producing a tap-output pair corresponding to a real double pole. However in the following examples a mixture of filter structures of Figures 4.1 and 4.2 is used in the case of both real and complex conjugate poles, where in the latter the allpass characteristics of the transversal blocks is restored by shifting the denominators one step to the right and by compensating for the change in the tap-output blocks. In an actual implementation the normalizing coefficients are obviously integrated into the tap-output coefficients.

As an example of the nature of Kautz filter responses with complex conjugate pole pairs, the magnitude spectrum of the tap-output impulse responses is presented in Figure 4.3. The chosen complex poles are depicted in Figure 4.3a: the real Kautz filter structure is actually induced by the complex conjugate symmetric pole set $\{z_i, z_i^*\}$. Figures 4.3b and 4.3c characterize the complementary orthogonal behavior of resonances for odd versus even tap output responses.

4.2 Strategies for choosing the Kautz filter poles

Contrary to some all-pole and all-zero modeling and filter design methods there are in general no analytical or global solutions for the optimal pole positions of a Kautz filter[5]. This is however a general feature of pole-zero or ARMA modeling. Kautz filter design can be seen as a two-fold procedure involving the choosing of a particular Kautz filter pole set and the evaluation of the corresponding filter weights. The fact that the latter task is much easier, better defined, and inherently least squares optimal makes it tempting to use sophisticated guesses and random or iterative search in the pole optimization. For a more analytic approach, the whole idea in the Kautz concept is how to incorporate desired *a priori* information to the Kautz filter. This may mean knowledge on system poles or resonant frequencies and corresponding time constants, or indirect means, such as any available all-pole or pole-zero modeling method to find potential Kautz filter poles.

A practical way to limit the "degrees of freedom" in filter design is to restrict to structures with identical blocks, that is, to use the same smaller set of poles repeatedly. The pole optimization and the model order selection problems are then essentially separated and various optimization methods can be applied to the substructure. Additionally, for the structure with identical blocks, a relation between optimal model parameters and error energy surface stationary points with respect to the poles may

---

[5]An iterative method (with respect to the pole set) based on a modified error criterion is presented in [Sarroukh et al., 2001]. However, the attained pole set seem to diverge quite rapidly (apparently due to the modification in optimization criterion) from the ones that are attained more genuinely in the LS sense, for example, using the method of Section 4.3.

a) log. spaced angles, r=0.97

(b) Magnitude responses of odd tap-output
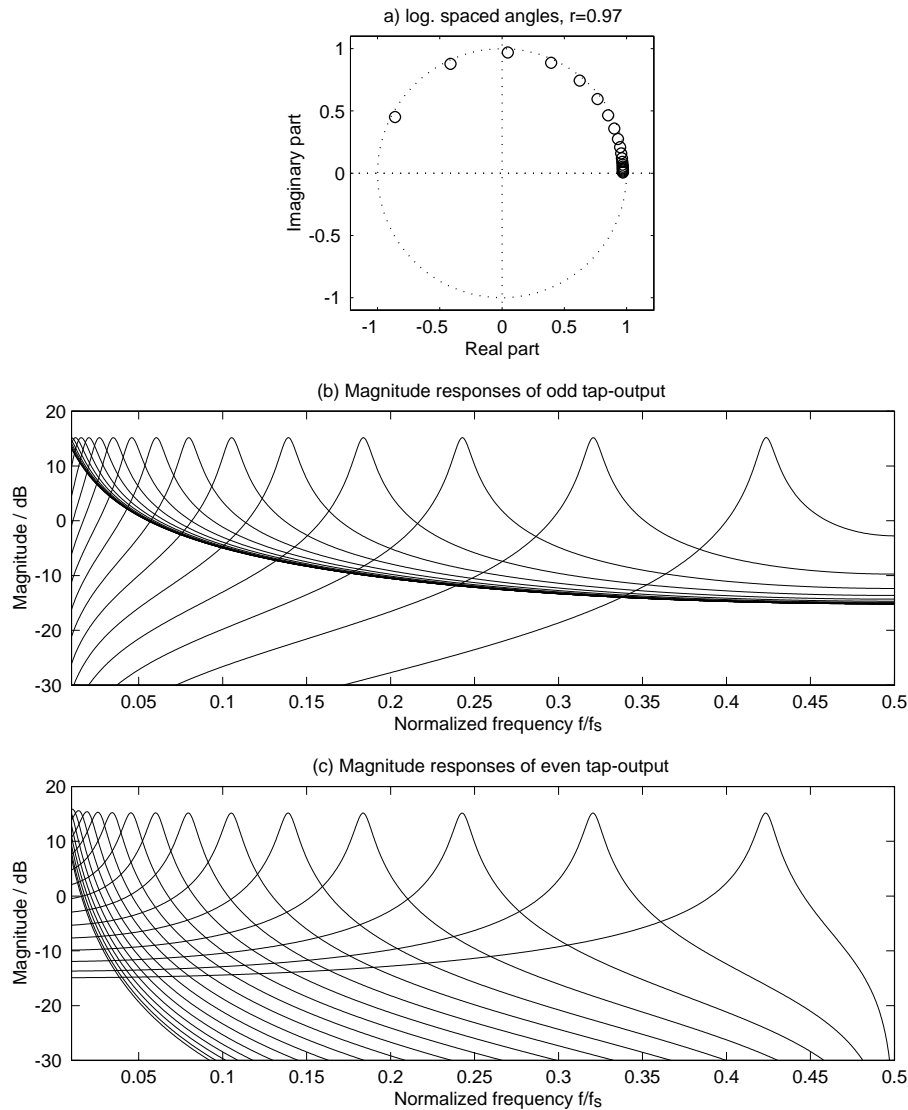
(c) Magnitude responses of even tap-output

Figure 4.3: An example of real-valued Kautz filter responses for complex conjugate pole pairs: (a) poles (only one per pair shown) positioned logarithmically in frequency (with respect to the pole angle), and with pole radius of 0.97. (b) Magnitude responses for odd tap outputs and (c) even tap outputs (see Fig. 4.2).

be utilized [den Brinker et al., 1996] as well as a classification of systems to associate systems and basis functions [Oliveira e Silva, 1994c] [Wahlberg and Mäkilä, 1996]. However, as it has been stated, the aim here is to genuinely compete with traditional pole-zero or ARMA modeling techniques, which calls for a method that produces large sets of distributed (and usually close to unstable) poles for the efficient modeling of challenging target responses. In the following some general aspects of the Kautz filter framework are first considered.

4.2.1 Generalized frequency resolution descriptions

As mentioned earlier, frequency-warped configurations in audio signal processing [Härmä et al., 2000] constitute a self-contained tradition originating from *warping effects* observed in analog-to-digital mappings and digital filter transformations [Constantinides, 1970]. The concept of a warped signal was introduced to compute non-uniform resolution Fourier transforms using the *Fast Fourier Transform* (FFT) [Oppenheim et al., 1971], and in a slightly different form, to compute warped autocorrelation terms for warped linear prediction [Strube, 1980]. The original idea of replacing a unit delay element with a first-order allpass operator in a transfer function, that is, to utilize the substitution

$$z^{-1} \leftarrow \frac{z^{-1} - \lambda}{1 - \lambda z^{-1}} \tag{4.9}$$

was subsequently restated and generalized to include warped counterparts of arbitrary linear filter structures [Steiglitz, 1980] [Imai,1983] [Karjalainen et al., 1996].

The warping effect or resolution description introduced by the first-order allpass warping is defined by the phase function of the allpass transfer function (4.9). The warping parameter λ can be chosen to approximate desired frequency scale mappings, such as the perceptually motivated Bark scale, with respect to different error criteria and sampling rates [Smith and Abel, 1999]. This is comprehensive in the sense that the first-order allpass element is the only (rational, stable and causal) filter having a one-to-one phase function mapping. Parallel structures can be constructed to approximate any kind of warping [Laine, 1992], including the logarithmic scale [Härmä and Paatero, 2001], but the aim of this consideration is to broaden the concept of frequency resolution description to account for the resolution allocation introduced by a Kautz filter. More on the topic can be found in [Paatero and Karjalainen, 2003].

The following is a short reminder of the conformal mappings and related isomorphisms (Section 2.4.1) involved in the change of basis representation. The phase function of the allpass operator defining the formal change of variable, $z^{-1} \leftrightarrow 1/A(z)$, is simply chosen as an interpretation of the frequency resolution mapping introduced by a particular Kautz filter. This phase function is a well-defined (monotonic and unique) function of the pole set (as a whole), which is consistent with the fact that the order of the poles in the generating allpass operator do not change the modeling characteristics of the particular orthonormal structure. If a mapping onto the "base interval", $[0, 2\pi]$, is desired, it may always be produced by unwrapping the phase function using the conformal mapping $z \mapsto z^{1/N}$, where N is the number of poles. This is analogous to the frequency resolution introduced by the conventional warping in the sense that the phase function of N cascaded identical first-order allpass functions, $e^{jNv(\omega)}$, is mapped back to $e^{jv(\omega)}$. In principle, a given phase response profile, considered as a desired frequency resolution mapping, may then be used to estimate the poles. However, designing allpass filters is in general a demanding task, and especially in the implied case of high order allpass filters with strongly resonant poles, this procedure is usually characterizable as hopeless [Härmä and Paatero, 2001].

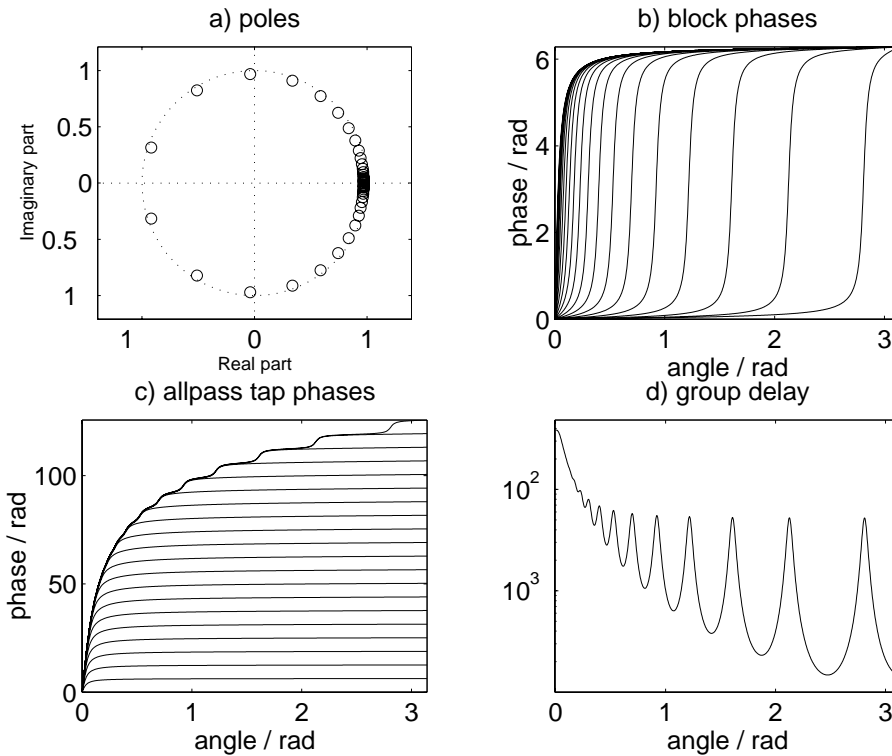The same Kautz filter as in Figure 4.3 is in the following used to demonstrate the

Figure 4.4: Phase and resolution behavior of the real Kautz filter characterized in Fig. 4.3: (a) pole positions, (b) phase functions of individual (second-order) allpass sections, (c) accumulated phase in allpass tap outputs, and (d) group delay (phase derivative).

proposed interpretation of the frequency resolution allocation introduced by the particular choice of model. Figure 4.4 characterizes the situation from the perspective of the allpass operator. Figure 4.4(b) shows the phase functions for the individual second-order allpass sections whereas Figure 4.4(c) displays the accumulated phases at each tap-output of the allpass chain. The accumulated phase of the whole allpass chain, that is, the uppermost curve in Figure 4.4(c) is proposed as the corresponding frequency scale mapping. The negated derivative of this mapping, the *group delay* $\tau_{\mathrm{g}} = -d\phi/d\omega$ of the allpass operator, is depicted in Figure 4.4(d). Similarly, the group delay can be interpreted as a measure of frequency resolution allocation as a function of the frequency. It illustrates well how the resolution is (in this case) highest at low frequencies and then decreases except locally at frequencies corresponding to the resonating pole pairs. It should be pointed out that this frequency resolution consideration is supposed to be purely descriptive. In particular, this is not proposed as a generalization of the warping concept, and any such implications should be used with caution.

A trivial way to attain a desired frequency resolution allocation (in an approximative sense) is to use suitable pole distributions. Sequences of (complex conjugate) poles may be placed with pole angle spacings corresponding to any chosen frequency resolution mapping. The choosing of pole radii is then also to be taken into account; from the allpass phase matching point of view this is a much easier optimization problem than trying to directly design a high order allpass filter. For example,

motivated by some experiments on producing warping on a logarithmic scale with parallel allpass structures, the choice of pole radii inversely proportional to the pole angles has proven to be a good *ad hoc* solution [Härmä and Paatero, 2001].

4.2.2 Manual fitting to a given response

It is always possible to simply adjust manually the Kautz filter poles through trial and error to produce a Kautz filter response matching, at least to some extent, a given target response. A Kautz filter impulse response is a weighted superposition of damped sinusoids which provide schemes for direct tuning of a set of resonant frequencies and corresponding decay time constants. This approach is demonstrated in the first case study in Section 4.4. However, this procedure is usually meaningful only on a smaller scale, in adjusting or appending a few poles into a larger set of poles that is attained using some "more automated" method. This hand-tuning scheme is nevertheless useful in correcting local deviations that are not captured by the overall model response.

By direct inspection of the time- and frequency-domain responses it is relatively easy to find useful pole sets by selecting a set of prominent resonances and by applying proper pole radius tuning. Choosing the complex conjugate pole angles is more critical than the pole radius selection in the sense that the filter coefficients perform automatic weighting of the sinusoidal components. It should be emphasized that in the real Kautz structure two orthogonal signal components are assigned to model the chosen resonance which result in relatively flexible modeling of the phase as well as more complex decay profiles of the resonance.

An obvious way to improve the overall modeling with a structure based on a fixed set of resonances is to use the corresponding generating substructure repetitively, producing a Kautz filter with identical blocks. There is no obligation to use the same multiplicity for all poles, but it makes model reduction easier. If a set of poles is assigned for substructures that is then used repetitively, some kind of damping by reduced pole radii should be applied to avoid "over modeling" the chosen resonances.

4.2.3 Utilizing the concept of complementary signals

The underlying assumption in utilizing orthonormal basis functions for approximating a given signal $h(n) \in \ell^2(\mathbb{N})$ or $H(z) \in H^2(\mathbb{E})$ is that there is a (possibly hypothetical) complete representations for the given signal,

$$h(n) = \sum_{i=0}^{\infty} c_i g_i(n) \quad \text{or} \quad H(z) = \sum_{i=0}^{\infty} c_i G_i(z), \qquad (4.10)$$

where $c_i = (h, g_i) = (H, G_i)$, $i = 0, 1, \ldots$, the inner product that provide the Fourier coefficients, define in fact an isomorphism (consisting of linear functionals) from $\ell^2(\mathbb{N})$ or $H^2(\mathbb{E})$ back to $\ell^2(\mathbb{N})$. It is also an isometry since $\|h\|^2 = \sum_{i=0}^{\infty} |c|^2 = \|H\|^2$ for all $h(n) \in \ell^2(\mathbb{N})$ or $H(z) \in H^2(\mathbb{E})$. Now from a more practical point of view the

relevant question is how the coefficient mapping performs energy compaction into the chosen truncation; for a chosen approximation order N, the approximation error energy is given by

$$E = \sum_{i=N+1}^{\infty} |c_i|^2 = H - \sum_{i=0}^{N} |c_i|^2, \qquad (4.11)$$

where $H = (h, h)$ is the energy of the target signal. Hence the energy of an infinite duration error signal is attained as a by-product from finite filtering operations, previously described for the evaluation of the filter weights. It will be demonstrated in the following that it is actually the allpass operator (defining the chosen Kautz filter) that performs the energy compaction as well as the orthogonal error division in Equation 4.11. This profound observation was made by Young and Huggins in their introduction to the concept of *complementary signals* [Young and Huggins, 1962].

An allpass filter $A(z)$ is *lossless* by definition, that is, the energy of the response $a(n) = A[x(n)]$ to any finite-energy input $x(n)$ is preserved as $\sum_{n=-\infty}^{\infty} |a(n)|^2 = \sum_{n=-\infty}^{\infty} |x(n)|^2$. However, the allpass operator introduces an interesting partition of the signal energy: if the time-reversed target signal $h(-n)$, $n = M, \ldots, 0$, is fed into the allpass filter defining the chosen Kautz filter (as it is done in the practical coefficient evaluation process), then the energy of the response $a(n) = A[h(-n)]$ is distributed as

$$\sum_{n=-\infty}^{\infty} |a(n)|^2 = \sum_{n=-M}^{\infty} |a(n)|^2 = \sum_{n=-M}^{0} |a(n)|^2 + \sum_{n=1}^{\infty} |a(n)|^2, \qquad (4.12)$$

where the first term in the sum is precisely the approximation error energy E whereas the second term is the energy of the approximation, $\hat{H} = (\hat{h}, \hat{h}) = \sum_{i=0}^{N} |c_i|^2$. This is a consequence of the complementary nature of representations (4.11) and (4.12): the infinite set of filter coefficients $\{c_i\}_{i=0}^{\infty}$ is attained by reading the Kautz filter tap-outputs at $n = 0$ (for the input $h(-n)$), which implies that the energy of the excessive coefficients, $\sum_{i=N+1}^{\infty} |c_i|^2$, the approximation error energy, must be equal to the energy of the allpass filter response in the time-interval $[-M, 0]$.[6] That is, the Kautz filter optimization problem reduces to the minimization of the energy of a finite duration signal, $a(n)$, $n = -M, \ldots, 0$, with respect to the poles.

To the author's knowledge, there have been only two prior attempts to utilize the concept of complementary signals in the pole position optimization of the Kautz filter [McDonough and Huggins, 1968] [Friedman, 1981]. McDonough and Huggins replaced the allpass numerator with a polynomial approximating the denominator mirror polynomial, to attain linear equations for the polynomial coefficients in an

---

[6]The time-reversed input signal is used to attain correspondence with the coefficient evaluation process; the actual energy partition may also be expressed in the "causal form", using the shifted filtering operation $a(n) = A[h(M-n)]$, $n = 0, 1, \ldots$, to produce

$$\sum_{n=0}^{M} |h(n)|^2 = \sum_{n=0}^{\infty} |a(n)|^2 = \sum_{n=0}^{M} |a(n)|^2 + \sum_{n=M+1}^{\infty} |a(n)|^2, \qquad (4.13)$$

where then correspondingly, $\sum_{n=0}^{M} |a(n)|^2$ is the approximation error energy, whereas $\sum_{n=M+1}^{\infty} |a(n)|^2$ is the energy of the approximation.

iteration scheme [McDonough and Huggins, 1968]. Friedman constructed a network structure for parallel calculations of all partial derivatives of the approximation error with respect to the real second-order polynomial coefficients (γ_i and ρ_i in (4.8)) that he then used in a gradient algorithm in search for the poles [Friedman, 1981]. These methods are described in more detail in [Paatero, 2000]; both constructions are interesting and illuminating from a principled point of view, but they were found to be far from applicable, at least in the implied case of modeling complex responses with relatively high order models.

4.3   The BU-method

In a search for an appropriate method for the optimization of the Kautz filter poles, the author was led to the more general topic of FIR-to-IIR filter conversion. The proposed methods are usually based on forming a state-space description for the FIR filter which is then reduced into a lower order IIR filter using balanced model reduction techniques [Belizynski et al., 1992] [Fahmy et al., 1994]. However, a method proposed by Brandenstein and Unbehauen [Brandenstein and Unbehauen, 1998] was immediately found to be interesting from the Kautz filter point of view. To begin with, it is more explicit than the state-space formulations, and thus easier to approach and size up. It also operates directly on the time-domain target signal, resembling a lot (the denominator part of) the more familiar Steiglitz-McBride method of pole-zero modeling [Steiglitz and McBride, 1965]. The most striking observations were however that Brandenstein and Unbehauen had found Walsh's interpolation results for the determination of the numerator polynomial and that they derived what is essentially the complementary error signal concept for the optimization of the denominator polynomial, all this without any reference to orthogonal filter structures. The method proposed by Brandenstein and Unbehauen for optimizing the poles of an allpass operator was readily adopted (with a minor modification) to the optimization of the the Kautz filter poles, and consequently, entitled as the *BU-method* [Paatero, 1999] [Paatero et al., 2001].

An outline of the BU-method for generating the Kautz filter poles is given in the following:

- The algorithm is based on approximating an allpass operator $A(z)$ of a given order N with

$$\hat{A}^{(k)}(z) = \frac{z^{-N} D^{(k)}(z^{-1})}{D^{(k-1)}(z)}, \qquad (4.14)$$

  where $\{1, D^{(1)}(z), D^{(2)}(z), \ldots\}$ are iteratively generated polynomials, restricted to the form

$$D^{(k)}(z) = 1 + \sum_{i=1}^{N} d_i^k z^{-i} = 1 + D_1^{(k)}(z), \qquad k = 1, 2, \ldots. \qquad (4.15)$$

- The ratio (4.14) converges to an allpass function, if $\|D^{(k)}(z) - D^{(k-1)}(z)\| \to 0$.

- The objective is to minimize the output of (4.14) to the input $X(z) = z^{-M}H(z^{-1})$ (the z-transform of $h(M-n)$, $n = 0, \cdots, M$). Denote $U^{(k)}(z) = \hat{A}^{(k)}[X(z)]$.

- Define $Y^{(k)}(z) = X(z)/D^{(k-1)}(z)$ (all-pole filtered $h(M-n)$, $n = 0, \cdots, M$).

- Now $U^{(k)}(z) = z^{-N}D^{(k)}(z^{-1})Y^{(k)}(z)$, and by substitution (4.15) and rearranging,

$$Y^{(k)}(z)z^{-(N-1)}D_1^{(k)}(z^{-1}) = U^{(k)}(z) - z^{-N}Y^{(k)}(z). \qquad (4.16)$$

- Collecting common polynomial terms into a matrix equation produces $\mathbf{A}^{(k)}\mathbf{d}^{(k)} = \mathbf{u}^{(k)} + \mathbf{b}^{(k)}$, where $\mathbf{d}^{(k)}$ and $\mathbf{u}^{(k)}$ are unknown. The solution of $\mathbf{A}^{(k)}\mathbf{d}^{(k)} = \mathbf{b}^{(k)}$ minimizes the square-norm of $\mathbf{u}^{(k)} = \mathbf{A}^{(k)}\mathbf{d}^{(k)} - \mathbf{b}^{(k)}$.

- The BU-algorithm:

  1. For $k = 1, 2, \ldots$, filter $h(M-n)$, $n = 0, \cdots, M$, by $1/D^{(k-1)}(z)$ to produce the elements of $\mathbf{A}^{(k)}$ and $\mathbf{b}^{(k)}$ (from samples $y^{(k)}(n)$, $n = 0, \cdots, M$).

  2. Solve $\mathbf{A}^{(k)}\mathbf{d}^{(k)} = \mathbf{b}^{(k)}$, $\mathbf{d}^{(k)} = \mathbf{A}^{(k)}\backslash\mathbf{b}^{(k)}$, the (mirror) polynomial coefficients of $D_1^{(k)}(z)$. Go to step 1.

  3. After a (sufficient) number of iterations, choose $D^{(k)}(z)$ that minimizes the true least-square error. The Kautz filter poles are the roots of $D^{(k)}(z)$.

The fact that the BU-algorithm operated directly on the time-domain signal makes it possible to utilize various manipulations of the target response in the design phase to emphasize desired features in the pole generation procedure. For example, time-domain windowing or frequency-domain weighting by suitable filtering can be applied to modify the target response. The optimization may also be divided into two or more parts using selective filters, which does not necessarily mean strict bandpass filtering: pole sets attained by different emphasizing strategies may be interlaced, cut and pasted quite freely. Obviously, the overall balance of such constructions has to be taken into account, for example, by attenuating the combined pole set. However, the actual Kautz filter weights are evaluated (in all subsequent examples) with respect to the original non-emphasized target response.

4.3.1   The BU-method and warping techniques

The following approach that combines the BU-method with frequency-warping techniques has proven to be very useful. In the *warped BU-method* (WBU), as described in more detail in [Paatero, 2002] [Paatero, 2003], the BU-method is applied to the frequency warped target response, and subsequently the produced poles are mapped back to the original frequency-domain using the corresponding inverse allpass mapping. An overview of various warping techniques is given in [Härmä et al., 2000], but it is strongly emphasized that the WBU-method is not warped filter design. The only situation when these methods coincide is when the BU-method produces an FIR filter in the warped domain, that is, forces all the poles to the origin, implying that the optimal Kautz filter is a Laguerre filter. This freak incident, however,

justifies the particular choice of warping in the WBU-method: the correct "transformation pair" in this case is to apply Laguerre-warping (in the time-domain) to the target response and to use allpass-warping in the frequency-domain to map the produced set of poles.

The Laguerre-warped target signal is attained as the Laguerre-Fourier coefficients, $c_i = (h, g_i)$, $i = 0, 1, \ldots, N$, of the target response h with respect to the choice of poles $a = z_i$, $i = 0, 1, \ldots, N$, in the Kautz filter of Figure 4.1. The Laguerre filter pole a, the warping parameter, should have a magnitude less than one to maintain stability; however in this consideration the Laguerre parameter is allowed to take complex values[7]. The number N is determined with respect to the "effective length" of the produced warped signal: the accuracy and efficiency of the BU-algorithm is affected by an unconsidered choice.

The BU-method is then applied to the warped target response $\tilde{h}(i) = c_i = (h, g_i)$, $i = 0, 1, \ldots, N$. Similar considerations as in the unwarped case may be used to determine a desired approximation order (number of poles), that is, utilizing inspection of the convergence of the BU-iterations and validation of the attained models in the warped domain with respect to different approximation orders. The produced pole set is then mapped back to the original frequency-domain according to the corresponding inverse allpass mapping

$$z \mapsto \frac{z + a^*}{1 + az}. \tag{4.17}$$

This procedure emphasizes a chosen frequency region in the choice of Kautz filter poles, but as stated earlier, the corresponding Kautz filter tap-output weights are evaluated (in the LS sense) with respect to the original target response. Here too, pole sets that are attained by different warping schemes may be interlaced or more selectively combined. The WBU-method is demonstrated in Section 4.4.2, where some particular aspects in choosing a complex warping parameter are also considered.

Figure 4.5 gives an example how a signal is transformed using real Laguerre-warping. The interpretation of Figure 4.5 is that the original signal is compressed to a smaller set of (Laguerre-Fourier) filter coefficients. For example, an 800th order Laguerre filter with $a = 0.7$ would exactly reproduce the original response with implied duration of 5000 samples. This is not however always the case, that is, the effective length of the produced warped signal may also exceed the length of the original signal.

4.3.2   Utilizing partitions in time- and frequency-domains

To put it bluntly, the pragmatic reason why sub-signal techniques are proposed here as a solution to the modeling of complicated and long impulse responses is that it

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[7]There are many practical ways to attain the warped signal, including Laguerre counterparts of de-warping techniques [Härmä et al., 2000]. In the complex case, however, when utilizing various interpretations it should be noticed that it is actually the complex conjugated Laguerre functions that are implied by the inner products (h, g_i).
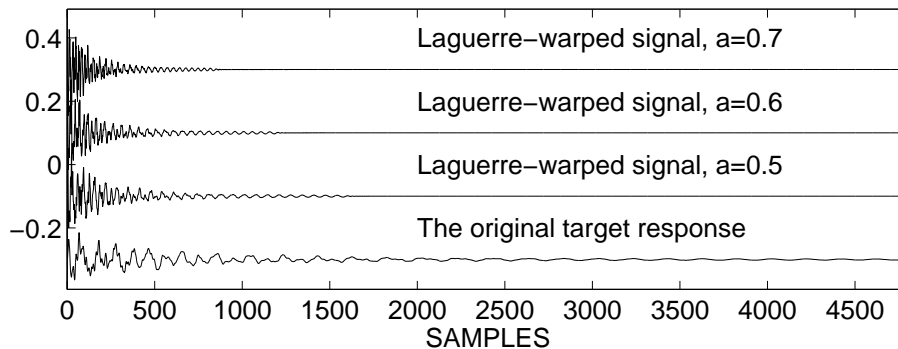
Figure 4.5: A measured acoustic guitar body impulse response and some Laguerre transformed counterparts.

is a way to split the original task into sub-model dimension, where the filter design algorithms work. In the case of Kautz filter design using the BU-method this means typically sub-signal lengths up to 10000 samples and maximum filter orders 200–300.

Various partitions into frequency subbands using filterbank techniques [Vaidyanathan 1993] may be used, where the poles and the corresponding Kautz models are generated with respect to the decimated sub-signals. As an alternative, the poles generated in a subband may also be mapped back to the original frequency-domain for the construction of an overall Kautz filter. This latter approach will be demonstrated in the case of low-frequency models for room responses in Section 4.4.3. A somewhat unorthodox mixture of subband and multirate techniques based on complex modulation and complex-valued models in the subbands has proven to be very useful in detailed modeling of challenging responses. This *frequency-zooming ARMA technique* [Karjalainen et al., 2002b] was generalized to Kautz filters in [Paatero and Karjalainen, 2002] and an example of this approach will be presented in Section 4.4.3. It seems to be a general feature that "non-perfect reconstruction" subband partitions are usually sufficient and even preferable, because it is not desirable that the model in the subband concentrates too much on the bandpass filtering. In the Kautz filter case, however, it is relatively simple to eliminate poles that are clearly dedicated to the cut-off frequencies of the chosen subband partition[8]. Many configurations are apparently plausible, but it is quite difficult to estimate on a general level the actual complexity of a particular implementation.

There are also more noble reasons for splitting a complicated modeling task into sub-problems. Our knowledge about the auditory perception may be used to include perceptually relevant criteria into the modeling. This complicated topic was addressed to some extent in [Paatero and Karjalainen, 2002]. In the case of modeling a room impulse response by subband techniques this may mean appropriate allocation of modeling resolution for different subbands, criteria on the subband segmentation itself, or separation into different modeling strategies, such as the use of artificial reverberation above a certain frequency. In the following, two elemen-

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[8]Conversely, Kautz filters that are designed with respect to the subband signals, including possible truncation of the pole sets at the band-edges, are genuinely bandpass in nature, although the stop-band characteristics may be somewhat arbitrary.
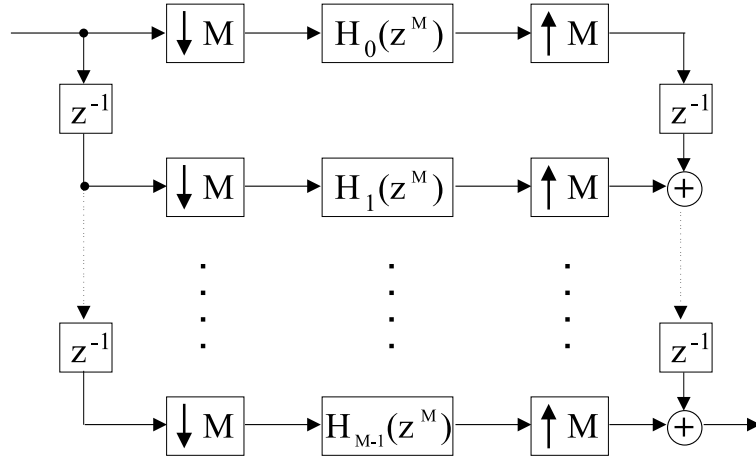
Figure 4.6: The polyphase realization corresponding to (4.18) and (4.18).

tary partitions in the time-domain are proposed, where the latter can be used in modeling, equalization and control of the early response.

An $N \times M$:th order FIR filter, corresponding to an impulse response $h(n)$, $n = 0, \ldots, NM$, can be decomposed into a polyphase form [Vaidyanathan 1993]

$$H(z) = \sum_{k=0}^{M-1} z^{-k} H_k(z^M), \qquad (4.18)$$

where the component filters $H_k(z)$ are related to the original response $h(n)$ trough

$$H_k(z) = \sum_{n=0}^{N-1} h(Mn + k) z^{-n}, \qquad k = 0, \ldots, M - 1. \qquad (4.19)$$

A direct representation of Eqs. (4.18) and (4.19) is depicted in Fig. 4.6. In practice, the combined operation of segmentation, decimation, expansion and reconstruction can be implemented in its commutator form [Vaidyanathan 1993]: the component signals are formed sequentially from the input signal and the reconstructed response in attained from the component responses by simply interlacing the samples.

Here the polyphase decomposition is utilized by approximating the component filters $H_k(z)$ using Kautz filters $\hat{H}_k(z)$. This configuration relies on the waveform matching property, that is, accurate phase as well as magnitude modeling capabilities of the Kautz filter, especially in the onset parts of the responses. The obvious idea is that an appropriate decomposition splits the original response into manageable portions, that is, into sub-signal lengths where the BU-method operates efficiently and accurately. Typically, at least for audio related responses, good approximations are attained with 10 to 20 times lower filter orders compared to the direct FIR filter implementation.

Another strategy in dividing a long target response into manageable portions is to partition it with respect to a chosen set of successive sample indexes, $0 < t_1 < \ldots < t_M < N$, where N is the length of the target response. The (Kautz) approximation
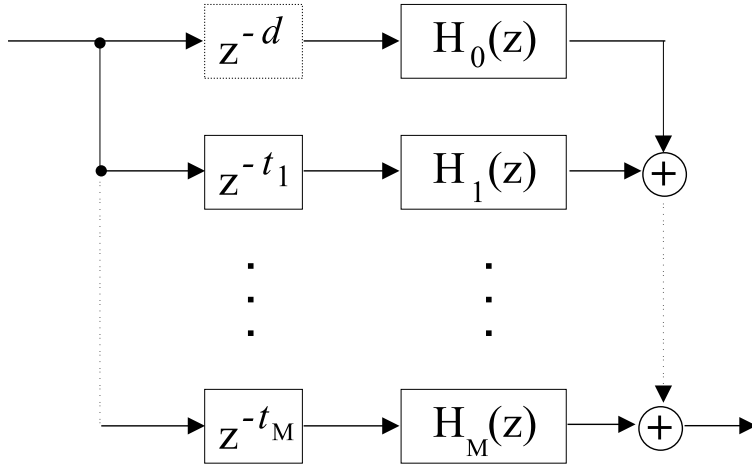
Figure 4.7: The proposed partition of a target transfer function: an approximation by a superposition of delayed Kautz filter responses.

of a target response $H(z)$ is then composed as

$$\hat{H}(z) = \sum_{k=0}^{M} z^{-t_k} H_k(z), \qquad (4.20)$$

where the first delay is either $t_0 = 0$ or some other non-negative integer d that represents an initial delay in the response. The scheme is illustrated in Fig. 4.7. Based on this decomposition of the response $h(n)$, $n = 0, \ldots, N$, the component Kautz filters are designed according to

step 1 choose the partitioning, $0 \le t_0 < t_1 < \ldots < t_M < t_{M+1} \equiv N$, and the initial sub-signal as $s(n) = h(n)$, $n = t_0, \ldots, t_1 - 1$

step 2 generate the poles with respect to s and a chosen (iterated) sub-filter order N_0

step 3 produce the Kautz filter impulse response $h_0(n)$, $n = 0, \ldots, N - t_0$, and form the next sub-signal by subtracting the "overflow", $s(n) = h(n) - h_0(n - t_0)$, $n = t_1, \ldots, t_2 - 1$

step 4 repeat from step 2 for t_k, $k = 1, \ldots, M - 1$. The approximation is $\hat{h}(n) = \sum_{k=0}^{M} h_k(n - t_k)$, $n = 0, \ldots, N$, where $h_k(n) = 0$ for $n < 0$

In step 3, the "overflow" means the tail of all preceding Kautz filter responses. In practice it is probably just the previous response that is not negligible. In principle, any response can be modeled exactly using a partition of the form (4.20), but the relevant applications of the proposed method are clearly related to the extraction of successive impulse-like bursts. One such application is proposed in Section 4.4.3, where the modeling of the early part of a room response is also suggested as a potential way to control the early reflections.

4.4 Audio Application Cases

In the following, the applicability of Kautz filter design is demonstrated using three audio-oriented applications. The first one is a loudspeaker equalization task where an overall model of the system is complemented to take into account local deviations in the frequency response. This low-order modeling example is included to explicate the manual tuning scheme of Section 4.2.2. In the second case, Kautz filters are used to model a measured acoustic guitar body (impulse) response. The aim is again somewhat methodological: utilization of the warped BU-method (Section 4.3.1) is demonstrated. In the third case, measured room impulse responses are modeled using Kautz filters to attain detailed models of the low-frequency part of the response, but also to demonstrate that brute force solutions for modeling very long complicated responses are possible using techniques of Section 4.3.2.

4.4.1 Case 1: Loudspeaker response equalization

An ideal loudspeaker has a flat magnitude response and a constant group delay. Simultaneous magnitude and phase equalization of a non-perfect loudspeaker would be achieved by modeling the response and inverting the model, or by identifying the overall system of the response and a chosen Kautz equalizer. However, here the use of Kautz filters is demonstrated in pure magnitude equalization based on an inverted minimum-phase target response. A small two-way active loudspeaker was selected for the equalization experiment due to its clear deviations from the ideal magnitude response. The measured response and a derived equalizer target response are included in Figure 4.8. The sample rate is 48 kHz.

Magnitude response equalization consists typically of compensating for three different types of phenomena: (a) slow trends in the response, (b) sharp and local deviations[9], and (c) correction of roll-offs at the band edges. This makes "blind equalization" methods, which do not utilize audio-specific knowledge, ineffective. The Kautz filter approach is proposed as an alternative between blind and hand-tuned parametric equalization, with an obvious abuse of terminology.

As is well known, FIR modeling and equalization has an inherent emphasis on high frequencies when considered on an auditorily motivated frequency scale. Warped FIR (or Laguerre) filters [Härmä et al., 2000] shift some of the resolution to the lower frequencies, providing a competitive performance with 5 to 10 times lower filter orders than the corresponding FIR filters [Karjalainen et al., 1999]. However, the filter order required to flatten the peaks at 1 kHz in this example is still high, of the order 200, and in practice warped FIR or Laguerre models up to the order 50 are able to model only slow trends in the response. In a recent publication [Tyril et al., 2001], a tapped cascade structure of real-pole allpass filters was suggested for the low-

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[9]In this study the equalization cases are presented as illustrative examples on the controllability of the modeling task rather than from the point of view of practicality of the results. It may not even be desirable to flatten sharp resonances in the main axis free-field response of a loudspeaker, since off-axis responses can become worse and degrade the overall quality of sound reproduction.
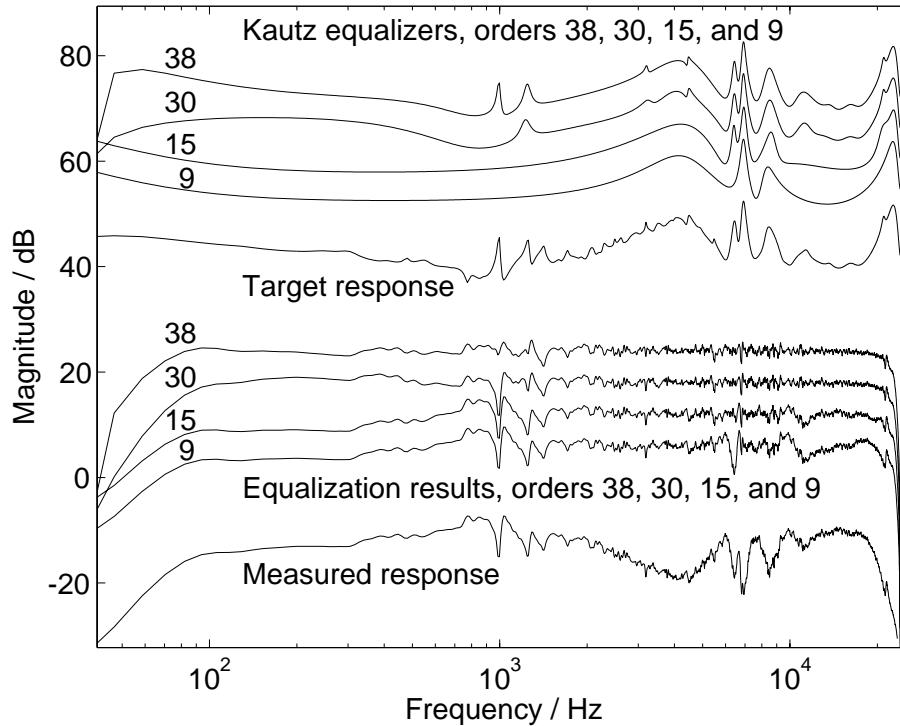
Figure 4.8: Measured magnitude spectrum of the loudspeaker under study (bottom), Kautz equalizer responses (top), and equalization results for Kautz filter orders 9, 15, 30, and 38, respectively. The Kautz equalizers are designed using the BU-method with respect to the target response (in the middle).

frequency equalization, although in practice, it was found too difficult to design. However, here we demonstrate efficient design methods for the orthonormal and complex-pole counterpart.

A simple way to focus on the 1 kHz region is to use a Kautz filter defined by a single complex conjugate pole pair that is used repeatedly, that is, a pole $a = z_i$, $i = 1, \dots, N$, in the Kautz filter structure of Figure 4.2 is chosen, where the phase angle of a corresponds to the region of interest. By tuning the pole radius a trade-off between the 1 kHz region and the overall modeling is introduced. Quite interestingly, as a good compromise, one ends up with a radius close to a typical warping parameter at this sampling rate, for example $\lambda = 0.76$ corresponding to the Bark-scale warping [Smith and Abel, 1999], and thus (maybe not anymore so) surprisingly, similar results are attained for the Laguerre and the two-pole Kautz equalizers for filter orders 50–200. Actually this simply means that in this case a perceptually motivated warping is also technically a good choice for the flattening of the 1 kHz region. This is demonstrated in Fig. 4.9.

The obvious way to proceed would be to add another pole pair corresponding to the 7 kHz region. However, in search for considerably lower-order Kautz filters, compared to FIR and Laguerre equalizers, the BU-method is utilized directly on the target response. The BU-method provides stable and reasonable pole sets for orders at least up to 40. Figure 4.8 presents the corresponding Kautz equalizers and equalization results for orders 9, 15, 30 and 38. These straightforward Kautz filter
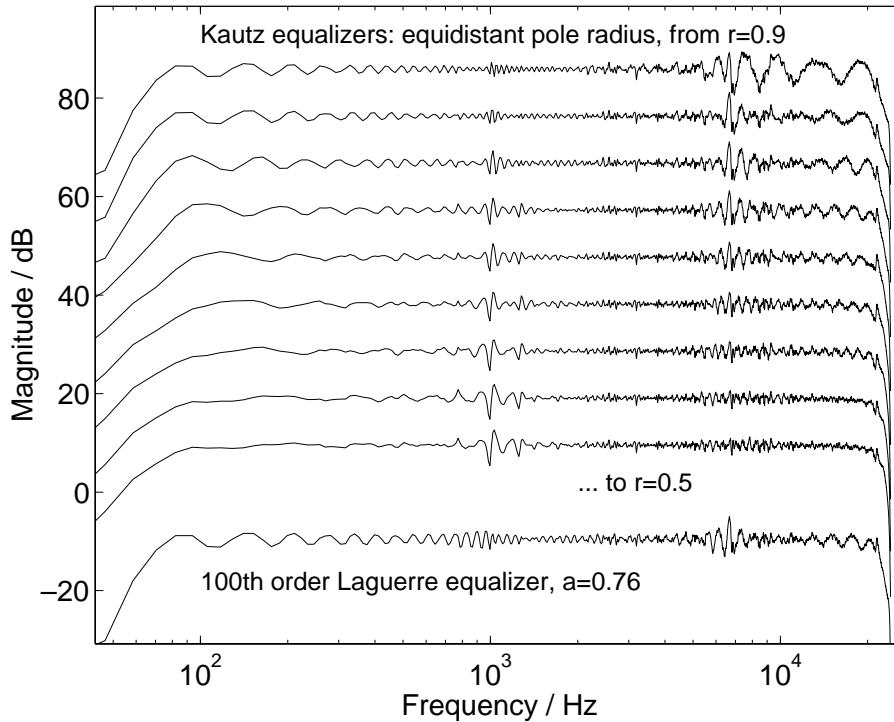
Figure 4.9: Comparison of 100th order Laguerre and Kautz equalization results; Kautz filters with 50 complex conjugate pole pairs corresponding to 1 kHz pole angles and pole radius varied from 0.5 to 0.9 in steps of 0.05.
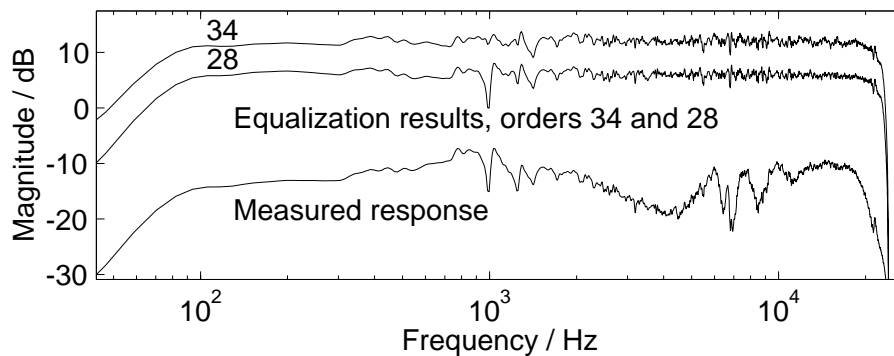

Figure 4.10: Kautz equalization results for orders 28 and 34, with pruned BU-poles, comparable with the Kautz equalizers of orders 30 and 38 in Figure 4.8.

constructions are already comparable with the FIR and Laguerre counterparts, but filter orders can be reduced further by omitting some of the poles. For example for orders above 15, the BU-method produces poles that are very close to $z = 1$ because of the low-frequency boost in the target response, and omitting some of these poles will actually tranquilize the whole low-frequency region. For example, in the equalization result of Figure 4.10 the Kautz filters of orders 28 and 34 are attained by omitting one and two pole pairs, respectively, from the original sets of 30 and 38 poles.

To improve the modeling at 1 kHz, three to four manually tuned pole pairs are
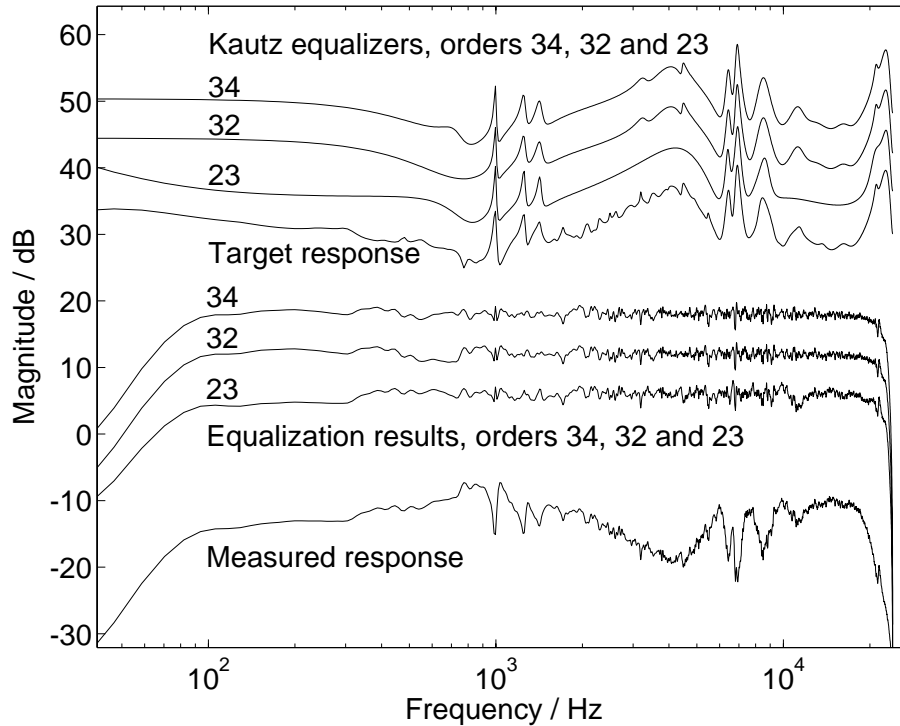
Figure 4.11: Kautz equalizers and equalization results for orders 23, 32, and 34, with combinations of manually tuned an BU-generated poles.

added to the BU-pole sets, corresponding to the resonances in the problematic area. This is actually not too hard since the 1 kHz region is quite well "isolated" from the dominant pole region, which allows for undisturbed tuning. The 15th and 30th order Kautz equalizers of Figure 4.8 were used as the starting points, omitting three pole pairs in the latter case. Three pole pairs were tuned directly to the three prominent resonances and one pole pair was assigned to improve the modeling below the 1 kHz region. Results for final filter orders 23, 32, and 34 are displayed in Figure 4.11, where the last two differ only in the above mentioned optional compensating pole pair.

Finally, as a mere example of the manual tuning scheme (Section 4.2.2), 10 pole pairs are tuned manually to a chosen set of prominent target response resonances. The design is based on 10 selected resonances, represented with 10 distinct pole pairs, chosen and tuned to fit the magnitude response by trial and error. This is of course somewhat arbitrary, but it seems to work. In Figure 4.12, along with the equalizer and target responses, are vertical lines indicating pole pair positions. This is clearly one form of "parametric equalization" with second order blocks since each resonance is represented with a single pole pair. However, using Kautz filters the choice of resonant frequencies is not as severe as it is for some other designs because the tap-output weights perform fine-tuning (to some extent) and overall modeling in the LS sense, and a misplaced pole (pair) usually just results in inefficiency.

Figure 4.13 compares some of the Kautz equalization results to those achieved with FIR and Laguerre equalizers of orders 200 and 100, respectively. To aid the comparison, at respective filter orders, the computational complexity of the Kautz filter
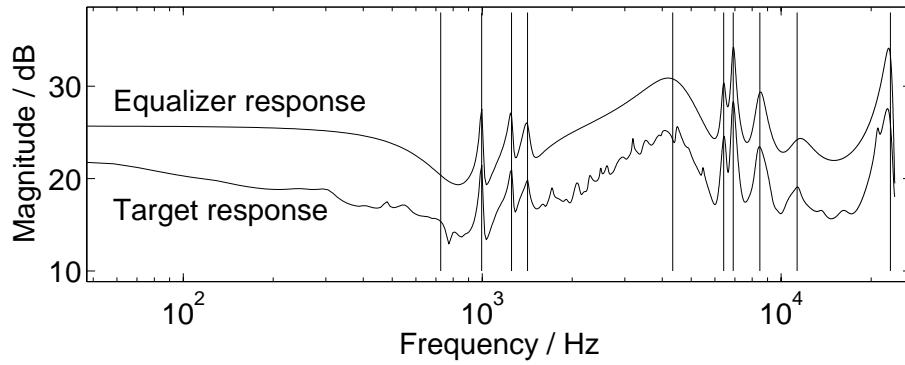
127

Figure 4.12: Manually tuned 20th order Kautz equalizer and target magnitude responses, with lines indicating pole pair positions.
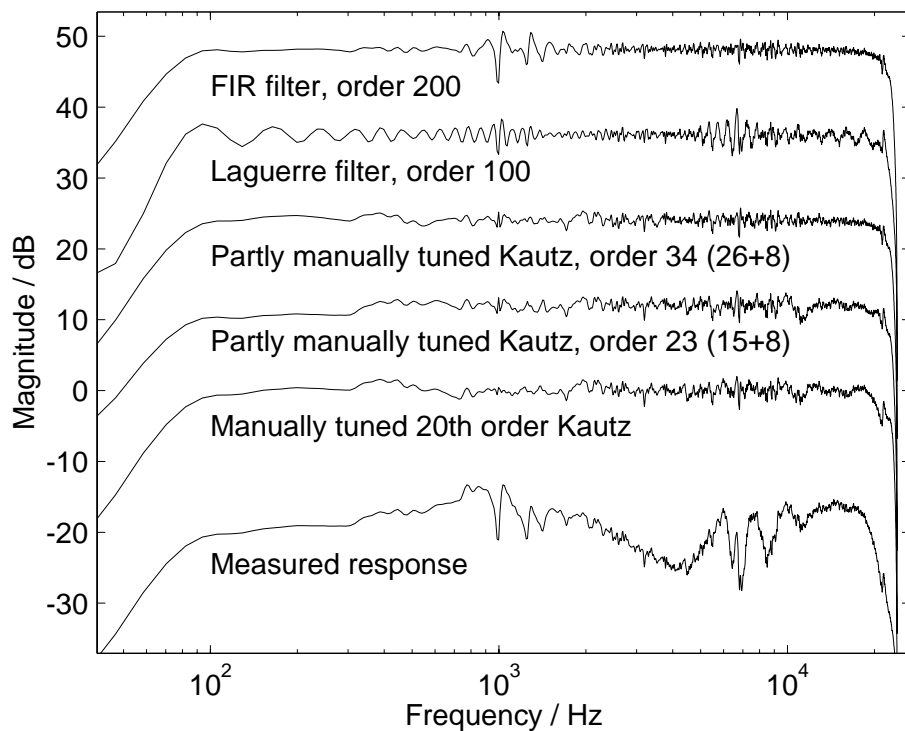

Figure 4.13: Comparison of FIR, Laguerre, and Kautz equalization results.

(measured simply as the number of arithmetic operations) is about 3 to 4 times higher than a direct FIR filter implementation using convolution. There are of course usually more efficient ways to implement an FIR filter, as there would in most cases also be to realize a particular Kautz filter, but this gives a picture of the tradeoff. The additional computational load for Kautz filters compared to warped FIR or Laguerre filters is somewhere between none and twice. The actual complexity depends on many details, but in any case, efficient equalizers are attained at comparatively low filter orders. Furthermore, the fact that the Kautz filters in this case are of relatively low order enables in principle filter transformations to other structures that are potentially more efficient, such as, various direct-form filter structures.
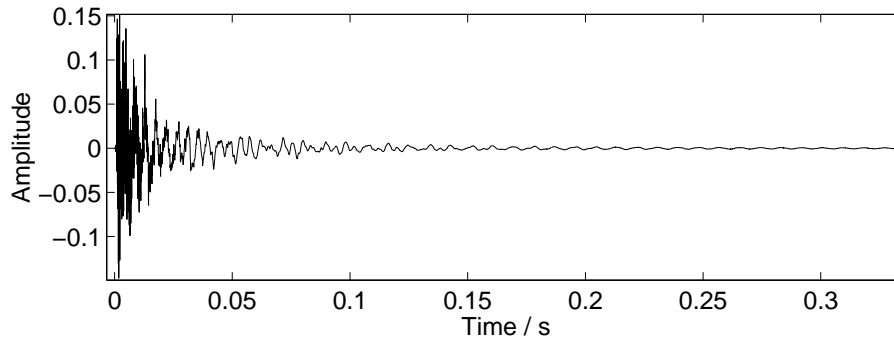
Figure 4.14: Measured impulse response of an acoustic guitar body.

4.4.2 Case 2: Guitar body modeling

As an example of higher-order Kautz modeling, a measured acoustic guitar body (impulse) response is modeled using various techniques, in particular the warped BU-method of Section 4.3.1. The target response is presented in Figure 4.14 and it was obtained by tapping the bridge of an acoustic guitar with an impulse hammer [Karjalainen and Smith, 1996] [Karjalainen et al., 2000], with strings damped[10]. The response of an acoustic guitar body is a challenging but well suited Kautz modeling task because the modes of the response are excited relatively simultaneously.

The obvious disadvantage of a straightforward FIR filter implementation for the body response is that modeling of the slowly decaying lowest resonances requires a very high filter order, in this case for example, orders up to thousands would be needed to capture the essential resonant structure. All-pole and pole-zero modeling are the traditional choices to improve the flexibility of the spectral representation. However, model orders remain problematically high and the basic design methods seem to work poorly[11]. A significantly better approach is to use separate IIR modeling for the slowly decaying lowest resonances combined with an FIR filter for the modeling of the rest of the response [Penttinen et al., 2000]. Perceptually motivated warped counterparts of all-pole and pole-zero modeling pay off, even in more technical terms [Karjalainen and Smith, 1996], which is encouraging also from the Kautz filter point of view[12].

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[10]A more 'natural' impulse response would be achieved by extracting the impulse response by deconvolution from an identification setup using spectrally rich real playing of the acoustic guitar as excitation [Penttinen et al., 2000], although the signal-to-noise ratio is better through an impact hammer measurement.

[11]This vague statement tries to characterize a general dilemma in utilizing conventional all-pole and pole-zero filter design methods. For example, all-pole modeling (using linear prediction) is inefficient in the sense that very high filter orders are required to "capture the essential resonant structure". Prony's method for pole-zero modeling is in principle a better choice, but in this case, the required filter orders "remain problematically high", because the "resonance modeling" is still essentially based on all-pole modeling. Pole-zero modeling using the Steiglitz-McBride method would be the correct choice (for modeling non-minimum phase resonant components), but the required filter orders are not just "problematic", but in fact, pathological for the stability of the Steiglitz-McBride method.

[12]What is tried to say is that focusing of modeling resolution, for example by utilizing intermediate warping, may, as it does here, in fact improve the model in the least-square sense.
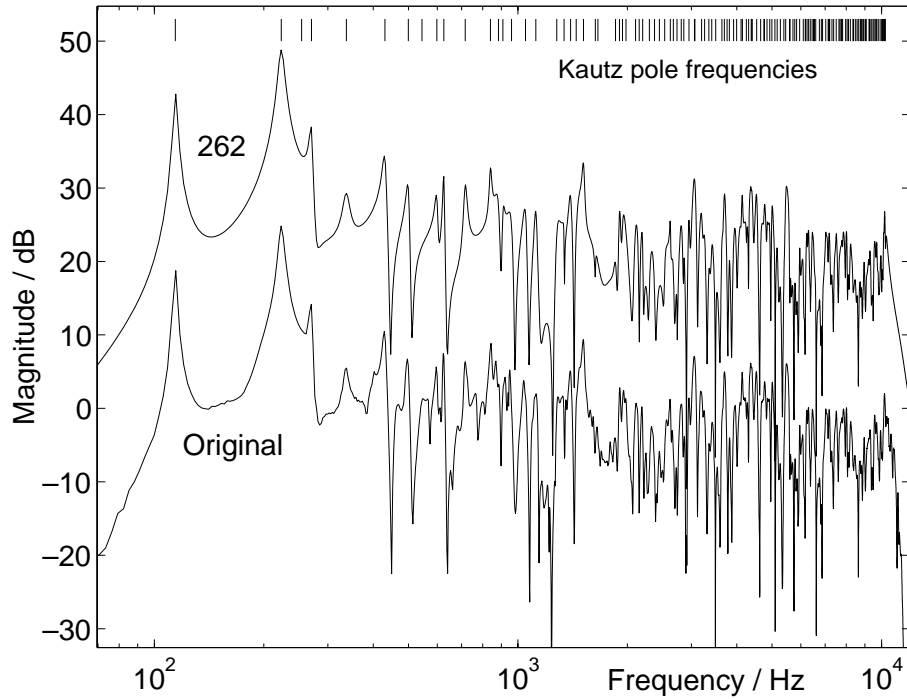
Figure 4.15: A 262th order Kautz model, displayed together with the target response. Vertical lines indicate pole pair positions, where the poles are obtained by direct application of the BU-method.

Figure 4.15 demonstrates that the proposed pole position optimization scheme, the BU-method, is able to capture essentially the whole resonance structure. The Kautz filter order is 262 and the poles are obtained from a 300th order BU-pole set, omitting some poles close to $z = -1$. In general, the BU-method works quite well at least up to an order of 300 and the lower limit for finding the chosen prominent resonances is about 100.

A relatively high filter order is still required to obtain good match of the lowest (and strongest) modal resonances. The direct application of the BU-method pays in average too much attention to the high frequencies compared to the importance of the lowest modes both physically and from a perceptual point of view. A better overall balance is achieved by applying the warped BU-method. Figure 4.16 depicts the magnitude response of a 120th order Kautz model and Figure 4.17 displays how the pole set is mapped from the warped domain back to the original frequency-domain, where the actual model is constructed. A (Laguerre) warping parameter $\lambda = 0.64$ was used, corresponding approximately to the Bark scale warping with respect to the sample rate 22050 Hz [Smith and Abel, 1999]. The model of Figure 4.16 is at least as good as the much higher-order model in Figure 4.15 for the low- and mid-frequencies, but the high-frequency region has obviously degenerated.

Figure 4.18 gives further information about the modeling power and characteristics of the warped BU-method by depicting magnitude responses for a collection of low order models (orders 10, 18, 39, 60, and 90), compared with the original response. As with any reverberant system, the magnitude response does not tell the full story how the response is perceived. Comparing time-frequency plots may be needed to
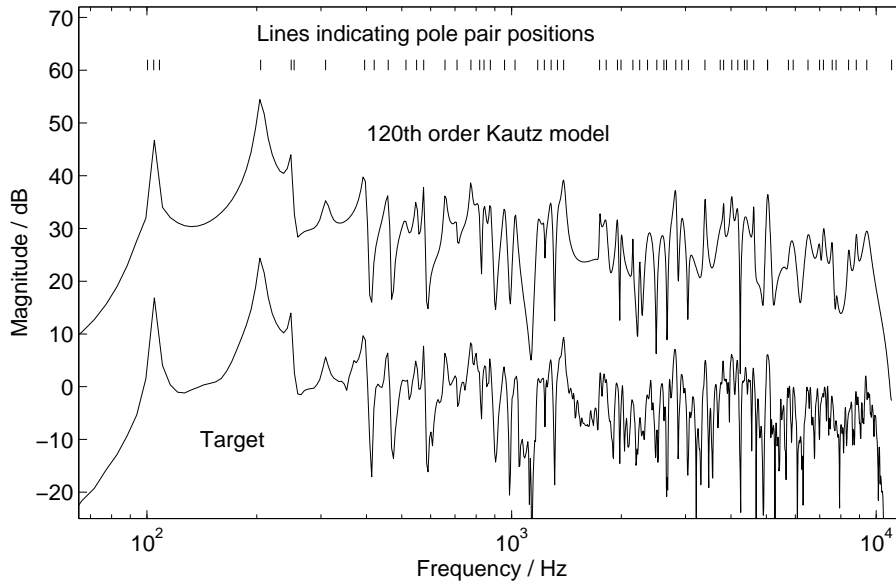
130

Figure 4.16: Magnitude spectra of the target response and an 120th order Kautz model corresponding to the pole set in Figure 4.17b.
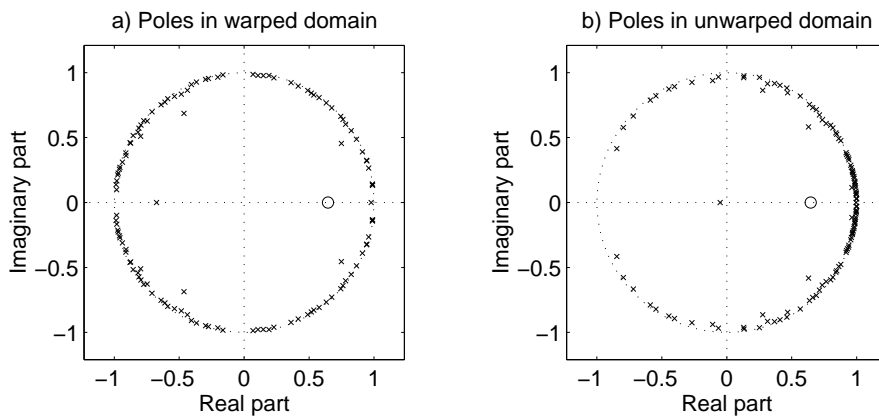

Figure 4.17: Kautz filter poles produced by the WBU-method, a) poles with respect to the Bark-warped target response, and b) corresponding poles in the original frequency-domain. The circle is the warping parameter.

evaluate both the temporal as well as the spectral evolution of model responses. The purpose of Figure 4.18 is mainly to demonstrate that the model genuinely clings to the resonances, distinctly with respect to the allowed number of poles, which is not a general feature of IIR or pole-zero modeling. Furthermore, in this case a virtually perfect match with the target response is attained at relatively low filter orders, as implied for example by the model of Figure 4.15 that deviates less than 1 % (in the square-error sense) from the original response. It is a safe estimate that in this case audibly transparent models are attained for various purposes with filter orders around 200, presuming that the modal density (in the higher-frequencies) is kept rich enough. This can be achieved for example by more moderate warping or by combining pole sets for different warping profiles.

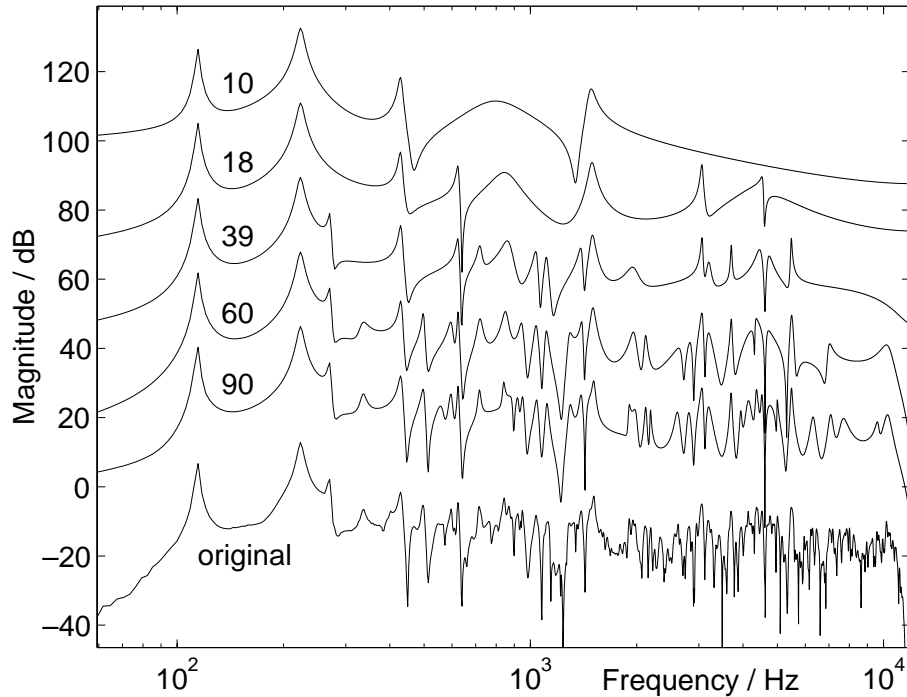This case study is concluded by demonstrating the idea of zooming by complex warp-

Figure 4.18: Kautz models of the guitar body response, displayed with offset from top to bottom: orders 10, 18, 39, 60, 90, and the target magnitude response.

ing, that is, the complex counterpart of the WBU-method. The target response is still the same as above: the CWBU-method can be used to focus the modeling resolution into a chosen frequency region. As an example, a complex warping parameter $a = 0.68 \ e^{j \ 0.43}$ is chosen, which corresponds approximately to the "zooming frequency" 1.5 kHz. The poles are generated using the BU method with respect to the complex Laguerre-warped target response. As in the WBU-method, the produced pole set is then mapped back to the original frequency-domain using the corresponding inverse allpass mapping. Figure 4.19 displays the two sets of poles for the model order 120. Unfortunately, the procedure is not accurate enough to produce real and complex-conjugate poles, as is seen from Figure 4.19b. This is due to inaccuracy of the model in the warped domain and not a principled defect of the complex warping procedure itself[13]. However, as expected, the achieved Kautz model response is "almost real" in the sense that the imaginary part is relatively negligible compared to the real part of the response. The target response and the real- and imaginary-parts of the model response are displayed in Figure 4.20a. The magnitude response of Figure 4.20b shows that the resolution of the modeling is indeed increased in the 1–3 kHz region.

The question remains: how to make it real? The evaluation of modeling errors reveals that the real part of the model response is actually a better model (in the LS

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[13]The Laguerre signal transformation is a complete orthogonal transformation, and thus, an exact representation (in terms of the generated poles and the implied rational orthonormal systems) in the warped domain would transfer into an exact, and consequently real, representation in the original signal domain. In practice, this ambiguity is indeed a defect of the proposed method, particularly in forming low-order approximations. However, the pole distribution attained by this zooming by complex warping procedure may always be utilized to construct or supplement a real rational construction.
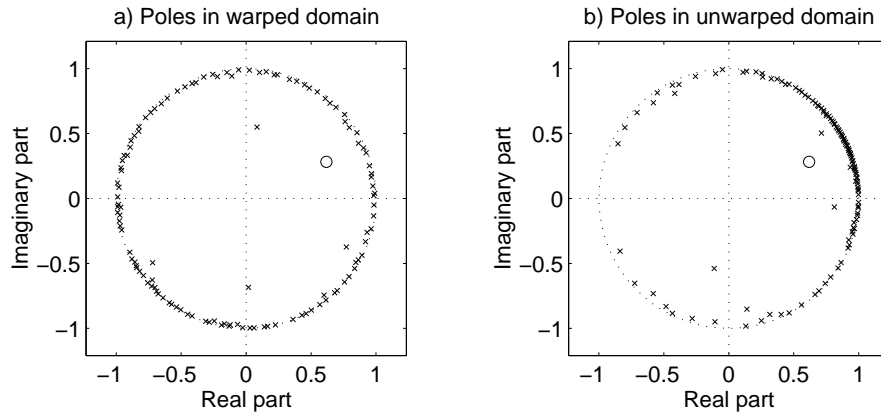
Figure 4.19: Kautz filter poles produced by the CWBU-method, a) poles with respect to the complex warped target response, and b) corresponding poles that are mapped back to the original frequency-domain. The circle is the complex warping parameter.
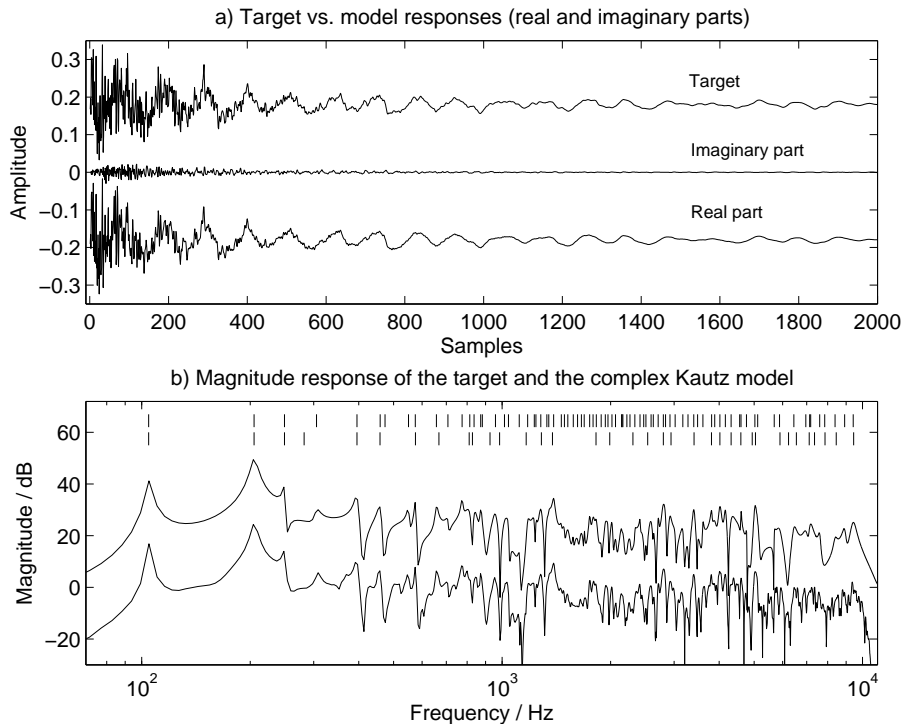

Figure 4.20: The target and the complex Kautz model responses, a) the real part fits well to the target and the imaginary part is relatively flat. In b) from bottom to top, target and model magnitude responses and lines that indicate poles in the lower and upper half of the unit circle, respectively.

sense) than the complex Kautz model response. However, the utilization of complex arithmetics may in practice be badly justifiable or simply impossible. There are two simple ways to form a conjugate symmetric pole set from the poles of Figure 4.19b, that is, the upper half-plane poles can be used as complex-conjugate (cc) pole pairs, or alternatively, the pole set may be expanded to include cc pairs of all the poles[14].

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[14]The latter scheme is equivalent to adding a pole set that is produced by the CWBU-method
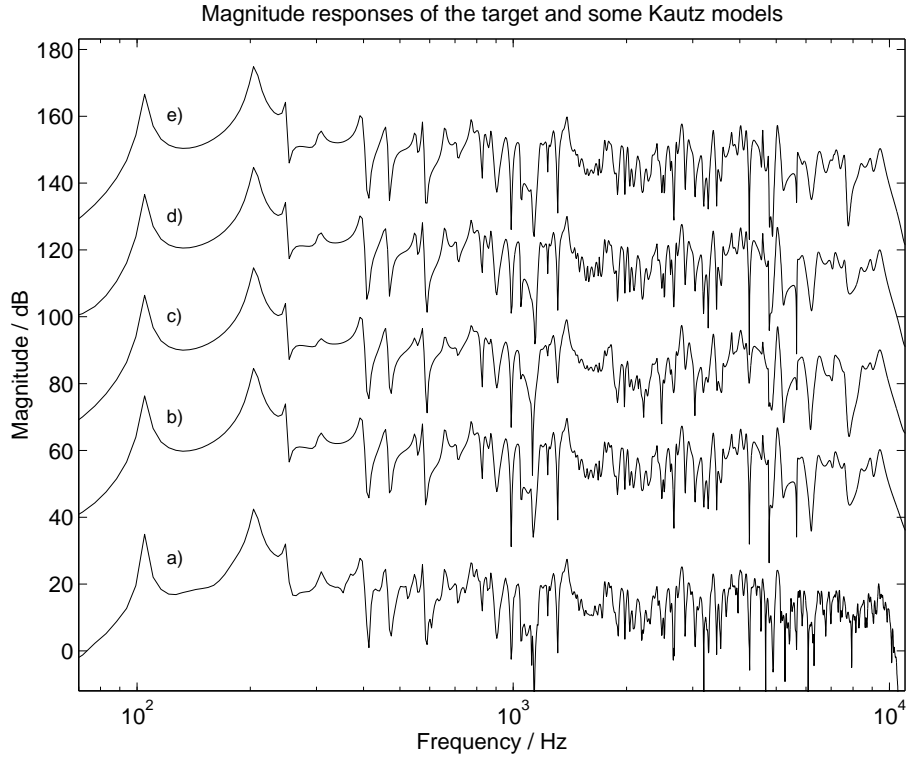
Figure 4.21: The target magnitude response (a) and various Kautz models generated from the pole set in Figure 4.19b, b) the complex model, c) its real part, d) real Kautz model with $\Im(z) > 0$ poles as cc-pairs, and e) real Kautz model with all poles as cc-pairs.

Actually, some kind of damping of the pole radii should be used to take into account for the increase in the model order. A comparison of magnitude response for different configurations of the Kautz models is depicted in Figure 4.21, where the orders of the real Kautz models d) and e) are 168 and 240, respectively. The main information of Figure 4.21 is that the proposed alternatives for producing a real Kautz model do not differ very much.

4.4.3   Case 3: Room response modeling

Models for a room response, that is, transfer function or impulse response representations from a sound source to an observation location in a room, are used for different purposes in audio signal processing, typically as a part of a larger system. Room response modeling may constitute a major computational burden, both because of the complexity of target responses and the difficulty of incorporating proper perceptual criteria in those models.

An obvious difficulty in modeling a room response is that the duration of the target response is usually long and that the temporal as well as spectral structure of the response can be very complex. Physically speaking, there are low-frequency modes

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with respect to the complex conjugate warping parameter a^* and the same chosen number of poles.
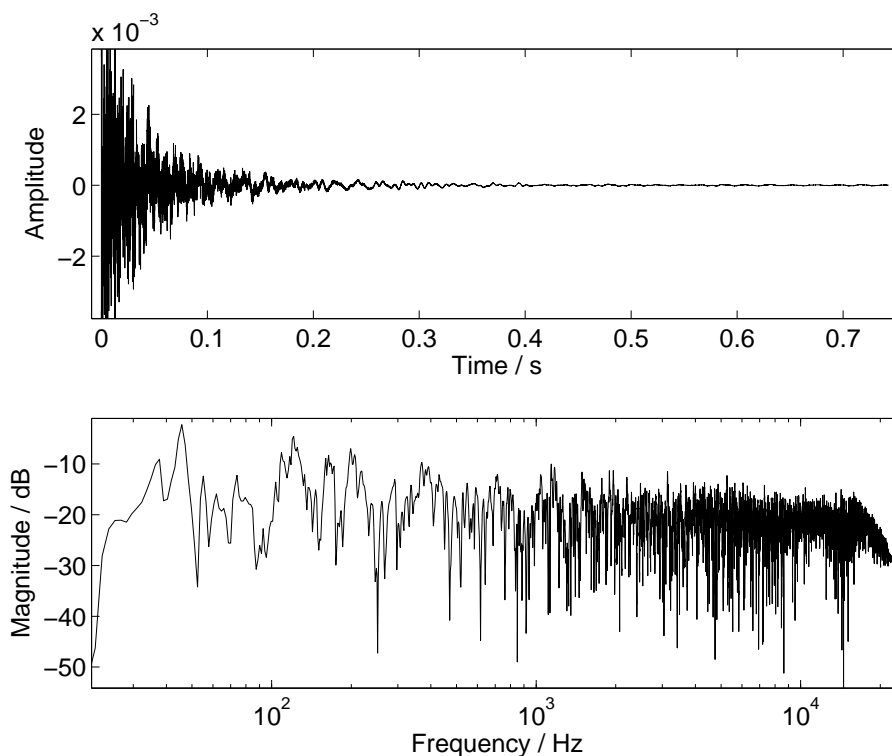
Figure 4.22: Time and magnitude responses of a measured room impulse response.

determined essentially by the room dimensions and on the other hand a reverberation structure produced by the multitude of reflections. There are methods proposed to take into account various time- and frequency-domain modeling aspects as LTI digital filter models [Mourjopoulos and Paraskevas, 1991, Haneda et al., 1994], including also reverb designs that approximate reflections and reverberation by complicated parallel and feedback structures [Gardner, 1998].

A measured impulse response in a medium size room[15] was chosen as an example for room response modeling using Kautz filters. The target response and its magnitude spectrum are displayed in Figure 4.22. (The sample rate is 44.1 kHz.) This is not an ideal Kautz modeling task since the response is not a superposition of coincident damped exponential components. The fact that the modes of a room response are not excited simultaneously will become even more apparent as the dimensions increase; a solution for partitioning the early response of a measured concert hall response is introduced in the end of this Section. Another immediate and general difficulty in modeling a room response is that the modal density is inherently very high, and although it is known that a more sparse representation of the modal structure in the high-frequency part would be sufficient from a perceptual point of view [Karjalainen and Järveläinen, 2001], it is in practice very difficult to incorporate such criteria into LS filter design. However, it is always possible to construct a (long and complicated enough) Kautz filter that is able to model all temporal details, though with an apparent inefficiency, for example, in producing delayed resonant components. Such an exact but extravagant model construction

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[15]The room has approximate dimensions of 5.5 x 6.5 x 2.7 m^3 and shows relatively strong modes with long decay times at low frequencies.
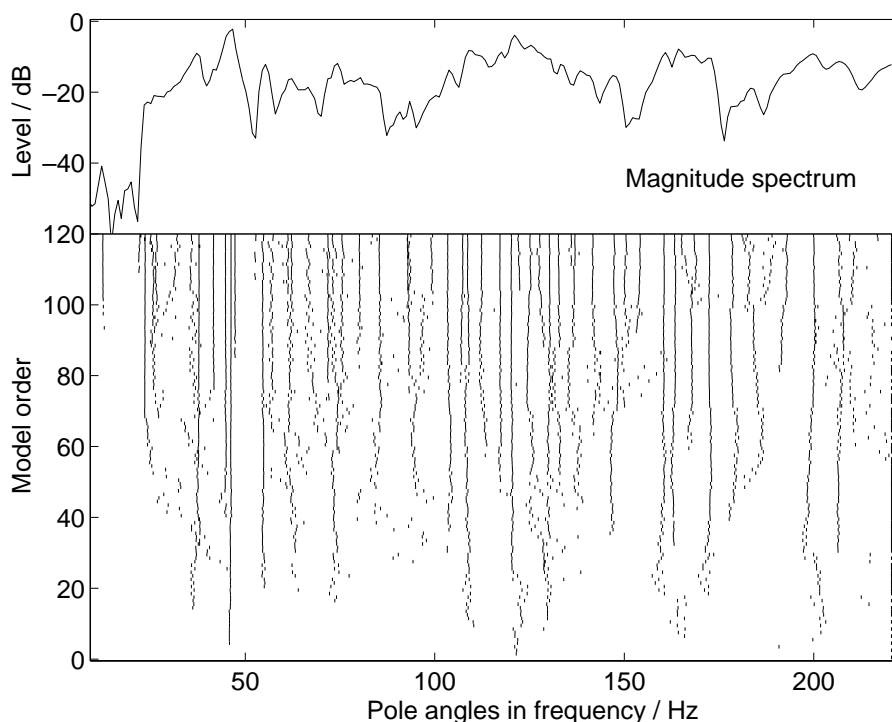
135

Figure 4.23: Frequency bins corresponding to poles angles produced by the BU-method for filter orders 1–120 and the magnitude spectrum of the room response.

could then in principle be reduced using technical and perceptual criteria; in practice, however, such pruning tasks may become overwhelming, or simply impossible to conduct in a meaningful way. In the following however, the applicability of Kautz filter techniques are considered mainly as supplementary or specialized elements in the construction of an overall model for such complicated target responses as room responses.

Detailed modeling of the low-frequencies is first considered by decimating the target response into the frequency region of about 0–220 Hz, where each prominent mode may have noticeable perceptual effect. The implied new target signal length is less than 1000 samples, which makes the BU-method for generating the poles operate very robustly. Figure 4.23 displays a pole angle bifurcation pattern for model orders 1–120 along with the magnitude spectrum of the room response up to 220 Hz at the top. It is noteworthy how the BU-method fixes the prominent resonances at a relatively low model order and that there is virtually no wobbling and splitting in the choice[16]. Figure 4.24 illustrates the accuracy of the model magnitude responses compared to the target response (top curve) for Kautz model orders 20–100 in steps of 5. For orders 100–120 the match is practically perfect.

The quality of time-frequency modeling can be checked by a waterfall plot (cu-

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[16]This figure does obviously not tell anything about the magnitude of the poles and how they evolve. Typically a resonance is represented by a relatively strong and fixed pole pair, whereas the supplementary poles, as the model order increases, are somewhat weaker. In this case it is also possible to push the process to the limit of the target signal length: the optimal limiting Kautz filter is indeed the FIR filter.
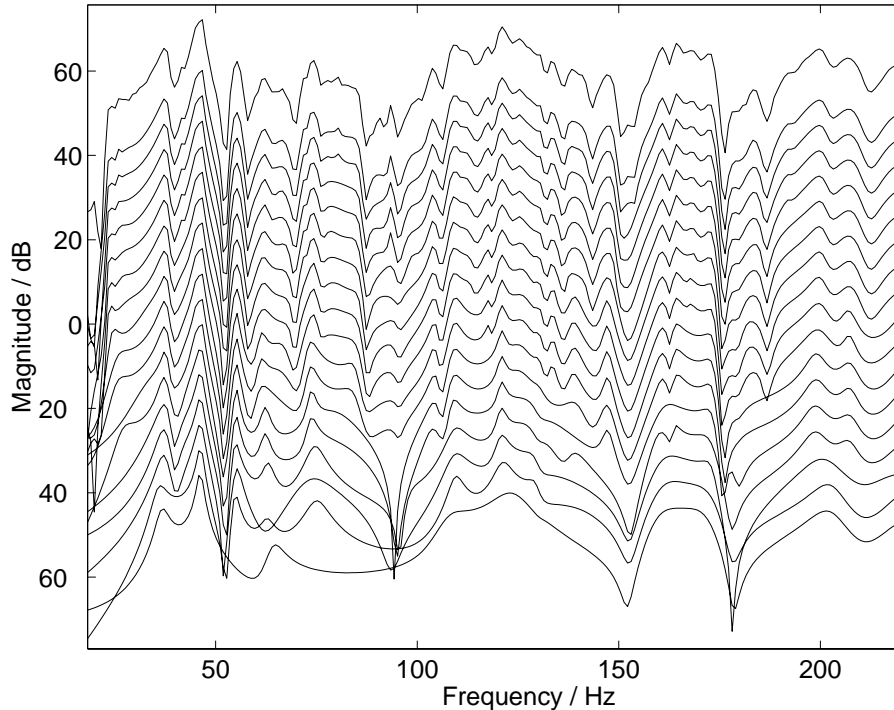
Figure 4.24: Magnitude responses of Kautz models of order 20–100, in steps of 5 from bottom up, compared to the target magnitude response at the top.

mulative spectral decay), as shown in Figure 4.25 for the target response and for a Kautz model of order 80. In careful comparison it can be noticed that some less prominent modal resonances are weaker and shorter in the model response. Increasing the filter order to 120 makes the model practically perfect also in a time-frequency plot. There are also redundant poles at the band edges so that the actual limit for perfect reconstruction is about 100. The attained low-frequency model can then also be transformed back to the original frequency-domain (or corresponding to any other chosen re-sampling) by mapping the poles with an appropriate complex exponent function and by evaluating the tap-output weights with respect to these poles and the original target response. This scheme results in a high concentration of poles, which is not a numerical problem in the Kautz filter case, in contrast to some other IIR filter designs, but it introduces a steep low-pass cutoff, which could and should be compensated or controlled by adding poles. This type of a model for the low-frequency part could then be complemented, for example, by appropriate artificial reverberation [Jot and Chaigne, 1991], as suggested in [Paatero and Karjalainen, 2001].

As stated earlier, full bandwidth[17] modeling using a single overall Kautz filter (or any other LTI filtering approach) is at the least very difficult, and in addition, inherently inefficient. As a somewhat more practical issue, the proposed pole optimization process, the BU-method, is really in trouble at implied filter orders above 400–600 for accurate modeling of the target response. If the modeling accuracy can be com-

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[17]In the following examples, the target response is re-sampled to 22050 Hz, providing a bandwidth of approximately 10 kHz, which is sufficient for most practical considerations of full bandwidth modeling of a room response.
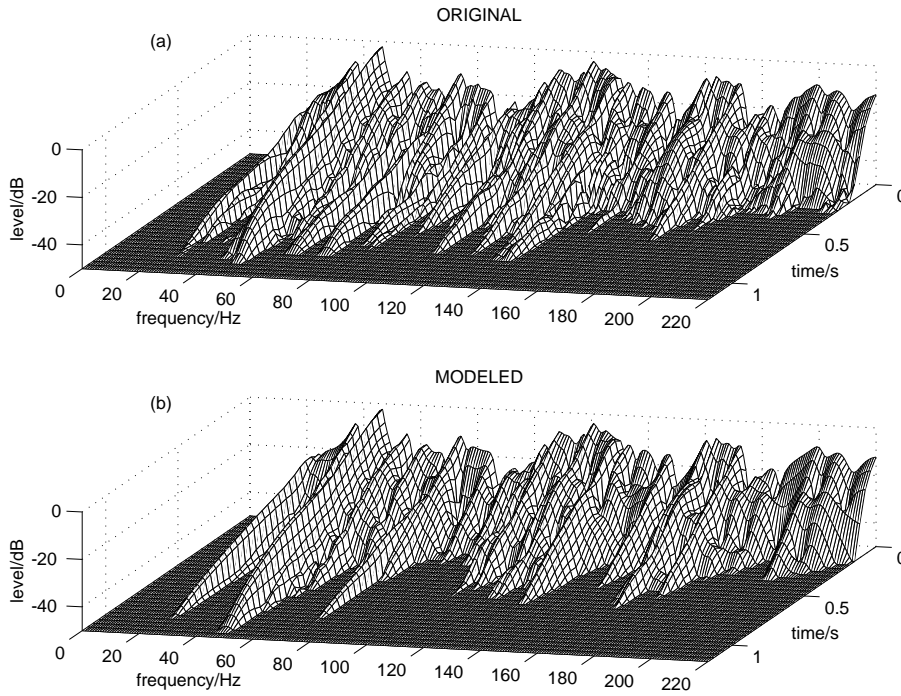
Figure 4.25: Cumulative spectral decay plot for (a) the target response and for (b) a 80th order Kautz model in frequency range 0–220 Hz.
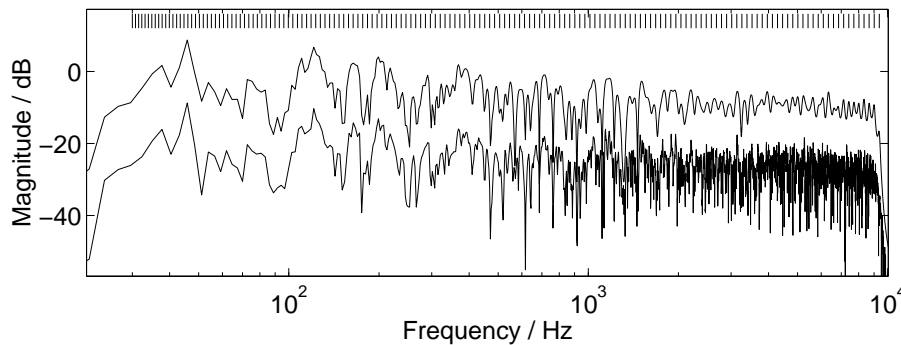

Figure 4.26: Magnitude responses for the room: (upper) 320th order Kautz model and (lower) target response, plotted with vertical offset. The poles are placed with logarithmically spaced angles and constant radius 0.98. The pole pair positions are indicated by vertical lines.

promised at some frequencies, or only a spectral envelope model is needed, several Kautz filter pole determination techniques can be applied. Figure 4.26 illustrates magnitude responses of a 320th order model and the target response, where the Kautz filter poles are simply positioned according to a logarithmic spacing in frequency and with constant pole radius of 0.98. At low frequencies below 200-300 Hz the magnitude response fit is quite complete, but the poor performance in modeling the dense resonance structure at higher frequencies is evident[18].

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[18] Alternatively, using a large set of weak poles (small radius), the Kautz filter acts as a "slightly recursive FIR filter", and there is a clear transition from this kind of FIR-type fit of the early response to representing resonances by corresponding pole pairs. For responses where the high-frequency components are short in time, some of the poles could actually be forced to $z_i = 0$, that
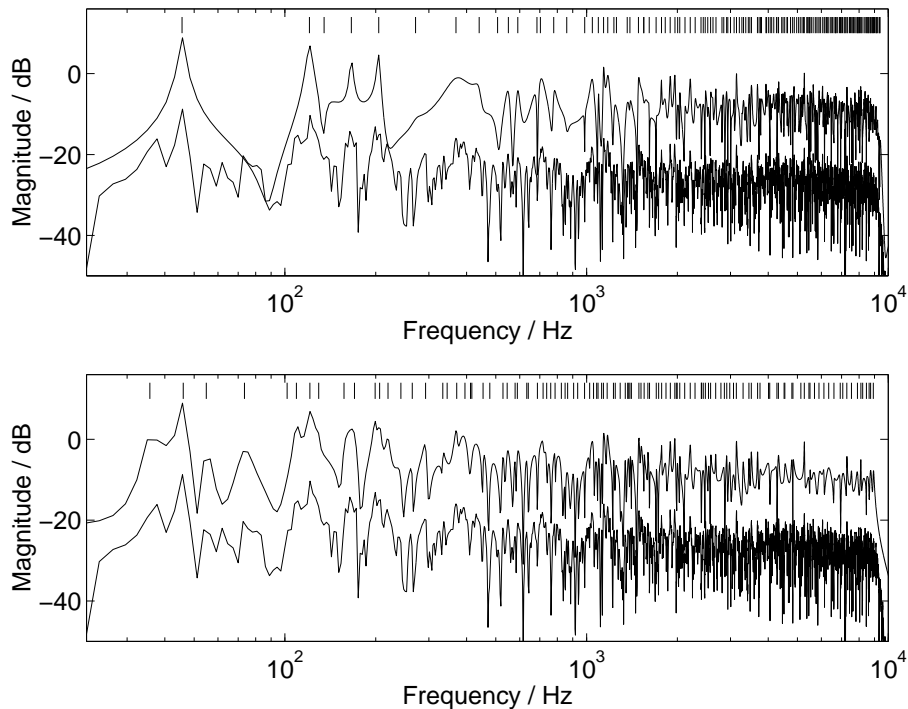
Figure 4.27: Kautz model magnitude responses, produced using the BU-method (with upward offset to target response): for filter order 318 (top), and by warping and back-mapping at the filter order 240 (bottom).

As examples of direct utilizations of the BU-method, two Kautz filter models for the room response are presented in Figure 4.27. The 318th order Kautz model, presented in the upper plot of Figure 4.27, is attained from a larger pole set by omitting some poles close to $z = -1$. Poor modeling of the lowest resonances is evident, but the high-frequency region is represented relatively well. In the lower pane of Figure 4.27 the warped BU-method is used to produce a 240th order Kautz filter, where the low-frequency region is emphasized by using a warping parameter $a = 0.64$. As a tradeoff, worse modeling of the higher frequencies is apparent. These model orders are already close to the "reliability limit" of the BU-method for the corresponding target response duration and characteristics. A better overall model would be achieved by combining the pole sets with respect to a chosen partition of the frequency range.

A more detailed and reliable description of the "mid-frequency region" is attained by further band-limiting (re-sampling to 11050 Hz) the target response; the magnitude response of a 500th order Kautz filter is displayed in Figure 4.28, for a change on a linear frequency scale. The WBU-method was used with a warping parameter $a = 0.4$, which is somewhat less than the corresponding Bark-scale warping parameter ($\lambda = 0.48$ with respect to the implied sample rate [Smith and Abel, 1999]). It is once more emphasized that these examples are merely demonstrations of the methodology and that they are not suggested as such for the full bandwidth modeling of the room

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is, a mixture of Kautz and FIR filter blocks could be utilized. It is noteworthy that this may be done in any order of the sections and that the same chosen filter tap-output weighting applies also to the FIR blocks. From a more practical point of view, the early response could be isolated as an FIR filter whereas the rest of the response would constitute the target for the Kautz model.
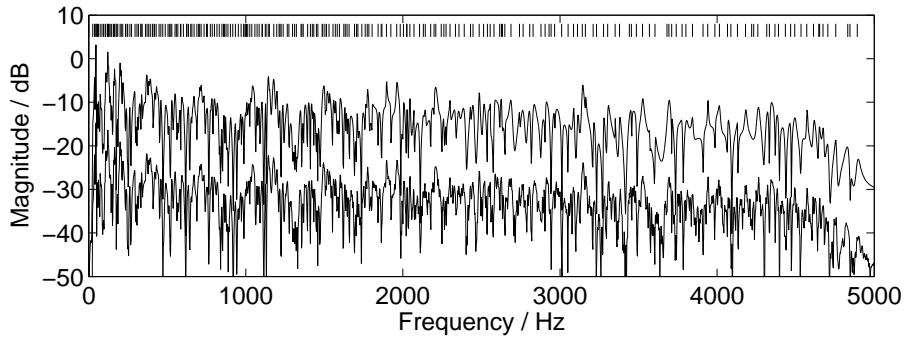
Figure 4.28: Magnitude responses on a linear scale: for the re-sampled target response (bottom), and the 500th order Kautz model (top). Vertical lines indicating complex conjugate pole pair positions.
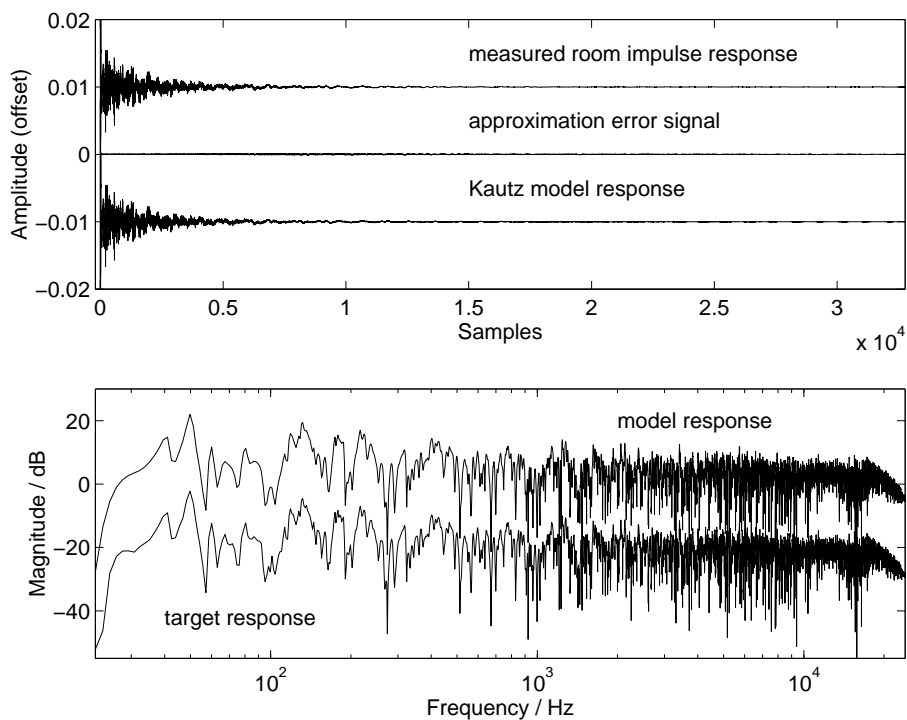
Figure 4.29: The target response versus a 64×60th order Kautz model.

response. However, these considerations provide detailed descriptions of the resonant frequencies, that is, parametrizations in terms of accurate complex pole positions that can be used in other and pragmatically better-justified modeling strategies, such as, parallel resonator design for the high-frequency part.

This case study is concluded by utilizing the model partition schemes of Section 4.3.2. As a brute force example, the original measured room response of Figure 4.22 (32768 samples, sample rate 44.1 kHz) is modeled using a 64×60 polyphase Kautz filter, which implies that the response is partitioned into 64 sub-signals of length 512 samples and that each of the sub-signals are approximated using a 60th order Kautz filter. The time- and frequency-domain fit to the target response is almost perfect, as can be deduced from Figure 4.29. A fixed sub-filter order is used, although a more sophisticated allocation of the filter orders would improve the model, or alternatively,

Figure 4.30: The early part of a measured concert hall impulse response (h), component Kautz models (a)–(e), with filter orders 40, 30, 60, 60, and 80, respectively, the composite model (f), and the modeling residual (g). The actual duration of the time-window is 77 ms.
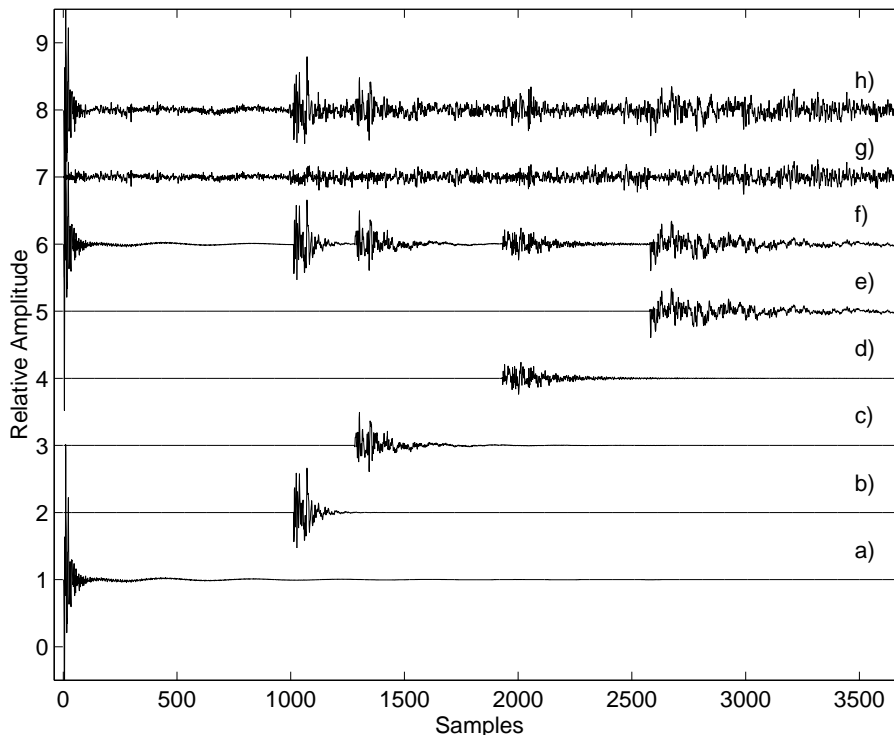
reduce the overall model order. Also in this approach, the low-frequency region could be emphasized by using warping in the pole generation process, naturally, at the expense of a decrease in the high-frequency mode density. The model response was found to be audibly transparent (by subjective listening) to the original target response and substantially lower order models were found to be tolerable for many purposes. The actual usefulness of this approach from the implementation point of view is not considered here and the only measure of computational complexity is with respect to the implied polyphase FIR filter implementation, that is, a 512th order FIR filter for the component signal is replaced with a 60th order Kautz filter.

From the time-domain perspective, the polyphase decomposition is an interlacing operation; as described in Section 4.3.2 it is also possible to utilize successive segmentation of the target response to attain partial model descriptions. As an example, the early part of a measured impulse response of a concert hall was modeled using the proposed Kautz filter configuration. Figure 4.30 presents a low-order approximation. More detailed models are achieved by increasing the filter orders. In principle, any response can be modeled exactly using a partition of this form, but the relevant applications of the proposed method are clearly related to the extraction of impulse-like bursts. As another aspect, the model of the early response provides a very compact parametrization that could be used in control and reproduction of the response, for example, to be utilized in various virtual acoustics applications.

4.4.4   Some comments on the application cases

Kautz filter design using the BU-method, its variants, and different rearrangements of the modeling task, can be seen as a particular IIR filter design technique. The motivation from an audio processing perspective is two-folded: many audio-related target responses can be well-modeled by a combination of distributed decaying exponential components, which is by definition what a Kautz filters does, but in an orthonormalized form, providing many favorable properties. On the other hand, Kautz filtering techniques provide many ways to incorporate auditorily meaningful allocation of frequency resolution to the modeling.

The above cases of modeling and equalization were taken as challenging examples in order to show the applicability of Kautz filter techniques. Many specific questions, such as the audio engineering relevance of modeling details, perceptual aspects of the designs, as well as computational robustness and expense have been addressed only briefly or not at all. In particular, an actual comparison between the proposed methods and the "conventional means in FIR and IIR filter design" is never conducted, although such expressions as "unattainable using traditional methods" are, at the least, implied every now and then. The confrontation with FIR filter counterparts is more or less self-explanatory, but especially the low-order Kautz models should be more genuinely placed into the scope of available IIR filtering methods. In the case of high-order models for complicated responses, the author is somewhat presumptuously relying on the readers experience and knowledge of the difficulties of IIR filter design, which is of course not a very good excuse. It is apparent that these details as well as the more fundamental questions call for further investigations.

The aim of this study was to show that it is possible to achieve good modeling or equalization results with lower Kautz filter orders than, for example, with warped (Laguerre) or traditional FIR and IIR filters. In the loudspeaker equalization case, Kautz filters of orders 20–30 can achieve similar results of flatness as warped IIR equalizers of orders 100–200. This reduction is due to well-controlled focusing of frequency resolution both on the global shape and on local deviations.

The guitar body response modeling case is probably the most straightforwardly applicable of the presented examples. The combination of the BU-method and warping techniques can be used quite flexibly to attain desired frequency resolution allocation into the modeling. Low-order Kautz filters are able to focus sharply on the perceptually significant low-frequency modes, showing advantage over warped, FIR, and other IIR filter designs. On the other hand, increasingly detailed models are obtained by increasing the model order: in this case, the target response is within the reach of perfect reconstruction.

The room response modeling case should be taken as a mere methodological study, except possibly regarding the usefulness of obtained low-frequency models. In particular, the only full bandwidth overall model for the room response, based on polyphase decomposition, was just a pompous tour-de-force experiment of the waveform matching capabilities of Kautz filters in the subbands. Nevertheless, detailed description of resonant frequencies in terms of accurate pole locations are attained

by various warping, mapping and decimation techniques that could then in turn be utilized in more sparse, pragmatic, and perceptually motivated schemes for the modeling of the reverberation structure. Alternatively, a Kautz model for the low-frequency part could be complemented with carefully designed artificial reverberation that would model the high-frequency part in a statistically and perceptually meaningful way. As a future case study, it would be interesting to try to model a room response by combining some of the proposed Kautz filter strategies: successive segmentation for the early response, accurate modeling of the prominent resonances, and a sparse but sufficiently rich resonator implementation of the late reverberation, based on the pole position analysis provided by the BU-method.

Chapter 5

Contributions of the thesis and concluding remarks

The outcome is seldom what was expected in the beginning of a process. This is particularly true in the case of this monograph which took its time to shape up. The original idea of more or less translating and updating the licentiate's thesis was completely altered along the way. The sudden burst in publishing activity in the years 2001–2002 brought the writing process to an almost standstill and it caused a dilemma between continuing with the draft of a monograph or to switch to the article bundle format, which could have been at the least a much faster alternative. The author is however reasonably happy with the outcome and particularly content with the decision of sticking to the monograph form.

The biggest contribution of this thesis, in the author's opinion, is in the way that various elements are combined and displayed. Even the somewhat anecdotal touch is considered as an achievement. This is reflected, for example, in the way mathematical concepts are treated as universally available, or as common property, with very few actual references. On the other hand, there are some genuinely new ideas, even in the mathematical part, that are not emphasized as such too clearly. This situation would be sufficient enough for the author, but as it is customary, the thesis is concluded by enumerating some of the more clear and distinct contributions.

The contribution of Section 2.1 is mainly that it provides a somewhat broader insight into representing signals, systems and related transformations, where in particular, the subsequent "signal space" framework is primed. Sections 2.2 and 2.3 are of course elementary and simplified from the point of view of mathematics, but it is actually relatively difficult to bring about an appropriate but compact presentation of the relevant concepts to the context of this thesis. As an example, the identification of the z-transform is in fact quite original. The concept of a base was also introduced more thoroughly than usual. Some of the denotations and definitions in Section 2.3.3 are genuinely new. However, Section 2.4 is clearly the most original part of Chapter 2. Utilization of the bilinear transformation in relating function spaces is not new, and on the other hand, there are much more profound considerations of the shift operator in the mathematical literature; however in Section 2.4,

conformal mapping techniques were introduced and utilized very pragmatically and explicitly, which has probably not been done before. The author is not too concerned about the mathematical correctness of this part of the thesis: the point is in the deduction and the validity of the outcome is assured by different means.

The GLM concept of Chapter 3 is obviously in principle new, although not particularly ingenious as an innovation in the sense that many others have considered, at least partly, similar aspects. Nevertheless, it has not been formulated before as such, that is, from the point of view of linear independency. It is more in the eye of the reader to decide how original the deductions of Sections 3.2 and 3.3 themselves are compared to their conventional counterparts. The implication should however be clear: linear independency is the driving force behind most familiar ways of solving least-square problems, and in addition, it is the source of potential orthogonality. The obvious vagueness of the construction is compensated by referring to proposed applications that can be considered as exemplifications of the GLM. Section 3.4 contains some genuinely new constructions and considerations. The GLM lattice structure is an authentic generalization, although it is still merely a conceptual construction, but in the author's opinion, it provides more insight to the actual problem than some other more explicit and specified generalizations. Similarly, Section 3.4.2 is not just an introduction to orthogonal filter structures; most of the presented aspects seem to be at least somewhat original, and for example, the basis function construction (3.81) is genuinely unique, although once more slightly artificial.

It is lot easier to point out the actual contributions of Chapter 4. In the methodological part, the generalized frequency resolution description of Section 4.2.1 is genuinely new, as well as, the rehabilitation of the concept of complementary signals in Section 4.2.3. The most important contribution is probably the introduction of the BU-method to the context of rational orthonormal function expansions. The BU-method and its variants are presented in Section 4.3. The case study part provides an entirely new perspective to the modeling of audio related responses. More generally, the proposed methodology introduces a new family of IIR filter design techniques that can be used in modeling very complicated responses. In particular, the combination of the BU-method and the Kautz filter structure can be used to produce such high-order and efficient pole-zero filters that are not usually attainable. The importance and originality of the proposed applications have also been evaluated in the form of a series of related publications.

The GLM concept could naturally be extended to include, for example, various weighting schemes or further considerations with respect to different norms. Alternatively, the GLM could be concretized by specifying families of particular model structures. The point of the whole construction was, however, in the simplicity, as well as, in the artificiality, and the author feels that this theme has been dragged along at least long enough. The Kautz filter approach to the modeling of audio related systems, on the other hand, is an ongoing process. As natural extensions to the modeling by synthesis framework, more genuine identification configurations have been considered for various equalization tasks. Another interesting topic is the utilization of Kautz filters in adaptive noise control or sound management systems. In conclusion, there is lot to be done.

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