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AFFINE EQUATIONS AS DYNAMIC VARIABLES TO OBTAIN ECONOMIC EQUILIBRIA

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Abstract: This thesis studies how economic equilibria can be reached with limited information by adjusting affine equations, and how collusion in oligopolistic markets can be maintained with affine reaction strategies. The first question is considered for exchange economies, contracting problems, and a two-party negotiation support method. The main contributions are new ways to reach and maintain equilibria.

For exchange economies the thesis shows that a modification of fixed-point iteration converges globally under conditions that are remarkably close to those required for the continuous time tâtonnement process presented by Arrow, Block, and Hurwicz. For the constraint proposal method for two-party negotiations this thesis provides answers for three major questions: does the method produce Pareto-optimal points, does it lead to a problem that has a solution, and can the solutions be found with fixed-point iteration. The thesis also shows that the complete information equilibrium of a contract design problem can be reached by adjusting the contract with fixed-point iteration. Finally, the thesis formulates a simple dynamic counterpart of a static reaction strategy by D. K. Osborne in a repeated oligopoly game with discounting. It is proven that these strategies lead to a subgame perfect equilibrium when the possible deviations are bounded and the proportional reaction strategies have sufficiently large slopes.

Keywords: exchange economies, tâtonnement, negotiation support, contract design, fixed-point iteration, collusion, oligopoly, subgame perfection

Academic dissertation

Systems Analysis Laboratory
Helsinki University of Technology

Affine Equations as Dynamic Variables to Obtain Economic Equilibria

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Preface

As this thesis is completed, I wish to acknowledge my gratitude to the people and institutions who have provided me guidance and support.

First, I want to express my warmest thanks to Professor Harri Ehtamo, the instructor and supervisor of this work. His ideas and initiatives have made this thesis possible. I have also greatly benefited from his extensive comments on my manuscripts. Most importantly, Ehtamo has offered me a great wisdom on scientific work in general.

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I am indebted to the personnel of the SAL for the extraordinary working atmosphere. Special thanks go to the people who have shared unforgettable moments with me either in travels, playing floorball, or in having quiz breaks. I also want to thank all who have collaborated with me on teaching or on research topics not included in this study.

I am grateful to all who have commented my writings and given me helpful suggestions. In particular, I thank Professor Hannu Salonen and Dr Pauli Murto for examining this thesis with great accuracy and for making improving suggestions. I also appreciate TKK Dissertations Committee for devoting special attention to the formal structure of this thesis. I hope that my study helps them in reaching equilibrium.

While making final corrections to this thesis I worked at MTT/Economic Research (MTTL). I am grateful to the people of MTTL, particularly to the Environmental Economics research group, for their kind support and hospitality.

My friends deserve special credits for adding fascinating dimensions to my otherwise rather affine life – I wish you peace! Finally, I want to express my gratitude to my parents and my brother for their continuous and (subgame) perfect support.

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Helsinki, December 2005
Mitri Kitti

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Chapter I:

Introduction and Summary

1 Introduction

This thesis studies the following questions: How economic equilibria can be reached with limited information by adjusting affine equations, and how collusion in oligopolistic markets can be maintained with affine reaction strategies. The main emphasis is on the first question, which is considered for exchange economies, contracting problems, and a two-party negotiation support method. In brief the main contributions are new ways to reach and maintain equilibria.

The common characteristic with the various models included in this work is that they all include agents who are making decisions with respect to affine constraints, i.e., constraints that include a linear part and a constant. For example, in an exchange economy there are consumers willing to acquire amounts of products that maximize their utility functions and do not exceed their budgets, which are affine constraints with respect to the amounts. An example of an affine contracting game is a buyer-seller game where the seller announces that the price of amounts of product to be sold depends on a unit price and a constant premium, i.e., the price is an affine mapping of the amount. Mathematically speaking, all the models considered in this thesis include convex optimization under affine equations. Furthermore, the models are dynamic and the affine equations are the variables that determine the dynamics.

Understanding the resource allocation among economic agents is the first step to understanding competitive markets. Exchange economies provide a simple framework for studying resource allocation in its purest form, i.e., in settings where there is no production but only consumers. The equilibrium of an exchange economy is the set of prices for which the total demand of commodities equals the consumers' total amounts of them. The main issues concerning the equilibrium are its existence, uniqueness, Pareto-optimality of the corresponding resource allocation, stability, and comparative statics. This thesis focuses on stability issue, more specifically on explaining how the equilibrium can be reached. Existence and Pareto-optimality are studied for a negotiation support model by Ehtamo et al. (1999a) that can be interpreted as a specific exchange economy.

Processes for reaching the equilibrium of exchange economies have been studied extensively in economics literature beginning from the works of Walras (1874), who introduced the concept of tâtonnement processes. The idea of tâtonnement processes is that prices are adjusted until the equilibrium is found but no trades are made before that. Samuelson (1947) was the first to suggest the use of differential equations to analyze tâtonnement stability. Arrow and Hurwicz (1958), and Arrow et al. (1959) proved the global stability of Samuelson's continuous time tâtonnement process, i.e., convergence to an equilibrium for all initial prices, under economically significant conditions such as gross substitutability and the weak axiom of revealed preferences.

Continuous time tâtonnement processes have the disadvantage that they cannot be used in solving resource allocation problems in any real world setting because the consumers' actions can only take place in discrete time instances. Therefore, the stability issue should rather be studied in discrete time. The results obtained within the last two decades show that most discrete time processes for reaching the equilibrium fail to converge under the stability conditions given by Arrow et al. (1959) for the continuous time process. This thesis answers to the need of discrete time tâtonnement process with satisfactory convergence properties by showing that a modification of fixed-point iteration converges under conditions that are remarkably close to the continuous time convergence conditions. Mathematically the main contribution is a new convergence result for fixed-point iteration in solving equations that are characterized by a property known in economics literature as Walras' law.

For the affine contract design problem this thesis shows that the complete information equilibrium can be reached when the contracting game with the same agents is played repeatedly and the contract is adjusted with fixed-point iteration. For example, if there is a seller and a population of buyers who all have similar preferences and each period one of them is to be served, then the seller can finally reach the price-amount tariff that gives him the optimal profits even though he does not have any prior information on the buyers' preferences. The process has two meanings. First, it gives a practical method for finding optimal contracts with limited information. Second, it could explain why complete information contracts are observed even though there was incomplete information in the market settings.

In this thesis, in addition to exchange economies and contracting games, the adjustment of affine equations is considered in finding Pareto-optimal points for two-party negotiations over continuous issues. For example, the negotiation could be on allocating resources, such as money and labor force, between two units of a company. The purpose of negotiation support methods in such settings is to locate Pareto-optimal points among which the decision makers, e.g., the units of a company, can choose their agreement. The Pareto-optimality of a point means that at least one of the parties is worse off at any other point.

The constraint proposal method, a negotiation support method by Ehtamo et al. (1999a) for finding Pareto-optimal points, is based on adjusting artificial budget constraints. It is one of the few negotiation support methods that does not require the parties to disclose any private information to each other. This thesis presents conditions under which the constraint proposal method leads to a system of equations that has Pareto-optimal solutions. Furthermore, convergence of fixed-point iteration in solving the problem is analyzed. The thesis also discusses how the setting in the constraint proposal method can be interpreted as an exchange economy where the parties are allocating their total dispute on the issues.

In the aforementioned models, the decision makers are assumed to be short-sighted in the sense that during the process they do not consider any other future periods than the current one, i.e., they are willing to reveal their best responses for given affine equations. When studying the question of maintaining cooperation in oligopolistic markets it is reasonable to assume that the firms are fully rational and do not behave short-sightedly as in these models. One purpose of this thesis is to explain tacit collusion as a result of rational behavior. This issue is studied in the framework of repeated games with discounting. It is well-known that cooperation can be maintained in such games with various trigger-type strategies when the players are patient enough, i.e., their discount factors are sufficiently large. This thesis takes a different approach: Instead of restricting the discount factors the possible deviations from cooperation are assumed to be bounded.

Usual trigger-type strategies are based on switching to a punishment phase after one of the firms has deviated. The punishment phase is the same regardless of the magnitude of deviation. Thus, these discontinuous strategies omit the continuous nature of firms' output variables and they cannot fully explain market behavior. As an alternative to trigger strategies economists have considered reaction function strategies, which presume that the firms' actions vary continuously to their rivals' actions. In this thesis a static reaction function strategy, proposed by Osborne (1976), is formulated in a repeated game. The strategy is based on changing the outputs in proportion to the deviation and hence it is called the *proportional strategy*.

It is shown that depending on the discount factors there are always intervals of deviations such that the proportional strategies constitute a subgame perfect equilibrium when restricting to these intervals. Subgame perfection means that it is optimal for the firms to follow the proportional strategies regardless of what has happened during the history of the game. In a duopoly case the equilibrium also satisfies an important equilibrium refinement called weak renegotiation proofness, which guarantees that the firms would not be willing to change their behavior if they could renegotiate their agreement anew in any contingencies, see Farrell and Maskin (1989). Furthermore, this thesis discusses the relationship of proportional strategies and conjectural variations models that are widely used, especially in empirical literature, to explain and measure incomplete competition.

The following three chapters of this thesis study the adjustment of affine equations in various economic settings. In chapters II and III the affine equations are budget constraints and in Chapter IV they are contracts. In all of these three models there is a coordinator, who adjusts the equations, and a group of decision makers, who solve optimization problems under these equations as constraints. In an exchange economy the coordinator is a Walrasian auctioneer, a hypothetical agent whose only task is to adjust the prices until equilibrium is reached. For the constraint proposal method a mediator acts as a coordinator and in a contract design game the principal takes the coordinator's position.

In addition to their structural similarities, the models in chapters II–IV are mathematically close to each other: They all result to equilibrium problems that are characterized by Walras' law and degree zero homogeneity. In the oligopolistic game of Chapter V there is no outside coordinator to drive the market to an equilibrium, but rather the collusive firms themselves act as coordinators. With this interpretation the proportional strategies play the role of dynamic coordination variables as the affine equations in chapters II–IV. The coordinator and the other parties in the models are summarized in Table 1.

Table 1: Summary of Contents

	Chapter II	Chapter III	Chapter IV	Chapter V
Setting	Exchange economy	Negotiation	Contract design	Cartel
Coordinator	Auctioneer	Mediator	Principal	Firms
Other parties	Consumers	Negotiators	Agent	None
Affine equations	Budget constraints	Artificial constraints	Contract	Reaction strategies

In chapters II–IV the two main assumptions are short-sighted behavior and limited information on the decision makers' preferences, whereas Chapter V assumes full rationality and complete information. On the other hand, chapters II–IV focus on reaching an equilibrium and Chapter V on maintaining a predetermined point as the equilibrium outcome of an oligopolistic game. The common characteristic of Chapters II–IV is that affine equations are exogenously taken as dynamic variables. In Chapter I budget constraints arise naturally as equations to be adjusted.

In the following sections I describe the various models of this thesis in more detail, shortly review the related literature, and present the main results. Finally I discuss some topics of further research.

2 Chapter II: Convergence of Non-Normalized Iterative Tâtonnement

In an exchange economy there is a group of consumers with initial endowments of commodities that have exogenously defined market prices. Although the market participants are called consumers, exchange economies are suitable models for quite general resource allocation problems that involve other economic agents than consumers and other resources than commodities as well.

In the following $n \geq 2$ is the number of commodities, m is the number of consumers, and $w^j = (w_1^j, \dots, w_n^j)$ is the consumer j's initial endowment of commodities. For given prices $p = (p_1, \dots, p_n)$ each consumer is willing to take the bundle of commodities that maximizes her utility with respect to the budget constraint, the meaning of which is that the monetary value of the bundle cannot exceed the value of consumer's initial endowment. In other words consumer j maximizes her utility function $u_j(x)$ subject to $p \cdot x \leq p \cdot w^j$ and $x_i \geq 0$ for all $i = 1, \dots, n$. Here $p \cdot x = \sum_i p_i x_i$. The result of consumer's maximization problem is the demand function $x^j(p)$. On the aggregate level the economy is characterized by the *excess demand function* $z(p)$ that is obtained by summing all the consumers' excess demands $x^j(p) - w^j$, i.e., $z(p) = \sum_j [x^j(p) - w^j]$. The properties that excess demand functions usually have are listed below.

- (P1) z is single valued and continuous for all $p > 0$.
- (P2) Monetary value of the excess demand is zero, i.e., z satisfies *Walras' law*: $p \cdot z(p) = 0$ for all $p > 0$.
- (P3) Only relative prices matter, i.e., the prices can be scaled with any positive multiplier without affecting the excess demand. Mathematically z is *homogeneous of degree zero*: $z(\alpha p) = z(p)$ for all $\alpha > 0$.
- (P4) Excess demand of each commodity is bounded from below, i.e., there is a scalar $\nu < 0$ such that $z_j(p) > \nu$ for all j and $p > 0$.
- (P5) All the commodities are *desirable* in the sense that when some of them become free, the excess demand becomes infinitely large at least for some of those commodities. Mathematically this means that

$$\lim_{p^k \to p} [\max_{j \in J_p} z_j(p^k)] = \infty,$$

  when $p^k > 0$, $p \neq 0$ and $J_p = \{j : p_j = 0\} \neq \emptyset$.

A price vector $p^* \in \mathbb{R}_+^n = \{p \in \mathbb{R}^n : p_j > 0 \ \forall j\}$ that clears the market, i.e., satisfies $z(p^*) = 0$, is the equilibrium of the economy. It can be shown that when the properties (P1)–(P5) hold there is always at least one equilibrium for an exchange economy. Moreover, (P1)–(P5) hold under quite loose assumptions on the consumers' utility functions. These assumptions also guarantee that the allocations $x^1(p^*), \ldots, x^m(p^*)$ corresponding to an equilibrium p^* are Pareto-optimal. See, e.g., Balasko (1988), Debreu (1959), Hildenbrand and Kirman (1988), Mas-Colell et al. (1995), Takayama (1974), and Varian (1992) for more on these properties and how they are obtained from the consumers' utility maximization problems.

Let us now go to the stability issue, which is the main concern in Chapter II. Samuelson (1947) was the first to formulate the continuous time tâtonnement process as the following differential equation

$$\dot{p}(t) = z(p(t)), \tag{1}$$

where $\dot{p}(t)$ is the time derivative of $p(t)$. Arrow and Hurwicz (1958), Arrow et al. (1959) and Arrow and Hurwicz (1960) were the first to show that the process (1) is globally stable, i.e., it converges to an equilibrium for any positive initial prices, under the following condition:

(C1) There is an equilibrium $p^* > 0$ that satisfies $p^* \cdot z(p) > 0$ for all $p > 0$ for which $z(p) \neq 0$.

Furthermore, Arrow et al. (1959) proved that (C1) holds when the excess demand function satisfies the gross substitutability condition or the weak axiom of revealed preferences. A differentiable excess demand function z is said to have the *gross substitute property* if $\partial z_j(p)/\partial p_i > 0$ for $j \neq i$. This property means that when the price of some commodity increases, the demand for other commodities grows. The weak axiom of revealed preferences is discussed in Section 2.1.

The process (1) as well as other tâtonnement processes are usually interpreted as auctions run by a fictitious agent, a Walrasian auctioneer, who sets the prices until an equilibrium is reached and the trades are made. When formulated in discrete time, such processes could provide practical auctioning methods for solving resource allocation problems. The more common purpose of tâtonnement processes is to explain how an economy comes to its equilibrium. However, as remarked by Balasko (1988, Section 1.7), economic theory does not propose a vector field defining the true dynamics leading to equilibrium. Hence, tâtonnement processes are merely approximations that simplify the formulation and analysis of market dynamics.

To be economically meaningful a tâtonnement process should have the following properties: Its use should not require other information than prices and the corresponding excess demand, it should satisfy the *law of demand*, according to which prices should increase for commodities with excess demand and fall in the opposite case, the process should converge under economically relevant conditions, and the process should be formulated in discrete time. Basically most discrete time processes are variations of fixed-point iteration $p^{k+1} = p^k + z(p^k)$, which is clearly the simplest discrete time process that could possibly satisfy the other aforementioned requirements.

One feature of all tâtonnement processes is that they do not describe strategic behavior because the consumers' are assumed to be price takers: They do not consider how their current demand affects the future prices. An explanation for this behavior could be that the economy is too large for any single consumer to be able to affect the prices by acting strategically, see Roberts and Postlewaite (1976). Moreover, since the equilibrium of an exchange economy is a result of price taking behavior by definition, it is natural to consider processes in which the consumers are price takers.

Uzawa (1960) was the first to provide convergence results for an iterative discrete time process with normalized prices, i.e., one of the prices is set to a constant and only the rest of them are adjusted. More recent studies, however, question the relevance of these results by showing that such discrete time processes often fail to converge and they may exhibit periodic or even chaotic behavior. Saari (1985) has shown that for any normalized iterative process there are always economies for which the process fails to converge. Furthermore, according to Goeree et al. (1998) a rather general class of normalized discrete time processes exhibits periodic and chaotic behavior. Tuinstra (2000) demonstrates similar results for a multiplicative process. For related studies see also Bala and Majumdar (1992), Day and Pianigiani (1991), Mukherji (1999), Saari (1995), and Saari and Simon (1978). Actually, one conclusion from Chapter II is that price normalization is the main reason for undesirable properties that occur in discrete time tâtonnement processes.

This far the stability result of the continuous time process (1) has not been extended for discrete time processes. Arrow and Hahn (1971, Section 12.8) argue that a discrete time version of the non-normalized process converges to any given neighborhood of the set of equilibria when the iteration parameter and initial prices are chosen appropriately. The corresponding continuous time process, however, has significantly better convergence properties. This thesis aims to remove the lack of a discrete time tâtonnement process with satisfactory convergence properties.

Chapter II shows that a simple iterative process, a modification of fixed-point iteration, converges globally under conditions that are only slightly stronger than those required for the continuous time tâtonnement process. The process is based on changing the prices in proportion to the excess demands of the commodities such that the new prices are positive and the difference between the old and the new prices is bounded. The process has minimal informational requirements, it satisfies the law of demand, and it can be interpreted as a discrete time alternative of the non-normalized continuous time tâtonnement process. It is not, however, based on approximating the corresponding continuous time process.

In practice, the process can be mplemented as follows

$$p^{k+1} = p^k + \mu_k z(p^k), \tag{2}$$

where the parameter μ_k is updated as follows:

Step 1 a scalar $\gamma_k > 0$ is chosen such that $p^k + \gamma_k z(p^k) > 0$, and $\gamma_k = \gamma_{k-1}$ for $k \geq 1$ if $p^k + \gamma_{k-1} z(p^k) > 0$,
Step 2 $\mu_k = \min\{\gamma_k, M/\|z(p^k)\|\}$, where $M > 0$.

The purpose of the first step is to guarantee that the new prices are positive. When $p^k > 0$ there is a positive number γ_k such that $p^k + \gamma_k z(p^k) > 0$. It follows from the first step that when the initial prices are positive, i.e., $p^0 > 0$, then all the prices obtained during the process are positive as well. The second step guarantees that $\mu_k z(p^k)$ is bounded in the Euclidean norm $\|\cdot\|$. As a result the change of the price vector is bounded, namely $\|p^{k+1} - p^k\| = \|\mu_k z(p^k)\| \leq M$, where M is an arbitrarily chosen positive number. Instead of step 2 we could assume that ever increasing price changes do not occur or prices cannot change arbitrarily fast between two periods. This is because step 2 is needed only to guarantee bounded price changes for the purposes of convergence analysis.

2.1 Convergence Results

Chapter II proves that the process (2) converges when z has the properties (P1)–(P5) and satisfies the condition (C1) required for the continuous time process together with (C2) as stated below. In the condition (C2) vector p^* is the same equilibrium vector for which (C1) holds and $E_\varepsilon = \{p \in \mathbb{R}^n_+ : \|z(p)\| < \varepsilon\}$. The convergence condition (C2) is the following:

(C2) There are positive scalars ε and σ such that $p^* \cdot z(p) \geq \sigma \|z(p)\|^2$ for all $p \in E_\varepsilon$.

The condition (C1) means that the hyperplane $\{x \in \mathbb{R}^n : p^* \cdot x = 0\}$ supports the *hypersurface* $\{x \in \mathbb{R}^n : x = z(p),\ p > 0\}$, see Figure 1. According to (C2) this hypersurface is around the origin inside a ball which has its center at the ray of solutions $\{p \in \mathbb{R}^n_+ : p = \lambda p^*,\ \lambda > 0\}$. In Figure 1 the center of the ball is located at p^*, i.e., $\lambda = 1$. As σ goes to zero, z is allowed to become flatter, i.e., (C1) is obtained as the limit from (C2). Hence, the difference between

(C1) and (C2) is conceptual rather than something that could have a major effect in computational or practical considerations.

For a mathematically sufficiently regular excess demand function, the condition (C2) means that the hypersurface obtained from the excess demand function is not too flat around the origin. Consequently, the iterative process converges when in addition to gross substitutability the hypersurface obtained from the excess demand function has positive normal curvature to all tangent directions around the origin. In Chapter II this property is demonstrated for Cobb-Douglas economies.

In addition to gross substitutability also another economically important condition called the weak axiom of revealed preferences implies the convergence of (2). An excess demand function is said to satisfy the *weak axiom of revealed preferences* if for any pair of price vectors p^1 and p^2 for which $z(p^1) \neq z(p^2)$ it holds that

$$p^1 \cdot z(p^2) \leq 0 \implies p^2 \cdot z(p^1) > 0.$$

The interpretation of the WA is that if p^1 is revealed preferred to p^2, which means that the value of $z(p^2)$ with prices p^1 is negative, then p^2 cannot be revealed preferred to p^1. Chapter II presents a new *second order weak axiom of revealed preferences* (SWA) that implies (C2) analogously as the WA implies (C1). An excess demand function z satisfies this condition if for any $p^2 > 0$ there is $\sigma > 0$ such that

$$p^1 \cdot z(p^2) \leq 0 \implies p^2 \cdot z(p^1) \geq \sigma \|z(p^1) - z(p^2)\|^2$$

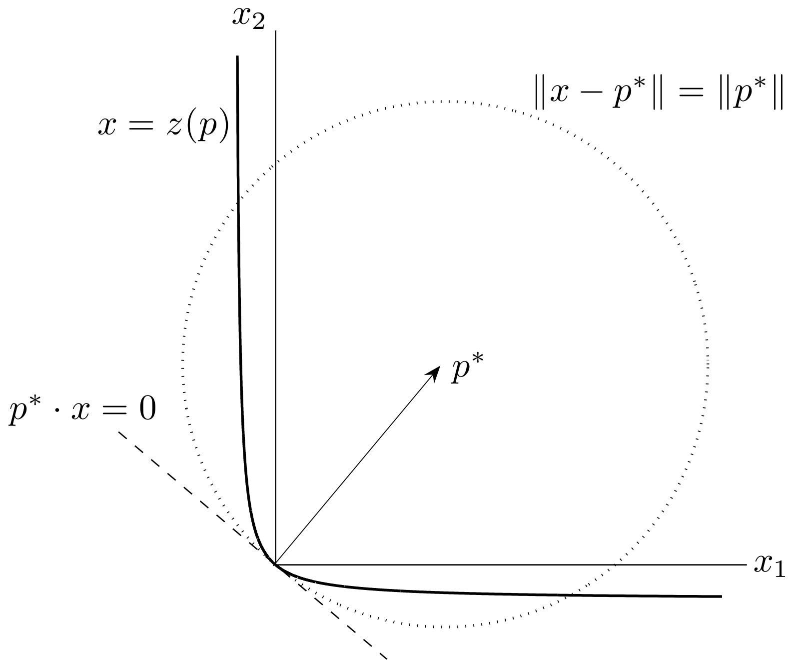

Figure 1. Illustration of (C2).

The SWA means that if p^1 is revealed preferred to p^2, then p^2 is not revealed preferred to p^1 and the value of $z(p^1)$ with prices p^2 is bounded from below in proportion to the differences of the excess demands $\|z(p^1) - z(p^2)\|^2$. Hence, the economic interpretation of the WA and the SWA are the same but the SWA also implies (C2). Consequently the iterative process converges under the SWA and (P1)–(P5. It is shown that the SWA holds when the excess demand function is Lipschitzian and strongly monotone, or the economy has a representative consumer with a strongly concave utility function. On the monotonicity of mappings and concavity of functions see, e.g., Aubin (1993) and Rockafellar and Wets (1998).

3 Chapter II: Analysis of the Constraint Proposal Method for Two-Party Negotiations

In the constraint proposal method an impartial mediator locates points where the parties' utility functions have joint tangent hyperplanes. The setting is very close to exchange economies since, as demonstrated in Chapter III, the mediator's task can be formulated as a resource allocation problem, where the total dispute is to be allocated between the parties.

The idea of locating Pareto solutions by finding the joint tangent was first presented for oligopoly games by Ehtamo et al. (1996), and Verkama et al. (1996). Ehtamo et al. (1999a) extend the approach to negotiation settings. The constraint proposal method is implemented in a negotiation support system RAMONA, which has been applied, e.g., to agricultural negotiations between Finnish Government and the Finnish Farmer's Union, see Teich et al. (1995).

From the practical point of view, the main benefit of the constraint proposal method is that the DMs' utility functions need not be elicited explicitly. Moreover, the method is informationally decentralized in the sense that the DMs do not have to disclose any private information to each other but only to the mediator. Other informationally decentralized methods include, e.g., the heuristic presented by Teich et al. (1996) and the Joint Gains method by Ehtamo et al. (1999b), and Ehtamo et al. (2001). These methods are based on seeking joint improvements from a tentative agreements. See Raiffa (1982) about the original discussion behind these methods.

Chapter III focuses on three major questions: Does the method produce Pareto-optimal points, does it lead to a problem that has a solution, and can the joint tangent hyperplanes be found with fixed-point iteration. These questions are essential for the method to be useful in practice. Moreover, the relationship to exchange economies is discussed as well as a possible way of using the method to generate a single Pareto-optimal point.

In practice the mediator adjusts a hyperplane going through a predetermined reference point until the DMs' most preferred alternatives on the hyperplane coincide. Mathematically, the mediator chooses a reference point r and defines a *hyperplane*

$$H(p, r) = \{x \in \mathbb{R}^n : p \cdot (x - r) = 0\}.$$

Here p denotes the normal of the hyperplane. The mediator asks the DMs to give their most preferred points on the hyperplane. These most preferred points solve optimization problems of the form

$$\max_{x \in H(p,r)} u_j(x), \tag{3}$$

where the subscript j refers to one of the parties. Knowing the optimal answers the mediator then updates the hyperplane. The procedure is repeated until the most preferred points coincide within some predetermined tolerance. Hence, the problem of locating Pareto-optimal points is decentralized to solving of individual optimization problems.

When the DMs' problems have unique optima, the mediator's problem can be formulated as a system of equations $F(p, r) = 0$ to be solved for p. The mapping F has some important similarities with excess demand functions: It satisfies Walras' law, i.e., $p \cdot F(p, r) = 0$, and it is degree zero homogeneous with respect to p. Actually, when r is chosen from the line connecting to the DMs' optimal global optima, F can be interpreted as an excess demand function for the exchange economy, where both DMs have a share of the total dispute, the difference of their global optima, as an initial endowment to be exchanged.

Chapter III shows that reference points chosen from the line connecting the DMs' global optima produce Pareto-optimal points. Moreover, under quite general assumptions on the DMs' utility functions, the mediator's problem has a solution. The essay also gives local convergence condition for fixed-point iteration as a scheme for adjusting the normal of the hyperplane constraint. This convergence condition is essentially a localized version of the condition obtained for exchange economies in Chapter II. Furthermore, Chapter III derives an algebraic convergence test that is based on examining whether the normal curvature of the hypersurface obtained from F for fixed r is positive to all tangent directions.

By solving $F(p, r) = 0$ for p with different reference points r, several Pareto solutions can be obtained. The DMs can then choose their agreement among these points. Another possible way to use the method is to generate a single Pareto-optimal point as a resolution, for example by first bargaining on a suitable reference point for the method and then using it to produce a single Pareto point.

As pointed out, the constraint proposal method is actually a specific exchange economy. There are, however, some important differences between the constraint proposal method and ordinary exchange economies. In an exchange economy the demand functions are not defined if some of the prices are negative. Moreover, the demand for a resource usually grows infinitely large as its prices go to zero. In the constraint proposal method p can have negative components and there is no reason to assume the DMs' responses to satisfy any boundary conditions for zero components of p. Due to these specific properties the analysis on Chapter III is not based on earlier results for exchange economies.

4 Chapter IV: Adjustment of an Affine Contract with Fixed-point Iteration

In an adverse selection game a principal designs the optimal contract for an agent without knowing the agent's preferences completely, for textbooks on contracting see, e.g., Macho-Stadler and Pérez-Castrillo (2001), and Salanié (1997). Usually it is assumed that the principal knows the agent's preferences except for a single type parameter θ that can take one of the values $\theta_1, \ldots, \theta_N$. The value of type parameter is the agent's private information. The principal, however, has a subjective probability distribution over the agent's possible types, which makes the game Bayesian, see Harsanyi (1967–68).

A typical example of a principal-agent game is a buyer-seller game, in which the seller does not know exactly how the buyers value the product he is selling. In the buyer-seller game, the seller acts as the principal who offers the buyer a price tariff that specifies the prices of the goods for any amounts to be bought, see Mirman and Sibley (1980), Roberts (1979), and Spence (1980).

Let $y \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$ denote the decision variables for the principal and the agent, respectively. Furthermore, let v and u be their utility functions, e.g., principal's utility from pair (x, y) is $v(x, y)$. In the general contract design game the principal offers the agent a menu of contracts $\gamma_i(x)$, $i = 1, \ldots, N$, and commits to make his decision according to the contract that the agent chooses. The agent may also reject all the contracts in which case he obtains his reservation utility $\bar{u}$. After the agent chooses to sign a contract $\gamma_i(x)$ he makes a decision x_γ that maximizes $u(x, \gamma_i(x))$. The principal then implements the contract, i.e., takes the action $y_\gamma = \gamma_i(x_\gamma)$, and the game ends. The agent is said to participate the game if he signs a contract instead of taking his reservation utility.

The principal's problem in a static setting is to design the menu of contracts that maximizes his expected utility such that the agent obtains at least her reservation utility and chooses the contract intended for his type. The expected utility is calculated with respect to the principal's subjective probability distribution over the types. If the the principal knew the agent's type, i.e., has complete information, then he could offer the agent a single contract that maximizes his utility with respect to the constraint that the agent should participate the game.

Using subjective probability distributions in modeling games of incomplete information has an alternative in economics literature. Namely, adjustment processes provide a way to reach the complete information equilibrium under limited preference information. Nevertheless, such processes have not raised any attention in contracting literature although various learning processes have recently been popular in other fields of game theory, see, e.g., Fudenberg and Levine (1999).

Instead of Bayesian approach and type a parameterization, it is assumed in Chapter IV that the game is played repeatedly with the same participants. Hence, the principal can offer the agent a single contract and adjust it according to observations on the agent's behavior. Furthermore, the agent is not fully rational in the sense that he does not consider the outcomes of future periods when making decisions. One explanation for such behavior is that there is a large population of similar agents and each round one of them is drawn randomly to play the game and it is unlikely that the same agent is chosen several times repeatedly. Repeated adverse selection games and long term contracting have also been studied with fully rational agents, see, e.g., Freixas et al. (1985) and Laffont and Tirole (1988). This essay, however, studies only how the complete information contract can be reached with limited information, and the issues on long term relationships are thus omitted.

Let (x^*, y^*) denote the principal's optimum under the agent's participation constraint. Note that when u is unknown to the principal he cannot necessarily have prior knowledge about the agent's participation constraint. Therefore, the principal is assumed to know that his global optimum (x^*, y^*) satisfies the agent's participation constraint, which is that the agents obtains at least $\bar{u}$ by signing the contract. Thus, the agent will always participate the game. For example, this can happen when it is common knowledge that the agent does not have a participation constraint at all. The essay also considers the general case where the principal does not know the participation constraint but can find the best point that satisfies this constraint through adjustments.

In Chapter IV the principal's contract design problem is defined as follows. Find an affine contract $\gamma(x) = y^* + L(x - x^*)$, i.e., mapping L, such that x^* maximizes principal's utility $v(x, \gamma(x))$. Affine or linear contracts are appealing mainly because of they are simply to administer and monitor. On related contracts see, e.g., Rasmusen (1989, Chapter 7). Recently Ehtamo et al. (2002) have presented a procedure for finding the optimal linear tariff in a wage-contracting game. Their procedure is close to the adjustment process presented in Chapter III: One step in the tariff adjustment process is to find the correct slope of the tariff as in Chapter III the task is to find the correct L to define the optimal contract.

To formulate the principal's problem as a system of equations L is parameterized with $p \in \mathbb{R}^{n+m}$, with $p = (p_x, p_y)$, $p_x \in \mathbb{R}^n$, $p_y \in \mathbb{R}^m$, such that the contract defines an affine subset on the hyperplane

$$p_x \cdot (x - x^*) + p_y \cdot (y - y^*) = 0.$$

The basic idea of the adjustment approach is that the principal tries to find p so that

$$d(p) = \begin{pmatrix} x(p) - x^* \\ y(p) - y^* \end{pmatrix} = 0,$$

where $x(p)$ denotes the solution of the agent's maximization problem and $y(p) = \gamma(x(p))$. An appropriate method for this task is fixed-point iteration $p^{k+1} = p^k + \mu d(p^k)$, where $\mu \neq 0$ is an iteration parameter. The advantage of fixed-point iteration is that it can be implemented in a repeated game where the principal does not know u. This is because the agent's response $x(p)$ is sufficient information for updating p. Hence, the iteration can be interpreted as a naive learning process similarly as, e.g., the Cournot adjustment process, in which the players use their best responses sequentially to their opponents' latest moves.

The sufficient condition for the convergence of fixed-point iteration is that the agent's preferences are characterized by a strongly concave utility function, i.e., there is $\sigma > 0$ such that $u(x, y) - \sigma(\|x\|^2 + \|y\|^2)$ is concave. More specifically, it is shown that for all initial parameters p^0 fixed-point iteration, with $\mu > 0$, either stalls at a point in which $p_y = 0$ or converges. Basically this result means that the complete information equilibrium can be reached by adjusting the contract with fixed-point iteration.

5 Chapter V: Osborne's Cartel Maintaining Rule Revisited

The main question on incomplete competition in oligopolistic markets is how collusion can be maintained without explicit cartel agreements. The following properties are desirable for models explaining this tacit collusion: Cooperation should be obtained as a result of rational behavior, the dynamic aspect of the firms' interaction and the continuous nature of their output variables should be accounted for.

The models of incomplete competition in oligopolistic markets can be divided roughly to three approaches: conjectural variations equilibrium, subgame perfect trigger strategies in repeated games, and reaction function strategies both in static and in repeated games. The advantage of latter two approaches, compared to conjectural variations equilibrium, is that they can be motivated by rational behavior. Moreover, reaction strategies are particularly attractive because they presume that the firms' actions vary continuously as a response to deviations. Hence, small deviations lead to small punishments. Trigger-type strategies are based on breaking cooperation at least temporarily, if one of the parties deviates from cooperation. Hence, the punishment is the same regardless of the magnitude of deviation, which is rather implausible in circumstances where the outputs are continuous.

Osborne (1976) made a significant finding by observing that keeping the market shares is a continuous equilibrium strategy that maintains collusion in a Cournot oligopoly. Actually, Osborne noticed that the firms' profit functions have a common tangent line at the joint profit maximum, and moving along this tangent line maintains cooperation and keeps the market shares constant. In economics literature the strategy of keeping the market shares constant is called Osborne's rule or Osborne's quota rule, see Rothschild (1981) and Phlips (1988).

Although Osborne's model is static, because the firms are assumed to react without time delay, it has given much insight to what might be the cartel maintaining mechanisms in practice, see Phlips (1988, Section 6.2) and Jacquemin and Slade (1989). One possible example where Osborne's rule could explain market behavior is the OPEC oil cartel, of which Osborne was mainly inspired when introducing his quota rule. Chapter IV formulates Osborne's rule in an infinitely repeated game with discounting and shows that it can maintain cooperation in such settings as well. The motivation for generalizing Osborne's rule to repeated games is that such reaction strategy could explain market behavior and incomplete competition better than trigger strategies. Furthermore, it is reasonable to assume that by keeping their market shares the firms could indeed sustain collusion.

In an infinitely repeated Cournot oligopoly firms are making decisions on production quantities and they face each others infinitely many times. In such setting the firms' one period profit functions are typically of the form

$$\pi_j(q) = P(q)q_j - C_j(q_j),$$

where j refers to a firm, q_j is firm j's output quantity, q is the vector of all firms' outputs, C_j is firm j's cost function, and P is the inverse demand function, i.e., price as the function of outputs. In the repeated game the firms are supposed to maximize their discounted cash flows; firm j maximizes $\sum_k \delta_j^k \pi_j(q^k)$ with respect to $q_j^1, q_j^2, \ldots$. Here k refers to the round and $\delta_j \in (0,1)$ is the firm's discount factor.

Osborne observed that the deviations should be bounded for his quota rule to be credible. Therefore, Chapter V takes a different approach to earlier repeated game models by bounding the allowed deviations that the firms can make instead of restricting the discount factors. Namely, for trigger-type strategies it is well-known that cooperation is the subgame perfect equilibrium outcome when the firms' discount factors are large enough, see, e.g., Friedman (1971), Abreu (1986), and Fudenberg and Maskin (1986). Subgame perfection means that it is always better for the firms to follow their strategies than to deviate from them. Friedman and Samuelson (1990, 1994a,b) construct reaction functions that lead to subgame perfect equilibria for large discount factors. For less elaborate continuous reaction functions see also Friedman (1968, 1973, 1976). However, according to Stanford (1986b) and Robson (1986) these reaction functions cannot maintain cooperation.

Linear reaction function strategies in dynamic games have been previously studied by Kalai and Stanford (1985), Stanford (1986a), Ehtamo and Hämäläinen (1989, 1993), and Ehtamo and Hämäläinen (1995). These strategies are close to the proportional reaction strategy presented in Chapter V but they all lack subgame perfection. Kalai and Stanford (1985), however, prove that linear strategies constitute an ε-perfect equilibrium when the reaction times are short enough. Stanford (1986a) shows that linear reaction functions lead to a subgame perfect equilibrium for a repeated duopoly with using the limit of the means evaluation criterion instead of discounting. Ehtamo and Hämäläinen (1993) consider linear reaction strategies in a continuous time natural resource model and introduce the concept of incentive equilibrium.

5.1 Results for Proportional Strategies

The dynamic counterpart of Osborne's rule in a repeated game is the following. After observing that one of the firms has unilaterally deviated, i.e., exceeded the tacitly agreed cooperative output, the other firms choose their total output in proportion to the deviation while the deviating firm returns to cooperation. Each punishing firm has a largest acceptable punishment output, which is the output they choose. If some of the punishing firms unilaterally exceeds the largest acceptable output, this firm is treated as a deviator. More formally the strategy can be formulated for firm j as the feedback law:

$$q_j^k(q^{k-1}) = \begin{cases} f_j(q_i^{k-1}) & \text{if in round } k-1 \text{ firm } i \neq j \text{ has unilaterally deviated or exceeded its maximal allowed punishment,} \\ q_j^\lambda & \text{otherwise.} \end{cases}$$

Here q_j^λ is the firm's cooperative output and f_j gives the maximal allowed punishment output for firm j. The firms' profits are assumed to depend on their own output and the other firms' total output. Moreover, the total punishment output depends affinely on the deviation, i.e., if firm j has deviated

$$\sum_{i \neq j} f_i(q_j) = L(q_j, \alpha_j) = \alpha_j(q_j - q_j^\lambda) + \sum_{i \neq j} q_i^\lambda.$$

In the following $L(q_j, \alpha_j)$ is referred to as the proportional scheme with slope α_j.

The above formulation of Osborne's rule in a repeated game differs from the usual reaction function models; the whole strategy is not assumed to be a continuous function. The deviating firm returns to cooperation and only the punishing firms adjust their output continuously. Hence, the strategy resembles more so-called tit-for-tat strategy, see Axelrod (1984), than usual reaction function strategies that lead to a sequence of consecutive deviations from the cooperative output after a deviation. The proportional scheme is also close to the measure-for-measure strategy observed in duopoly experiments by Selten et al. (1997). These experiments suggest that people tend to respond in a continuous manner to each others' moves when the variables are continuous.

The main result of Chapter V is that cooperative play is the subgame perfect equilibrium outcome of the game when (*i*) the profit functions satisfy certain rather general assumptions, (*ii*) the functions giving the maximal allowed punishments satisfy certain regularity assumptions, (*iii*) the slopes of the proportional schemes are large enough, and (*iv*) the possibility of large deviations does not affect the firms' behavior. The latter requirement means that subgame perfection requires that the possible deviations should be bounded. Due to this boundedness the cooperative point is said to be supportable as a *locally subgame perfect equilibrium outcome*. Figure 2 illustrates the possible intervals of deviations on which proportional strategies constitute a subgame perfect equilibrium.

In practice deviations could be bounded due to technological limitations on making large output adjustments. On the other hand, the upper bounds of allowed deviations reflect how trustful the firms should be in order to sustain cooperation. Namely, subgame perfection requires that the possibility of large deviations does not affect the firms' behavior, i.e., the firms should trust that the deviations will stay below certain upper bounds. It is demonstrated with a duopoly example that these upper bounds can be quite large. Furthermore, it is shown that cooperative points that are below the firms' best response functions are supportable as locally subgame perfect equilibrium outcomes. In Figure 2 the shaded region represents these supportable points.

The lower bounds for the slopes required for subgame perfection are the slopes that maintain cooperation in the static setting divided with the firms' discount factors. In particular, when the joint profit maximizing point is maintained as

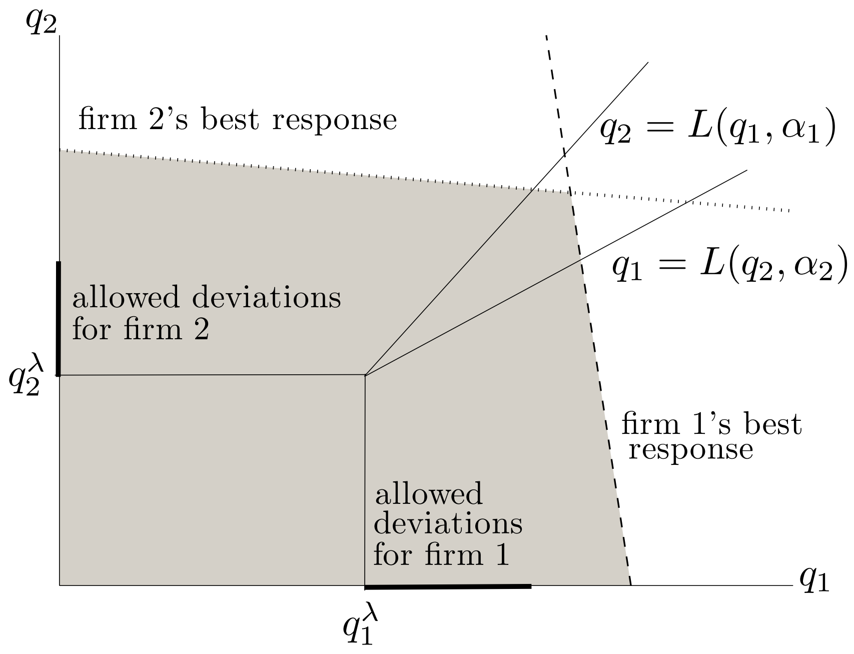

Figure 2. Illustration of the Region of Supportable Points in a Repeated Duopoly.

the equilibrium, the static Osborne's rule can be obtained in the limit of the proportional strategies as discount factors tend to one. Hence, maintaining the market shares has a new motivation as a way to sustain collusion.

In addition to subgame perfection it is shown that in a duopoly setting, which the essay mainly considers, the equilibrium is *weakly renegotiation proof*. This property means that none of the continuation payoffs with both firms using proportional strategies is Pareto dominated by any other continuation payoff. See Farrell and Maskin (1989) for the concept of weak renegotiation proofness and Bernheim and Ray (1989), and van Damme (1989) for related concepts. Continuation payoffs are the discounted profits that the firms obtain when they follow their proportional strategies beginning from a given history of the play. Hence, weak renegotiation proofness means that the firms would not change their behavior even if they could renegotiate the original agreement anew in any contingencies.

The subgame perfection of proportional strategies could explain conjectural variations equilibrium as a result of rational behavior. The main idea of conjectural variations models originating from Bowley (1924), is that each firm believes that the quantities chosen by its rivals depend on the firm's own output. These behavioral assumptions are captured in parameters called the conjectural variations. By identifying the conjectural variations as the slopes of the proportional schemes, a conjectural variations equilibrium, with sufficiently large conjectural variations, corresponds to a locally subgame perfect equilibrium under proportional strategies. When this occurs the conjectural variations are said to be *strategically consistent*. I also present some implications of strategic consistency to Lerner indices, which are popular measures of incomplete competition in empirical literature, see, e.g., Bresnahan (1989).

6 Future Research Directions

Chapters II–IV of this thesis deal with reaching an equilibrium while Chapter V considers maintaining a desired outcome as an equilibrium. Although this thesis takes mostly a theoretical approach to these topics, the results could prove useful in real world applications. For example, the iterative tâtonnement process could be applied as an auctioning procedure to solve resource allocation problems in practice. Indeed, there are some experimental results on the use of tâtonnement processes, see, e.g., Biais et al. (1999), Bronfman et al. (1996), Anderson et al. (2004), Myagkov and Plott (1997), and Smith (1965). One topic of further theoretical work on excess demand functions is the relationship of different monotonicity concepts and the various forms of the weak axiom of revealed preferences.

In Chapter III the number of negotiators is limited to two. There is, however, a possible extension for the constraint proposal method to multi-party settings, see Heiskanen et al. (2001) and Heiskanen (2001). This generalization leads to a mathematically different problem than the two-party case. Hence, existence, Pareto-optimality, and convergence issues are still open for the multi-party constraint proposal method.

In Chapter IV there is only a single agent with unknown utility function from the principal's view. The computational challenges on multi-agent contracting problems, especially when there are more than to dimensions, are still to be resolved. In particular, an economically meaningful learning process for finding price-amount tariffs is lacking for buyer-seller games in presence of different types of buyers.

An interesting question for further research on proportional strategies, Chapter V, is whether they predict the bounds where cooperation collapses in experimental oligopoly games. Namely, we can assume that people, and other economic agents as well, react to small deviations with small punishments and the cooperation collapses only when the deviations are large, in which case some kind of trigger is launched. Testing this hypothesis experimentally would be of further interest. A theoretical topic of additional research on proportional strategies is to study them under incomplete price information, for such a model see Green and Porter (1984). Namely, in practice the firms may not observe each others' outputs directly but only through market prices that may contain uncertain or stochastic elements.

In conclusion, this thesis provides both theoretical explanations for market behavior and practical methods for reaching equilibria and maintaining cooperation.

Contributions of the Author

Mitri Kitti is solely responsible for the writing of this monograph. The ideas presented in Chapter II, as well as the mathematical formulation are Kitti's. The ideas presented in Chapters III-V have been developed jointly with Harri Ehtamo, and the mathematical formulation and analysis in these chapters is Kitti's. The research initiative for Chapters IV and V is from Ehtamo. Currently, the author is preparing publication manuscripts on the results of this study, see Kitti (2005), Kitti and Ehtamo (2005a,b), Ehtamo and Kitti (2005). Manuscripts on chapters III–V are joint works with Harri Ehtamo. Manuscript Kitti and Ehtamo (2005b) has been submitted for possible publication.

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Chapter II:

Convergence of Non-Normalized Iterative Tâtonnement

Abstract

This chapter gives global convergence conditions for iterative tâtonnement with the additional requirements that prices stay strictly positive and their changes are bounded. Convergence is shown when the excess demand function has the gross substitute property and curves appropriately around the equilibrium. Furthermore, a new, second order, form of the weak axiom of revealed preferences is introduced; a condition which also implies convergence. It is shown that this condition holds when the excess demand function is strongly monotone or has a representative consumer with strongly concave utility function.

Key words: equilibrium, iteration, tâtonnement, convergence

1 Introduction

The latest results on discrete time price adjustment processes are mostly negative: Discrete time processes may fail to converge and they may exhibit periodic or even chaotic behavior, see Bala and Majumdar (1992), Day and Pianigiani (1991), Goeree et al. (1998), Mukherji (1999), and Tuinstra (2000). This paper shows that a simple iterative process avoids these phenomena and converges globally under conditions that are only slightly stronger than those required for the continuous time tâtonnement process.

Tâtonnement processes are usually interpreted as auctions, where a fictitious agent, Walrasian auctioneer, sets the prices until an equilibrium is reached and the trades are made. The main purpose of such processes is to explain how an economy comes to its equilibrium. In addition to this, a discrete time process could provide a practical auctioning method for solving resource allocation problems.

The need for analyzing discrete time price adjustment processes has been long recognized. Samuelson, who formulated the tâtonnement process in continuous time with a set of differential equations, observes the following, see Samuelson (1947, p. 286):

> “ The types of functional equations which have been most studied are those defined by differential equations, difference equations, and integral equations, and mixed varieties. The first of these possesses the most highly developed theory and provides valuable examples of various principles. Since economic observations consist essentially of series defined for integral values of time, the second category of difference equations is perhaps of greatest interest to the theoretical economist.”

Some discrete time alternatives for the continuous time tâtonnement process have been suggested in the economics literature. Uzawa (1960) has analyzed an iterative process for the normalized excess demand, where the price of one of the commodities, numéraire, is set to a constant and only the rest of the prices are adjusted. There are, however, some negative results on normalized processes. Saari (1985) has shown that for any normalized iterative process there are always economies for which the process fails to converge. Furthermore, according to Goeree et al. (1998) a rather general class of normalized discrete time processes exhibits periodic and chaotic behavior. Tuinstra (2000) demonstrates similar results for a multiplicative process.

Several authors have noticed that in most cases the results on the continuous time process do not hold for the discrete time process. Arrow and Hahn (1971, Section 12.8) argue that a discrete time version of the non-normalized process converges to any given neighborhood of the set of equilibria when the iteration parameter and initial prices are chosen appropriately. The corresponding continuous time process, however, has significantly better convergence properties. Indeed, satisfactory convergence results for non-normalized discrete time tâtonnement are lacking.

This paper studies fixed-point iteration with the additional requirements that prices stay strictly positive and the difference between the old and the new prices is bounded. Since only the value of the excess demand function is used in updating the prices, the process has minimal informational requirements. Moreover, the process has the property that if a commodity has positive excess demand, its price rises, and if the excess demand is negative the price falls. Hence, the process can be interpreted as a discrete time alternative of the non-normalized continuous time tâtonnement process. The process is not, however, based on approximating the continuous time process.

It is well known that the continuous time process converges globally under the gross substitution property and the weak axiom of revealed preferences, see Arrow et al. (1959) and Arrow and Hurwicz (1958). Here it is shown that the iterative process converges when in addition to gross substitutability the hypersurface obtained from the excess demand function has positive normal curvature to all tangent directions. Furthermore, a second order form of the weak axiom of revealed preferences is introduced. Together with some common properties of excess demand functions this condition implies the convergence of the iterative process, too. It will be shown that the second order weak axiom holds when the excess demand function is Lipschitzian and strongly monotone, or the economy has a representative consumer with a strongly concave utility function.

The paper is organized as follows. Section 2 presents the model for an exchange economy and the iterative adjustment process. The global convergence of the process is analyzed in Section 3. In Section 4 the conditions of Section 3 are applied to show convergence for economies that satisfy the gross substitute property and curve appropriately around the equilibrium. As an example, I demonstrate that Cobb-Douglas economies satisfy these conditions. The convergence of the process is shown under the second order weak axiom of revealed preferences in Section 5. The results are discussed in Section 6.

2 The Model

2.1 Excess Demand Function

An exchange economy with m consumers and n commodities, $m, n \geq 2$, is described by preference relations $\succeq^i$, $i = 1, \ldots, m$, defined on $\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x \geq 0\}$, and endowment vectors $w^i = (w_1^i, \ldots, w_n^i) \geq 0$, $i = 1, \ldots, m$. In this model, subscript denotes the component of a vector and superscript denotes the consumer. Furthermore, $x \geq 0$ means that $x_j \geq 0$ for all j.

Given a price vector $p > 0$, that is $p \in \mathbb{R}^n_+ = \{p \in \mathbb{R}^n : p_j > 0 \, \forall j\}$, the demand function $x^i(p)$ for consumer i is the maximizer of $\succeq^i$ over the budget set, which means that $x^i(p) \succeq^i x$ holds for all $x \in \{x \in \mathbb{R}^n : p \cdot x \leq p \cdot w^i, \; x \geq 0\}$, where $p \cdot w^i = \sum_j p_j w_j^i$. The equilibrium of the economy is a price vector p^* for which

$$z(p^*) = \sum_{i=1}^{m} [x^i(p^*) - w^i] = 0. \tag{1}$$

The mapping z is called the excess demand function of the economy and it will be assumed to have the following properties:

(P1) z is single valued and continuous for all $p > 0$.
(P2) z satisfies Walras' law: $p \cdot z(p) = 0$ for all $p > 0$.
(P3) z is homogeneous of degree zero: $z(\alpha p) = z(p)$ for all $\alpha > 0$ and $p > 0$.
(P4) There is a scalar $\nu < 0$ such that $z_j(p) > \nu$ for all j and $p > 0$.
(P5) It holds that

$$\lim_{p^k \to p} [\max_{j \in J_p} z_j(p^k)] = \infty,$$

when $p^k > 0$, $p \neq 0$ and $J_p = \{j : p_j = 0\} \neq \emptyset$.

Homogeneity is an elementary property that an excess demand function has because the consumers' budget sets stay the same when the budget constraints are multiplied with positive constants. Walras' law and continuity result from the consumers' maximization problems when the preferences are strictly convex and locally non-satiated.

The property (P4) means that all the component functions of z are bounded from below on $\mathbb{R}^n_+$. An excess demand function has this property because the consumers' net supply of any commodity cannot exceed the total endowment. According to (P5) all the commodities are desirable in the sense that when some of them become free, the excess demand becomes infinitely large at least for some of those commodities. This is the case, for example, when there is a positive total amount of all the commodities and the consumers have strongly monotone preferences. When z has the properties (P1)–(P5), the economy has at least a ray of equilibrium prices. See, e.g., Mas-Colell et al. (1995, Chapter 17) for more about the properties of excess demand functions.

2.2 Iterative Price Adjustment Processes

To be economically meaningful a tâtonnement process should not require other information than prices and the corresponding excess demand, it should satisfy the *law of demand*, according to which prices should increase for commodities with excess demand and fall in the opposite case, and the process should converge under economically relevant conditions. The simplest continuous time process that satisfies these properties was introduced by Samuelson (1947) and is described by the differential equation

$$\dot{p}(t) = z(p(t)), \tag{2}$$

where $\dot{p}(t)$ is the time derivative of $p(t)$. This process is usually interpreted as an auction run by a fictitious agent, a Walrasian auctioneer, who sets the prices until an equilibrium is reached and the trades are made.

It can be shown that under the following condition (C1), the process (2) is globally stable, i.e., it converges to an equilibrium for any positive initial prices. The stability condition can be stated as follows:

(C1) There is $p^* > 0$ that solves (1) and satisfies $p^* \cdot z(p) > 0$ for all $p > 0$ for which $z(p) \neq 0$.

This convergence condition was first presented by Arrow and Hurwicz (1958) and Arrow et al. (1959), who assumed the set of equilibria to be unique up to a positive scalar multiple, i.e., a unique ray. It was further shown Arrow and Hurwicz (1960) that (C1) implies the convergence of the continuous time process even though the set of equilibria is not a unique ray. For a detailed analysis of the corresponding normalized process see, e.g., Balasko (1988, Appendices to Chapters I and III).

The convergence condition (C1) can be interpreted as the weak axiom of revealed preferences between the equilibrium p^* and any disequilibrium price vector. An excess demand function satisfies this condition in three important cases: (*i*) when there is no trade at equilibrium, (*ii*) when the excess demand function satisfies the weak axiom of revealed preferences for any pair of price vectors, or (*iii*) when it has the gross substitute property. The latter two properties will be discussed in detail in Sections 4 and 5.

The simplest discrete time alternative for the process (2) is the fixed-point iteration

$$p^{k+1} = p^k + z(p^k), \tag{3}$$

where k is the iteration index that corresponds to the time instants at which the prices are adjusted. The main argument for analyzing (3) instead of (2) is that the auction, which a price adjustment process aims to characterize, proceeds in discrete time instants. This paper studies (3) with the additional assumptions that prices stay positive and their changes are bounded. A way to implement such process in practice is given in the following section.

To obtain non-negative prices we could update p_j^k as follows

$$p_j^{k+1} = \max\{0, p_j^k + \mu z_j(p^k)\}, \tag{4}$$

where μ is a positive constant. The convergence of this process has been analyzed by Uzawa (1960) when the prices are normalized so that the price of one commodity is set to a constant and only the prices of other commodities are adjusted. In essence, it has been shown that under gross substitution there is a choice of μ such that the process converges. The corresponding non-normalized process converges to any given neighborhood of the equilibrium ray with some choice of μ and with p^0 chosen such that the prices remain strictly positive during the process, see Arrow and Hahn (1971, Section 12.8). In addition to the limitations on the choice of μ and p^0, the drawback of the process (4) is that due to (P5) the excess demand function is not finite if some prices become zero.

2.3 Fixed-Point Iteration with Positive Prices

It is commonly known that the discrete time process (4) does not converge under the same assumptions as the continuous time process (2). For example, the convergence of the process (4) depends on the choice of parameter μ, whereas the convergence of the process (2) is not dependent on any additional parameter. Moreover, the normalized discrete time processes tend to exhibit chaotic behavior. The aim of this paper is to show that a modification of fixed-point iteration (3) converges under condition that are remarkably close to the convergence conditions of the continuous time process. Indeed, for numerical or computational considerations the difference of the convergence conditions for the iterative process studied in this paper and the conditions for the process (2) are negligible.

As mentioned earlier, prices should stay strictly positive. Other requirement we need is that their changes are bounded, i.e., there is $M > 0$ such that $\|p^{k+1} - p^k\| \leq M$, where $\|\cdot\|$ denotes the Euclidean norm. This assumption is needed to show the convergence and it is quite reasonable. Namely, it means that ever increasing price changes do not occur, or prices cannot change arbitrarily fast between two periods. Note that this condition does not mean that prices should be bounded themselves. Moreover, the bound M can be arbitrarily large.

A process that satisfies the two aforementioned requirements can be defined by the following formula

$$p^{k+1} = p^k + \mu_k z(p^k), \tag{5}$$

where the parameter μ_k is updated as follows:

Step 1 a scalar $\gamma_k > 0$ is chosen such that $p^k + \gamma_k z(p^k) > 0$, and $\gamma_k = \gamma_{k-1}$ for $k \geq 1$ if $p^k + \gamma_{k-1} z(p^k) > 0$,
Step 2 $\mu_k = \min\{\gamma_k, M/\|z(p^k)\|\}$, where $M > 0$.

The first step guarantees that the new prices are positive and the second step guarantees bounded price changes. When $p^k > 0$ there is a positive number γ_k such that $p^k + \gamma_k z(p^k) > 0$. One way to find an appropriate γ_k in numerical considerations is to choose $\gamma_k = (1/2)^l$ where l is the smallest integer for which $p^k + (1/2)^l z(p^k) > 0$. It follows from the first step that when the initial prices are positive, i.e., $p^0 > 0$, then all the prices obtained during the process are positive as well. The second step guarantees that $\mu_k z(p^k)$ is bounded. As a result the change of the price vector is bounded, namely $\|p^{k+1} - p^k\| = \|\mu_k z(p^k)\| \leq M$. Note that according to the two steps, μ_k is updated only if it is necessary for obtaining positive prices or for keeping the changes bounded by M. Hence, it may well happen that these steps are never implemented during the actual process.

The process (5) satisfies the law of demand and prices are adjusted in proportion to their excess demands in a similar way as in the process (2). There is, however, an important difference between the prices obtained from the two processes. Namely, it follows from Walras' law that for the process (5) we have $\|p^{k+1}\| > \|p^k\|$ when $z(p^k) \neq 0$, whereas $\|p(t)\| = \|p(0)\|$ for the process (2).

If γ_k went to zero, then the sequence of prices obtained from (5) could become arbitrarily close to the path obtained from (2). If this happened, the process (5) would be an approximation of the process (2) for large k, and we could expect the two processes to converge under the same conditions. In the following section we shall see that μ_k does not converge to zero when the process (5) converges, which means that the process (5) does not approximate (2). The convergence conditions of the two processes are, however, very close to each other.

The rest of the chapter focuses on the convergence of the process (5) although we could be speaking about fixed-point iteration with positive prices and bounded price changes.

3 Convergence Analysis

This section gives general convergence conditions for the process (5). These conditions will be applied in Sections 4 and 5 to show convergence when z has some more specific economic properties. The main result is that the process (5) converges when z has the properties (P1)–(P5) and satisfies (C1), see Section 2.2, together with (C2) as stated below. In the condition (C2) vector p^* is the same equilibrium vector for which (C1) holds and $E_\varepsilon = \{p \in \mathbb{R}^n_+ : \|z(p)\| < \varepsilon\}$. The convergence condition (C2) is stated as follows:

(C2) there are positive scalars ε and σ such that $p^* \cdot z(p) \geq \sigma\|z(p)\|^2$ for all $p \in E_\varepsilon$.

Section 5 introduces a slightly strengthened form of the weak axiom of revealed preferences and shows that it implies (C2) analogously as the weak axiom implies (C1).

Let us next examine the geometrical interpretation of conditions (C1) and (C2). The condition (C1) means that the hyperplane $\{x \in \mathbb{R}^n : p^* \cdot x = 0\}$ supports the set $\{x \in \mathbb{R}^n : x = z(p),\ p > 0\}$, see Figure 1. The condition (C2) means that this set is at least locally, around the origin, inside a ball which has its center at the ray of solutions $\{p : p = \lambda p^*,\ \lambda > 0\}$. This can be seen by writing $p^* \cdot z \geq \sigma\|z\|^2$ equivalently as $\|p^*/(2\sigma) - z\|^2 \leq \|p^*/(2\sigma)\|^2$. In Section 3.1 we show that for a regular economy (C2) means that the hypersurface obtained from the excess demand function is not too flat around the origin. Indeed, as σ goes to zero, z is allowed to become flatter, i.e., (C1) is obtained as the limit from (C2).

The way in which the parameter μ_k is updated guarantees that the norm of the scaled excess demand $\mu_k z$ is bounded by the constant M. As a result the scaled excess demand is for all $p > 0$ inside a ball centered at the ray of solutions. These geometrical ideas are illustrated in Figure 1, where $\sigma = 1/2$ and $\lambda = 1$.

Let us state the main convergence theorem that will be used in showing the other convergence results of this paper.

Theorem 1. *Let z have the properties (P1)–(P5) and satisfy the conditions (C1)–(C2). Then the process (5) converges to an equilibrium for any $p^0 > 0$. If there is a unique ray of equilibria, then there is $N \geq 0$ such that convergence is monotonical when $k \geq N$.*

The monotonical convergence of the sequence $\{p^k\}_k$ to $\tilde{p}$ means that $\|p^k - \tilde{p}\| \to 0$, when $k \to \infty$, and if $p^k \neq \tilde{p}$, then $\|p^{k+1} - \tilde{p}\| < \|p^k - \tilde{p}\|$.

The following lemmas are used in the proof of Theorem 1. Here we let $B(p^*, \varepsilon)$ denote the closed ball with radius $\varepsilon > 0$ centered at p^*, i.e., $B(p^*, \varepsilon) = \{x \in \mathbb{R}^n : \|x - p^*\| \leq \varepsilon\}$.

Lemma 1. *Let the continuous mapping $z : B(p^*, r) \mapsto \mathbb{R}^n$ satisfy Walras' law for all $p \in B(p^*, r)$, and let the inequality $p^* \cdot z(p) \geq \|z(p)\|^2$ hold for all $p \in B(p^*, r)$. If $p^0 \in B(p^*, r)$ and $\mu_k \leq 1$ for all k, then the iteration $p^{k+1} = p^k + \mu_k z(p^k)$ converges. When there is $\bar{\mu}$ such that $0 < \bar{\mu} \leq \mu_k$, the iteration converges to a solution of $z(p) = 0$.*

The proof of Lemma 1 is presented in Appendix. The following lemma is for showing that convergence is monotonical when there is a unique ray of equilibria. For the proof of Lemma 2 see Chapter IV.

Lemma 2. *Let z satisfy the same conditions as in Lemma 1 and let the iteration $p^{k+1} = p^k + \mu z(p^k)$, $\mu > 0$, converge to a solution $\tilde{p}$ for which there is $\alpha > 0$ such that*

$$\|z(p)\|^2 \leq 2\alpha z(p) \cdot \tilde{p}$$

for all $p \in B(p^*, r)$. Then convergence is monotonical.

The following lemma shows essentially that the convergence condition of Lemma 1 holds for the scaled excess demand that is obtained by adjusting the parameter μ_k as described in steps 1 and 2. The proof is presented in Appendix.

Lemma 3. *If z has the properties (P1), (P3)–(P5), and satisfies (C1)–(C2),*

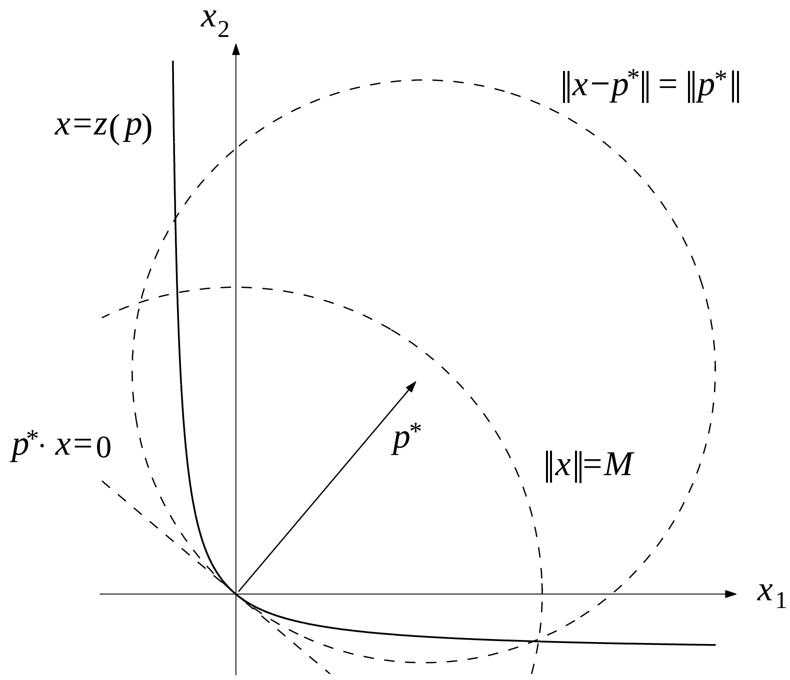

Figure 1. Illustration of the convergence conditions.

then there is $\sigma > 0$ *such that* $p^* \cdot \hat{z}(p) \geq \sigma \|\hat{z}(p)\|^2$ *for all* $p > 0$, *where*

$$\hat{z}(p) = \begin{cases} Mz(p)/\|z(p)\| & \textit{if } \|z(p)\| \geq M, \\ z(p) & \textit{otherwise.} \end{cases}$$

With the lemmas 1–3 we can prove Theorem 1.

Proof of Theorem 1. Let us first note that the process (5) can be expressed with the formula

$$p^{k+1} = p^k + \lambda_k \hat{z}(p^k),$$

where $\lambda_k = \min\{\gamma_k, 1\}$, and $\hat{z}$ is as defined in Lemma 3. When z has the properties (P1)–(P4) so does $\hat{z}$, and (P5) implies that $\hat{z}$ has the property

(P5') $\lim_{p^k \to p} [\max_{j \in J_p} \hat{z}_j(p^k)] > 0$, when $p \neq 0$, and $J_p = \{j : p_j = 0\} \neq \emptyset$.

Moreover, it is known from Lemma 3 that $p^* \cdot \hat{z}(p) \geq \sigma \|\hat{z}(p)\|^2$ holds for all $p > 0$ when z satisfies (C1)–(C2). Due to homogeneity p^* can be replaced by p^*/σ in (C1) and (C2); hence, without loss of generality we may suppose that $\sigma = 1$. It follows then from Lemma 1 that the iteration converges.

Let us show that due to (P5') the parameter λ_k has a positive lower bound that is required in Lemma 1 to obtain convergence to a solution of (1). On the contrary, suppose that $\{\lambda_k\}_k$ has a subsequence which converges to zero. Since the sequence is decreasing this means that the whole sequence converges to zero. It then follows that $p^k \to p$, where some components of p are zero, i.e., $J_p \neq \emptyset$. This can be shown by observing that p^k cannot converge to a positive price vector if λ_k is updated infinitely many times, which is the case as $\lambda_k \to 0$. Namely, assume that $\{p^k\}_k$ converges to a point $p > 0$. First, note that there has to be at least one commodity l for which there is a negative subsequence of $\{\hat{z}_l(p^k)\}_k$. Otherwise λ_k would be updated only finitely many times and it could not converge to zero. Because $p > 0$, for all $\varepsilon \in (0, p_l)$ there is $N_\varepsilon \geq 0$ such that $p_l^k > \varepsilon$ for all $k \geq N_\varepsilon$. By the iteration formula we have $p_l^k + \lambda_k \hat{z}_l(p^k) > \varepsilon$ for all $k \geq N_\varepsilon$ and consequently $\lambda_k > (\varepsilon - p_l^k)/\hat{z}_l(p^k)$ when $\hat{z}_l(p^k) \neq 0$ and $k \geq N_\varepsilon$. For the iteration indices i corresponding to the negative subsequence of $\{\hat{z}_l(p^k)\}_k$ we have $0 < (\varepsilon - p_l^i)/\hat{z}_l(p^i) < \lambda_i \to 0$. Then either $p_l^i \to \varepsilon$ or $\hat{z}_l(p^i) \to -\infty$. The first is a contradiction with $p_l^k \to p_l$ and the latter is a contradiction with the convergence of $\{p^k\}_k$ and the continuity of $\hat{z}$. Hence, we have $J_p \neq \emptyset$ and $p_j^k \to 0$ for all $j \in J_p$. Thus, by the continuity of $\hat{z}$ and (P5') there are $l \in J_p$ and $N \geq 0$ such that $p_l^k \to 0$, and $\hat{z}_l(p^k) > 0$ for all $k \geq N$. Now we get from the iteration formula that $p_l^{k+1} > p_l^k$ for all $k \geq N$, which contradicts $p_l^k \to 0$. Hence, λ_k has a positive lower bound and convergence to a solution of (1) follows from Lemma 1.

Let us assume that there is a unique ray of solutions for (1). Then the process (5) converges to a point $\tilde{p} = \beta p^*$, where $\beta > 0$. From Lemma 3 we see that there is $\alpha > 0$ such that for $\alpha\tilde{p}$ we have $2\alpha\tilde{p} \cdot \hat{z}(p) \geq \|\hat{z}(p)\|^2$ for all $p > 0$. We can also note that λ_k is updated only finitely many times, since as shown above p^k cannot converge to a positive price vector if λ_k is updated infinitely many times. Hence, there is N such that $\lambda_k = \lambda_N$ for all $k \geq N$. Lemma 2 then implies monotonical convergence for $k \geq N$. □

Let us make some observations on the proof of Theorem 1. First, it was shown that the parameter γ_k does not converge to zero, which essentially means that the process (5) does not approximate (2) for large k.

Second, suppose the condition

$$p^* \cdot z(p) \geq \sigma \|z(p)\|^2 \tag{6}$$

holds for all $p > 0$ and z has the properties (P1)–(P3) and (P5'), then Lemma 3 is not needed in showing the convergence of the process (5). Moreover, in that case we can set $\mu_k = \gamma_k$ in step 2, because z is bounded due to (6), namely $\|z(p)\| \leq \|p^*\|/\sigma$. Boundedness is, however, in contradiction with (P5), according to which the excess demand becomes infinitely large when some of the commodities become free. Therefore, it is reasonable to suppose that (6) holds only locally; that is exactly what the condition (C2) says.

Third, constructing an example where the process (5) fails to converge should be rather easy since there are such examples for the process (2), see, e.g., Scarf (1960). More interesting question is whether there are excess demand functions that satisfy (C1) but for which the iterative process does not converge. Conditions (C1) and (C2) guarantee that the sequence of prices obtained from the process (5) is bounded. Hence, we could expect the sequence of prices to be unbounded if only (C1) holds. This would be natural in view of results by Arrow and Hahn (1971), according to which (C1) implies convergence to any given neighborhood of the equilibrium ray but not necessarily to an equilibrium.

3.1 Curvature and Convergence

Theorem 1 shows that the process (5) converges when the set $\{x \in \mathbb{R}^n : x = z(p),\ p > 0\}$ is included in a specific ball at least around the origin. This property holds in Figure 1 because this set is not too flat around the origin. This section characterizes more closely the relationship between the convergence and the geometry of the hypersurface defined by a regular excess demand function.

Let us first define a parameterized hypersurface that can be obtained from an excess demand function z. Because z is homogeneous, one of the commodities, e.g., the last one, can be selected as a numéraire, which means that the price of this commodity is set to a constant and the other prices are considered as relative prices with respect to the price of this commodity. Let $\bar{p} \in \mathbb{R}^{n-1}$ denote the price vector that is obtained by dropping the last price of p. As a result we can define a mapping $\bar{z} : \mathbb{R}^{n-1}_{+} \mapsto \mathbb{R}^{n}$ by setting $\bar{z}(\bar{p}) = z(\bar{p}, 1)$. This mapping is a parameterized hypersurface in $\mathbb{R}^n$ and $\{x \in \mathbb{R}^n : x = \bar{z}(\bar{p}),\ \bar{p} > 0\}$ is the actual hypersurface obtained from z. Note that $z(p) = \bar{z}(\bar{p})$ when $p = (\bar{p}, 1)$, but due to homogeneity z as such is not an appropriate parameterized hypersurface.

In the rest of this section it will be assumed that $\bar{z}$ is twice continuously differentiable. Let $\nabla_j \bar{z}(\bar{p})$ denote the vector that is obtained by differentiating the component functions of $\bar{z}$ with respect to j'th argument. These vectors are the row vectors of the Jacobian matrix $\nabla \bar{z}(\bar{p})$ and we use them to define the regular points of the parameterized hypersurface $\bar{z}$.

Definition 1. Point $\bar{p}$ is a regular point of $\bar{z}$ if $\nabla_1 \bar{z}(\bar{p}), \ldots, \nabla_{n-1} \bar{z}(\bar{p})$ are linearly independent. A parameterized hypersurface $\bar{z}$ is said to be regular if all points $\bar{p} > 0$ are regular points of $\bar{z}$.

Let $N(\bar{p})$ be the unit normal of the tangent space of $\bar{z}$ at $\bar{p}$, i.e., the normal of the set $\{x : x = \nabla \bar{z}(\bar{p}) d,\ d \in \mathbb{R}^{n-1}\}$. At a regular point $\bar{p}$, the tangent space of $\bar{z}$ is $n-1$ dimensional subspace, a hyperplane, spanned by the vectors $\nabla_j \bar{z}(\bar{p})$, $j = 1, \ldots, n-1$. It follows from Walras' law that $N(\bar{p}^*) = p^*/\|p^*\|$, where $\bar{p}^* = (p_1^*/p_n^*, \ldots, p_{n-1}^*/p_n^*)$. Namely, Walras' law implies that $p \cdot \nabla z(p) = -z(p)$, which gives that $p^* \cdot [\nabla z(p^*) d] = -p^* \cdot z(p^*) = 0$, i.e., p^* is perpendicular to all tangent directions at p^*.

The normal curvature of a parameterized hypersurface can be defined as follows.

Definition 2. Let $\bar{p} > 0$ be a regular point of a parameterized hypersurface $\bar{z}$. The normal curvature of $\bar{z}$ at $\bar{p}$ to a tangent direction $\nabla \bar{z}(\bar{p}) d$, $d \neq 0$, is

$$\kappa(d; \bar{p}) = \sum_{k=1}^{n} \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \left[N_k(\bar{p}) \frac{\partial^2 \bar{z}_k(\bar{p})}{\partial \bar{p}_i \partial \bar{p}_j} d_i d_j \right] / \|d\|^2. \tag{7}$$

Normal curvature measures how the normal direction of the hypersurface changes when moving from $\bar{z}(\bar{p})$ to a tangent direction. The change of the normal direction describes how the hypersurface curves at $\bar{z}(\bar{p})$. See, e.g., Spivak (1979, Sections 7.C–D) on deriving (7) from the basics of differential geometry.[1] In this paper (7) is taken as the definition of normal curvature.

[1] In fact $\kappa(d; \bar{p}) = II(v, v)/I(v)$, where I and II are the first and the second

The following lemma shows that the positive normal curvature of $\bar{z}$ to all tangent directions at equilibrium $\bar{p}^*$ is necessary and sufficient condition for z to satisfy (C2), when the regular parameterized hypersurface $\bar{z}$ is twice continuously differentiable. The proof is given in Appendix.

Lemma 4. *Let z be a twice continuously differentiable excess demand function having the properties (P1)–(P3) and an equilibrium at $p^* = (\bar{p}^*, 1)$, and let $\bar{z}$ be a regular parameterized hypersurface. Then z satisfies (C2) if and only if $\bar{z}$ has positive normal curvature at $\bar{p}^*$ to all tangent directions.*

From Theorem 1 and Lemma 4 we can prove the following convergence result according to which (P1)–(P5) together with positive normal curvature of $\bar{z}$ at $\bar{p}^*$ to all tangent directions guarantees the convergence of the process (5). Note that due to regularity and (C1) there is a unique ray of equilibria, which implies monotonical convergence.

Theorem 2. *Let z be twice continuously differentiable regular excess demand function that has the properties (P1)–(P5) and satisfies (C1) for $p^* = (\bar{p}^*, 1)$. Furthermore, let the normal curvature of the regular parameterized hypersurface $\bar{z}$ be positive at $\bar{p}^*$ to all tangent directions. Then the process (5) converges to an equilibrium for any $p^0 > 0$ and there is N such that convergence is monotonical when $k \geq N$.*

4 Convergence under the Gross Substitute Property

A differentiable excess demand function z is said to have the gross substitute property if $\partial z_j(p)/\partial p_i > 0$ for $j \neq i$. This property means that when the price of some commodity increases, the demand for other commodities grows. For such an excess demand function it can be shown that if p^* is an equilibrium then $p^* > 0$. Moreover, the set of equilibria is a unique ray, see, e.g., Arrow et al. (1959, Lemma 4).

It can be shown that under the gross substitute property the excess demand function satisfies (C1), see, e.g., Arrow et al. (1959, Lemma 5). It follows that the continuous time process (2) converges when the excess demand function has the gross substitute property. The following proposition shows a related result on the convergence of the process (5) when z satisfies (C2) in addition to having the gross substitute property. The condition (C2) can be replaced by the assumption that the parameterized hypersurface $\bar{z}$ has positive normal curvature at equilibrium to all tangent directions.

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fundamental forms, respectively, and $v = \nabla \bar{z}(\bar{p})d$. The expression (7) for the normal curvature follows from the properties of the second fundamental form.

Proposition 1. *Let z be a differentiable excess demand function with the properties (P2)–(P4) and the gross substitute property. Let $p^* = (\bar{p}^*, 1)$ be an equilibrium.*

(a) If z satisfies (C2), then the process (5) converges to an equilibrium for any $p^0 > 0$.
(b) If $\bar{z}$ is twice continuously differentiable and has positive normal curvature to all tangent directions at $\bar{p}^*$, then the process (5) converges to an equilibrium for any $p^0 > 0$.

In both cases there is N such that convergence is monotonical when $k \geq N$.

Proposition 1 is based on theorems 1 and 2 and the following lemmas. Lemma 5 shows that in the gross substitute case z has the property (P5). The proof of Lemma 5 is presented in Appendix.

Lemma 5. *Suppose that z is homogeneous, satisfies Walras' law, and has the gross substitute property. Then z has the property (P5).*

Lemma 6 shows that in the gross substitute case $\bar{z}$ is regular. The result follows from the well known fact that the rank of the Jacobian $\nabla z(p)$ is $n-1$ for all $p > 0$ when z has the gross substitute property, see, e.g., Hildenbrand and Kirman (1988, Section 6.4).

Lemma 6. *When z has the gross substitute property, all points $\bar{p} > 0$ are regular points of $\bar{z}$.*

The result (a) of Proposition 1 follows from Theorem 1 and Lemma 5. The result (b) follows from Theorem 2 and Lemma 6. Furthermore, under gross substitution there is a unique ray of solutions so that convergence is monotonical in both cases.

4.1 Cobb-Douglas Economy

In this section the convergence of the process (5) is explicitly shown for an economy in which the consumers' preferences are characterized by Cobb-Douglas utility functions that are of the form

$$u_i(x) = \prod_{j=1}^{n} x_j^{a_{i,j}},$$

where $a_{i,j} > 0$ and $\sum_j a_{i,j} = 1$ for all $i = 1, \ldots, m$. It follows from each consumer's optimization problem that the j'th component of the consumer i's demand function is $x_j^i(p) = a_{i,j}(p \cdot w^i)/p_j$. Thus, the excess demand for the j'th commodity is $z_j(p) = (p \cdot q^j)/p_j - t_j$, where $q^j = \sum_i a_{i,j} w^i$ and $t_j = \sum_i w_j^i$.

Let us suppose that $q^i_j > 0$ for all i, j, for example because $w^j_i > 0$ for all i, j. It can be seen that $\partial z_j(p)/\partial p_i = q^j_i/p_j > 0$ when $i \neq j$, i.e., the excess demand function of the Cobb-Douglas economy z has the the gross substitute property. Moreover, z has the properties (P1)–(P4).

For the convergence of the process (5) to an equilibrium, we need to show that the normal curvature of $\bar{z}$ is positive at $\bar{p}^*$ to all tangent directions. Let us begin with deriving the derivatives of $\bar{z}$ up to second order. The first derivatives of $\bar{z}$ at $\bar{p}$ are

$$\frac{\partial \bar{z}_j(\bar{p})}{\partial \bar{p}_k} = \begin{cases} q^n_k & \text{if } j = n, \\ (q^j_j \bar{p}_j - p \cdot q_j)/(\bar{p}_j)^2 & \text{if } k = j < n, \\ q^j_k/\bar{p}_k & \text{if } k \neq j < n, \end{cases}$$

where $p = (\bar{p}, 1)$, and the second derivatives are

$$\frac{\partial^2 \bar{z}_j(\bar{p})}{\partial \bar{p}_k \partial \bar{p}_l} = \begin{cases} 0 & \text{if } k,\ l \neq j \text{ or } j = n, \\ -2(q^j_j \bar{p}_j - p \cdot q_k)/(\bar{p}_j)^3 & \text{if } k = l = j < n, \\ -q^j_l/(\bar{p}_j)^2 & \text{if } k = j,\ l \neq j < n, \\ -q^j_k/(\bar{p}_j)^2 & \text{if } l = j,\ k \neq j < n. \end{cases}$$

Let us assume for simplicity that the unique ray of equilibria is $\{\lambda(1, \ldots, 1) : \lambda > 0\} \subset \mathbb{R}^n_+$. It can be shown that the general case, where (C1) holds for some equilibrium p^*, can be transformed such that (C1) holds for the transformed excess demand function with $\lambda(1, \ldots, 1)$ in place of p^*, see Arrow et al. (1959, Section 3.1.1.0). Let us denote $p^* = (1, \ldots, 1) = (\bar{p}^*, 1)$. It follows that $t_j = p^* \cdot q^j = \sum_{k=1}^n q^j_k$.

The normal curvature of $\bar{z}$ at $\bar{p}^*$ to a tangent direction defined by $d \in \mathbb{R}^{n-1}$, $d \neq 0$, is

$$\begin{aligned} \kappa(d; \bar{p}^*) &= 2\sum_{j=1}^{n-1} \left[(d_j)^2 t_j - \sum_{k=1}^{n-1} (q^j_k d_j d_k) \right] / \left[(n-1)^{1/2} \|d\|^2 \right] \\ &= 2f(d) / \left[(n-1)^{1/2} \|d\|^2 \right]. \end{aligned}$$

To prove that κ is positive it is enough to show that the function

$$f(d) = \sum_j \left[(d_j)^2 t_j - \sum_k (q^j_k d_j d_k) \right]$$

is positive for all $d \in \mathbb{R}^{n-1}$ for which $\|d\| = \rho$. It turns out that the unique minimizer of f is $d = 0$ and $f(d) > f(0) = 0$ when $d \neq 0$.

The necessary condition for the minimum of f over $\mathbb{R}^{n-1}$ is

$$\frac{\partial f(d)}{\partial d_j} = 2(t_j - q_j^j)d_j - \sum_{\substack{k=1\\k\neq j}}^{n-1}(q_k^j d_k) = 0$$

for all j. Clearly, $d = 0$ satisfies this condition. Let us now show that $f(d)$ is strictly convex function by which it follows that the necessary condition is sufficient and $f(d) > f(0) = 0$ for all $d \neq 0$.

To see that $f(d)$ is strictly convex it is enough to show that its Hessian matrix is positive definite. The entry in the j'th row and k'th column of the Hessian is

$$b_{j,k} = \frac{\partial^2 f(d)}{\partial d_j \partial d_k} = \begin{cases} 2(t_j - q_j^j) & \text{if } j = k, \\ -q_k^j & \text{if } j \neq k. \end{cases} \tag{8}$$

The positive definiteness of the Hessian matrix follows from the observation that the Hessian is strictly positively diagonally dominant, which means that $b_{j,j} > 0$ and $|b_{j,j}| > \sum_{k\neq j}|b_{j,k}|$ for all $j = 1,\ldots,n-1$. From (8) we see that $|b_{j,k}| = q_k^j$ for $j \neq k$, and $b_{j,j} = |b_{j,j}| = 2(t_j - q_j^j)$, so that the Hessian has positive diagonal entries. Furthermore, it can be seen that the Hessian is, indeed, diagonally dominant:

$$|b_{j,j}| - \sum_{k\neq j}|b_{j,k}| = 2\left[(\sum_{k=1}^{n} q_k^j) - q_j^j\right] - \sum_{\substack{k=1\\k\neq j}}^{n-1} q_k^j = 2q_n^j + \sum_{\substack{k=1\\k\neq j}}^{n-1} q_k^j > 0,$$

because $q_j^k > 0$ for all j and k. As a conclusion $\kappa(d;\bar{p}^*) > 0$ holds for all $d \neq 0$. The convergence of the process (5) for a Cobb-Douglas economy follows then from Proposition 1.

5 Second Order Weak Axiom of Revealed Preferences

An excess demand function is said to satisfy the weak axiom of revealed preferences if for any pair of price vectors p^1 and p^2 for which $z(p^1) \neq z(p^2)$ it holds that

$$p^1 \cdot z(p^2) \leq 0 \implies p^2 \cdot z(p^1) > 0.$$

The interpretation of the WA is that if p^1 is revealed preferred to p^2, which means that the value of $z(p^2)$ with prices p^1 is negative, then p^2 cannot be revealed preferred to p^1.

It can be seen that the WA implies (C1); hence, also the stability of the continuous time process (2). As was seen in Section 3 the excess demand function has to satisfy (C2) to obtain convergence for the process (5). Hence, we need to define a strengthened form of the WA, called the second order weak axiom of revealed preferences, which implies (C2) analogously as the WA implies (C1).

Definition 3. An excess demand function z satisfies the second order weak axiom of revealed preferences (SWA) if for any $p^2 > 0$ there is $\sigma > 0$ such that

$$p^1 \cdot z(p^2) \leq 0 \implies p^2 \cdot z(p^1) \geq \sigma \|z(p^1) - z(p^2)\|^2$$

The SWA means that if p^1 is revealed preferred to p^2, then p^2 is not revealed preferred to p^1 and the value of $z(p^1)$ with prices p^2 is bounded from below in proportion to the differences of the excess demands $\|z(p^1) - z(p^2)\|^2$. Note that in the definition of the SWA the constant σ depends on p^2.

It can be seen that the SWA implies (6) for all $p > 0$. Thus, we could say that (6) means that the second order weak axiom holds between the equilibrium vector p^* and any other price vector. As explained in Section 3, if z satisfies (6) for all $p > 0$, then it is bounded. Because excess demand functions are not necessarily bounded it is reasonable to assume that the SWA holds only around the equilibria. We say that z satisfies the SWA on E_ε if the SWA holds for all $p^1,\ p^2 \in E_\varepsilon$.

As a corollary of Theorem 1 we obtain the following convergence result for economies that satisfy the SWA.

Proposition 2. *Let z be an excess demand function that has the properties (P1)–(P5), and let p^* be an equilibrium. If z satisfies the WA for all $p > 0$ and the SWA on E_ε, then the process (5) converges to an equilibrium for any $p^0 > 0$.*

5.1 Strongly Monotone Mappings and the SWA

It is well known that the WA holds when the excess demand function is monotone or has a representative consumer with an appropriate preference relation. This arises the question whether there are similar economic conditions which imply the SWA. This section shows that if the excess demand function is a strongly monotone and Lipschitz continuous, then it satisfies the SWA. Furthermore, if the economy has a representative consumer, whose preferences are characterized by a strongly concave utility function, then the excess demand function satisfies the SWA. The latter result is based on the strong monotonicity of the gradient mapping of a strongly concave function. It follows from Proposition 2 that when z has the properties (P1)–(P5) and satisfies one of the conditions presented in this section, then the process (5) converges globally to an equilibrium.

We first define some monotonicity concepts. Below, I denotes the $n \times n$ identity matrix.

Definition 4. Let S be a convex set in $\mathbb{R}^n$. Mapping $F : S \mapsto \mathbb{R}^n$ is monotone on S if the inequality $(p^1 - p^2) \cdot [F(p^1) - F(p^2)] < 0$ holds for all p^1, $p^2 \in S$ whenever $F(p^1) \neq F(p^2)$. If there is $\sigma > 0$ such that $F + \sigma I$ is monotone on S, then F is said to be strongly monotone on S.

Because excess demand functions are homogeneous, it is reasonable to define monotonicity for them by restricting the monotonicity condition to those price vectors that are somehow comparable to each other. An appropriate monotonicity concept for excess demand functions is obtained by requiring that the monotonicity condition holds for z with a pair of prices p^1 and p^2 if for some vector $y > 0$ we have $p^1 - p^2 \in T_y = \{x \in \mathbb{R}^n : y \cdot x = 0\}$. The condition $p^1 - p^2 \in T_y$ means that the value of commodity bundle y is the same for prices p^1 and p^2. Geometrically monotonicity means that the vector of price changes and the vector of demand changes point to the opposite half spaces. It can be shown that the excess demand function of a large economy, in which there is a continuum of consumers, is monotone when the income distribution of the economy has certain properties, see Hildenbrand (1983).

It is well known that when z is monotone in the sense that the monotonicity condition holds when $p^1 - p^2 \in T_y$, then z satisfies the WA. The SWA is related to the strong monotonicity of the excess demand function analogously, which is shown in Proposition 3, where in addition to monotonicity, z is assumed to be Lipschitz continuous in the sense of the following definition. Note that due to homogeneity z cannot satisfy the ordinary Lipschitz condition $\|z(p^1) - z(p^2)\| \leq L\|p^1 - p^2\|$.

Definition 5. An excess demand function z is Lipschitz continuous on the cone $C \subset \mathbb{R}^n_+$ relative to vector $y > 0$ if there is a constant $L > 0$ such that the inequality $\|z(p^1) - z(p^2)\| \leq L\|\alpha_1 p^1 - \alpha_2 p^2\|$ holds for all p^1, $p^2 \in C$ when α_1, $\alpha_2 > 0$ satisfy $\alpha_k p^k - y \in T_y$ for $k = 1, 2$.

The following proposition shows that Lipschitz continuity and strong monotonicity imply the SWA.

Proposition 3. *Let the excess demand function z be Lipschitz continuous on E_ε relative to $y > 0$, and strongly monotone for all p^1, $p^2 \in E_\varepsilon$ that satisfy $p^1 - p^2 \in T_y$. Then z satisfies the SWA on E_ε.*

Proof. Let p^1, $p^2 \in E_\varepsilon$, $p^1 \neq p^2$, and $p^1 \cdot z(p^2) \leq 0$. Moreover, let the positive coefficients α_1 and α_2 be such that $\alpha_k p^k - y \in T_y$ for $k = 1, 2$. In that case we have $\alpha_1 p^1 - \alpha_2 p^2 \in T_y$ and $\alpha_1 p^1$, $\alpha_2 p^2 \in E_\varepsilon$. Note that E_ε is a cone, i.e.,

$\alpha p \in E_\varepsilon$ for all $\alpha > 0$. By homogeneity it holds that $z(\alpha_k p^k) = z(p^k)$ for $k = 1, 2$. From strong monotonicity and Walras' law we obtain

$$\begin{aligned} -\alpha_1 p^1 \cdot z(p^2) - \alpha_2 p^2 \cdot z(p^1) &= (\alpha_1 p^1 - \alpha_2 p^2) \cdot [z(p^1) - z(p^2)] \\ &\leq -\sigma \|\alpha_1 p^1 - \alpha_2 p^2\|^2. \end{aligned}$$

It follows that

$$\alpha_2 p^2 \cdot z(p^1) - \sigma \|\alpha_1 p^1 - \alpha_2 p^2\|^2 \geq -\alpha_1 p^1 \cdot z(p^2) \geq 0.$$

From the Lipschitz continuity relative to y we get

$$\sigma \|z(p^1) - z(p^2)\|^2 / L^2 \leq \sigma \|\alpha_1 p^1 - \alpha_2 p^2\|^2 \leq \alpha_2 p^2 \cdot z(p^1).$$

Hence, the SWA condition holds with the constant $\sigma/(L^2 \alpha_2)$. □

Due to the result of Proposition 3 it would be natural to call the SWA as the strong axiom of revealed preferences. Strong axiom, however, usually refers to the following indirect form of the WA: for any $N \geq 2$ the inequalities $p^k \cdot z(p^{k+1}) \leq 0$, $k = 1, \ldots, N-1$, imply that $p^N \cdot z(p^1) > 0$.

There is another relationship between the SWA and strongly monotone mappings in addition to the one described above. Namely, if the economy has a representative consumer whose preferences can be characterized by a locally strongly concave utility function u (see the definition below), then the economy satisfies the SWA around the equilibrium ray. A representative consumer means a preference relation for which $\sum_i x^i(p)$ equals the demand function obtained by maximizing this preference relation under the budget constraint $p \cdot (x - \sum_i w^i) \leq 0$.

Definition 6. A differentiable function u is strongly concave on a convex set S if ∇u is strongly monotone on S. [2]

See, e.g., Rockafellar and Wets (1998, Section 12.H) for more about strongly monotone mappings and convex functions. In the framework of exchange economies Shannon and Zame (2002) have utilized strong concavity to show determinacy of equilibrium. [3]

The relationship of the SWA and the representative consumer with a strongly concave utility function is stated in the following proposition. In addition to strong concavity we need local nonsatiation, which means that in any environment of a commodity bundle there are more desirable bundles. This condition guarantees that Walras' law is satisfied.

[2] In a non-differentiable case the definition is the same except that the gradient is replaced with subgradient. Differentiability is assumed here for simplicity.

[3] Shannon and Zame (2002) call strong concavity as quadratic concavity.

Proposition 4. *Let a locally nonsatiated preference relation be characterized by a strictly concave utility function* u *that is strongly concave on* $B(\sum_i w^i, \delta)$*. Let* $x(p)$ *be the demand function that is obtained by maximizing* $u(x)$ *subject to the budget constraint* $p \cdot (x - \sum_i w^i) \leq 0$ *and let the prices be positive at equilibrium. Then the excess demand function* $z(p) = x(p) - \sum_i w^i$ *satisfies the SWA on* E_ε *for some* $\varepsilon > 0$*.*

Proof. First, note that the demand function $x(p)$ is continuous because u is a strictly concave function, see, e.g., Hildenbrand and Kirman (1988, Proposition 3.1). Under local nonsatiation and concavity, the necessary and sufficient optimality condition for maximizing u over the budget $p^k \cdot (x - \sum_i w^i) \leq 0$ is $\nabla u(x(p)) = \lambda_k p^k$ for some $\lambda_k > 0$ when $x(p^k) > 0$. Because prices are positive at the equilibrium, i.e., $p^* > 0$, there is $\bar{\varepsilon} > 0$ such that $x(p) > 0$ when $p \in E_{\bar{\varepsilon}}$. Note that we have $x(p^1) - x(p^2) = z(p^1) - z(p^2)$, and local nonsatiation implies Walras' law. These facts and strong concavity yield

$$\begin{aligned}&\Big[\nabla u\Big(x(p^1)\Big) - \nabla u\Big(x(p^2)\Big)\Big] \cdot \Big[x(p^1) - x(p^2)\Big] = \\ &\qquad\qquad \Big(\lambda_1 p^1 - \lambda_2 p^2\Big) \cdot \Big[z(p^1) - z(p^2)\Big] = \\ &\qquad\qquad\quad -\lambda_1 p^1 \cdot z(p^2) - \lambda_2 p^2 \cdot z(p^1) \leq -\sigma \|z(p^1) - z(p^2)\|^2.\end{aligned}$$

By rearranging the terms in the bottom line and dividing with λ_2 we obtain

$$p^2 \cdot z(p^1) - (\sigma/\lambda_2)\|z(p^1) - z(p^2)\|^2 \geq -(\lambda_1/\lambda_2) p^1 \cdot z(p^2) \geq 0,$$

where the latter inequality holds when p^1 is revealed preferred to p^2. Thus, z satisfies the SWA when p^1 and p^2 are chosen such that $x(p^1),\ x(p^2) \in B(\sum_i w^i, \delta)$ and $x(p^1),\ x(p^2) > 0$. It follows from this result and the continuity of x that there is $\varepsilon \leq \bar{\varepsilon}$ such that z satisfies the SWA on $p \in E_\varepsilon$. □

6 Conclusion

An extensive part of literature has concentrated on normalized tâtonnement processes, i.e., processes in which one of the prices is selected as a numéraire and only the rest of them are adjusted. This paper shows that a non-normalized process $p^{k+1} = p^k + z(p^k)$, with additional requirements on positive prices and bounded price changes, converges under conditions that are remarkably close to the classical convergence conditions by Arrow et al. (1959) and Arrow and Hurwicz (1958) for the continuous time process $\dot{p} = z(p)$. Indeed, price normalization seems to lead to chaotic behavior whereas non-normalized process has better convergence properties.

For practical or numerical considerations the difference between the convergence conditions of the usual continuous time process and the conditions obtained in this paper are quite negligible. This is because the continuous time convergence condition is obtained as the limit from the discrete time convergence condition as the hypersurface defined by the excess demand function becomes flatter.

This paper has also introduced a second order form of the weak axiom of revealed preferences that implies convergence of iterative tâtonnement. This condition has the same economic interpretation as the ordinary weak axiom of revealed preferences but the condition is mathematically more stringent. Actually, the ordinary weak axiom is obtained as a limiting case from the second order version. The second order weak axiom holds in two specific cases: when the excess demand function is strongly monotone, or the economy has a representative consumer with locally strongly concave utility function.

Appendix: Proofs of the Lemmas

Proof of Lemma 1. Let us first observe that

$$\mu_k p^* \cdot z(p) \geq \|\mu_k z(p)\|^2, \tag{9}$$

when $\mu_k \leq 1$. This can be seen by multiplying both sides of $p^* \cdot z(p) \geq \|z(p)\|^2$ with μ_k^2 and noticing that $\mu_k^2 p^* \cdot z(p) \leq \mu_k p^* \cdot z(p)$ because $\mu_k \leq 1$.

From (9) and Walras' law we have

$$\|p^{k+1} - p^*\|^2 = \|p_k + \mu_k z(p^k) - p^*\|^2 =$$
$$\|\mu_k z(p^k)\|^2 - 2\mu_k z(p^k) \cdot p^* + \|p^k - p^*\|^2 \leq \|p^k - p^*\|^2.$$

Note that p^k belongs to $B(p^*, r)$ for all $k = 0, 1, \ldots$, when $p^0 \in B(p^*, r)$. Therefore, the sequence $\{\|p^k - p^*\|\}_k$ converges and as a result the sequence $\{\|p^k\|\}_k$ is bounded. From Walras' law it follows that

$$\|p^k\|^2 = \|p^0\|^2 + \sum_{i=0}^{k-1} \mu_i^2 \|z(p^i)\|^2,$$

so that $\{\|p^k\|\}_k$ is a growing and bounded sequence and hence convergent. The iteration formula yields

$$p^k = p^0 + \sum_{i=0}^{k-1} \mu_i z(p^i).$$

Hence, $\|p^0 + \sum_{i=0}^{k-1} \mu_i z(p^i)\|$ converges, too. From the triangular inequality we get

$$\|p^0 + \sum_{i=0}^{k+l} \mu_i z(p^i)\| \geq \left| \|p^0 + \sum_{i=0}^{k} \mu_i z(p^i)\| - \| \sum_{i=k+1}^{k+l} \mu_i z(p^i)\| \right|$$

and we obtain

$$\|p^{k+l} - p^k\| = \| \sum_{i=k+1}^{k+l} \mu_i z(p^i)\| \to 0, \tag{10}$$

when $k \to \infty$ and $l \geq 1$. Thus, $\{p^k\}_k$ is a Cauchy sequence and hence convergent. Let $\tilde{p}$ denote the limit point of this Cauchy sequence.

Let us now show that when $0 < \bar{\mu} \leq \mu_k$ the sequence $\{p^k\}_k$ converges to a solution of $z(p) = 0$. By setting $l = 1$ it follows from (10) that $\mu_k \|z(p^k)\| \to 0$. Because it holds that $\bar{\mu}\|z(p^k)\| \leq \mu_k \|z(p^k)\|$ and z is continuous, we see that $\tilde{p}$ is a solution of $z(p) = 0$. □

Proof of Lemma 3. Let z satisfy (C2) on $E_{\bar{\varepsilon}} = \{p \in \mathbb{R}^n_+ : \|z(p)\| < \bar{\varepsilon}\}$ with constant $\bar{\sigma}$. By the homogeneity of excess demand we know that $\hat{z}$ obtains all its values on the unit simplex $\Delta = \{p \in \mathbb{R}^n_+ : \sum_j p_j = 1\}$. Because of (P4) and (P5) it can be seen that $p^* \cdot z(p^k) \to \infty$, when $p^k \to p$ and $J_p \neq \emptyset$. As a result, we have

$$\lim_{p^k \to p} p^* \cdot \hat{z}(p^k) > 0,$$

when $J_p \neq \emptyset$. From this property, continuity, and (C1), it follows that there is $\delta > 0$ such that $p^* \cdot \hat{z}(p) \geq \delta$ for all $p \in \Delta \setminus S$, where $S = \{p \in \Delta : p_j > \varepsilon'$ $\forall j = 1, \ldots, n\}$ and $\varepsilon' > 0$ is chosen such that $E_{\bar{\varepsilon}} \cap \Delta \subset S$.

Clearly, the infimum of $p^* \cdot \hat{z}(p)$ over $S \setminus E_{\bar{\varepsilon}}$ is positive, since otherwise $\hat{z}$ would violate (C1). Let $\alpha > 0$ denote this infimum. We have $p^* \cdot \hat{z}(p) \geq \min\{\delta, \alpha\}$ for all $p \in \Delta \setminus E_{\bar{\varepsilon}}$. Because $\|\hat{z}(p)\| \leq M$ we get $p^* \cdot \hat{z}(p) \geq \hat{\sigma}\|\hat{z}(p)\|^2$ for all $p \in \Delta \setminus E_{\bar{\varepsilon}}$ by choosing $\hat{\sigma} < \min\{\delta, \alpha\}/M^2$. The result follows by choosing $\sigma = \min\{\bar{\sigma}, \hat{\sigma}\}$. □

Proof of Lemma 4. From Taylor's formula we get

$$\bar{z}_k(\bar{p}^* + d) = \bar{z}_k(\bar{p}^*) + \nabla \bar{z}_k(\bar{p}^*) \cdot d + \frac{1}{2} d \cdot \nabla^2 \bar{z}_k(\bar{p}^*) \cdot d + o(\|d\|^2), \tag{11}$$

where $d \in \mathbb{R}^{n-1}$ is such that $\bar{p}^* + d > 0$ and $o(\|d\|^2)/\|d\|^2 \to 0$ as $\|d\| \to 0$. Here $\nabla \bar{z}_k$ denotes the gradient of k'th component function of $\bar{z}$. Furthermore, vectors are considered as column vectors and x' denotes the transpose of vector x.

Recall that Walras' law gives $N(\bar{p}^*) = p^*/\|p^*\|$. Furthermore, we have

$$\nabla \bar{z}(\bar{p}) = \begin{bmatrix} \nabla \bar{z}_1(\bar{p})' \\ \vdots \\ \nabla \bar{z}_n(\bar{p})' \end{bmatrix} = \begin{bmatrix} \nabla_1 \bar{z}(\bar{p}) & \cdots & \nabla_{n-1} \bar{z}(\bar{p}) \end{bmatrix},$$

and because $\nabla \bar{z}(\bar{p})d$ is a tangent direction of the parameterized hypersurface $\bar{z}$ at $\bar{z}(\bar{p})$, we get $p^* \cdot [\nabla \bar{z}(\bar{p}^*)d] = 0$. From this together with (11) and $\bar{z}(\bar{p}^*) = 0$ we obtain

$$2p^* \cdot \bar{z}(\bar{p}^* + d) = \sum_{i,j,k} p_k^* \frac{\partial \bar{z}_k(\bar{p}^*)}{\partial \bar{p}_i \partial \bar{p}_j} d_i d_j + o(\|d\|^2),$$

where in the summation i and j run from 1 to $n-1$ and k runs from 1 to n. By the definition of normal curvature this can be written as

$$2p^* \cdot \bar{z}(\bar{p}^* + d) = \|p^*\| \kappa(d; \bar{p}^*) \|d\|^2 + o(\|d\|^2). \tag{12}$$

From Taylor's formula (11) we also get

$$\|\bar{z}(\bar{p}^* + d)\|^2 = \bar{z}(\bar{p}^* + d) \cdot \bar{z}(\bar{p}^* + d) = d \cdot [\nabla \bar{z}(\bar{p}^*)' \nabla \bar{z}(\bar{p}^*) d] + o(\|d\|^2), \tag{13}$$

where $\nabla \bar{z}(\bar{p}^*)'$ is the transpose of the Jacobian matrix.

Because $\bar{p}^*$ is a regular point of $\bar{z}$, the Jacobian is full rank matrix. Therefore, the matrix $A = \nabla \bar{z}(\bar{p}^*)' \nabla \bar{z}(\bar{p}^*)$ is positive definite. It is known from linear algebra that for a symmetric matrix A it holds that

$$\beta_L \|d\|^2 \leq d \cdot (Ad) \leq \beta_U \|d\|^2, \tag{14}$$

where β_L and β_U are the minimal and maximal eigenvalues of A, respectively.

When the curvature is positive to all directions, we have

$$\kappa^* = \min_{\|d\|=\rho} \kappa(d; \bar{p}^*) > 0,$$

because as a continuous function κ attains its minimum over $\partial B(0, \rho) = \{d \in \mathbb{R}^{n-1} : \|d\| = \rho\}$, where $\rho > 0$ is chosen such that $\bar{p}^* + d > 0$ for all $d \in \partial B(0, \rho)$. An appropriate ρ can be found because $p^* > 0$. By choosing $\alpha > \beta_U/(\|p^*\|\kappa^*)$ we obtain the following inequality from (12) and (13):

$$2\alpha p^* \cdot \bar{z}(\bar{p}^* + d) \geq \|\bar{z}(\bar{p}^* + d)\|^2,$$

when $\|d\| \leq \rho$. Hence, z satisfies (C2) around the equilibrium ray with the constant $\sigma = 1/(2\alpha)$.

To conclude the proof it needs to be shown that (C2) implies that $\bar{z}$ has positive normal curvature at $\bar{p}^*$ to all tangent directions. Without loss of generality we may assume that $\sigma = 1$ in (C2). It follows that there is $\rho > 0$ such that $2p^* \cdot \bar{z}(\bar{p}) \geq \|\bar{z}(\bar{p})\|^2$ for all $\bar{p} \in B(\bar{p}^*, \rho)$. From (12), (13), and (14) we get

$$\|p^*\|\kappa(d; \bar{p}^*)\|d\|^2 + o(\|d\|^2) \geq \beta_L \|d\|^2,$$

for all $d \in B(0, \rho)$, and consequently $\kappa(d; \bar{p}^*) \geq \beta_L / \|p^*\| > 0$, i.e., the normal curvature of $\bar{z}$ is positive to all tangent directions at $\bar{p}^*$. □

Proof of Lemma 5. Let us suppose that $p^k \to p$ as $k \to \infty$. Without loss of generality we may suppose that the first l prices of p are zero, i.e., $J_p = \{1, \ldots, l\}$, $l < n$. By homogeneity we can choose an equilibrium vector p^* such that $p_j > p_j^*$ for all $j \notin J_p$. Moreover, there is N such that when $k \geq N$, we have $p_j^k < p_j^*$ for all $j \in J_p$, and $p_j^k > p^*$ for all $j \notin J_p$.

According to Walras' law

$$\sum_{j=1}^{l} p_j^* z_j(p_1^*, \ldots, p_l^*, p_{l+1}^k, \ldots, p_n^k) = - \sum_{j=l+1}^{n} p_j^k z_j(p_1^*, \ldots, p_l^*, p_{l+1}^k, \ldots, p_n^k). \quad (15)$$

The gross substitute property implies that $z_j(p^k) < z_j(p_1^*, \ldots, p_l^*, p_{l+1}^k, \ldots, p_n^k)$ for $j \notin J_p$ and $k \geq N$, because $p_j^k < p_j^*$, $j \in J_p$. Thus, from (15) we obtain

$$\sum_{j=1}^{l} p_j^* z_j(p_1^*, \ldots, p_l^*, p_{l+1}^k, \ldots, p_n^k) < - \sum_{j=l+1}^{n} p_j^k z_j(p^k) = \sum_{j=1}^{l} p_j^k z_j(p^k), \quad (16)$$

where the last equality is from Walras' law.

Let us make a counter assumption that $z_1(p^k), \ldots, z_l(p^k)$ are bounded above. Taking limits from both sides of (16) as $k \to \infty$ yields

$$\sum_{j=1}^{l} p_j^* z_j(p_1^*, \ldots, p_l^*, p_{l+1}, \ldots, p_n) \leq 0, \quad (17)$$

because $\lim_{k\to\infty} \sum_{j=1}^{l} p_j^k z_j(p^k) \leq 0$ by the boundedness. From the gross substitute property, on the other hand, it follows that when $j \in J_p$ we have

$$z_j(p_1^*, \ldots, p_l^*, p_{l+1}, \ldots, p_n) > z_j(p^*) = 0. \quad (18)$$

Recall that p^* was chosen such that $p_j > p_j^*$ for all $j \in J_p$. Clearly, (17) leads to contradiction with (18). Thus, at least one of $z_1(p^k), \ldots, z_l(p^k)$ becomes infinitely large as $p_j^k \to 0$ for all $j \in J_p$. □

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Chapter III:

Analysis of the Constraint Proposal Method for Two-Party Negotiations

Abstract

In the constraint proposal method a mediator locates points at which the two decision makers have joint tangent hyperplanes. This chapter gives conditions under which these points are Pareto optimal and proves the mediator's problem has a solution. In practice, the mediator adjusts a hyperplane going through a reference point until the decision makers' most preferred alternatives on the hyperplane coincide. Local convergence conditions for fixed-point iteration as an adjustment process are given. I also discuss the relationship of exchange economies and the constraint proposal method, and the possible ways of using the method.

Key words: group decisions and negotiations, negotiation support method, Pareto optimality, existence of solution, fixed-point iteration

1 Introduction

This chapter considers a two-party negotiation over two or more continuous issues. For example, the negotiation could be on allocating resources, such as money and labor force, between two units of a company. The purpose of negotiation support methods in such settings is to locate Pareto optimal points among which the decision makers (DMs), e.g., the units of a company, can choose an agreement. Ehtamo et al. (1999a) have recently formalized an interactive method for finding Pareto points by means of joint tangent hyperplanes. The method is called the constraint proposal method. This chapter focuses on three major questions: Does the method produce Pareto-optimal points, does it lead to a problem that has a solution, and can the joint tangent hyperplanes be found with fixed-point iteration.

The idea of locating Pareto solutions by finding the joint tangent was first presented for oligopoly games by Ehtamo et al. (1996) and Verkama et al. (1996). Teich et al. (1995), Ehtamo et al. (1999a), and Heiskanen et al. (2001) extend the approach to negotiation settings, where an impartial mediator tries to find joint tangent hyperplanes. The method is based on the geometrical observation that under some concavity assumptions there is a jointly tangential hyperplane for the DMs' indifference contours at a Pareto optimal point.

In practice, the mediator adjusts a hyperplane going through a predetermined reference point until the DMs' most preferred alternatives on the hyperplane coincide. I show that reference points chosen from the line connecting the DMs' global optima produce Pareto optimal points, and the mediator's problem has a solution.

In the theory of oligopolistic markets the joint tangent can be interpreted as a mechanism according to which the members of a cartel can punish each others from deviating the joint optimum, see Osborne (1976). This idea is further generalized to a dynamic resource management problem by Ehtamo and Hämäläinen (1993) and Ehtamo and Hämäläinen (1995), where the parties safeguard themselves with linear strategies against any attempts by the other party to break an agreement. See also Chapter V where Osborne's approach is extended to repeated oligopoly games.

From the negotiation support point of view, the main benefit of the constraint proposal method is that the DMs' utility functions need not be elicited. Second, the method is informationally decentralized in the sense that the DMs do not have to disclose any private information to each other. Other methods with similar properties include, e.g., the heuristic presented by Teich et al. (1996) and the Joint Gains method by Ehtamo et al. (1999b) and Ehtamo et al. (2001). These methods are based on seeking joint improvements from a tentative agreements; an approach, which was first suggested by Raiffa (1982).

The constraint proposal method is implemented in a negotiation support system RAMONA, which has been applied, e.g., to agricultural negotiations between Finnish Government and the Finnish Farmer's Union, see Teich et al. (1995). In RAMONA the hyperplane, on which the DMs are asked their most preferred points, is interpreted as a budget constraint. This interpretation relates the method to exchange economies. I shall briefly discuss the similarities and differences of exchange economies and the constraint proposal method.

The chapter is organized as follows. In Section 2 I describe the mediator's problem as a system of equations to be solved and make some observations on the properties of the system. Section 3 studies the choice of the reference point and Pareto optimality of the solution of the mediator's problem. Conditions for the existence of solution of the mediator's problem are analyzed in Section 4. Adjustment of hyperplane constraint with fixed-point iteration is studied in Section 5. Section 6 discusses the relationship of the constraint proposal method and exchange economies. In Section 7 I make some concluding remarks and discuss the possible ways of using the method.

2 Constraint Proposal Method

There are two DMs, a and b, who negotiate over $n \geq 2$ continuous issues. Let the real numbers $x_1, \ldots, x_n$ denote the values of these issues and let $x = (x_1, \ldots, x_n)$. The DMs' preferences are characterized with the utility functions $u_a,\ u_b : \mathbb{R}^n \mapsto \mathbb{R}$. These are needed in the mathematical analysis, but the negotiation method itself does not require these functions to be explicitly known. The utility functions' sublevel sets at y are denoted by $S_i(y) = \{x \in \mathbb{R}^n : u_i(x) \geq u_i(y)\}$, $i = a, b$. In this chapter we need the following assumptions on the DMs' value functions:

(A1) u_a and u_b have unique global optima at $\bar{x}^a$ and $\bar{x}^b$, respectively, and $\bar{x}^a \neq \bar{x}^b$,
(A2) u_a and u_b are continuous,
(A3) u_a and u_b are quasiconcave,
(A4) u_a and u_b are locally nonsatiated on $S_a(\bar{x}^b) \cap S_b(\bar{x}^a)$ with the exception that local nonsatiation is not required at DMs' own optima,
(A5) u_a and u_b are strictly quasiconcave.

The global optima are used for constructing appropriate reference points for the constraint proposal method. We may assume that these optimal points are different since otherwise there would be no need to negotiate at all. The continuity of utility functions is crucial when studying the existence of solution for the mediator's problem of finding joint tangent hyperplanes.

Quasiconcavity of function u_i means that the set $S_i(y)$ is convex for all $y \in \mathbb{R}^n$. The local nonsatiation of u_i at y can be formulated mathematically as follows: for all $\rho > 0$ there is $x' \in B(y, \rho) = \{x \in \mathbb{R}^n : \|x - y\| \leq \rho\}$ such that $u_i(x') > u_i(y)$. This condition is needed within the region $S_a(\bar{x}^b) \cap S_b(\bar{x}^a)$ where the Pareto-optimal points are located. Here $\|\cdot\|$ is the Euclidean norm and $B(y, \rho)$ is the ball centered at y having radius ρ. The strict quasiconcavity of u_i means that for each x^1, $x^2 \in \mathbb{R}^n$, $x^1 \neq x^2$, we have $u_i\left(\lambda x^1 + (1-\lambda)x^2\right) > \min\{u_i(x^1), u_i(x^2)\}$ for all $\lambda \in (0, 1)$. Strict quasiconcavity implies both quasiconcavity and local nonsatiation. It also assures the uniqueness of the global maximum, see, e.g., Mas-Colell et al. (1995, Section 3.D).

The purpose of the constraint proposal method is to locate Pareto optimal solutions that are points where it is not possible to move to any other point without worsening one of the DMs value. Formally, Pareto optimality of point x^* means that there is no x for which

$$u_i(x) \geq u_i(x^*)$$

for $i = a, b$ and the inequality is strict for at least one i.

In the constraint proposal method an impartial mediator tries to locate a hyperplane going through a given reference point such that the DMs' most preferred alternatives on that hyperplane coincide. When this happens the hyperplane is tangential to the DMs' indifference curves at the point in question; see Figure 1, where the hyperplane is simply a line. If all the DMs' more preferred points are on the opposite sides of the hyperplane as in Figure 1, then the point is Pareto optimal.

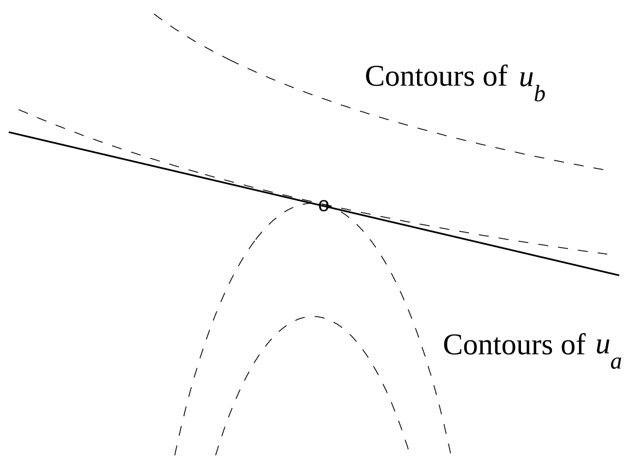

Figure 1. A Pareto optimal point and a joint tangent hyperplane.

Let us now formulate the mediator's problem mathematically. First, the mediator chooses a reference point r and defines a hyperplane

$$H(p, r) = \{x \in \mathbb{R}^n : p \cdot (x - r) = 0\}$$

going through the reference point. The normal of the hyperplane is denoted by p and $p \cdot x$ denotes the usual inner product of vectors p and x. The mediator asks the DMs to give their most preferred points on the hyperplane. These points solve

$$\max_{x \in H(p,r)} u_i(x), \ \ i = a, b. \tag{1}$$

Knowing the optimal answers the mediator then updates the hyperplane. The procedure is repeated until the most preferred points coincide within some predetermined tolerance. I will turn back to the adjustment of the hyperplane in Section 5.

Let $X_i(p, r)$, $i = a, b$, denote the solutions to (1). The mediator's problem can be formulated as follows: for fixed r find p such that

$$X_a(p, r) \cap X_b(p, r) \neq \emptyset. \tag{2}$$

When (1) has a unique solution, i.e., X_a and X_b consist of single points, then the mediator's problem can be formulated equivalently as the following system of equations to be solved for p:

$$F(p) = x^a(p, r) - x^b(p, r) = 0, \tag{3}$$

where $x^i(p, r)$, $i = a, b$, denotes the unique solution of (1). Recall that under strict quasiconcavity the solution of (1) is unique.

By solving (3) with different reference points, different Pareto solutions can be obtained. This can be done in practice, e.g., by sliding the reference point as suggested by Ehtamo et al. (1999a). Under some concavity assumptions for u_a and u_b the resulting Pareto optimal points vary lower semicontinuously as the reference point is changed, see Heiskanen et al. (2001, Theorem 5).

This study assumes that when solving for their most preferred points the DMs do not have other constraints than the hyperplane given by the mediator. There could be some other constraints as well, e.g., in a resource allocation problem the amounts of the resources could be limited. Nevertheless, adding the same compact and convex constraint set to the DMs' optimization problems would not affect the mathematical properties of the problem. To ease the notation I neglect all these additional constraints. See Heiskanen (2001) for the use of the constraint proposal method in negotiations with additional constraints.

Let us now make observations on the properties of F. These properties are needed in the following sections. First, because any parallel normal vectors define the same hyperplane, F is degree zero homogeneous, i.e.,

(P1) $F(p) = F(\alpha p)$ for all $\alpha \neq 0$ and p.

In particular, if $F(p^*) = 0$ then $F(\alpha p^*) = 0$ for all $\alpha \neq 0$, which means that the mediator's problem has at least a ray of solutions if it has one solution. This holds for both formulations (2) and (3) of the mediator's problem.

Second, since $x^i(p,r) \in H(p,r)$ for $i = a, b$, it follows that F satisfies a condition which is known as Walras' law in microeconomics literature:

(P2) $p \cdot F(p) = 0$ for all $p \neq 0$.

We shall see that Walras' law plays an important role in the analysis of the constraint proposal method. It is also a property that does not hold for the multi-party generalization of the method considered by Heiskanen et al. (2001) and Heiskanen (2001). Hence, most of the results of this chapter cannot be generalized to a multi-party setting with the same techniques as used in this chapter. The interpretation of Walras' law is further discussed in Section 6.

2.1 Example: Quadratic Utility Functions

Let us assume that the utility functions are of the form

$$u_i(x) = -\sum_{j=1}^{n} \alpha_j^i (x_j - \bar{x}_j^i)^2,$$

where $\alpha_j^i > 0$ for $j = 1, \ldots, n$ and $i = a, b$. By solving the optimality conditions of (1) we get that the DM i's responses for given hyperplane constraint are

$$x_j^i(p,r) = [p \cdot (r - \bar{x}^i)]p_j / [\alpha_j^i \sum_k (p_k^2/\alpha_k^i)] + \bar{x}_j^i, \tag{4}$$

for $j = 1, \ldots, n$ and $i = a, b$.

To illustrate the geometrical ideas behind the constraint proposal method let us now consider the two dimensional case and set $\alpha_1^a = \alpha_2^b = 15$, $\alpha_2^a = \alpha_1^b = 1$, $\bar{x}^a = (0,0)$, $\bar{x}^b = (2,2)$, and let us choose the reference point $r = (2,0)$. The contours of the utility functions are illustrated in the left part of Figure 2; dotted lines represent the contours of u_a and dashed lines represent the contours of u_b. The resulting optimal solution functions $x^a(p,r)$ and $x^b(p,r)$ are illustrated in the figure by solid lines. The optimal solution functions, given by (4), are

$$x^a(p,r) = (2p_1^2, 30p_1p_2)/(p_1^2 + 15p_2^2),$$
$$x^b(p,r) = \left(30(p_1^2 - p_1p_2) + 2p_2^2, 30p_1^2\right)/(15p_1^2 + p_2^2).$$

The resulting F, defined by (3), is drawn in the right part of Figure 2.

There are three solution rays to (3):

$$R_1 = \{(p_1,p_2) : (p_1,p_2) = \lambda(1,1),\ \lambda \neq 0\},$$
$$R_2 = \{(p_1,p_2) : (p_1,p_2) = \lambda\left((2-\sqrt{3})/(2+\sqrt{3}), 1\right),\ \lambda \neq 0\},$$
$$R_3 = \{(p_1,p_2) : (p_1,p_2) = \lambda\left(1, (2-\sqrt{3})/(2+\sqrt{3})\right),\ \lambda \neq 0\}.$$

These rays are illustrated with dashed lines in the right part of Figure 2. A hyperplane going through the reference point $(2,0)$ and the normal in R_1, R_2, or R_3 gives the joint tangential points $(1,15)/8$, $(0.00, 0.14)$, and $(1.86, 2.00)$, respectively. These points are the intersection points of the solid lines in the left part of Figure 2, the points marked with circles. All these points are Pareto optimal. In the right part of Figure 2 we also see that Walras' law, (P2), means that $F(p)$ is perpendicular to its argument p.

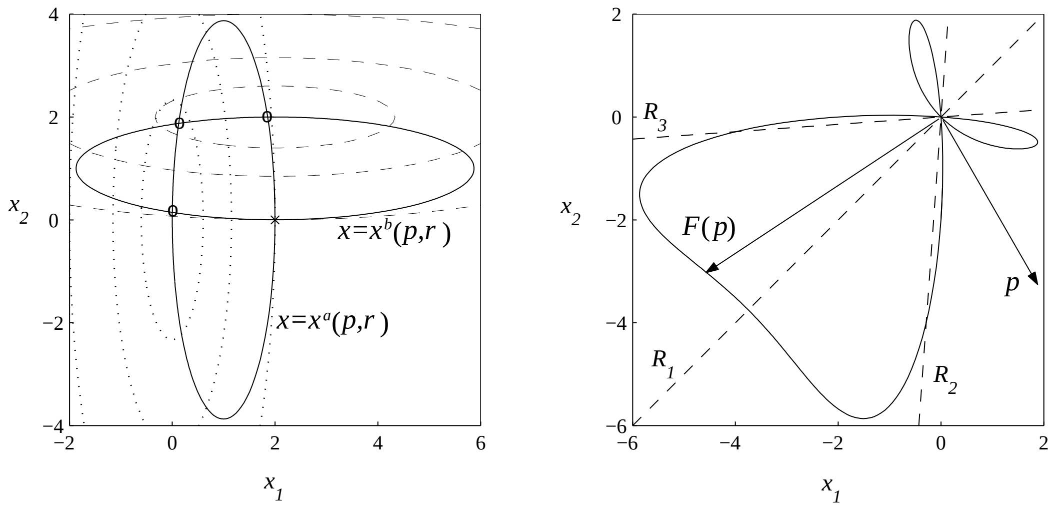

Figure 2. Illustration of $x^a(p,r)$, $x^b(p,r)$, and $F(p)$.

3 Pareto Optimality and the Choice of Reference Points

In this section I first show that under the assumptions (A1)–(A4) all the solutions of the mediator's problem (2) are Pareto optimal when the reference point is chosen from the line connecting the DMs' optima. I also show that all the Pareto points can be obtained by choosing the reference points in this manner.

Let us begin with showing in Lemma 1 that Pareto optimality can be characterized with jointly supporting hyperplanes when the value functions satisfy (A1)–(A4). See, e.g., Yu (1985, Section 3.4) for other Pareto optimality conditions. The following notation is adopted

$$H^{+}(p,r) = \{x \in \mathbb{R}^n : p \cdot (x - r) \geq 0\},$$
$$H^{-}(p,r) = \{x \in \mathbb{R}^n : p \cdot (x - r) \leq 0\}.$$

The proof of Lemma 1 is presented in the appendix.

Lemma 1. *Let the assumptions (A1)–(A4) hold. Then x^* is Pareto optimal if and only if there is $H(p,x^*)$ such that $S_a(x^*) \subset H^{+}(p,x^*)$ and $S_b(x^*) \subset H^{-}(p,x^*)$.*

The part (a) of the following theorem tells that the solutions of (2), if there are such, are Pareto optimal when the reference point is chosen from the line connecting the DMs' global optima. The part (b) of the theorem has been proven in Ehtamo et al. (1999a) in the case of differentiable quasiconcave utility functions. The meaning of this results is that all the Pareto points can be obtained by taking reference points from the line connecting the decision makers' global optima. Related results are also given by Heiskanen (2001) for strictly pseudoconcave utility functions. [1]

Theorem 1. *Let the assumptions (A1)–(A4) hold.*

(a) Let $r = \lambda \bar{x}^a + (1-\lambda)\bar{x}^b$, $\lambda \in [0,1]$. If $x^* \in X_a(p,r) \cap X_b(p,r)$, then x^* is Pareto optimal.
(b) If x^* is Pareto optimal, then there are $r = \lambda \bar{x}^a + (1-\lambda)\bar{x}^b$, $\lambda \in [0,1]$, and p such that $x^* \in X_a(p,r) \cap X_b(p,r)$.

Proof. Let us begin with the part (a). If $\lambda = 0$ or $\lambda = 1$ the result is obvious. Thus we may suppose that $\lambda \in (0,1)$. Let us first observe that under the quasiconcavity and the local nonsatiation of u_i, $x^* \in X_i(p,r)$ implies that $H(p,r)$ is tangential to $S_i(x^*)$. Namely, $S_i'(x^*) = \{x \in \mathbb{R}^n : u_i(x) > u_i(x^*)\}$ has to belong either to $H^{+}(p,r)$ or to $H^{-}(p,r)$. If this was not the case, the local nonsatiation would imply that there are x^1 and x^2 that belong to opposite half spaces and yield larger utility than x^*. Quasiconcavity then assures that points along the line $\lambda x^1 + (1-\lambda)x^2$ are better than x^*, too. Since this line goes through $H(p,r)$, the point x^* cannot be optimal on $H(p,r)$. As shown in

[1] Strictly pseudoconcave functions are strictly quasiconcave. Hence, Theorem 1 gives a more general result for two-party negotiations than the results of Heiskanen (2001).

the proof of Lemma 1 under (A4) we have $S_i(x^*) = \text{cl}S_i'(x^*)$ when x^* is not the global optimum of u_i. Here cl denotes the closure of a set. Hence $S_i(x^*)$, $i = a, b$, belong to the same halfspaces as $S_i'(x^*)$, $i = a, b$.

It follows from the choice of the reference point that the halfspaces containing $S_a(x^*)$ and $S_b(x^*)$ are opposite. For example, let us suppose that $S_a(x^*) \subset H^+(p, r)$, i.e., $p \cdot (x - r) \geq 0$ for all $x \in S_a(x^*)$. In particular we have $p \cdot (\bar{x}^a - r) \geq 0$. Observing that

$$\bar{x}^a - r = (1 - \lambda)(\bar{x}^a - \bar{x}^b) = (1 - \lambda)(r - \bar{x}^b)/\lambda,$$

it follows that $p \cdot (\bar{x}^b - r) \leq 0$. Then $S_a(x^*)$ and $S_b(x^*)$ belong to the opposite halfspaces and Pareto optimality follows from Lemma 1.

Let us now show the part (b). By Lemma 1, Pareto optimality means that there is a hyperplane $H(p, x^*)$ such that $S_a(x^*) \subset H^+(p, x^*)$, and $S_b(x^*) \subset H^-(p, x^*)$. An appropriate r is now obtained by taking the intersection of the line $\lambda\bar{x}^a + (1 - \lambda)\bar{x}^b$, $\lambda \in [0, 1]$, and $H(p, x^*)$ as the reference point. Namely, for such r we have $H(p, r) = H(p, x^*)$ and thus x^* would be optimal choice for both DMs under the constraint $x \in H(p, r)$. Hence, we need to show that there is such an intersection point. Let us denote

$$f(\lambda) = p \cdot [\lambda\bar{x}^a + (1 - \lambda)\bar{x}^b - x^*].$$

Because $S_i(x^*)$, $i = a, b$, are convex sets and $\bar{x}^i \in S_i(x^*)$, $i = a, b$, and because $S_a(x^*) \subset H^+(p, x^*)$ and $S_b(x^*) \subset H^-(p, x^*)$, we know that there are $\delta_1,\ \delta_2 \in [0, 1]$, $\delta_2 \leq \delta_1$, such that $f(\lambda) \geq 0$ for all $\lambda \in [\delta_1, 1]$, and $f(\lambda) \leq 0$ for all $\lambda \in [0, \delta_2]$. Clearly f is a continuous function, so that there is $\lambda^* \in [0, 1]$ such that $f(\lambda^*) = 0$. The result follows by taking $r = \lambda^*\bar{x}^a + (1 - \lambda^*)\bar{x}^b$. $\square$

Under assumption (A5), Theorem 1 implies that a Pareto point other than one of the global optima $\bar{x}^a$, $\bar{x}^b$ should have a reference point other than one of these optima. The result does not, however, guarantee that there is always a solution for the mediator's problem even though the reference point is chosen from the line connecting the DMs' optima.

Since any point on a given hyperplane can be taken as a new reference point defining the same hyperplane, the solutions of (2) can be Pareto optimal even though the reference point is not chosen from the line connecting the DMs' optima. It is also easy to find reference points such that at least some of the solutions of (2) fail to be Pareto optimal.

4 Existence of Solution

In this section I show that under the assumptions (A1), (A2), and (A5), the mediator's problem, i.e., equation (3) since (A5) holds, has a solution for any reference point. In Figure 2 we have an example where the system has three solutions, which are all Pareto optimal; the points market by circles in the left part of Figure 2.

Let us begin with a general existence result for $F(p) = 0$. The proof of the following lemma is given in Appendix and it is based on a fixed-point theorem according to which a continuous mapping from a unit sphere to itself has either a fixed-point or it maps some point to its antipode when n is odd, see Dugundji (1966, Corollary 3.3 in Chapter XIV).

Lemma 2. *Let the continuous mapping $F : \mathbb{R}^n \setminus \{0\} \mapsto \mathbb{R}^n$, $n \geq 2$, have the properties (P1) and (P2). Then $F(p) = 0$ has at least a ray of solutions.*

Recall that a ray of solution mean that if $F(p^*) = 0$ then $F(\alpha p^*) = 0$ for all $\alpha \neq 0$. Due to homogeneity of $x^a(p, r)$ and $x^b(p, r)$ with respect to their first argument there is at least a ray of solutions for (3) if there is one solution. Similar existence results as that given by Lemma 2 can be found in economics literature, where F is the excess demand function of an exchange economy. In that framework the solution is called competitive equilibrium. There is, however, a significant difference between the results on economic equilibria and the results of this chapter. Namely, for exchange economies vector p represents prices and they are assumed to be positive. Moreover, $\|F(p)\|$ becomes infinitely large when some components of p converge to zero. Because of these specific properties, the existence results for exchange economies are based on different deduction than the result of this section, see, e.g., Mas-Colell et al. (1995, Chapter 3). The relationship of exchange economies and the constraint proposal method is discussed in detail in Section 6.

To be able to use Lemma 2 for the mediator's problem we need to show that x^a and x^b are continuous with respect to $p \neq 0$. Lemma 3 gives this result when u_a and u_b are strictly quasiconcave functions with the property that the sets that are preferred to r are compact. These properties hold, e.g., for strictly concave functions that attain their optima. The main characteristics of the problem that guarantee the continuity of F are the continuity of the utility functions, the single valuedness of the optimal solutions due to strict quasiconcavity, and the continuity of the hyperplane constraint with respect to its normal. The proof of Lemma 3 is given in the appendix.

Lemma 3. *For $i = a, b$ let u_i be a strictly quasiconcave continuous function and let $S_i(r)$ be a compact set. Then $x^i(p, r)$ is continuous with respect to its first argument for all $p \neq 0$.*

When x^a and x^b are continuous with respect to p, then F is continuous, too. The following theorem tells essentially that for the constraint proposal method with two negotiators $F(p) = 0$ has a solution for any reference point. The theorem follows immediately from Lemma 2 and Lemma 3.

Theorem 2. *Let the assumptions (A2) and (A5) hold, and let $S_a(r)$ and $S_b(r)$ be compact sets. Then there is $p^* \neq 0$ such that $x^a(p^*, r) = x^b(p^*, r)$.*

Finally, let us notice that the existence result does not generalize as such to the multi-DM setting because of the structural differences of these problems with the two DM case. For example, in multi-DM setting equation (3) is not defined for linearly dependent parameter vectors, and therefore the resulting F is not continuous.

5 Adjustment of the Hyperplane Constraint

The basic idea of the constraint proposal method is that the mediator proposes the negotiators a hyperplane and asks their optimal points on the plane. If the points are significantly different the mediator updates the normal of the hyperplane with using the DMs' current and possibly other previous optimal choices. Ehtamo et al. (1999a) have suggested fixed-point iteration for updating the normal of the hyperplane constraint. The main advantage of this iteration is that the mediator can adjust the hyperplane on the basis of the DMs' optimal answers for given normal. For example, the derivatives of the mapping F need not be approximated. Although fixed-point iteration has been successfully applied by several authors dealing with the constraint proposal method, e.g., Ehtamo et al. (1999a), an explicit convergence proof is lacking. This section aims to remedy this matter.

In fixed-point iteration the normal p^k is updated in proportion to the value of F as follows:

$$p^{k+1} = p^k + \mu F(p^k), \tag{5}$$

where $\mu > 0$ is a fixed parameter. If the difference of two successive normals is small, then F is close to zero and an approximate solution has been found. Due to the properties (P1) and (P2) fixed-point iteration can also be applied to a normalized system, where one of the components of p is set to a non-zero constant and only the rest of the components are updated. I do not, however, consider the normalized procedure in this chapter because it is not clear whether it makes the process more stable or not. The results for the non-normalized process do not hold for the normalized one because Walras' law does not hold if one of the equations is dropped.

The following result on the convergence of fixed-point iteration is shown in

Lemma 4. *Let the continuous mapping* $F : B(p^*, \rho) \mapsto \mathbb{R}^n$, $\rho > 0$, *satisfy (P2) and the inequality*

$$p^* \cdot F(p) \geq \|F(p)\|^2 \tag{6}$$

for all $p \in B(p^*, \rho)$. *If* $p^0 \in B(p^*, \rho)$ *then (5) converges to a solution of (3). If (5) converges to a solution* $\tilde{p}$ *for which there is* $\alpha > 0$ *such that*

$$\|F(p)\|^2 \leq 2\alpha F(p) \cdot \tilde{p}$$

for all $p \in B(p^*, \rho)$, *then the convergence is monotonical.*

The above lemma assumes continuity, (P2), and condition (6). As shown earlier F is continuous when the utility functions are continuous and strictly quasiconcave. Hence, an additional property to obtain convergence is the inequality (6), which means geometrically that the hypersurface $\{x \in \mathbb{R}^n : x = F(p), p \neq 0\}$ curves enough at the origin. More specifically, (6) is equivalent to

$$\|p^*/2 - F(p)\| \leq \|p^*/2\|.$$

Thus, $F(p)$ is inside a ball centered at the ray defined by p^*. This is illustrated in Figure 3, where $F(p)$ is indeed inside a ball, represented with the dashed line, for p chosen from the vicinity of $p^*/2$.

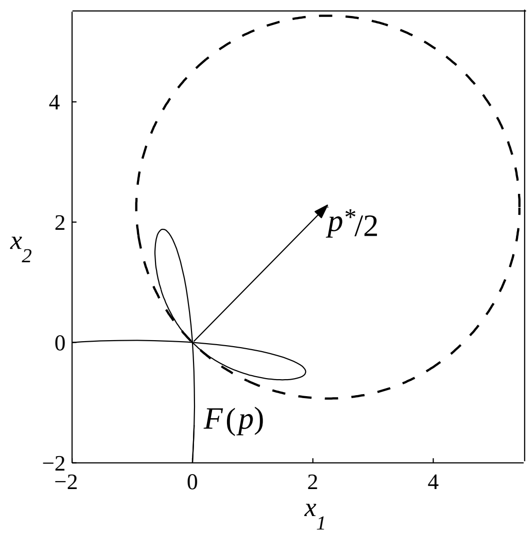

Figure 3. Illustration of convergence condition (6).

Unfortunately, the concavity assumptions do not imply (6) even though (6) seems to be a generic feature. Chapter II shows that when the parameterized hypersurface obtained from F has non-zero normal curvature to all its tangent directions, then (6) is is satisfied. This curvature condition can be formulated for the second derivatives of F as is done in Section 5.1. The above Lemma on the convergence of (5) gives a local convergence result. Nevertheless, numerical tests suggest that the iteration converges globally, i.e., for all initial normals. A possible explanation is that $\mathbb{R}^n$ can be divided into regions corresponding to different solution rays and in these regions (6) holds.

5.1 Convergence Test

In this section I give a more detailed characterization for condition (6) and derive a simple algebraic test for the convergence of (5). The test is based on examining whether the normal curvature of the hypersurface obtained from F is positive to all tangent directions.

Let us first define some basic concepts of differential geometry. Let us assume that the last component of p is equal to one, i.e., $p_n = 1$. Let us denote $p = (\bar{p}, 1)$, where $\bar{p} \in \mathbb{R}^{n-1}$, and set $\bar{F}(\bar{p}) = F(\bar{p}, 1)$. Mapping $\bar{F} : \mathbb{R}^{n-1} \mapsto \mathbb{R}^n$ is the parameterized hypersurface obtained from F. To define the normal curvature of $\bar{F}$ at $\bar{p}$ we need to assume that it is twice continuously differentiable and regular. Regularity means that the vectors $\nabla_1 \bar{F}(\bar{p}), \ldots, \nabla_{n-1} \bar{F}(\bar{p})$ are linearly independent. Here $\nabla_j \bar{F}(\bar{p})$ denotes the vector that is obtained by differentiating the component functions of $\bar{F}$ with respect to j'th argument. Furthermore, we let $N(\bar{p})$ denote the normal of the hypersurface at $\bar{p}$. It follows from Walras' law that $N(\bar{p}^*) = p^*/\|p^*\|$, when $F(p^*) = 0$ and $p^* = (\bar{p^*}, 1)$, see, e.g., Section 3.1 of Chapter II.

The normal curvature of $\bar{F}$ at $\bar{p}$ to a tangent direction $\bar{F}(\bar{p})d$, $d \neq 0$, is a function $\kappa(d; \bar{p}) = [(Ld) \cdot d]/\|d\|^2$ that depends on $N(\bar{p})$ and the second derivatives of $\bar{F}$. Here L denotes the matrix $L = AB^{-1}$, where $A = [N(\bar{p}) \cdot \nabla_{ij} \bar{F}(\bar{p})]_{i,j}$, $B = [\nabla_i \bar{F}(\bar{p}) \cdot \nabla_j \bar{F}(\bar{p})]_{i,j}$, and $\nabla_{i,j} \bar{F}$ denotes the vector obtained by differentiating the component functions of $\bar{F}$ with p_i and p_j. The notation $[a_{i,j}]_{i,j}$ for a matrix means that the component of the matrix in i'th row of j'th column is $a_{i,j}$. For the derivation of the formula for the normal curvature see, e.g., Spivak (1979, Section 7.C–D). It is shown in Lemma 4 of Chapter II that (6) holds around $p^* = (\bar{p}^*, 1)$ if $\kappa(d; \bar{p}^*) > 0$ for $d \neq 0$. Hence, we can formulate the following theorem.

Theorem 3. *Let F satisfy (P1) and (P2), and let $\bar{F}$ be regular and twice continuously differentiable. Then F satisfies (6) around $p^* = (\bar{p}^*, 1)$, $F(p^*) = 0$, if and only if $\bar{F}$ has positive normal curvature to all tangent directions at $\bar{p}^*$.*

We can use the result of Theorem 3 to derive an algebraic test for the convergence. Namely, if we find the minimal value of the normal curvature over the unit sphere we can see from its sign whether (6) holds. Indeed, the critical points of κ over the unit sphere correspond to so called principal curvatures. These critical points are exactly the eigenvectors of L and the eigenvalues are the principal curvatures. Hence, we can test (6) numerically by computing the eigenvalues of L. If these eigenvalues are positive we know that (6) holds for p^*, and if they are negative then (6) holds for $-p^*$. Notice, however, that this test requires that p^* is known.

For example, in the two-dimensional quadratic case of Section 2.1 we have the following positive principal curvatures corresponding to the three ray of solutions: $\kappa_1 = 16\sqrt{2}/45$, $\kappa_2 = \kappa_3 = \sqrt{14}/45$. Hence, condition (6) holds when p is chosen close enough to any of the the three solution rays illustrated in Figure 2.

6 Constraint Proposal Method and Exchange Economies

In this section I discuss the relationship of the constraint proposal method and exchange economies. In an exchange economy there is a number of consumers with initial allocations of some resources. Given prices for the resources each consumer is willing to buy a bundle that maximizes his utility under his budget, which is the monetary value of his initial bundle. The maximizing bundle is called the consumer's demand function. A vector of prices is an equilibrium if the total demand equals the total supply of the resources which is simply the sum of the initial allocations. Under some economic conditions the equilibrium prices can be found by a simple auctioning process, where an auctioneer adjusts the prices until an equilibrium is reached but no trades are made during the adjustment process. See, e.g., Mas-Colell et al. (1995, Part IV) for more about the basic properties of exchange economies.

The problem of finding a Pareto solution for the negotiation can be interpreted as a resource allocation problem, where the decision makers are sharing their total dispute $w = \bar{x}^a - \bar{x}^b$. The initial allocation of the total dispute is defined by the reference point $r = \lambda\bar{x}^a + (1-\lambda)\bar{x}^b$, $\lambda \in [0, 1]$; the proportion of the total dispute for the first DM is λ and $(1-\lambda)$ for the second DM. Moreover, decision maker i gets at the least value $u_i(r)$ as the outcome from the negotiation.

The constraint proposal method can be interpreted as an auctioning process, where the mediator acts as an auctioneer who tries to find a Pareto optimal allocation of the total dispute w. The relationship to resource allocation can be explicitly seen by making the following transform of variables: $y^a = x - \bar{x}^a$, $y^b = \bar{x}^b - x$. The DMs' optimization problems are then of the form

$$\max_{y^i} U_i(y^i) \text{ s.t. } p \cdot (y^i - \lambda_i w) = 0, \; i = a, b, \tag{7}$$

where $U_a(y^a) = u_a(y^a + \bar{x}^a)$, $U_b(y^b) = u_b(\bar{x}^b - y^b)$, and $\lambda_a + \lambda_b = 1$, $\lambda_i \in [0, 1]$ for $i = a, b$.

Let p denote a price vector of n-resources, which correspond to the issues, and let $\lambda_i w$ denote the initial endowment that the DM i has. Then we can interpret the linear constraint in (7) as a budget identity. Moreover, the point $y^i(p)$, $i = a, b$, that solves (7) is the DM's demand function for the resources, and $\sum_i (y^i(p) - \lambda_i w)$ is the excess demand of the resources. Similarly as $F(p)$, the excess demand satisfies Walras' law, which means now that the monetary value of the excess demand is zero. Homogeneity of the excess demand function means that only the relative prices of the resources matter.

As the above discussion demonstrates the mediator's problem in the constraint proposal method is remarkably close to the resource allocation problems of exchange economies. Indeed, the part (a) of Theorem 1 corresponds to the first fundamental welfare theorem in microeconomics and the part (b) corresponds to the second fundamental welfare theorem. According to the first fundamental theorem a price equilibrium is Pareto optimal and according to the latter there is a price equilibrium corresponding to a Pareto solution, see Mas-Colell et al. (1995).

There are some important differences between the constraint proposal method and exchange economies. In an exchange economy the demand functions are not defined if some of the prices are negative. Moreover, the demand for a resource usually grows infinitely large as its prices go to zero, i.e., the utility functions do not have global optima and (A1) does not hold. In the constraint proposal method p can have negative components as well and there is no reason to assume the DMs' responses to satisfy any boundary conditions for zero components of p.

Due to the aforementioned differences, the results for exchange economies are not applicable for the constraint proposal method. Pareto optimality results of Section 3 are also based on different assumptions than the welfare theorems for exchange economies.

7 Discussion

7.1 General Remarks

In this chapter I have analyzed the choice of the reference point in the constraint proposal method. I have shown that the method produces Pareto optimal points when the mediator chooses the reference point from the line connecting the DMs' optima and all the Pareto points can be produced in this manner. Moreover, I have proven that the mediator's problem has always a solution under reasonable assumptions on the DMs' utility functions. In essence, these results mean that the constraint proposal method is not just a heuristic approach for finding Pareto solutions, but it indeed gives Pareto optimal points.

To find a joint tangent hyperplane the mediator has to solve a system of equations. A suitable method for that purpose is fixed-point iteration, which requires only the DMs' last optima to update the hyperplane. This study gives local convergence conditions for fixed-point iteration and a numerical convergence test.

In addition to the aforementioned results, I have discussed the relationship of the constraint proposal method and exchange economies. I have shown that the mediator's problem in the constraint proposal method can be transformed to a resource allocation problem where the total resource to be shared is the difference of the DMs' optima. I have also pointed out some differences between the economic resource allocation model and the mediator's problem. Due to these differences the results for the constraint proposal method are based on different assumptions and techniques as those for exchange economies.

7.2 Ways of Using the Constraint Proposal Method

The constraint proposal method can be applied in a variety of ways. One way, as suggested by Ehtamo et al. (1999a), is to use the method for finding an approximation for the whole Pareto frontier. The negotiation then becomes distributive along the frontier. The method can also be used in a kind of "post-settlement settlement" fashion; this method was suggested by Raiffa (1982). First the parties negotiate unaided and reach a tentative solution point, not necessarily Pareto optimal, after which they search for a jointly beneficial Pareto optimal solution using one of the available methods, e.g., the constraint proposal method.

Yet, there is at least a third possible way of using the constraint proposal method. Namely, that of first bargaining on a suitable reference point for the method and then using it. I describe such a process briefly. In particular, the bargaining could be restricted to the reference points on the line connecting the DMs' optima. This problem is one dimensional since it is over the choice of parameter λ that defines a point $r(\lambda) = \lambda\bar{x}^a + (1-\lambda)\bar{x}^b$. Note, however, that the resulting utility points $v_i(\lambda) = u_i(r(\lambda))$, $i = 1, 2$, do not form a line in u_1, u_2-plane; rather they form a rough approximation of the Pareto frontier.

If the negotiation over the "approximate" Pareto frontier results in $r(\bar{\lambda})$, then the DM i is guaranteed to have at least the value $v_i(\bar{\lambda})$ after applying the constraint proposal method with this reference point. Indeed, the constraint proposal method applied with reference point $r(\lambda)$ gives a point that both

DMs prefer to it.

The negotiation over the reference points can be considered as a bargaining problem. For example, one may use the axiomatic approach to bargaining initiated by Nash (1950). Nash bargaining solution is obtained by maximizing the product

$$[v_a(\lambda) - d_a] \cdot [v_b(\lambda) - d_b], \tag{8}$$

where d_i is the threat point, which gives the value for the DM i if the bargaining fails. For example, we may take the threat point according to the worst case scenario, where the d_i is chosen to be the value at the other party's optimum, i.e., $d_a = v_a(0)$, $d_b = v_b(1)$. Even though it is not necessarily possible to give both DMs their worst case outcomes, these values can be taken as the threat points.

In practice, the bargaining solution $\bar{\lambda} \in [0, 1]$ can be found approximately by first eliciting the utility functions $v_a(\lambda)$ and $v_b(\lambda)$ within some accuracy. See von Winterfeldt and Edwards (1986, Section 7.3.) for methods of estimating utility functions, such as v_a and v_b, that depend on a single parameter. There is a plethora of efficient methods to perform this task. After having found the approximations of utility functions the bargaining solution can be computed numerically by maximizing (8).

Let us sum up the process of finding a single Pareto optimal point for the negotiation problem:

1. The reference point $r(\bar{\lambda})$ is chosen according to Nash bargaining solution, e.g., by a sequential bargaining process.
2. The mediator finds one solution for (2) with the reference point $r(\bar{\lambda})$ and suggest this point to the DMs.

Theorem 1 guarantees that the above procedure gives a Pareto optimal point if the mediator finds a solution for (2).

As an example, let us consider the same utility functions as in the two dimensional example of Section 2.1. We now obtain $v_a(\lambda) = -64(1-\lambda)^2$, $v_b(\lambda) = -64\lambda^2$, $d_a = v_a(0) = 0$, and $d_b = v_b(1) = 0$. The optimum of (8) is obtained at $\bar{\lambda} = 1/2$ and $r(\bar{\lambda}) = (1, 1)$, i.e., the reference point is chosen exactly from the middle of the DMs' optima. With this reference point the mediator's problem has three solutions giving the Pareto optimal points $(1, 15)/8$, $(1.74, 1.99)$, and $(0.22, 1.16)$, the latter two being approximate values.

Appendix: Proofs of the Lemmas

Proof of Lemma 1:

Let us denote $S_i'(x^*) = \{x \in \mathbb{R}^n : u_i(x^*) > u_i(x)\}$. To obtain the result we need the property that $S_i'(x^*) = \text{int}S_i(x^*)$ or equivalently $\text{cl}S_i'(x^*) = S_i(x^*)$ when x^* is not the global optimum of u_i. Here int stands for the interior and cl for the closure of a set. Note that $S_i(x^*)$ is a closed set and $S_i'(x^*)$ is an open set by the continuity of u_i. Moreover, $S_i'(x^*)$ is assumed to be a nonempty set, which will be the case because x^* belongs to a region $\{x \in \mathbb{R}^n : u_a(x) \geq u_a(\bar{x}^b),\ u_b(x) \geq u_b(\bar{x}^b),\ x \neq \bar{x}^i\}$ where u_i is locally nonsatiated. Let us denote $I_i(x^*) = \{x \in \mathbb{R}^n_+ : u_i(x) = u_i(x^*)\}$ and observe that $S_i(x^*) = S_i'(x^*) \cup I_i(x^*)$. By the continuity we have $\text{cl}S_i'(x^*) \subset S_i(x^*)$ and it only needs to be shown that $S_i(x^*) \subset \text{cl}S_i'(x^*)$, i.e., $I_i(x^*) \subset \text{cl}S_i'(x^*)$. Now take $x \in I_i(x^*)$ and note that $x \neq \bar{x}^i$ as $x^* \neq \bar{x}^i$. By (A4) we find for all $k = 1, 2, \ldots$ points x^k such that $u_i(x^k) > u_i(x)$ and $\|x^k - x\| \leq 1/k$. By this construction $x^k \to x$ which means that $x \in \text{cl}S_i'(x^*)$.

Let us first assume that $x^* = \bar{x}^a$, which is a Pareto optimal point, and show that there is a joint tangent hyperplane at this point. Note that the deduction is similar for $x^* = \bar{x}^b$. We have $S_a(x^*) = \{x^*\}$ and by the convexity of $S_b(x^*)$ and $x^* \in \partial S_b(x^*) = I_b(x^*)$ there is a hyperplane $H(p, x^*)$ such that $S_b(x^*) \subset H^-(p, x^*)$. Here ∂ denotes the boundary of a set. Because $x^* \in H(p, x^*)$ we see that $S_a(x^*) \subset H(p, x^*) \subset H^+(p, x^*)$.

Let us now show that there is a joint tangent hyperplane for a Pareto optimal point $x^* \neq \bar{x}^i$, $i = a, b$. Recall that such point belongs to $\{x \in \mathbb{R}^n : u_a(x) \geq u_a(\bar{x}^b),\ u_b(x) \geq u_b(\bar{x}^a)\}$. By the definition of Pareto optimality, we have

$$S_a'(x^*) \cap \mathrm{S}_b(x^*) = \emptyset \text{ and } S_a(x^*) \cap S_b'(x^*) = \emptyset, \tag{9}$$

since the other DM should be worse of at any other point that the other prefers to x^*.

Because $S_a'(x^*) \cap S_b(x^*) = \emptyset$, there is a hyperplane $H(p, x^*)$ such that $S_a'(x^*) \subset H^+(p, x^*)$ and $S_b(x^*) \subset H^-(p, x^*)$, see, e.g., Bazaraa et al. (1993, Theorem 2.4.8). Since the halfspaces are closed sets and $S_i(x^*) = \text{cl}S_i'(x^*)$ we have $S_a(x^*) \subset H^+(p, x^*)$ and hence there is joint tangent hyperplane at x^*

Let us now assume that there is a hyperplane $H(p, x^*)$ such that $S_a(x^*) \subset H^+(p, x^*)$ and $S_b(x^*) \subset H^-(p, x^*)$. The purpose is to show that x^* is Pareto optimal. We may assume that $x \neq \bar{x}$, $i = a, b$, since otherwise the result would be obvious. Let us assume that x^* was not Pareto optimal and at least one of the DMs prefer x to x^*. We may suppose that $x \in S_a'(x^*) \cap S_b(x^*)$, i.e, $u_a(x) > u_a(x^*)$ and $u_b(x) \geq u_b(x^*)$. Moreover, $S_a'(x^*) = \text{int}S_a(x^*)$ as shown above and therefore $S'_a(x^*) \subset \text{int}H^+(p, x^*) = \{x \in \mathbb{R}^n : p \cdot (x - x^*) > 0\}$. On the other hand $x \in H^-(p, x^*)$ by the separation of $S_a(x^*)$ and $S_b(x^*)$. Since $\text{int}H^+(p, x^*) \cap H^-(p, x^*) = \emptyset$ we have obtained contradiction. Thus x is Pareto optimal. □

Proof of Lemma 2: Let us define a mapping $G : \partial B(0, 1) \mapsto \partial B(0, 1)$ by setting

$$G(p) = \frac{p + F(p)}{(1 + \|F(p)\|^2)^{1/2}}.$$

It follows from (P2) that $\|G(p)\| = 1$ so that the image of $\partial B(0, 1)$ under G belongs to $\partial B(0, 1)$ itself.

For any $p \in \partial B(0, 1)$ the mapping is continuous since F is continuous. Thus, either G has a fixed point or it sends some point to its antipode when n is odd, which follows from a corollary of Poincaré-Brouwer theorem, see, e.g., Dugundji (1966, Corollary 3.3 in Chapter XIV). Let us assume that $n \geq 3$ is odd. Hence, there is $p^* \in \partial B(0, 1)$ such that $p^* = G(p^*)$ or $p^* = -G(p^*)$. By taking inner product of both sides of these equations with respect to $(1 + \|F(p^*)\|^2)^{1/2} F(p^*)$ and applying (P2) we get $\|F(p^*)\|^2 = 0$ or $-\|F(p^*)\|^2 = 0$, which implies that $F(p^*) = 0$.

Let us now assume that $n \geq 2$ is even. Then we can define a continuous mapping $\tilde{F} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with properties (P1) and (P2) as follows: $\tilde{F}(p, q) = (F(p), 0)$, where $p \in \mathbb{R}^n$ and $q \in \mathbb{R}$. As F is continuous, satisfies (P1) and (P2), $\tilde{F}$ has all these properties, too. Since $n + 1$ is odd, $\tilde{F}(p, q) = 0$ has a solution by the above deduction, i.e., there is $(p^*, q^*) \in \mathbb{R}^{n+1}$, with $\|p^*\|^2 + (q^*)^2 = 1$, such that $\tilde{F}(p^*, q^*) = 0$. Consequently, we have $F(p^*) = 0$. □

Proof of Lemma 3:

Let us first notice that

$$X_i(p, r) = \arg \max_{x \in \phi_i(p,r)} u_i(x), \; i = a, b,$$

where $\phi_i(p, r) = S_i(r) \cap H(p, r)$, i.e., the constraint $x \in H(p, r)$ can be replaced with $x \in \phi_i(p, r)$. This is because the maximization problem has a unique solution that belongs to $S_i(r)$, which is the set of points that are at least as good as r. Note also that ϕ_i is non-empty and because S_i is compact valued so is ϕ_i.

Let us first show that ϕ_i is lower hemicontinuous with respect to $p \neq 0$, i.e., $p^k \to \bar{p}$, $x^k \in \phi_i(p^k, r)$, imply that there is a subsequence $\{x^{k_j}\}_j$ such that $x^{k_j} \to \bar{x} \in \phi_i(\bar{p}, r)$. This is because due to compactness of $S_i(r)$ the sequence $\{x^k\}_k$ has a convergent subsequence and because $p \cdot (x - r)$ is a continuous function, the limit of the subsequence belongs to $\phi_i(\bar{p}, r)$. Second, ϕ_i is upper hemicontinuous with respect to $p \neq 0$, i.e., $p^k \to \bar{p}$, $\bar{x} \in \phi_i(\bar{p}, r)$, imply that there is a sequence $\{x^k\}_k$ with $x^k \in \phi_i(p^k, r)$ for all k such that $x^k \to \bar{x}$. Indeed, such a sequence can be constructed by setting $x^k = \arg\min_{x \in \phi_i(p^k, r)} \|x - \bar{x}\|$.

Because ϕ_i is both upper and lower hemicontinuous, it is continuous. By the Berge's theorem X_i is a closed and upper hemicontinuous set-valued mapping for $p \neq 0$, because it is the set of points that maximize a continuous function u_i over a compact-valued continuous mapping ϕ_i, see, e.g., Border (1985, Theorem 12.1). Strict quasiconcavity implies that X_i is a singleton, and as a single valued upper hemicontinuous mapping X_i is continuous, see Border (1985, Proposition 11.9 (d)). Hence, $x^i(p, r)$ is continuous with respect to its first argument when $p \neq 0$. □

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Chapter IV:

Adjustment of an Affine Contract with Fixed-Point Iteration

Abstract

This chapter studies a principal-agent game where the principal commits to an affine contract. The principal has incomplete information but he can adjust the contract according to the myopically behaving agent's reactions when the game is played repeatedly. The adjustment process can be considered as a learning model. I derive convergence conditions for fixed-point iteration as an adjustment scheme and study a related continuous time process. The analysis is based on parameterizing the problem such that we obtain a degree zero homogeneous system of equations, where the nonlinear mapping satisfies Walras' law.

Key words: principal-agent problems, contracts, fixed-point iteration, convergence, adjustment, learning

1 Introduction

In an adverse selection game there is a principal who's task is to design an optimal contract for a number of agents without knowing their preferences completely. An example of an adverse selection game is a nonlinear pricing model where the principal is a seller who designs a price-amount tariff under uncertainty on byers' preferences. For an overview on contract design with asymmetric information see, e.g., Macho-Stadler and Pérez-Castrillo (2001) and Salanié (1997).

There are two basic ways to tackle incomplete information in economic models. The first way is to characterize the equilibrium by assuming probability distributions over the incompletely known variables. The second way is to consider dynamic processes that lead to the complete information equilibrium even with limited information. The latter approach has long traditions in the stability analysis of economic equilibria, e.g., in the fields of oligopolistic markets and exchange economies. In the recent literature of game theory, reaching the equilibrium has been considered from the learning point of view, see Fudenberg and Levine (1999). A simple example of a learning scheme is the Cournot process in which the players use their best responses sequentially to their opponents' latest moves.

Adverse selection problems have been mostly studied in a static Bayesian framework and the question of reaching the complete information equilibrium has not raised much attention. See, however, Ehtamo et al. (2002) on adjusting a linear wage contract in a simple principal-agent setting by using a three-phase procedure. This study takes the stability approach to the contracting problem and shows that the complete information equilibrium, the principal's optimum, can be reached with a simple adjustment process beginning from any disequilibrium contract.

This chapter considers a two-player game, where the principal commits to a contract that is an affine mapping of the agent's actions. The principal has incomplete information but the game with the same players is played repeatedly so that the principal can adjust the contract according to the observations on the agent's behavior. Mathematically the problem is on finding linear equation constraints such that the agent's optimum under these constraints equals a predetermined point, the principal's optimum. I show how this complete information equilibrium can be reached by adjusting the contract with fixed-point iteration when supposing that the agent acts myopically, i.e., the agent does not consider any other future periods than the next one. One reason for this behavior could be that there is a large number similar agent's and each round a randomly chosen agent plays the game against the principal.

In the literature on repeated adverse selection models the focus has been on commitment and renegotiation issues and the analysis is usually Bayesian, i.e., the agent's utility function is assumed to be known except for one parameter over which there is a probability distribution. In a repeated game with this kind of parameterization there is no need for adjustment because the principal knows the agent's utility function completely after the first round. This study, however, is not considering long the term contracting problem, but the question is rather on the stability of the complete information equilibrium — one of the main questions for any economic equilibrium. The major differences to the usual Bayesian approach on adverse selection are that no type parameterization is assumed and the contracting problem may involve more than just two variables.

The main result on the convergence of the adjustment process is based on a new convergence theorem for fixed-point iteration. Namely, I show that fixed-point iteration converges when the system of equations is characterized by Walras' law and an additional condition that is remarkably close to the classical stability condition of Walrasian tâtonnement process given in Arrow et al. (1959). I will briefly discuss how the adjustment of an affine contract presented here is related to the stability of Walrasian equilibrium.

The contents of the chapter are as follows. Section 2 presents the principal-agent game with complete information and discuss the existence of solution for the principal's contract design problem. In Section 3 the game is parameterized such that the contract design problem can be formulated as a system of equations to be solved. Furthermore, I study the properties of the parameterized problem. The results of sections 2 and 3 are based on concavity properties of the agent's utility function.

Section 4 shows how fixed-point iteration can be used in adjusting an affine contract when the two-player game defined in sections 2 and 3 is played repeatedly and the principal is supposed to have incomplete information. In Section 5 I discuss the corresponding continuous time adjustment process and show the similarities of the contract design problem and price adjustment in exchange economies. In sections 4 and 5 it is assumed that the principal's optimum is such that the agent always participates the game when the contract passes through that point. In Section 6 I briefly characterize processes that work for finding the principal's optimum in the case where the agent does not always participate the game.

2 Contract Design Game and the Complete Information Solution

In this section I define the principal-agent game and derive conditions for the existence of an affine solution for a contract design game with complete information. Although principal-agent terminology is adopted the words principal and agent do not refer to any specific agency problem.

There are two utility maximizing players, a principal and an agent with utility functions v, $u : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}$, respectively. The principal's decision variable is $y \in \mathbb{R}^m$ and the agent's is $x \in \mathbb{R}^n$, $\| \cdot \|$ denotes the Euclidean norm in $\mathbb{R}^m$, $\mathbb{R}^n$, as well as in their product space.

In a game of incomplete information the principal does not know the agent's utility function. Usually it is assumed that the agent's utility function is determined by a type parameter $\theta \in \{\theta_1, \ldots, \theta_N\}$ that is unknown for the principal, who, however, has a probability distribution over the possible values of the parameter. The agent knows his own type. There are two interpretations for using type parameters. One is that there is a population of agents whose types are drawn from a probability distribution. Alternatively we could assume that there is only a single agent whose type is random from the principal's view. This study does not use type parameterization but the approach is close to the latter one because it will be assumes that the principal meets the same or similar agent when the game is played repeatedly.

In the contract design game the principal offers the agent a menu of contracts $\gamma_i(x)$, $i = 1, \ldots, N$, and commits to make his decision according to the contract that the agent chooses. The agent may also reject all the contracts in which case he obtains his reservation utility $\bar{u}$. After the agent chooses to sign a contract γ_i he makes a decision x' and the principal implements the contract, i.e., chooses the action $y' = \gamma_i(x')$, and the game ends. The principal's problem is to design the menu of contracts that maximizes his expected utility such that the agent obtains at least his reservation utility and chooses the contract intended for his type.

There is a wide variety of applications for contract design games. For example, the principal could be a seller who offers a buyer a price tariff that specifies the prices y of the goods for any amounts x to be bought. Nonlinear pricing with a multi-product monopoly has been studied, e.g., in Roberts (1979), Mirman and Sibley (1980), Spence (1980).

In this study the principal observes perfectly the agent's move. Furthermore, in this and the next section the principal is assumed to have complete information, that is to say he knows the agent's utility function or type. I show under mild technical assumptions that the principal gets his optimum with a single affine contract. Affine contracts are often considered as simple to monitor and implement. Compared to discontinuous contracts, e.g., various threshold contracts, affine contracts have the advantage that the risk of losses becomes low for small deviations from the principal's optimum. In the aforementioned buyer-seller game an affine price tariff could be implemented by specifying the unit prices plus possible fixed prices for each good to be sold.

The contract design problem is defined as follows. Find a contract $\gamma : \mathbb{R}^n \mapsto \mathbb{R}^m$ that maximizes $v(x_\gamma, \gamma(x_\gamma))$ over all feasible contract mappings γ, where x_γ solves

$$\max_x u(x, \gamma(x)) \tag{1}$$

and

$$u(x_\gamma, \gamma(x_\gamma)) \geq \bar{u}. \tag{2}$$

Equation (2) is called the agent's participation constraint. I next show that there is an affine contract of the form

$$\gamma(x) = y_0 + Lx, \tag{3}$$

where L is a linear mapping ($m \times n$ matrix) from $\mathbb{R}^n$ into $\mathbb{R}^m$, and y_0 is a fixed vector in $\mathbb{R}^m$, that solves the contract design problem.

Suppose that x^*, y^* solves

$$\max_{(x,y)\in D} \quad v(x, y),$$

where D is the set of points that satisfy the agent's participation constraint $u(x, y) \geq \bar{u}$. Because the pair x^*, y^* is the best outcome the principal can hope to get in the game, we can restrict our attention to those affine contracts that pass through x^*, y^*, i.e., $y^* = \gamma(x^*)$. Thus for any contract of the form (3) we should have $y_0 = y^* - Lx^*$. Hence, we can without loss of generality assume that the principal gives his contracts in the form

$$\gamma(x) = y^* + L(x - x^*). \tag{4}$$

Note that since the contract goes through x^*, y^*, which belongs to D, the agent will get at least his reservation utility when accepting the contract. Thus he will always participate the game. It is assumed that there are no constraints other than the participation constraint for the principal's and the agent's decisions.

The affine contract design problem can now be defined as follows. Find a contract of the form (4), i.e., find an $m \times n$ matrix L such that x^*, y^* solves

$$\max_{x,y} \quad u(x, y) \tag{5}$$

$$\text{s.t.} \quad y = y^* + L(x - x^*). \tag{6}$$

The contract γ that solves the above problem is called a contract at x^*, y^* and the pair (x^*, γ) is a Nash equilibrium for the game.

Let us suppose that the agent's objective function u is concave and let us denote the set of all subgradients of u at x^*, y^*, i.e., the subdifferential of u at x^*, y^*, by $\partial u(x^*, y^*)$. A subgradient at x^*, y^* means a pair $(\xi_x, \xi_y) \in \mathbb{R}^{n+m}$ that satisfies

$$u(x^*, y^*) - u(x, y) \geq \xi_x^T (x - x^*) + \xi_y^T (y - y^*) \ \ \forall (x, y) \in \mathbb{R}^{n+m},$$

where the superscript T denotes the transpose of a vector or a matrix. The subdifferential of a concave function u is a non-empty set and if the function is differentiable at x^*, y^* then the subdifferential is a singleton and equals the gradient of u at x^*, y^*.

For given L necessary and sufficient optimality condition for (5), (6) at x^*, y^* is that

$$\partial u(x^*, y^*) \cap \{(\xi_x, \xi_y) \in \mathbb{R}^{n+m} \mid \xi_x + L^T \xi_y = 0\} \neq \emptyset. \tag{7}$$

Geometrical interpretation of condition (7) is that there is a subgradient of u at x^*, y^* that is normal to the affine set defined by (6).

Now, suppose that $(\xi_x, \xi_y) \in \partial u(x^*, y^*)$ is such that $\xi_y \neq 0$. Then there is a contract of the form (6) at x^*, y^*. One possible L satisfying (7) is given by

$$L = -\xi_y \xi_x^T / \|\xi_y\|^2. \tag{8}$$

Usually there are also other affine contracts than the one defined by (8). Suppose, e.g., that the dimension of x equals the dimension of y, i.e., $m = n$, and suppose $(\xi_x, \xi_y) \in \partial u(x^*, y^*)$ is such that all components of ξ_y are nonzero. Then we can choose L to be a diagonal matrix. In a multi-product buyer-seller situation the corresponding tariff can be specified by giving a unit price for each good to be sold.

I collect the essential of the above discussion to the following.

Theorem 1 *If u is concave and has a subgradient ξ_x, ξ_y at x^*, y^* such that $\xi_y \neq 0$, then there is a solution to the affine contract design problem. Furthermore, a mapping of the form (4) is a contract at x^*, y^* if and only if the x, y points satisfying (6) belong to the hyperplane*

$$\{(x, y) \in \mathbb{R}^{n+m} \mid \xi_x^T (x - x^*) + \xi_y^T (y - y^*) = 0\} \tag{9}$$

for some $(\xi_x, \xi_y) \in \partial u(x^*, y^*)$.

The latter part of the theorem is equivalent to the necessary and sufficient optimality condition (7). Also note that if u is differentiable the requirement

$\xi_y \neq 0$ becomes $\nabla_y u(x^*, y^*) \neq 0$, which means that the agent's utility is sensitive for the changes of y around x^*, y^*.

The rather general formulation of the contract design problem in this way is inspired by some early papers in the field of control theory and differential games. Affine contract design problems, or affine incentive design problems as they are called in these papers, and their relation to incentive problems in economics is discussed in Ho et al. (1982). For mathematical analysis of affine incentive design problems in dynamic game settings of complete information see Başar (1984) and Ehtamo and Hämäläinen (1993).

3 Parameterization of the Problem

Theorem 1 suggests us to parameterize the principal's problem. Let us denote the subgradients of u appearing in (9) by parameter vectors p_x and p_y, and denote the column vector composed of p_x and p_y by p. The contract design problem can then be formulated as follows: Find $p \in \mathbb{R}^{n+m}$, $p_y \neq 0$, such that x^*, y^* solves

$$\begin{aligned} &\max_{x,y} \;\; u(x, y) \\ &\text{s.t.} \;\; y = y^* + L(p)(x - x^*), \end{aligned} \tag{10}$$

where the matrix $L(p)$ is chosen such that the contract defines an affine subset on the hyperplane

$$p_x^T(x - x^*) + p_y^T(y - y^*) = 0. \tag{11}$$

An appropriate parameterization for L is given by

$$L(p) = -p_y p_x^T / \|p_y\|^2, \tag{12}$$

for $p_y \neq 0$. Because the contract is chosen to satisfy (11), L becomes, regardless of its explicit form, degree zero homogeneous, i.e., $L(\alpha p) = L(p)$ for $\alpha \neq 0$. This is because αp defines the same hyperplane as p.

Let $S(p) \subset \mathbb{R}^{n+m}$ denote the set of solutions to (10) for given p. Then the contract design problem above is to find p so that

$$(x^*, y^*) \in S(p). \tag{13}$$

Theorem 1 gives conditions for the existence of a solution for (13), and due to the degree zero homogeneity of L, there is at least a ray of solutions if the conditions of Theorem 1 hold. One can always obtain a system with a unique solution from a homogeneous system that has a unique ray of solutions, e.g., by setting one of the components of p to a nonzero constant and dropping the corresponding equation from the system. However, here we do not have any need to do so. Furthermore, we do not necessarily have a unique ray of solutions.

Note that if $(x(p), y(p)) \in S(p)$, then $x(p)$ is the agent's reaction for the contract parameterized by p. Furthermore, as a solution set of a convex optimization problem, $S(p)$ if non-empty, is a convex set. The other properties of S are summarized in the following theorem, which readily follows from Theorem 6 and Corollary 1 presented in Appendix A.

Theorem 2 *If u is concave, L is continuous at p and $S(p) \neq \emptyset$, then the set-valued mapping S is closed at p. If u is strictly concave and D is compact, then S is single-valued and continuous at p, $p_y \neq 0$.*

Notice that the compactness of a level set, e.g., the set D, of a concave function is equivalent with the compactness of all the level sets, see, e.g., Corollary 8.7.1 in Rockafellar (1970). Obviously strongly concave functions satisfy conditions of Theorem 2. Strong concavity of u means that $-\partial u$ is a strongly monotone mapping, i.e., there is a constant $\sigma > 0$ such that

$$(\xi_{x_1} - \xi_{x_0})^T(x_0 - x_1) + (\xi_{y_1} - \xi_{y_0})^T(y_0 - y_1) \geq \sigma(\|x_1 - x_0\|^2 + \|y_1 - y_0\|^2)$$

for all (x_1, y_1),(x_0, y_0) and $(\xi_{x_i}, \xi_{y_i}) \in \partial u(x_i, y_i)$, $i = 0, 1$. If u is twice continuously differentiable, then strong concavity is equivalent with the negative semidefiniteness of $\nabla^2 u(x, y) + \sigma I$ for every x, y pair, where $\nabla^2 u$ denotes the Hessian of u with respect to x and y and I is an identity matrix. We shall see in the following section that strong concavity is essential for the convergence of fixed-point adjustment.

A two-dimensional example of points $x(p), y(p)$ with $L(p)$ defined by (12) is presented in Figure 1. In the figure the agent's optimum with a given p and the corresponding contract line (dashed line) is a point where the contract line is tangent to one of the contours (dotted lines) of u. The solid line represents the locus of all $x(p), y(p)$ points. In the figure K denotes the (negative) cone of solutions of (13). The opposite directions are solutions as well.

4 Adjustment with Fixed-Point Iteration

In a game of incomplete information the explicit form of u is unknown for the principal. As explained earlier, the principal-agent game with incomplete information is usually formulated as a Bayesian game by supposing that the principal knows the form of u except for one parameter and has a probability distribution over the possible values of that parameter.

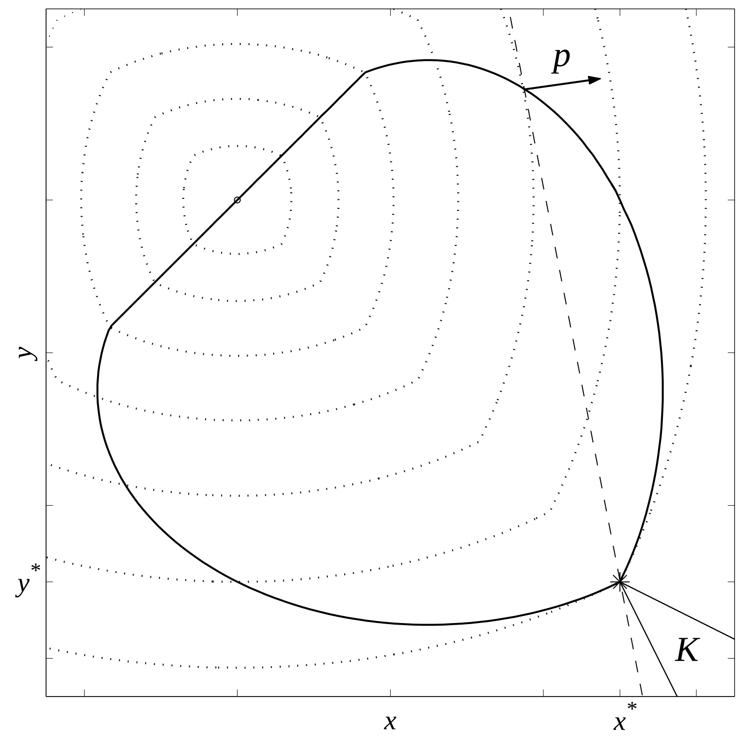

Figure 1. Two-dimensional illustration: the agent's contours and reactions.

Instead of Bayesian approach the game of incomplete information is played repeatedly and the principal can adjust the affine contract according to observations on the agent's actions, i.e., the principal is committed to a contract only for one period at a time and faces the same agent in each round. Note that when u is unknown to the principal he cannot necessarily have prior knowledge about the agent's participation constraint; hence he cannot use D when solving for (x^*, y^*) as in Section 2. We shall therefore assume here that the principal knows that his global optimum (x^*, y^*), solving $\max v(x, y)$ over $\mathbb{R}^{n+m}$, belongs to D. Thus the agent will always participate the game, recall the discussion in Section 2 below (4). For example, this can happen when it is common knowledge that the agent does not have a participation constraint at all. In Section 6 I study the general case where the principal does not know D but can find the best point in D through adjustments.

The basic idea of the adjustment approach is that the principal tries to find a solution p so that (13) holds. An appropriate method for this task is fixed-point iteration

$$p^{k+1} = p^k + \mu d(p^k), \tag{14}$$

where d denotes the mapping

$$d(p) = \begin{pmatrix} x(p) - x^* \\ y(p) - y^* \end{pmatrix},$$

$(x(p), y(p)) \in S(p)$, and $\mu \neq 0$ is a fixed parameter. The advantage of fixed-point iteration is that it can be implemented in a repeated game where the principal does not know u. This is because the agent's response $x(p)$ is sufficient information for updating p by (14). Note that $y(p)$ is defined through the affine contract given $x(p)$.

An interpretation for the above adjustment scheme is that it describes a learning model, where the iteration specifies the principal's learning rule. Furthermore, the agent is assumed to be myopic in the sense that he does not consider outcomes of other games than the current one. An explanation for myopic behavior is that the agent's discount factor is small compared to the speed at which the learning rule converges. Another argument for myopic learning comes from matching models where there are a great number of players and in each period the players match their strategies with different opponents. Since the same players are unlikely to meet anew they tend to play myopically. In the principal-agent game there could be a large number of similar agents and in each round one agent is chosen randomly to play the game.

4.1 Convergence Analysis

The convergence analysis of this section is based on properties of the parameterized problem (13). I show essentially that the strong concavity of the agent's utility function is required for the convergence of (14).

Because L is degree zero homogeneous, S and d are homogeneous, too. Moreover, $d(p)$ is perpendicular to p, i.e., $d(p)^T p = 0$, because the contract satisfies (11). This property is known as Walras' law and it generally holds for excess demand functions of exchange economies. In the following lemmas I give general convergence conditions and characterize the convergence properties of fixed-point iteration in a problem of finding a solution for a system of equations, where the nonlinear mapping satisfies Walras' law together with an additional condition. The proofs of the lemmas can be found in Appendix B.

Lemma 1 *Let the continuous mapping* $F : B(p^*, r) \mapsto \mathbb{R}^N$, $B(p^*, r) = \{p \in \mathbb{R}^N \mid \|p - p^*\| < r\}$, $r > 0$, *satisfy the following conditions:*

1. $F(p)^T p = 0 \ \forall p \in B(p^*, r)$,
2. $\|F(p)\|^2 \leq 2F(p)^T p^* \ \forall p \in B(p^*, r)$.

Then fixed-point iteration $p^{k+1} = p^k + \mu F(p^k)$, $\mu \neq 0$, *converges to a solution of* $F(p) = 0$ *when* $p^0 \in B(p^*, r)$. *Moreover* p^* *is a solution.*

Lemma 2 *Let conditions 1 and 2 of Lemma 1 hold for* F *and let fixed-point* *iteration* $p^{k+1} = p^k + \mu F(p^k)$ *converge to a solution* $\bar{p}$ *that satisfies*

$$\|F(p)\|^2 \le 2\alpha F(p)^T \bar{p} \quad \forall p \in B(p^*, r) \tag{15}$$

for some $\alpha > 0$*. Then the iteration converges monotonically, i.e.,* $\|p^{k+1} - \bar{p}\| < \|p^k - \bar{p}\|$*.*

The latter lemma tells that the convergence is monotonic if the sequence of parameters p converges to a solution, e.g., to p^* that satisfies the second condition in Lemma 1. This can be guaranteed in some specific cases as will be seen in the example of Section 4.2.

Using Lemmas 1 and 2 we obtain the following convergence theorem for the adjustment of an affine contract using fixed-point iteration.

Theorem 3 *If* u *is strongly concave and assumptions of Theorem 1 hold, then fixed-point iteration (14), for* $p_y^0 \neq 0$*, either converges to a solution of (13) or* $p_y^k = 0$ *for some* k*. If the iteration converges to a subgradient direction of* u *at* x^*, y^**, then it converges monotonically.*

Proof. As a strongly concave function u satisfies the assumptions of Theorem 2 and it follows that d is continuous when $p_y \neq 0$. Clearly condition 1 of Lemma 1 holds for all p with $p_y \neq 0$. Therefore we need to show only that condition 2 holds. Without loss of generality we can choose $\mu = 1$. Let $(x, y) \in S(p)$, $(\xi_{x^*}, \xi_{y^*}) \in \partial u(x^*, y^*)$ as in Theorem 1 and $(\xi_x, \xi_y) \in \partial u(x, y)$. From strong concavity we have

$$(\xi_x - \xi_{x^*})^T (x^* - x) + (\xi_y - \xi_{y^*})^T (y^* - y) \ge \sigma(\|x - x^*\|^2 + \|y - y^*\|^2), \tag{16}$$

where $\sigma > 0$. By plugging the contract in place of y we get from the second term on the left-hand side of (16)

$$\xi_y^T L(p)(x^* - x) - \xi_{y^*}^T (y^* - y),$$

and hence the left-hand side of (16) equals

$$(\xi_x + L(p)^T \xi_y)^T (x^* - x) - \xi_{x^*}^T (x^* - x) - \xi_{y^*}^T (y - y^*).$$

From the optimality condition it follows that $\xi_x + L(p)^T \xi_y = 0$; hence the left-hand side of (16) is equal to

$$\xi_{x^*}^T (x - x^*) + \xi_{y^*}^T (y - y^*) = d(p)^T p^*$$

where $p^* \neq 0$ denotes a vector that is composed of ξ_{x^*} and ξ_{y^*}. Note that the right-hand side of (16) is equal to $\sigma \|d(p)\|^2$. Thus, condition 2 holds for d when $p_y \neq 0$. It follows from Lemma 1 that if $p_y^k \neq 0 \; \forall k = 1, 2, \ldots$, then the iteration converges. Otherwise $p_y^k = 0$ for some k. Because condition 2 holds when p^* is any subgradient direction, Lemma 2 implies monotonic convergence. □

Theorem 3 shows that fixed-point iteration is an appropriate method to compute the solution for the contract design problem. Moreover, in a repeated game the iteration describes a convergent learning rule for the principal and we may say that the equilibrium of the game is stable. Note that it is obvious from Lemma 1 and Theorem 3 that when the initial parameter vector p^0 is chosen close enough to a solution the iteration does not stall.

4.2 Example

The purpose of this example is to illustrate the geometrical ideas of the convergence analysis. Let us assume that the agent's utility function is

$$u(x, y) = \min\{-x^2/2 - y^2, -x^2 - y^2/2\},$$

and the principal's optimum is achieved at $x^* = 1$, $y^* = -1$. As a minimum of two strongly concave functions u is also strongly concave but it is not differentiable for all x, y. Contours of u are illustrated in Figure 1. Because the example is only two-dimensional (11) defines the contract uniquely by (12).

The graph of S consists of four parts given below:

	(p_x, p_y)-region	Image under S
I:	$1/2 \leq p_x/p_y \leq 2$	$y = x, \ -1/3 \leq x \leq 1/3$
II:	$\|p_x/p_y\| \leq 1/2$	$(y + 1/2)^2 + (x - 1/2)^2/2 = 3/8,$
		$x \leq 1, \ y \leq -1/3$
III:	$2 \leq \|p_x/p_y\|$	$(y + 1/2)^2/2 + (x - 1/2)^2 = 3/8,$
		$1/3 \leq x, \ -1 \leq y, x \neq 1, y \neq 0$
IV:	$-2 \leq p_x/p_y \leq -1/2$	$x^*, y^*.$

The first three parts in x, y-plane are marked in Figure 2. Part IV is the point $(1, -1)$. Notice that S and d are not defined when $p_y = 0$ and therefore there is a discontinuity at $x = 1$, $y = 0$. The discontinuity is not, however, illustrated in the figure.

The first convergence condition of Lemma 1, namely that $d(p)$ is perpendicular to p, was explained in the second paragraph of Section 4.1 and it is illustrated in Figure 1. The geometric interpretation of the second convergence condition

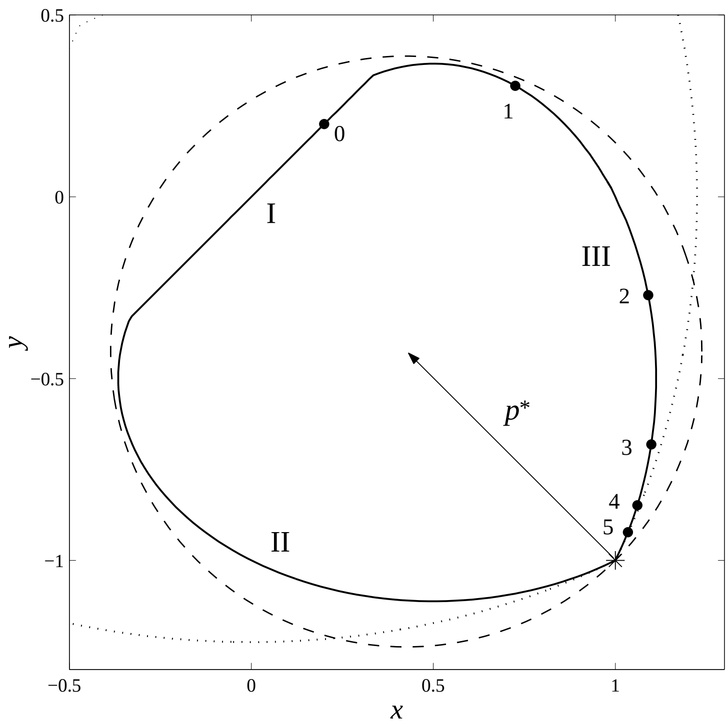

Figure 2. Two-dimensional illustration of the second convergence condition and an iteration.

in Lemma 1 for d is that the image of $\{(p_x, p_y) \in \mathbb{R}^{n+m} \mid p_y \neq 0\}$ under d is contained in a ball. This can be seen by writing the condition in an equivalent form

$$\|d(p) - p^*\|^2 \leq \|p^*\|^2,$$

i.e., the ball is centered at p^* and has radius $\|p^*\|$. This condition is now satisfied and the dashed line in Figure 2 illustrates one appropriate ball when the origin is transformed to $(1, -1)$.

In general, strong concavity of u implies that the level set $\{(x, y) \in \mathbb{R}^{n+m} \mid u(x, y) \geq u(x^*, y^*)\}$ is contained in a sufficiently large ball that goes through x^*, y^*. The region inside the dotted line in Figure 2 belongs to the aforementioned level set. Because the points $(x(p), y(p))$ are inside the level set and $d(p)$ is the difference vector of (x^*, y^*) and this point, d is inside the ball obtained from the one that contains the level set by transforming the origin to x^*, y^*. Notice that the condition 2 of Lemma 1 has both global and local interpretation. Globally the condition implies that $d(p)$ belongs to a compact set and in the vicinity of p^* it means that d is not too flat.

In this example y is one-dimensional and therefore all the solutions of (13) are subgradient directions. Hence, fixed-point iteration converges monotonically, see Theorem 3. The first six (x, y) points of an iteration with $\mu = 1$ and initial parameter vector $p^0 = [3 \;\; 2]^T$ are illustrated with numbered dots in Figure 2.

5 Continuous Time Process

In this section I focus on a continuous time process for adjusting the affine contract. Continuous time approach is commonly used in the stability analysis of Walrasian equilibrium and it turns out that the famous stability result by Arrow et al. (1959) is related to the adjustment of an affine contract.

In this section $p(t)$ denotes the parameter vector at time t and we suppose that p is differentiable with respect to t and denote its derivative by $\dot{p}$. If we set $p(t^k) = p^k$, $p(t^{k+1}) = p^{k+1}$ and assume that $\mu = t^{k+1} - t^k$, we get the process

$$\dot{p} = d(p) \tag{17}$$

as a limit from (14) when $\mu \to 0$. This process can not be implemented in a repeated game but it works as an idealization for the discrete time process where the principal reacts arbitrarily fast. Hence, the process describes a continuous time learning model.

The following lemma gives convergence conditions for a continuous time adjustment process for a system that satisfies Walras' law. The lemma is a modification of the stability theorem for Walrasian equilibrium by Arrow et al. (1959). The formulation and proof follow the presentation of Theorem 3.E.1 in Takayama (1974) with the difference that there is no unique ray of solutions. A similar result for exchange economies with multiple equilibria is given in Arrow and Hurwicz (1960).

Notice that in Lemma 3 we need to assume existence of a solution for the equation, which was not assumed in Lemma 1. However, the condition that is required in addition to Walras' law is less stringent than condition 2 of Lemma 1. The proof is presented in Appendix B.

Lemma 3 *Let $K \subset \Omega$, $K \neq \emptyset$ be the set of solutions of $F(p) = 0$ and let the continuous mapping $F : \Omega \mapsto \mathbb{R}^N$, $\Omega \subset \mathbb{R}^N$, satisfy Walras' law and*

$$F(p)^T p^* > 0 \quad \forall p \in \Omega \setminus K \tag{18}$$

for some $p^* \in K$. Then the process $\dot{p} = F(p)$ converges monotonically to a solution of $F(p) = 0$ when $p(0) \in B(p^*, r) \subset \Omega$.

The geometric interpretation of (18) is that points $F(p)$, $p \in \Omega \setminus K$, are on the half-space defined by the hyperplane with normal p^*. Clearly F has this property if it satisfies condition 2 of Lemma 1. Using Lemma 3 I can prove the following convergence theorem for the continuous time adjustment process (17).

Theorem 4 *If assumptions of Theorem 1 and Theorem 2 hold, then the* *process (17), for* $p_y(0) \neq 0$*, either converges to a solution or stalls at a point in which* $p_y = 0$.

Proof. From Theorem 1 it follows that the set of solutions K of $d(p) = 0$, is non-empty and clearly condition 1 of Lemma 1 holds. Moreover, from Theorem 2 we know that d is continuous when $p_y \neq 0$. Now let us suppose that p is not a solution of (13) and let $(x, y) \in S(p)$. Let $(\xi_{x^*}, \xi_{y^*}) \in \partial u(x^*, y^*)$ as in Theorem 1 and $(\xi_x, \xi_y) \in \partial u(x, y)$. From strict concavity of u we get

$$\begin{aligned}
&(\xi_x - \xi_{x^*})^T(x^* - x) + (\xi_y - \xi_{y^*})^T(y^* - y) = \\
&(\xi_x + L(p)^T \xi_y)^T(x^* - x) - \xi_{x^*}^T(x^* - x) - \xi_{y^*}^T(y^* - y) = \\
&\xi_{x^*}^T(x - x^*) + \xi_{y^*}^T(y - y^*) = d(p)^T p^* > 0,
\end{aligned}$$

where $p^* \neq 0$ is composed of ξ_{x^*} and ξ_{y^*}. Hence the conditions of Lemma 3 are satisfied for $F = d$ with $\Omega = \mathbb{R}^{n+m} \setminus \{p \in \mathbb{R}^{n+m} \mid p_y \neq 0\}$. If we get during the process a point $p(t)$ such that $p_y(t) = 0$, the iteration stalls since d is not defined at such a point.

□

Compared to Theorem 3, the convergence conditions in Theorem 4 are weaker because instead of strong concavity only strict concavity is required. Similarly as for the discrete time adjustment, the process does not stall when initial parameter vector $p(0)$ is chosen close enough to a solution.

The essential properties of d are its homogeneity and that it satisfies Walras' law and these properties are also typical for excess demand functions of exchange economies. According to the Sonnenschein-Mantel-Debreu theorem Sonnenschein (1973), Mantel (1974), Debreu (1974) any continuous function that satisfies Walras' law for $p \geq 0$ is an excess demand function for some economy.

Our two-player model is, indeed, similar to an exchange economy with only one consumer and a Walrasian auctioneer. This can be seen from (10) and (11) by making the following alternative interpretation. The variables x and y represent the amounts of commodities, p_x and p_y are the corresponding prices, (x^*, y^*) is the consumer's initial bundle, (11) gives a budget constraint for the consumer, and $x(p), y(p)$ are the amounts that the consumer is willing to buy for given prices. Furthermore, process (17) describes a price adjustment scheme that drives the excess demand of commodities, $d(p)$, to zero. The principal acts as an auctioneer who sets the prices according to the excess demand but no trade occurs until an equilibrium is reached, i.e., (17) is a tâtonnement process.

The analog to exchange economies leads us to ask whether fixed-point iteration works in finding a Walrasian equilibrium. The convergence conditions for fixed-point iteration are, however, more stringent than those required for the convergence of the continuous time process. Particularly the second condition of Lemma 1 does not necessarily hold for excess demand functions of exchange economies. The reason for this is that when d is an excess demand mapping it usually has the following property: when price of some commodity goes to zero, the excess demand for that commodity grows infinitely large given that the other prices are fixed. Therefore, d cannot be inside any ball and condition 2 of Lemma 1 does not hold. Though we cannot generalize the convergence result as such to exchange economies, it is possible to modify the iteration to obtain a convergent tâtonnement process as is shown in Chapter II.

6 Processes for Finding the Principal's Optimum

So far I have assumed that the agent will always participate the game during the process, i.e., the principal's global optimum (x^*, y^*) belongs to D, recall the discussion in the beginning of Section 4. If (x^*, y^*) does not belong to D, the agent does not necessarily participate the game. In this case the contract design problem is to find the optimal point for v over D and a corresponding contract at that point. In this section I characterize procedures for finding the principal's optimum assuming that a contract can be found at any given $(x^*, y^*) \in D$, e.g., with the adjustment process described previously in this chapter.

Let $w = (\bar{x}, \bar{y}) \in \mathbb{R}^{n+m}$, I call w a reference point, and let $S(p, w)$ denote the set of solutions to

$$\begin{aligned} \max_{x,y} \quad & u(x, y) \\ \text{s.t.} \quad & y = \bar{y} + L(p)(x - \bar{x}), \end{aligned} \tag{19}$$

for given p and w, where $L(p)$ is as in (10). Now the contract design problem is to find w^* such that $v(w^*) = \max_{w \in D} v(w)$ and $p^* \in G(w^*)$, where $G(w) = \{p \mid w \in S(p, w)\}$. We assume now that given any $w \in D$, a parameter vector p that defines the contract at w, i.e., $p \in G(w)$, can be found. Namely, given that u is strongly concave, and for all $w \in D$ there is $\xi_y \neq 0$, $(\xi_x, \xi_y) \in \partial u(w)$, the convergence result of Theorem 3 holds for any reference point $w \in D$. In addition, I make the following assumptions for the adjustment of reference point w:

1. The reference point $w^k \in D$ is updated only if a strictly better point $w^{k+1} \in D$ can be found.
2. If it holds that $w^k \in D$ and $w^k \neq w^*$, then $w^{k+1} \in D$ such that $v(w^{k+1}) > v(w^k)$ can be found.

The first assumption is natural because there is no reason for the principal to expect less utility from the future rounds than from the previous ones. Furthermore, it is rather easy to generate better reference points, so that the second assumption is reasonable, too. For example when the current reference point is in the interior of D, one can find a better reference point that is also in the interior of D by taking an appropriate step to an improving direction of v at the current reference point. It is also worth noticing that if the reference point is taken outside of D, an affine contract going through that point will usually have points in common with D. Nevertheless, the agent will reject the contract at some stage of the updating procedure. In that case some of the agent's previous choices can be taken as a new reference point.

We further need the concept of complete process. We say that a process is incomplete if the sequence of reference points converges to $\bar{w}$ but there is a non-trivial process that starts from $\bar{w}$ and satisfies assumptions 1 and 2. By a non-trivial process we mean a sequence $\{\bar{w}^k\}_k$ generated by the process with $\bar{w}^0 = \bar{w}$, such that $\bar{w}^k \neq \bar{w}$ at least for some k. Process that is not incomplete is called complete. Similar concept of a complete process in the price adjustment framework has been used in Smale (1976).

The following theorem shows that a complete adjustment processes, which satisfies the above assumptions, converges and there is a subsequence of contracts converging to the solution of the contract design problem.

Theorem 5 *When v is strictly concave, then for any complete process, which begins from $w^0 \in D$ and satisfies assumptions 1 and 2, the sequence of reference points converges to w^*. Furthermore, when u is strictly concave, and for all $w^k \in D$ there is $(\xi_x^k, \xi_y^k) \in \partial u(w^k)$ such that $\xi_y^k \neq 0$, then there is a subsequence of parameters $p^k \in G(w^k)$ converging to $p^* \in G(w^*)$ given that $G(w^*) \neq \emptyset$.*

Proof. Let us first observe that due to assumption 2 and the assumption that $w^0 \in D$ the sequence of reference points $\{w^k\}_k$, obtained during the process, belongs to D. Let us now show that this sequence converges to the principal's optimum over D, i.e., to point w^*. There are two possibilities: either $w^k = w^*$ when $k > N$, or for every k we have $v(w^{k+1}) > v(w^k)$. In both cases v is a Lyapunov function for the subsequence. Hence the subsequence converges and the limit is in D, which is a closed set because u is continuous. Notice also that v is strictly concave and w^* is its unique maximizer over D, so that it is an appropriate Lyapunov function. From the completeness assumption it follows that the limit is w^*.

It follows from the continuity of S, see Corollary 1, that $G(w)$ is a closed mapping. Because of homogeneity of S we may choose a bounded sequence $\{p^k\}_k$, $p^k \in G(w^k)$, where $w^k \in D$, e.g., we may set $\|p^k\| = 1$. This sequence has a convergent subsequence and from the closedness of G it follows that the limit is in $G(w^*)$ when $G(w^*) \neq \emptyset$. Hence we have the result. □

In view of Theorem 5, iteration (14) can be started using any reference point $w \in D$, and as I discussed it is rather easy to generate reference points. When during an iteration a point $(x(p), y(p))$ is encountered, giving the principal a better outcome than the current reference point, it can be taken as a new reference point in iteration (14), which can be continued from the current parameter vector.

7 Conclusion

In this chapter I have presented a new adjustment approach for an affine contract design problem. When a principal-agent game with incomplete information is played repeatedly, the principal can adjust his contract according to agent's previous move. The adjustment procedure is based on parameterizing the problem appropriately and updating the parameters with fixed-point iteration.

The parameterization of the contract design problem results to a degree zero homogeneous system of equations, where the mapping satisfies Walras' law. I showed that the iteration converges when an additional condition, the condition 2 of Lemma 1, holds for the system. As a result I obtained a convergence result for a principal-agent game where the agent has a strongly concave utility function. In addition to fixed-point iteration I have studied a related continuous time adjustment process.

The idea of using linear constraints in coordinating decision makers to a desired outcome has been used in the context of Walrasian tâtonnement and recently in negotiation analysis. In Ehtamo et al. (1999) a method of finding a Pareto-optimal solution for a two-party negotiation is formulated as a problem of searching for a joint tangent hyperplane for the parties utility functions. The problem results to a degree zero homogeneous system of equations that satisfies Walras' law. The search of the joint tangent hyperplane is done interactively between a mediator, using the method, and the parties. In this framework fixed-point iteration has been used successfully in numerical experiments. As chapters II and III show, the convergence results presented in this chapter are useful for adjustment of hyperplane constraints in finding Pareto optimal solutions or more generally adjustment of linear budget constraints for exchange economies.

A Continuity Properties of the Optimal Set Mapping

Here $S(p, w) \subset \mathbb{R}^{n+m}$ denotes the set of solutions for (19) for given p and $w = (\bar{x}, \bar{y})$. The following theorem characterizes the continuity of S with respect to p and w.

Theorem 6 *If u is concave, $L(p)$ is continuous at p and $S(p, w) \neq \emptyset$, then the set-valued mapping S is closed at (p, w).*

Proof. Let us assume that $S(p, w) \neq \emptyset$, $p^k \to p$, $w^k \to w$ and $(x^k, y^k) \to (x, y)$, where $(x^k, y^k) \in S(p^k, w^k) \neq \emptyset$. We denote the set of feasible points of problem (19), i.e., the set of points satisfying the linear contract, with $C(p, w)$ and the normal cone of the feasible set, $\{(\xi_x, \xi_y) \in \mathbb{R}^{n+m} \mid \xi_x + L(p)^T \xi_y = 0\}$, with $N(p)$. Let us first note that $C(p, w)$ is a closed mapping with respect to (p, w), because L is continuous.

According to sufficient optimality conditions $(x^k, y^k) \in S(p^k, w^k)$ if and only if $(x^k, y^k) \in C(p^k, w^k)$ and

$$\partial u(x^k, y^k) \cap N(p^k) \neq \emptyset.$$

From continuity of L it follows that N is a closed mapping. Concavity of u implies upper hemi-continuity of ∂u. Moreover,

$$\cup_{(x,y) \in \{(x^k, y^k)\}_k} \partial u(x, y)$$

is bounded, see, e.g., Prop. 6.2.1 and 6.2.2 in Section 6 of Hiriart-Urruty and Lemaréchal (1993). Hence, there is a convergent sequence $\{\xi_i\}_i$ such that

$$\xi_i \in \partial u(x^{k_i}, y^{k_i}) \cap N(p^{k_i}).$$

It follows that $\lim_{i \to \infty} \xi_i \in \partial u(x, y) \cap N(p)$. Because C is a closed mapping we have $(x, y) \in C(p, w)$. Thus, sufficient optimality conditions are satisfied and $(x, y) \in S(p, w)$.

□

Corollary 1 *If u is strictly concave with compact level sets and $L(p)$ is continuous at p, then S is single-valued and continuous at (p, w), $w \in D$.*

Proof. From the compactness of the level sets we know that D is compact and $S(p, w) \neq \emptyset$. The latter follows from Weierstrass theorem. Strict concavity of u implies that $S(p, w)$ is a singleton. Furthermore, since D is compact there is $\bar{w} = (\bar{x}, \bar{y})$ such that $u(\bar{x}, \bar{y}) = \max_{(x,y) \in D} u(x, y)$ and clearly $S(p, w) \subset \{(x, y) \in \mathbb{R}^{n+m} \mid u(x, y) \geq u(\bar{x}, \bar{y})\}$ so that S is a closed mapping into a compact set. Hence S is upper hemi-continuous, see, e.g., Prop. 11.9 (c) in Border (1985).

Continuity follows from upper hemi-continuity and single-valuedness, see prop 11.9 (d) in Border (1985).

□

B Proofs of the Lemmas

Proof of Lemma 1: If function F satisfies the conditions 1 and 2, then they hold also for μF with $\mu \neq 0$. Hence, without loss of generality we can prove the convergence with $\mu = 1$. Let p^* be as in condition 2, then

$$\begin{aligned}&\|p^{k+1} - p^*\|^2 = \|p^k + F(p^k) - p^*\|^2 = \\ &\|F(p^k)\|^2 - 2F(p^k)^T p^* + \|p^k - p^*\|^2 \leq \|p^k - p^*\|^2.\end{aligned}$$

Note that $p^k \in B(p^*, r)\ \forall k = 0, 1, \ldots,$ when $p^0 \in B(p^*, r)$. Therefore the sequence $\{\|p^k - p^*\|\}_k$ converges and the sequence $\{\|p^k\|\}_k$ is bounded. From condition 1 it follows that

$$\|p^k\|^2 = \|p^0\|^2 + \sum_{i=0}^{k-1} \|F(p^i)\|^2,$$

so that $\{\|p^k\|\}_k$ is a growing and bounded sequence and hence convergent. From the iteration formula we have

$$p^k = p^0 + \sum_{i=0}^{k-1} F(p^i).$$

Hence $\|p^0 + \sum_{i=0}^{k-1} F(p^i)\|$ converges, too. From triangular inequality we get

$$\|p^0 + \sum_{i=0}^{k+l} F(p^i)\| \geq \left| \|p^0 + \sum_{i=0}^{k} F(p^i)\| - \|\sum_{i=k+1}^{k+l} F(p^i)\| \right|$$

and it follows that

$$\|p^{k+l} - p^k\| = \|\sum_{i=k+1}^{k+l} F(p^i)\| \to 0, \qquad \text{(B.1)}$$

when $k \to \infty$ and $l \geq 1$. Thus $\{p^k\}_k$ is a Cauchy sequence and hence convergent; let $\bar{p}$ denote its limit point. Moreover, from (B.1) we get by setting $l = 1$ that $\|F(p^k)\| \to 0$, and from the continuity of F we have $F(\bar{p}) = 0$.

We can construct a sequence of solutions converging to p^* by taking neighborhoods $B(p^*, r^k)$ with $r \geq r^0 > r^1 > ... > r^k \to 0$. There is a solution $\bar{p}^k$ in each of these neighborhoods, and $\bar{p}^k \to p^*$ since $r^k \to 0$. From the continuity of F we have $F(p^*) = 0$. $\square$

Proof of Lemma 2: If (15) holds for $\alpha > 0$ then it holds for any $\bar{\alpha} > \alpha$. Specifically, we can choose $\bar{\alpha} > 0$ such that (15) holds for $p^* = \bar{\alpha}\bar{p} - 2\bar{p}$ instead of $\alpha\bar{p}$. Moreover we can take α such that $\|F(p)\|^2 < 2\alpha F(p)^T\bar{p}$ if p is not a solution. Similarly as in Lemma 1 we can deduce that $\|p^{k+1} - p^*\|^2 < \|p^k - p^*\|^2$, and $\|p^{k+1} - \alpha\bar{p}\|^2 < \|p^k - \alpha\bar{p}\|^2$ when p^k is not a solution. From parallelogram law we get

$$\|p^k - \alpha\bar{p}\|^2 + \|p^k - p^*\|^2 = 2\|p^k - \bar{p}\|^2 + 2|1 - \alpha| \cdot \|\bar{p}\|^2.$$

By rearranging the terms we have

$$\begin{aligned} 2\|p^k - \bar{p}\|^2 &= \|p^k - \alpha\bar{p}\|^2 + \|p^k - p^*\|^2 - 2|\alpha - 1| \cdot \|\bar{p}\|^2 \\ &> \|p^{k+1} - \alpha\bar{p}\|^2 + \|p^{k+1} - p^*\|^2 - 2|\alpha - 1| \cdot \|\bar{p}\|^2 = 2\|p^{k+1} - \bar{p}\|, \end{aligned}$$

and hence $\{p^k\}_k$ converges monotonically to $\bar{p}$. $\square$

Proof of Lemma 3: From the first condition it follows that $\|p(t)\| = \|p(0)\|$, because

$$d\|p(t)\|^2/dt = 2p(t)^T\dot{p} = 2p(t)^T F(p) = 0,$$

i.e., $\|p(t)\|$ is constant. Let us choose p^* such that $\|p^*\| = \|p(0)\|$, and differentiate $D(t) = \|p(t) - p^*\|^2$, where p^* is as required in the assumptions:

$$\begin{aligned} dD(t)/dt &= d\|p(t) - p^*\|^2/dt = 2\dot{p}^T(p(t) - p^*) \\ &= 2F(p)^T(p(t) - p^*) = -2F(p)^T p^* < 0, \end{aligned} \tag{B.2}$$

when $p(t)$ is not a solution. Hence, $p(t) \in \Omega$ and $p(t)$ moves monotonically towards p^*. We need to show that $p(t)$ is not bounded away from K, i.e., the process converges to a solution. Let us suppose that $p(t)$ is bounded away from K, i.e., there is $\varepsilon > 0$ such that

$$p(t) \in S = B(p^*, r) \setminus \{p \mid \|p - \bar{p}\| < \varepsilon \ \text{ for some } \ \bar{p} \in K\}$$

for all t. Note that $S \neq \emptyset$ because $p^* \in K$. From continuity of F it follows that $f(p) = -2F(p)^T p^*$ is a continuous function. From Weierstrass theorem we know that $f(p)$ achieves its maximum $-\delta < 0$ in the compact set S. Hence, we have $dD/dt = -2F\left(p(t)\right)^T p^* \leq -\delta < 0$. Integrating both sides from 0 to t and rearranging the terms we get

$$D(t) < D(0) - \delta t.$$

For t large enough we have $D(t) < 0$, which is a contradiction with non-negativity of the norm. □

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Chapter V:

Osborne's Cartel Maintaining Rule Revisited

Abstract

This chapter formulates a proportional reaction strategy in a repeated oligopoly game with discounting. The strategy is based on increasing the total output quantity in proportion to deviations from a cartel point. Such strategy was originally proposed by Osborne (1976) for a static oligopoly. I show that the resulting equilibrium is subgame perfect when the possible deviations are bounded and the proportional reactions have sufficiently large slopes. The lower bounds for the slopes depend on profit functions and discount factors in a simple way. In a duopoly setting, which the chapter mainly considers, the equilibrium is weakly renegotiation proof. A possible strategic explanation for conjectural variations equilibrium is discussed as well.

Key words: oligopoly, collusion, proportional strategy, renegotiation proofness, conjectural variations

1 Introduction

Tacit collusion in oligopolistic markets can be explained with subgame perfect equilibrium strategies in the framework of repeated games. In particular, there is a long tradition of studying strategies in which a firm's actions vary continuously as a response to its rivals' behavior. Following this tradition I consider specific linear punishment strategies for infinitely repeated Cournot oligopolies with discounting, the main emphasis being on duopoly settings. These strategies are called proportional strategies and this chapter shows that they provide subgame perfect equilibrium when restricting to sufficiently small deviations.

Continuous reaction functions in repeated games as mechanisms to sustain cooperation were first suggested by Friedman in a series of papers, see Friedman (1968, 1973, 1976). The main motivation for continuous strategies is that they are more plausible in many circumstances than discontinuous strategies, because with them small deviations lead only to small punishments rather than to the collapse of collusion. Experiments by Selten et al. (1997) suggest that people actually tend to behave in a continuous manner in duopoly games.

Among continuous strategies, linear reaction strategies are particularly interesting because they are simple but their subgame perfection is a non-trivial question. Moreover, in oligopolistic games such strategies are closely related to conjectural variations models that are widely used in empirical literature on imperfect competition. This far linear reaction strategies have lacked subgame perfection when using discounted profits as the evaluation criterion. For nonlinear reaction functions subgame perfection can be obtained, see Friedman and Samuelson (1990, 1994a,b). Kalai and Stanford (1985) prove that linear strategies give rise to an ε-perfect equilibrium when reaction times are short enough. Furthermore, Stanford (1986a) shows that using linear reaction functions leads to a subgame perfect equilibrium for a repeated duopoly when using the limit of the means evaluation criterion instead of discounting. Ehtamo and Hämäläinen (1993) consider linear reaction strategies in a continuous time natural resource model and study the credibility of these strategies. They call the corresponding equilibrium the incentive equilibrium, because, as they argue, punishing only slightly from small deviations is apt to encourage cooperation.

This chapter studies linear strategies in which unilateral deviations from the cartel point are punished in proportion to the deviation. The whole strategy is not continuous since the deviating firm is assumed to return to cooperation immediately after the deviation. Hence, the strategy re-establishes cooperation and does not lead to a sequence of consecutive deviations from the cooperative outputs as reaction function models usually do. The reason for this formulation is that subgame perfection cannot be obtained with reaction functions that give a firm's output as a function on the other players' last period moves regardless of who was the first to deviate, see Stanford (1986b) and Robson (1986).

It is shown that using proportional strategies is an equilibrium and it is subgame perfect when excluding the possibility of too large deviations. Nevertheless, subgame perfection for all deviations can be obtained, for example, by combining proportional strategies with trigger strategies. Local subgame perfection requires the slopes of the punishment lines to be steep enough depending on the marginal profits and discount factors in a simple way. In a duopoly setting the equilibrium is also weakly renegotiation proof in the sense of Farrell and Maskin (1989). Compared to the results of Kalai and Stanford (1985), Stanford (1986a), and Ehtamo and Hämäläinen (1993), new local properties for the equilibrium are obtained in this study.

The local nature of our analysis owes to observations made by Osborne (1976) in a static oligopoly setting. Osborne was inspired by the rather long lasting stability of the OPEC oil cartel and he suggested that maintaining the firms' market shares sustains the cartel in practice. In the literature this linear strategy has been called Osborne's rule or Osborne's quota rule, see Phlips (1988, Section 6.2) and Jacquemin and Slade (1989, Section 3.1.1) for discussion on this strategy. Osborne realized that the quota rule is also credible when deviations are sufficiently small.[1] The proportional strategies are studied in repeated games in a similar way as Osborne analyzed his quota rule in a static setting. In particular, the quota rule can be obtained as a limiting case from proportional strategies when Osborne's assumptions hold and the discount factors approach to one.

The proportional strategy is related to conjectural variations models as well; the slope of the proportional scheme can be understood as a conjectural variation parameter. Moreover, similarly as conjectural variations are often interpreted as the slopes of firms' reaction functions, the proportional scheme can be obtained as a linearization from a more general non-linear equilibrium strategy. In view of this interpretation it is natural to study local properties of proportional strategies. The results also have some relevance for conjectural variations. Namely, subgame perfection of proportional strategies could explain conjectural variations equilibrium as a result of rational behavior: Conjectural variations equilibrium can be maintained as an equilibrium with proportional strategies having the conjectural variations as their slopes.

[1] By credibility Osborne refers to the property that the punishing firm is better off by following the linear punishment than by ignoring the deviation. In the literature Credibility often refers to subgame perfection. In this paper credibility is not used in this meaning.

This paper is structured as follows. In Section 2 I discuss the Osborne's rule for a static duopoly model. In Section 3 I define the proportional scheme with time delay and analyze its properties for bounded deviations. The strategy is formulated for oligopolies with more than two companies in Section 4 where some implications for conjectural variations equilibria are also derived. Section 5 discusses the results.

2 Osborne's Rule for a Static Cournot Duopoly

This section goes through the static Osborne's rule in an appropriate extent. For simplicity I first consider a duopoly setting although. Originally Osborne (1976) presented his quota rule in the context of more general oligopolies. The extension will be discussed in Section 4.

The firms are indexed with i, $q = (q_1, q_2)$ denotes the pair of output quantities, and π_i is firm i's profit function. The subscript $-i$ refers to i's rival. The use of index i without specifying its values, means that we are considering either of the two firms.

The first assumption we need on a profit function is the following:

(A1) π_i is differentiable, strictly concave, and strictly decreasing in q_{-i} when $q_i > 0$.

Since π_i is a strictly concave function, the maximum of π_i under convex constraints is unique. This property will be needed when analyzing proportional reaction strategies. Our second assumption is that at the tacitly agreed cooperative point firm i's profit function is increasing with respect to q_i. Let q^λ denote the cooperative point, and let us assume that at this point the production quantities and the profits are positive. The second assumption can be written for π_i and q^λ as

(A2) $\partial\pi_i(q^\lambda)/\partial q_i > 0$.

Osborne's quota rule is based on the observation that when q^λ is a Pareto-optimal point and profit functions satisfy (A1) and (A2), then there is a joint tangent line to the contours of the profit functions. This tangent line is defined by

$$\nabla\pi_i(q^\lambda) \cdot (q - q^\lambda) = 0, \tag{1}$$

where the dot denotes the usual inner product. The joint tangency property is illustrated in Figure 1, where the solid contours are for π_1 and the dashed contours are for π_2. Although in Figure 1 the line also goes through the origin, this need not be the case in general. This property is discussed below. When q^λ

is not Pareto-optimal then the line given by (1) is tangent only to the contour of π_i at q^λ but not necessarily to the corresponding contour of π_{-i}.

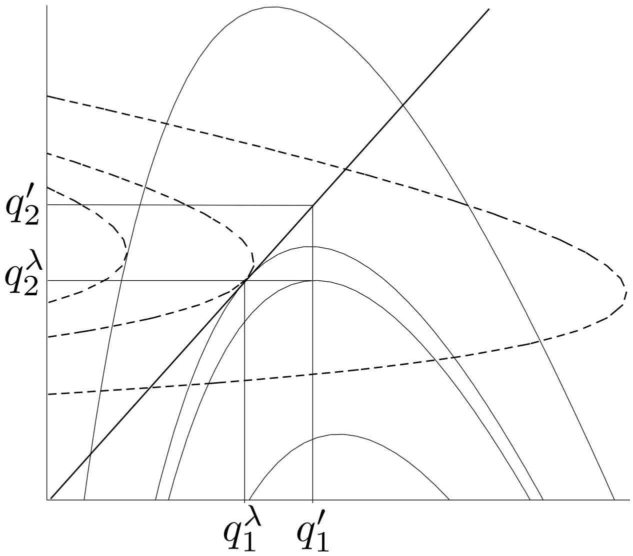

Figure 1. A Pareto-optimal point and a joint tangent line.

The tangent line (1) takes the form

$$q_{-i} = L(q_i, \alpha_i^\lambda) = q_{-i}^\lambda + \alpha_i^\lambda \cdot (q_i - q_i^\lambda),$$

where

$$\alpha_i^\lambda = -\frac{\partial \pi_i(q^\lambda)/\partial q_i}{\partial \pi_i(q^\lambda)/\partial q_{-i}}. \tag{2}$$

Under (A1) and (A2) we have $\alpha_i^\lambda > 0$. We also assume that $q_i^\lambda > 0$, $i = 1, 2$, in which case (A1) gives $\partial \pi_i(q^\lambda)/\partial q_{-i} > 0$, and thus $\alpha_i^\lambda < \infty$.

Osborne interpreted the joint tangent line strategically by assuming that firm i's rival responds to its actions according to the rule

$$q_{-i} = r_{-i}(q_i, \alpha_i^\lambda) = \max\{q_{-i}^\lambda, L(q_i, \alpha_i^\lambda)\}. \tag{3}$$

In other words, when firm i pushes its production from q_i^λ to increase its profit, the rival reacts by keeping the joint production on the tangent line, hence actually decreasing firm i's profit. For example, if firm 1 moves to q_1' as in Figure, then firm 2 raises its output to q_2', which decreases firm 1's profit. As Osborne showed, following this reaction rule suffices for an equilibrium when reactions are instantaneous. Note that it is not profitable for firm $-i$ to punish its rival from decreasing its output below q_i^λ.

Osborne showed the equilibrium property of the linear reaction rules for n firms, which is not as obvious result as in the duopoly case. Spence (1978) extends the result by characterizing a more general class of reaction functions that give rise to efficient outcomes. The following theorem is a reformulation of Osborne's result, and it shows that α_i^λ is a lower bound for the slope of L in r_{-i} such that q_i^λ becomes firm i's optimal choice. The result presented here is slightly different from that of Osborne (1976). In particular, the Pareto-optimality of q^λ is not required. The proof is is given in Appendix.

Theorem 1. *Let us assume that* $\alpha_i \geq \alpha_i^\lambda$ *and assumptions (A1) and (A2) hold for* π_i *and* q^λ. *Then* q_i^λ *maximizes* $\pi_i\left(q_i, r_{-i}(q_i, \alpha_i)\right)$.

When profit functions and the slopes of their tangent lines satisfy the assumptions made in Theorem 1, then q^λ becomes the equilibrium outcome under the proportional reaction strategies. Osborne further showed that the tangent line defined by (1) has the *ray property* if q^λ is the joint profit maximum and, in addition to satisfying (A1) and (A2), the profits are of the form

$$\pi_i(q) = P(q)q_i - C_i(q_i), \tag{4}$$

for $i = 1, 2$. Here C_i is the cost function and P is the inverse demand function that satisfies $\partial P(q)/\partial q_1 = \partial P(q)/\partial q_2$. The ray property says that the common tangent line $q_{-i} = L(q_i, \alpha_i^\lambda)$, $i = 1, 2$, also passes through the origin, see Figure 1. The economic interpretation of this property is that by reacting according to the rule (3) the firms automatically preserve their market shares at $q_i^\lambda/(q_1^\lambda+q_2^\lambda)$, $i = 1, 2$. Or putting it in another way, by always reacting so that the market shares remain constant, the firms move along the joint tangent line and thus maintain cooperation.

Finally, Osborne discusses the *credibility* of the quota rule in his paper. The credibility refers to the property that the punishing firm is better off, at least for small deviations, by following the rule than by just ignoring the deviation. In this paper this property is tightened; credibility means that it is better to follow the punishment line than to choose any other output below it as the punishment. Note that by (A1), a deviating firm i would not mind if it is punished less than the proportional scheme $L(q_i, \alpha_i^\lambda)$ suggests. This property will be the main ingredient of the equilibrium strategy in dynamic setting.

In Figure 2 we see that for firm 1's deviation q_1' firm 2 would choose the punishment $q_2' = L(q_1', \alpha_1^\lambda)$ among all quantities below this output (vertical line segment from 0 to q_2'). Actually, firm 2 would prefer quantities above q_2', the optimal output being on the best response line (dark circle). For firm 1's outputs larger than q_1^L, e.g., for q_1'', firm 2 would rather choose an output below the punishment line. To be more specific, firm 2 would choose the point from the best response line. Hence, in case of large deviations firm 1 has no reason to believe that firm 2 would actually follow the proportional scheme, if also punishment outputs below it were possible; i.e., the proportional strategy is not credible for deviations larger than q_1^L. Note that q_1^L is the point in which firm 2's best response function (the decreasing line) crosses the punishment line. In the following section I discuss the credibility in the dynamic setting but the meaning will essentially remain the same.

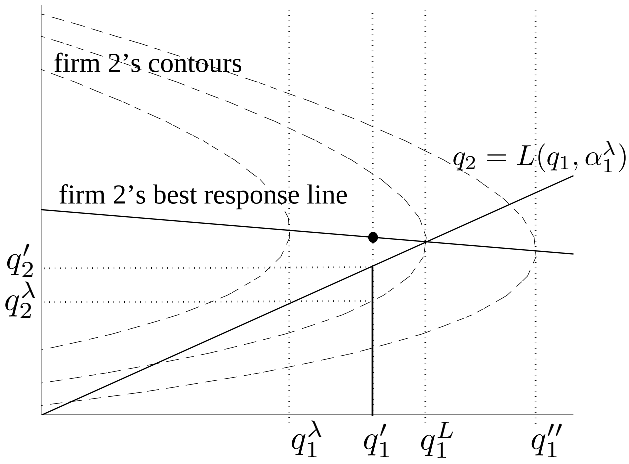

Figure 2. Credibility in a static setting. The further on the left the contour is, the higher the profit is.

Osborne's rule, as presented in this section, is static as the reaction to deviation is assumed to be instantaneous. The idea of punishing from deviations, however, implies the passage of time as one of the firms should first deviate and the other should subsequently react to this deviation. In the following section the proportional reaction strategy is formulated in a repeated game setting to account for the time delay between the observation of deviations and the resulting actions. As in the static setting of this section, we obtain lower bounds for the slopes of the proportional schemes such that cooperative play becomes the equilibrium outcome of the game.

3 Osborne's Rule with Time Delay

In this section the duopoly game is played repeatedly infinitely many times and the firms observe all each others' previous actions. Furthermore, the firms maximize their discounted profits, i.e., firm i maximizes

$$\sum_{k=0}^{\infty} \delta_i^k \pi_i(q^k),$$

where k refers to the period and $\delta_i \in (0,1)$ is the firm's discount factor. The deviation from cooperation is observed immediately so that the firms can react to the deviations in the next period.

The proportional scheme works in the repeated setting as follows. After having observed that the other firm has deviated, i.e., exceeded the cooperative output, the firm punishes the deviator by choosing the output in the next period according to the proportional scheme. Simultaneous deviations are, however, neglected and the firms continue as if none of them had deviated. Moreover, the firms accept that their deviations are punished but the punishment output should not exceed the output given by the proportional scheme. The deviating firm should return to cooperation after a deviation, i.e., it should choose the cooperative output. If the punishment output has, however, exceeded the proportional scheme, the roles of the original deviator and the punisher are changed; i.e., exceeding the proportional scheme is interpreted as a deviation from the cooperative point q^λ.

We say that a firm has played *conventionally* if its behavior does not give a reason to punish it. The only reasons to punish firm i are that it has unilaterally exceeded the cooperative output q_i^λ, or it has punished the other firm too harshly, i.e., $q_i^k > L(q_{-i}^{k-1}, \alpha_{-i}) = q_i^\lambda + \alpha_{-i} \cdot (q_{-i}^{k-1} - q_{-i}^\lambda)$. Formally, the strategy is to choose $q_i^0 = q_i^\lambda$ and for $k \geq 1$ play as follows:

i) $q_i^k = q_i^\lambda$, if in period $k-1$ both firms have played conventionally, deviated simultaneously, or only firm i has not played conventionally.
ii) $q_i^k = L(q_{-i}^{k-1}, \alpha_{-i})$, if the other firm has not played conventionally in period $k-1$.

The above strategy for firm i with slope α_{-i} is denoted by $\omega_i(\alpha_{-i})$. The strategy profile with both using proportional schemes is $\omega(\alpha) = (\omega_1(\alpha_2), \omega_2(\alpha_1))$ with $\alpha = (\alpha_1, \alpha_2)$. The slopes of the proportional schemes are common knowledge, i.e., both firms know each other's punishment scheme.

Ehtamo and Hämäläinen (1993) study a similar strategy as $\omega(\alpha)$ in a continuous time setting in maintaining Pareto-optimal solution as the outcome of a resource management game. However, according to their formulation of the proportional scheme, the players do not return to cooperation immediately after they have deviated. This is also the case with the linear reaction strategies studied by Kalai and Stanford (1985) and Stanford (1986a). Therefore, rather than being a reaction function strategy, $\omega_i(\alpha_{-i})$ can be seen as a continuous alternative of tit-for-tat strategy, according to which a player conditions his next action to the other player's previous move such that the deviator is punished as long as he keeps deviating.

The following Lemma shows that $q_i^k = q_i^\lambda$, for all k, is the optimal choice of actions for firm i when the other player uses $\omega_{-i}(\alpha_i)$ and the slope α_i is steep enough. Hence, the equilibrium outcome of the game with $\omega(\alpha)$ is *cooperative play*, i.e., $q^k = q^\lambda$ for all k. The proof of Lemma 1 is given in Appendix.

Lemma 1. *Let us assume that (A1) and (A2) hold for π_i. Then the optimal sequence of actions for firm i is to choose $q_i^k = q_i^\lambda$ for all $k \geq 0$, when the other firm follows $\omega_{-i}(\alpha_i)$ with $\alpha_i \geq \alpha_i^\lambda/\delta_i$.*

The lower bound that is obtained for the slope of the proportional scheme for firm $-i$ is the slope of the static equilibrium strategy divided by firm i's discount factor. This means that in presence of time lag, the deviator has to be punished stronger than the static equilibrium strategy would suggest. Furthermore, the lower bound depends only on the deviating firm's discount factor and the point q^λ. The larger the discount factor is, the less the equilibrium slope of the static setting has to be increased. This is natural since the smaller the discount factor is, the more the deviator should suffer from the current punishment.

By following $\omega_i(\alpha_{-i})$, with $\alpha_{-i} > 0$, firm i increases its output when the other firm has deviated. If we had $\alpha_{-i} < 0$, then following $\omega_i(\alpha_{-i})$ would decrease firm i's profits while deviating would be profitable for the other firm since by (A1) firm $-i$'s profits are decreasing with respect to q_i. It follows that $\omega(\alpha)$ cannot sustain cooperation when $\partial\pi_i(q^\lambda)/\partial q_i < 0$ for either of the firms since the lower bound $\alpha_i^\lambda/\delta_i$ depends on this partial derivative.

3.1 Credibility

It follows from Lemma 1 that the equilibrium outcome of the game is cooperative play, when the proportional schemes have sufficiently steep slopes. However, this does not guarantee that it would actually be optimal for the punishing firm to follow the proportional scheme, or the deviator to return to cooperation.

Following Osborne's original idea we say that $\omega(\alpha)$ is credible for firm i if it is optimal for firm i to follow $\omega_i(\alpha_{-i})$ when the deviator returns to cooperation and accepts punishments not above the line $L(q_i, \alpha_i)$, i.e., firm $-i$ follows $\omega_{-i}(\alpha_i)$. Consequently, if $\omega(\alpha)$ is credible for the punishing firm, the other firm knows that the deviations are really punished according to the proportional scheme. The notion of credibility, in the sense it is used here, has also been discussed by Holahan (1978), Rothschild (1981), and Ehtamo and Hämäläinen (1993). However, in these papers a firm does not need to worry about the rival's future actions and credibility rather means that it is better to follow the proportional scheme than to do nothing.

In the rest of the paper, the interval of *acceptable deviations* is denoted by $I_i(\bar{q}_i) = (q_i^\lambda, \bar{q}_i]$, $\bar{q}_i > q_i^\lambda$, and the interval of *acceptable punishments* is denoted by $I_i^L(q_{-i}) = [0, L(q_{-i}, \alpha_{-i})]$. We need the interval $I_i(\bar{q})$ because credibility as well as subgame perfection will be obtained only when restricting to sufficiently small deviations, here $\bar{q}$ denotes the upper bound of acceptable deviations. Formally the credibility of $\omega(\alpha)$ for firm i is defined as follows.

Definition 1. $\omega(\alpha)$ is credible for firm i on $I_{-i}(\bar{q}_{-i})$ if it is optimal for firm i to follow $\omega_i(\alpha_{-i})$ after any unilateral deviation by firm $-i$ on $I_{-i}(\bar{q}_{-i})$, assuming that firm $-i$ follows $\omega_{-i}(\alpha_i)$ after the deviation.

The credibility of $\omega(\alpha)$ for firm i means two things: Within the acceptable range of punishment outputs $I_i^L(q_{-i})$, where q_{-i} is the deviation in the prior period, it is optimal for firm i to choose the output given by the proportional rule, and it is better to follow $\omega_i(\alpha_{-i})$ than to choose the maximal deviation and then to be punished. In particular, credibility requires that the firm that has first deviated punishes the other firm for its unfair punishment outputs strongly enough. The condition for the optimality of the proportional scheme within the range of acceptable punishment outputs $I_i^L(q_{-i})$ is

$$\pi_i\left(L(q_{-i},\alpha_{-i}),q_{-i}^{\lambda}\right)=\max_{x\in I_i^L(q_{-i})}\pi_i(x,q_{-i}^{\lambda}). \tag{5}$$

Let us further define

$$q_{-i}^L=\sup\left\{x\in\mathbb{R}:(5)\text{ holds for all }q_{-i}\in[q_{-i}^{\lambda},x]\right\},$$

which gives an upper bound for the largest interval where $\omega(\alpha)$ is credible for firm i. When the deviation exceeds q_i^L the punishing firm would rather choose a smaller quantity than the one given by the proportional scheme. Hence, the greatest upper bound for the credibility interval is obtained from the crossing point of the best response function and the proportional strategy. This is as in the static setting of Section 2, see also Figure 2.

The following lemma shows that $\omega(\alpha)$ is credible for firm i when the deviations do not exceed q_{-i}^L and the punishment outputs that exceed $L(q_{-i},\alpha_{-i})$ are treated as deviations. The proof is given in Appendix and technically it is similar to that of Lemma 1.

Lemma 2. *Let us assume that π_i satisfies (A1) and (A2), and $q_{-i}^L>q_{-i}^{\lambda}$. Then the strategy $\omega(\alpha)$ is credible for firm i on $I_{-i}(q_{-i}^L)$ when $\alpha_i\geq\alpha_i^{\lambda}/\delta_i$ and $\alpha_{-i}>0$.*

We can observe that the farther the best response function R_i crosses the line $q_i=L(q_{-i},\alpha_{-i})$, the greater q_i^L becomes. Furthermore, when the slope α_{-i} decreases, the upper bound increases. In particular, the slope may decrease as δ_i increases. Finally, it is worth noticing that it may happen that firm's Cournot quantity is less than q_i^L, i.e., in some cases even quite large deviations can be punished credibly with the proportional scheme.

3.2 Re-establishing Cooperation

In addition to credibility, it should be optimal for the deviator to return to cooperation when the retaliation follows the proportional scheme. In that case the proportional scheme prevents further deviations from cooperative play. More formally this property is defined as follows.

Definition 2. $\omega(\alpha)$ returns firm i to cooperation on $I_i(\bar{q}_i)$, if it is optimal for the firm to follow $\omega_i(\alpha_{-i})$ after any of its own unilateral deviations on $I_i(\bar{q}_i)$, assuming that firm $-i$ follows $\omega_{-i}(\alpha_i)$.

The strategy profile can be shown to return a firm to cooperation for bounded deviations when the firm's marginal profit is decreasing with respect to the other firm's output. This condition can be formulated for π_i as follows:

(A3) $\partial \pi_i(q_i^\lambda, q_{-i})/\partial q_i$ is decreasing with respect to q_{-i}.

In the specific case when π_i is of the form (4), the assumption (A3) holds when $\partial P(q_i^\lambda, q_{-i})/\partial q_i$ and $P(q_i^\lambda, q_{-i})$ are decreasing with respect to q_{-i}.

The result on re-establishing cooperation is formulated in the following lemma, the proof of which is partly based on Lemma 1 and is given in Appendix.

Lemma 3. *Let us assume that π_i satisfies (A1)–(A3), and $\alpha_i \geq \alpha_i^\lambda/\delta_i$. Then $\omega(\alpha)$ returns firm i to cooperation on $I_i(q_i^+)$ with*

$$q_i^+ = \sup \left\{ x \in \mathbb{R} : \partial \pi_i \left(q_i^\lambda, L(q_i, \alpha_i) \right) / \partial q_i \geq 0 \ \forall q_i \in [q_i^\lambda, x] \right\},$$

assuming that $q_i^+ > q_i^\lambda$.

The explanation for the upper bound q_i^+ is that if the deviation q_i^k is too large, it becomes optimal for firm i to choose $q_i^{k+1} < q_i^\lambda$. This happens because by decreasing the output quantity in the period after the deviation, the firm can compensate the punishment, which loses its effect as a sufficient threat to prevent further deviations.

3.3 Subgame Perfection for Bounded Deviations

We have observed that for sufficiently small deviations the proportional scheme is credible, i.e., it is optimal for the punishing firm to follow the proportional strategy. Moreover, it returns firms to cooperation for bounded deviations when marginal profits are decreasing, i.e., it prevents further deviations from cooperation after sufficiently small unilateral deviations. When these properties hold simultaneously the strategy profile $\omega(\alpha)$ is a subgame perfect equilibrium (SPE) for bounded deviations, which means that if the deviations have been and will be small enough during the history of the play, it is optimal for both firms to follow $\omega(\alpha)$ when they know that the other firm will follow it, too.

Within the range of deviations where the equilibrium is subgame perfect, the strategy profile is also a weakly renegotiation proof equilibrium (WRPE), which means that in addition to subgame perfection none of the continuation payoffs of $\omega(\alpha)$ is Pareto dominated by any other continuation payoff of $\omega(\alpha)$. See Farrell and Maskin (1989) for the concept of WRPE strategies. Continuation payoffs are the discounted profits that the firms obtain when they follow $\omega(\alpha)$ beginning from a given history of the play. Hence, weak renegotiation proofness can be interpreted as the result of the firms negotiating the original agreement anew in any contingencies.

The following theorem summarizes the assumptions on the profit functions and the resulting properties of $\omega(\alpha)$. Here we denote $q_i^\alpha = \min\{q_i^L, q_i^+\}$.

Theorem 2. *Let us assume that (A1)–(A3) hold for both profit functions and* $\alpha_i \geq \alpha_i^\lambda/\delta_i$ *for* $i = 1, 2$. *Then* $\omega(\alpha)$ *is a WRPE, when the unilateral deviations do not exceed* q_i^α *for* $i = 1, 2$.

Proof. If none of the firms has deviated from q_i^λ, i.e., $q_i^{k-1} \leq q_i^\lambda$, $i = 1, 2$, or they have both deviated simultaneously from q_i^λ, i.e., $q_i^{k-1} > q_i^\lambda$, $i = 1, 2$, then by Lemma 1 it is optimal for the players to follow $\omega(\alpha)$. If firm i has deviated in period $k-1$ while the other firm has played conventionally and $q_i^{k-1} \leq q_i^+$, then by Lemma 3 it is optimal for firm i to return to cooperation as suggested by $\omega_i(\alpha_{-i})$. On the other hand, by Lemma 2 it is optimal for the punishing firm to choose the output according to $\omega_i(\alpha_{-i})$ when $q_{-i} \leq q_{-i}^L$. Thus, $\omega(\alpha)$ is a SPE.

Weak renegotiation proofness holds because the deviator's losses increase as the deviation gets larger whereas the other firm's profit increases. Hence, when comparing the continuation payoffs, one of the firms is always worse off and one of them is better off after unilateral deviations. Therefore, no continuation payoffs of $\omega(\alpha)$ dominate any other. □

According to Theorem 2, deviations have to be punished in proportion to the α_i^λ obtained from the static Osborne's rule times the inverse of the discount factor δ_i. The slope of the static equilibrium strategy is obtained as the limit of the lower bound of the proportional scheme when the discount factor goes to one. In particular, the line of constant market shares is the limit of both firms' proportional schemes when the joint profit maximizing point is to be supported as the equilibrium outcome. Obtaining the static case in the limit is natural, since large discount factor could be interpreted as an implication of an ability to react rapidly to rivals' output changes. See Kalai and Stanford (1985) for another approach to consider reaction times.

One interpretation of Theorem 2 is that the equilibrium is subgame perfect even though large deviations were possible but the firms trust that the other firm will not make such deviations intentionally. Actually, we could assume that only large deviations break the collusion and lead to a launching of a trigger and small deviations are punished in a continuous manner. Indeed, if we have a strategy profile that is subgame perfect for all deviations, then switching to this strategy profile after deviations that exceed q_i^α, $i = 1, 2$, and using proportional strategies for smaller deviations, is a possible way to sustain q^λ as SPE outcome for all deviations. For example, switching to Cournot quantities after deviations larger than q_i^α, $i = 1, 2$, would work as a way to obtain subgame perfection for all deviations. Cournot-trigger is known to be SPE when $\pi_i(q^\lambda)$, $i = 1, 2$, are greater than profits at the Cournot-point and the discount factors are large enough. The combination of a proportional strategy and a trigger strategy would be subgame perfect for exactly the same discount factors as the trigger strategy.

Let us now go to the characterization of those cooperative points that can be supported as WRPE outcomes for bounded deviations with proportional schemes. If q^λ is a WRPE outcome for bounded deviations under $\omega(\alpha)$, we say that it can be supported locally with $\omega(\alpha)$. More formally this property is defined below.

Definition 3. q^λ is locally supportable as a WRPE outcome with proportional strategies if there are $\alpha > 0$ and intervals of deviations and punishments such that $\omega(\alpha)$ is WRPE when restricting the possible deviations and punishments to these intervals.

Note that, as punishments depend continuously on deviations, they are bounded whenever deviations are bounded. Hence, we could simply define local subgame perfection by requiring the deviations to be bounded. However, even though deviations should not exceed q_i^α, the punishment outputs may exceed this upper bound, i.e., it may happen that $L(q_{-i}^\alpha, \alpha_{-i}) > q_i^\alpha$. The above definition emphasizes that the deviations and punishments may have different upper bounds.

The following lemma shows that the upper bound of allowed deviations is larger than q_i^λ when the proportional scheme has a positive slope and π_i is continuously differentiable. Hence, when both firms' profit functions and proportional schemes satisfy these conditions, then for any discount factors $\delta_i \in (0, 1)$, $i = 1, 2$, there are intervals of deviations on which $\omega(\alpha)$ is a WRPE. The proof of the lemma is given in Appendix.

Lemma 4. *Let us assume that π_i, $i = 1, 2$, satisfy (A2) and (A3), are continuously differentiable, and $\alpha_i > 0$ for $i = 1, 2$. Then $q_i^\alpha > q_i^\lambda$ for $i = 1, 2$.*

The set of supportable cooperative points depends on assumptions (A2) and (A3). Namely, whenever these conditions hold for profit functions at q^λ, the result of Lemma 4 is valid. Hence, $\omega(\alpha)$ supports q^λ as a locally WRPE outcome if the firms' marginal profits are decreasing with respect to each others outputs, $\partial\pi_i(q^\lambda)/\partial q_i > 0$, for $i = 1, 2$, and the slopes α_i, $i = 1, 2$, are steep enough. Recall that the positivity of the partial derivative is required for α^λ to be positive.

The set of points on which (A2) holds is actually the part of q_1, q_2-plane that is below the best response functions, i.e., $q_i^\lambda < R_i(q_{-i}^\lambda)$, for $i = 1, 2$. This is shown in the following lemma, the proof of which is presented in Appendix.

Lemma 5. *When π_i satisfies (A1), then (A2) is equivalent to $q_i^\lambda < R_i(q_{-i}^\lambda)$.*

The above result is intuitive. Namely, when the firm's cooperative output is below the best response to the other firm's output, the firm would like to increase the output, which means that (A2) holds.

By combining lemmas 4 and 5, and Theorem 2 we obtain the following "folk theorem" for locally supportable points. Here (A3) is assumed to hold for all quantity pairs, which means that at any output level the firms' marginal profits are decreasing with respect to rival's output.

Theorem 3. *Let us assume that both profit functions are continuously differentiable, satisfy (A1), and $\partial\pi_i(q)/\partial q_i$ is decreasing with respect to q_{-i} for all $q_i > 0$ and $i = 1, 2$. Then any q^λ with $0 < q_i^\lambda < R_i(q_{-i}^\lambda)$, for $i = 1, 2$, is locally supportable as a WRPE outcome with proportional strategies.*

It follows from Theorem 3 that the set of locally supportable outcomes is nonempty when the Cournot quantities are positive. Namely, at the Cournot point the firms' best response functions cross and at least the points that both prefer to the Cournot point are locally supportable. Moreover, all Pareto-optimal points, except for the firms' global optima, belong to the set of locally supportable points. The global optima cannot necessarily be supported because at these points $q_i^\lambda = 0$ for either of the firms, and then α_i^λ may become infinitely large.

3.4 Example: Symmetric Duopoly with Quadratic Profits

This example illustrates how the upper bound of the allowed deviations is determined. Let us assume that $\delta_i = \delta$ and $\pi_i(q) = (a - q_i - q_{-i})q_i$ for $i = 1, 2$, which satisfy (A1). The profits are of this form when the inverse demand function P and the cost functions C_1 and C_2 are linear. Here $a - q_1 - q_2$ is assumed to be positive so that profits are positive.

The slope of the tangent line for π_i at the cooperative point q^λ, as defined in (2), is $\alpha_i^\lambda = (a - 2q_1^\lambda - q_{-i}^\lambda)/q_i^\lambda$. We can see that when q_{-i}^λ is kept fixed and q_i^λ is increased α_i^λ decreases, which means that firm i's deviations become easier to prevent with $\omega(\alpha)$. As q_i^λ goes to zero the slope α_i^λ becomes infinitely large, i.e., the deviations become more difficult to punish. In particular, no proportional scheme prevents firm i' deviations from a point in which it produces nothing.

Let us assume that q^λ is the joint profits maximizing point, i.e., $q^\lambda = (a/4, a/4)$. Note that assumptions (A2) and (A3) are satisfied at this point. The slopes of the tangent lines are $\alpha_i^\lambda = 1$, $i = 1, 2$. Hence, we should have $\alpha_i \geq 1/\delta$, $i = 1, 2$, for $\omega(\alpha)$ to be a SPE. Let us take $\alpha_i = 1/\delta$ for $i = 1, 2$. Then the proportional scheme for firm i is

$$L(q_{-i}, \alpha_{-i}^\lambda) = (1 - 1/\delta)a/4 + q_{-i}/\delta.$$

Now q_{-i}^L is obtained at the intersection of the best response function

$$R_i(q_{-i}) = (a - q_{-i})/2$$

and the line of punishment outputs. The intersection point is at $q_{-i}^L = q_i^L = (\delta + 1)a/(2\delta + 4)$. Furthermore, the upper bound q_i^+ of Lemma 3 is obtained from

$$\partial\pi_i\left(q_i^\lambda, L(q_i, \alpha_i)\right)/\partial q_i = a - 2q_i^\lambda - L(q_i, \alpha_i) \geq 0,$$

which gives $q_i^+ = (1 - \delta/2)a$. Noticing that $q_i^+ \geq q_i^L$ we have $q_i^\alpha = q_i^L$.

Now we can see that the more patients the firms are, the more tolerant they become to deviations, i.e., the larger q_i^α, $i = 1, 2$, become. Furthermore, as $\delta \to 1$ we have $q_i^\alpha \to a/3$, which equals the firms' Cournot quantities.

4 Proportional Strategy for General Oligopolies

This section formulates the proportional reaction strategy in oligopolistic settings for $n > 2$ firms. The following assumption on profit functions makes this setting similar to the duopoly case:

(A4) firm i's profit is a function of q_i and $q_{-i} = \sum_{j \neq i} q_j$.

As earlier, a firm has played conventionally if it has not deviated unilaterally or made an unreasonably large punishment after one of the firms has deviated. In the following $f_{i,j}(q_j)$ denotes the largest acceptable punishment for firm i after firm j has deviated. The functions $f_{i,j}$ are assumed to be continuous, increasing, and they satisfy $f_{i,j}(q_j^\lambda) = q_i^\lambda$, and $\sum_{k \neq j} f_{k,j}(q_j) = L(q_j, \alpha_j)$. These assumptions mean that the total punishment output equals the one given by the proportional scheme, and the share of outputs is given by functions $f_{i,j}$. For example, the total punishment could be shared in proportion to firms' market shares. Let us denote $f_{-i,j}(q_j) = q_j^\lambda + \sum_{k \neq i,j} f_k(q_j, \alpha_j)$, i.e., $f_{-i,j}(q_j)$ is the total output of other punishing firms than firm i.

The strategy for firm i is to choose $q_i^0 = q_i^\lambda$ and for $k \geq 1$ play as follows:

i) $q_i^k = q_i^\lambda$, if in period $k-1$ all firms have played conventionally, two or more have deviated simultaneously, or only firm i has not played conventionally.
ii) $q_i^k = f_{i,j}(q_j^{k-1})$ if j has unilaterally deviated in period $k-1$.

The above strategy for i with slopes $\alpha = (\alpha_1, \ldots, \alpha_n)$ is denoted by $\omega_i(\alpha)$. The strategy profile with all firms using proportional schemes is denoted by $\omega(\alpha)$.

Under assumptions (A1)–(A4), lemmas 1 and 3 go through as in the two-firm case. Moreover, q_i^+ can be defined as before and the strategy re-establishes cooperation for bounded deviations. The result on credibility, Lemma 2, also generalizes to the n-firm case, but the proof is different. Now the upper bound of firm j's deviations that firm i can credibly punish, denoted by $q_{j,i}^L$, is defined as the largest quantity x for which

$$\partial \pi_i \left(f_{i,j}(q_j), f_{-i,j}(q_j) \right) / \partial q_i \geq 0 \text{ for all } q_j \leq x. \tag{6}$$

Moreover, we denote $q_j^L = \min_i q_{j,i}^L$. Lemma 6 shows that (A1)–(A4) imply the credibility of $\omega_i(\alpha)$ for firm j's deviations on $I_j(q_{j,i}^L)$. The proof is given in Appendix and it is similar to that of Lemma 3.

Lemma 6. *Let us assume that π_i satisfies (A1)–(A4) and $q_{j,i}^L > q_j^\lambda$, $j \neq i$. Then $\omega_i(\alpha)$ is credible for firm i on $I_j(q_{j,i}^L)$ when $\alpha_i \geq \alpha_i^\lambda / \delta_i$.*

Lemmas 3 and 6 together imply that $\omega(\alpha) = (\omega_1(\alpha_1), \ldots, \omega_n(\alpha_n))$ is a SPE when $\alpha_i \geq \alpha^\lambda / \delta_i$ for all i and the deviations are small enough. The characterization of supportable points is essentially the same as in the duopoly setting. These results are collected to the following theorem.

Theorem 4. *Let q^λ satisfy $0 < q_i^\lambda < R_i(q_{-i}^\lambda)$, for $i = 1, \ldots, n$. Let the profit functions be continuously differentiable, and satisfy (A1), (A3), (A4). Then q^λ is locally supportable as a SPE outcome with $\omega(\alpha)$.*

Proof. What is left to be shown is that $q_i^\alpha = \min\{q_i^+, q_i^L\} > q_i^\lambda$ for all i, because then $\omega(\alpha)$ is a SPE when $q_i^k \in I_i(q_i^\alpha)$ for all i and $\alpha_i \geq \alpha_i^\lambda / \delta_i$. Here, q_i^α, $i = 1, \ldots, n$, are defined as in the two firm case.

As in the proof of Lemma 4, it can be shown that $q_i^+ > q_i^\lambda$ for all i. By Lemma 5 we know that condition $q_i^\lambda < R_i(q_{-i}^\lambda)$ is equivalent with (A2). By (A2) and continuous differentiability, there are $q_{j,i}^L > q_j^\lambda$, for all $i, j = 1, \ldots, n$, $i \neq j$, such that (6) holds for all $i, j = 1, \ldots, n$, $i \neq j$. Hence, we have $q_i^\alpha > q_i^\lambda$ for all i, which concludes the proof. □

If $\pi_i(f_{i,j}(q_j), f_{-i,j}(q_j))$, $i, j = 1, \ldots, n$, $i \neq j$, were increasing with respect to q_j, then the strategy profile would also be a WRPE, since in that case no continuation payoffs would dominate each other as in the duopoly setting. However, these profits are not necessarily increasing because it is not clear whether increasing a punishing firm's own output compensates the effect that comes from the other punishing firms increasing their outputs as well.

4.1 Conjectural Variations Equilibria

In this section I discuss a possible way to detect empirically whether an observed market situation can be interpreted as a collusion with proportional strategies as the supporting mechanisms. This discussion is based on the observation that proportional strategies are linked to conjectural variations models of Bowley (1924). The main idea of these models is that each firm believes that the quantities chosen by its rivals depend on the firm's own output. Hence, firm i is assumed to have a conjecture on rivals' reactions around the cartel point. This behavioral assumption is captured in the conjectural variation parameter $\nu_i = dq_{-i}(q_i^\lambda)/dq_i$.

As the static Osborne's rule, conjectural variations models assume that the response for deviations is instantaneous. Indeed, in the static setting the slope of the punishment line $L(q_i, \alpha_i)$ plays exactly the same role as ν_i. Hence, α_i can be identified with ν_i. Consequently, proportional strategies in repeated game give a rational justification for conjectural variations models; when the conjectural variations are large enough, a conjectural variations equilibrium corresponds to a locally subgame perfect equilibrium under proportional strategies. More specifically, a conjectural variations equilibrium can be interpreted as a locally SPE with the slopes of proportional schemes equaling the conjectural variations. Recall, however, that the assumptions (A1)–(A4) should hold for all firms.

We say that conjectural variations that lead to a local SPE are strategically consistent. More formally this consistency is defined as follows.

Definition 4. Firm i's conjectural variation is *strategically consistent* if $\nu_i \geq \alpha_i^\lambda/\delta_i$. If q^λ is below the reaction functions and $\nu_1, \ldots, v_n$ are strategically consistent, then the conjectural variations equilibrium is strategically consistent.

Other dynamic interpretations of conjectural variations and various consistency concepts have been discussed, e.g., in Figuières et al. (2004, Chapters 2 and 3).

Let us assume that firm i's profit function is of the form (4) and $P(q) = P(\sum_j q_j)$. With the conjectural variation ν_i, the necessary conditions for firm

i's static profit maximization problem can be written as

$$1 - C_i'(q_i^\lambda)/P(q^\lambda) = 1/|\eta|(1 + \nu_i), \tag{7}$$

where $|\eta| = -[\partial q_i/q_i]/[\partial P(q)/P(q)]$, i.e., $|\eta|$ is the absolute value of the elasticity of demand. The left hand side of (7) is the firm's Lerner index and it is denoted by LE_i. This index measures the competitiveness of an oligopolistic market; the larger the Lerner index is, the less competitive the oligopoly is.

Note that as we are not considering a static oligopoly game, condition (7) need not hold for $\nu_i = \alpha_i$. However, we can derive another relationship for ν_i, LE_i, and η to obtain strategic consistency. Namely, it can be seen that $\alpha_i^\lambda = LE_i/|\eta| - 1$. Hence, strategic consistency, i.e., $\nu_i \geq \alpha_i^\lambda/\delta_i$, requires that $\delta_i \nu_i \geq LE_i/|\eta| - 1$. By using this inequality we could estimate the lowest strategically consistent conjectures for given discount factors, demand elasticities, and Lerner indices. Let us also observe that (A2) can be written equivalently as $LE_i > 1/|\eta|$, when $P(q^\lambda) > 0$. Hence, empirically observed Lerner indices that are greater than $1/|\eta|$ could be due to tacit collusion with proportional strategies.

5 Discussion

This study shows that for sufficiently regular profit functions, using proportional reaction strategies sustains cooperation as a subgame perfect outcome when the deviations are small enough. Using the strategies is also Nash equilibrium for all deviations. The slopes of the proportional strategies and the ranges of acceptable deviations depend on the profit functions and discount factors in a simple way. Moreover, the cooperative point that is to be supported as the equilibrium outcome should be in the region below the firms' best response functions.

The main motivation for the local nature of the results of this paper is that a linear strategy can be obtained as a linearization from a more general nonlinear equilibrium strategy. Since linearization is reasonably accurate only in the neighborhood of the cartel point, it is natural that the properties of linear strategy are also local. Local properties of linear strategies were first analyzed by Osborne (1976) in static settings. The linearization idea on the other hand appears in the conjectural variations literature, where the conjectural variations are explained as slopes of linearized reactions. Indeed, the results of this paper give new motivation for both Osborne's model and conjectural variations. Osborne's quota rule is obtained as a limiting case from the proportional equilibrium strategies in repeated game as discount factors approach to one. For conjectural variations equilibrium these strategies give a possible explanation: With large enough conjectures the corresponding equilibrium becomes locally subgame perfect with proportional strategies, where the slopes equal the conjectural variations.

The local results of this paper show that using linear strategies provides a simple way to punish small unintentional errors or trembles so that collusion can be re-established. Moreover, in a duopoly setting the equilibrium is weakly renegotiation proof, which means that there is no need to renegotiate the cooperation anew after small deviations. To obtain subgame perfection for all deviations the proportional scheme can be combined with other equilibrium strategies, e.g., with trigger strategies such that only large deviations lead to collapse of the cartel and launch the trigger. In the spirit of forward induction, deviations can be interpreted as strategic signals: A small deviation signals an accidental error but a large deviation is a signal of breaking the collusion.

A possible extension for the use of the proportional strategy is to consider a situation of imperfect monitoring, in which only past prices with possible random disturbances are observed instead of all production quantities, see, e.g., Green and Porter (1984) and Abreu et al. (1986). The proportional scheme could provide a cartel maintaining strategy in such games, too.

Appendix: Auxiliary Proofs

In this appendix $I_i^\lambda = [0, q_i^\lambda]$ denotes the interval of acceptable outputs for firm i, given that the other firm has played conventionally. Recall that $I_i(\bar{q}_i) = (q_i^\lambda, \bar{q}_i]$ and $I_i^L(q_{-i}) = [0, L(q_{-i}, \alpha_{-i})]$ is firm i's set of acceptable punishment outputs after a deviation q_{-i} by the other firm.

Proof of Theorem 1: Let us first observe that the maximization problem

$$\max_{q_i \geq 0} \pi_i \left(q_i, r_{-i}(q_i, \alpha_i)\right),$$

where $r_{-i}(q_i, \alpha_i) = \max\{q_{-i}^\lambda, L(q_i, \alpha_i)\}$, has exactly the same solution as the convex optimization problem

$$\begin{aligned} &\max \pi_i(q) \\ &\text{s.t. } q \in \left\{\hat{q} \in \mathbb{R}^2 : \hat{q}_i \geq 0,\ \hat{q}_{-i} \geq \max\{q_{-i}^\lambda, L(\hat{q}_i, \alpha_i)\right\}\}. \end{aligned}$$

This is because π_i is decreasing with respect to q_{-i} so that at the optimum we have $q_i = \max\{q_{-i}^\lambda, L(q_i, \alpha_i))$ even though inequality was allowed. The necessary and sufficient condition of this problem at q^λ is that the below variational inequality holds for all feasible q:

$$\nabla \pi_i(q^\lambda) \cdot (q - q^\lambda) \leq 0, \tag{8}$$

i.e.,

$$[\partial \pi_i(q^\lambda)/\partial q_i](q_i - q_i^\lambda) + [\partial \pi_i(q^\lambda)/\partial q_{-i}](q_{-i} - q_{-i}^\lambda) \leq 0.$$

For $q_i > q_i^\lambda$ condition (8) holds when

$$[\partial \pi_i(q^\lambda)/\partial q_i](q_i - q_i^\lambda) + \alpha_i[\partial \pi_i(q^\lambda)/\partial q_{-i}](q_i - q_i^\lambda) \leq 0,$$

because now $q_{-i} \geq L(q_i, \alpha_i)$. By (A1) and (A2) $\alpha_i \geq \alpha_i^\lambda > 0$, and by (A1) $\partial \pi_i(q^\lambda)/\partial q_{-i} < 0$. Thus,

$$\alpha_i \cdot (q_i - q_i^\lambda) \cdot \partial \pi_i(q^\lambda)/\partial q_{-i} \leq \alpha_i^\lambda \cdot (q_i - q_i^\lambda) \cdot \partial \pi_i(q^\lambda)/\partial,$$

which gives the optimality condition.

For $q_i \leq q_i^\lambda$ we have $q_{-i} \geq q_{-i}^\lambda$. Hence, the left hand side of condition (8) is less than zero if $[\partial \pi_i(q^\lambda)/\partial q_i](q_i - q_i^\lambda)$ is less than zero. It follows from (A2) that π_i is decreasing with respect to q_i. Hence, we obtain the inequality $[\partial \pi_i(q^\lambda)/\partial q_i](q_i - q_i^\lambda) \leq 0$, i.e., (8) holds for $q_i \leq q_i^\lambda$. □

Proof of Lemma 1: Let $\{q^k\}_k$ be a sequence of output quantity pairs and $\Pi_i(\{q^k\}_k) = \sum_k \delta_i^k \pi_i(q^k)$. The assumption is that $q_{-i}^0 = q_{-i}^\lambda$, and we need to show that then it is optimal for firm i to choose $q_i^k = q_i^\lambda$ for all $k \geq 0$, which means that $q^k = q^\lambda$ for all k is the optimal choice of output quantity pairs for firm i.

As in Theorem 1, the choice of the output can be written as a convex optimization problem

$$\begin{aligned} &\max \Pi_i(\{q^k\}_k) \\ &\text{s.t. } \{q^k\}_k \in F(\alpha_i), \end{aligned}$$

where

$$\begin{aligned} F(\alpha_i) = \Big\{ &\{q^k\}_k : q_i^k \geq 0 \ \forall k \geq 0, \ q_{-i}^0 = q_{-i}^\lambda, \\ &q_{-i}^k \geq \max\{q_{-i}^\lambda, L(q_i^{k-1}, \alpha_i)\} \ \forall k \geq 1 \Big\}. \end{aligned}$$

We assume that in the above optimization problem $\{q^k\}_k \in l_\infty \times l_\infty$, which is a Banach-space with the norm $\|\{q^k\}_k\| = \max_k |q_1^k| + \max_k |q_2^k|$. Hence, the sequence $\{q^k\}_k$ should be bounded, which is not too restrictive assumption since choosing large outputs usually causes losses for the firms. Moreover, the results that are based on this lemma require that the outputs stay within certain ranges.

Analogously to the static case, the sufficient and necessary condition for the optimality of $\{q^\lambda\}_k$ is that the variational inequality

$$\nabla\Pi_i(\{q^\lambda\}_k)(\{q^k\}_k - \{q^\lambda\}) \leq 0 \tag{9}$$

holds for all $\{q^k\}_k \in F(\alpha_i)$, see, e.g., Aubin (1993, Section 9.8.). Here $\nabla\Pi_i(\{q^\lambda\}_k)$ denotes the Fréchet-differential of Π_i at $\{q^\lambda\}_k$. It can be seen that

$$\nabla\Pi_i(\{q^\lambda\}_k)\{q^k\}_k = \sum_k \delta_i^k \nabla\pi_i(q^\lambda) \cdot q^k.$$

In the following we denote $I_+ = \{k : q_i^k > q_i^\lambda\}$ and $I_- = \{k : q_i^k \leq q_i^\lambda\}$. Now for $k \in I_-$ we have $q_{-i}^k \geq q_{-i}^\lambda$ and for $k \in I_+$ we have $q_{-i}^k \geq L(q_i, \alpha)$. The latter yields $q_{-i}^{k+1} - q_{-i}^\lambda \geq \alpha_i(q_i^k - q_i^\lambda)$ for $k \in I_+$. Let us now consider (9) in more detail:

$$\begin{aligned}
&\nabla\Pi_i(\{q^\lambda\}_k)(\{q^k\}_k - \{q^\lambda\}) = \sum_k \delta_i^k \nabla\pi_i(q^\lambda) \cdot (q^k - q^\lambda) \\
&= \sum_k \delta_i^k [(q_i^k - q_i^\lambda)\partial\pi_i(q^\lambda)/\partial q_i + (q_{-i}^k - q_{-i}^\lambda)\partial\pi_i(q^\lambda)/\partial q_{-i}] \\
&\leq \sum_{k\in I_+} \delta_i^k (q_i^k - q_i^\lambda)\partial\pi_i(q^\lambda)/\partial q_i + \sum_{k\in I_+} \alpha_i \delta_i^{k+1}(q_i^k - q_i^\lambda)\partial\pi_i(q^\lambda)/\partial q_{-i} \\
&+ \sum_{k\in I_-} \delta_i^k (q_i^k - q_i^\lambda)\partial\pi_i(q^\lambda)/\partial q_i + \sum_{k\in I_-} \delta_i^k (q_{-i}^k - q_{-i}^\lambda)\partial\pi_i(q^\lambda)/\partial q_{-i} \\
&\leq \sum_{k\in I_+} \delta_i^k (q_i^k - q_i^\lambda)\partial\pi_i(q^\lambda)/\partial q_i + \sum_{k\in I_+} (\alpha_i^\lambda/\delta_i)\delta_i^{k+1}(q_i^k - q_i^\lambda)\partial\pi_i(q^\lambda)/\partial q_{-i} = 0,
\end{aligned} \tag{10}$$

where the last equality follows by plugging α_i^λ from (2) in the equation. The second last inequality holds because $\alpha_i \geq \alpha_i^\lambda/\delta_i$ and the terms of the sum with $k \in I_-$ are negative. The latter is true because $\partial\pi_i(q^\lambda)/\partial q_{-i} < 0$ by (A1), and $\partial\pi_i(q^\lambda)/\partial q_i > 0$ by (A2). The above deduction shows the optimality of the sequence $q_i^k = q_i^\lambda$ for all k. □

Proof of Lemma 2: Because $q_i^\alpha \leq q_i^L$, it is optimal to choose $q_i^1 = L(q_{-i}^0, \alpha_{-i})$ within the acceptable range of punishments $I_i^L(q_{-i}^0)$, when $q_{-i}^0 \in I_{-i}(q_{-i}^L)$ and $q_i^0 \in I_i^\lambda$. Hence, we only need to show that it is not optimal to make the maximal deviation from $\omega_i(\alpha_{-i})$ and then to be punished. This is the case because

$$\max_{\{q^k\}\in F(\alpha_i)} \Pi_i\left(\{q^k\}_k\right) = \pi_i^\lambda/(1-\delta_i) \leq \pi_i\left(L(q_{-i}^0, \alpha_{-i}), q_{-i}^\lambda\right) + \delta_i\pi_i^\lambda/(1-\delta_i),$$

where the first equality follows from Lemma 1, and the inequality holds because

$$\pi_i\left(L(q_{-i}^0, \alpha_{-i}), q_{-i}^\lambda\right) \geq \pi_i^\lambda$$

by the choice $q^0_{-i} \in I_{-i}(q^L_{-i})$. Note that by making an unreasonably large punishment the firm cannot exceed the profits that maximize $\Pi_i\left(\{q^k\}_k\right)$ subject to $\{q^k\} \in F(\alpha_i)$ Thus, $\omega(\alpha)$ is credible for firm i. □

Proof of Lemma 3:

Let us denote $\tilde{q}_{-i} = L(q^0_i, \alpha_i)$ and $q' = (q^\lambda_i, \tilde{q}_{-i})$. It is optimal to choose $q^k_i = q^\lambda_i$ for all $k \geq 1$, if the variational inequality

$$\nabla \pi_i(q') \cdot (q^1 - q') + \sum_{k \geq 2} \delta_i^{k-1} \nabla \pi_i(q^\lambda) \cdot (q^k - q^\lambda) \leq 0 \tag{11}$$

holds for all feasible sequences $\{q^k\}_k$, similarly as in the proof of Lemma 1. This condition can be written as:

$$S_1 + S_2 \leq 0,$$

where S_1 contains the terms that include q^1_i and q^1_{-i}, and S_2 contains the rest of the sum. As in the necessary condition (9), we have:

$$\begin{aligned} S_2 \leq & \sum_{k \in I_+} \delta_i^{k-1}(q^k_i - q^\lambda_i)[\partial \pi_i(q^\lambda)/\partial q_i] + \sum_{k \in I_+} \alpha_i \delta_i^k (q^k_i - q^\lambda_i)[\partial \pi_i(q^\lambda)/\partial q_{-i}] \\ & + \sum_{k \in I_-} \delta_i^{k-1}(q^k_i - q^\lambda_i)[\partial \pi_i(q^\lambda)/\partial q_i] + \sum_{k \in I_-} \delta_i^{k-1}(q^k_{-i} - q^\lambda_{-i})[\partial \pi_i(q^\lambda)/\partial q_{-i}], \end{aligned}$$

where $I_+ = \{k \geq 2 : q^k_i > q^\lambda_i\}$ and $I_- = \{k \geq 2 : q^k_i \geq q^\lambda_i\}$. As in the case $q^0_i = q^\lambda_i$ in the proof of Lemma 1, it can be shown that $S_2 \leq 0$ when $\alpha_i \geq \alpha^\lambda_i / \delta_i$.

Hence, to obtain (11) we need to show that $S_1 \leq 0$. Let us first note that

$$\nabla \pi_i(q') \cdot (q^1 - q') = [\partial \pi_i(q')/\partial q_i](q^1_i - q^\lambda_i)$$

because $q^1_{-i} = \tilde{q}_{-i}$ according to $\omega_{-i}(\alpha_i)$. From this and the proportional scheme we obtain

$$S_1 = \begin{cases} [\partial \pi_i(q')/\partial q_i](q^1_i - q^\lambda_i) + \alpha_i \delta_i (q^1_i - q^\lambda_i)[\partial \pi_i(q^\lambda)/\partial q_{-i}] & \text{if } q^1_i > q^\lambda_i, \\ [\partial \pi_i(q')/\partial q_i](q^1_i - q^\lambda_i) & \text{if } q^1_i \leq q^\lambda_i. \end{cases}$$

Let us assume that $q^1_i > q^\lambda_i$. By the assumption (A3) we have $\partial \pi_i(q')/\partial q_i \leq \partial \pi_i(q^\lambda)/\partial q_i$ and hence

$$[\partial \pi_i(q')/\partial q_i](q^1_i - q^\lambda_i) \leq [\partial \pi_i(q^\lambda)/\partial q_i](q^1_i - q^\lambda_i).$$

It follows that

$$
\begin{aligned}
&[\partial\pi_i(q')/\partial q_i](q_i^1 - q_i^\lambda) + \alpha_i\delta_i(q_i^1 - q_i^\lambda)[\partial\pi_i(q^\lambda)/\partial q_{-i}] \leq \\
&[\partial\pi_i(q^\lambda)/\partial q_i](q_i^1 - q_i^\lambda) + \alpha_i\delta_i(q_i^1 - q_i^\lambda)[\partial\pi_i(q^\lambda)/\partial q_{-i}] \leq \\
&[\partial\pi_i(q^\lambda)/\partial q_i](q_i^1 - q_i^\lambda) + \alpha_i^\lambda(q_i^1 - q_i^\lambda)[\partial\pi_i(q^\lambda)/\partial q_{-i}] = 0
\end{aligned}
$$

where the last equality is obtained by plugging α_i^λ in the equation. Hence, $S_1 \leq 0$ and consequently (11) holds.

Because $[\partial\pi_i(q')/\partial q_i] \geq 0$ by $q_i^0 \leq q_i^+$, we have $S_1 \leq 0$ also for $q_i^1 \leq q_i^\lambda$. Thus, (11) holds for all feasible sequences. □

Proof of Lemma 4: The continuity of $\partial\pi_i(q_i^\lambda, q_{-i})/\partial q_i$, (A2), and (A3) imply that $q_i^+ > q_i^\lambda$. By (A2) we know that π_i is growing at q^λ with respect to its first argument. It then follows from the continuity of the derivative that there is $\tilde{q}_i > q_i^\lambda$ such that for all $q_i^1, q_i^2 \in [q_i^\lambda, \tilde{q}_i]$, with $q_i^1 \geq q_i^2$, we have $\pi_i(q_i^1, q_{-i}^\lambda) \geq \pi_i(q_i^2, q_{-i}^\lambda)$, i.e., π_i is growing on $[q_i^\lambda, \tilde{q}_i]$ with respect to q_i.

Since $\alpha_i > 0$, there is $\hat{q}_{-i} > q_{-i}^\lambda$ such that $\tilde{q}_i = L(\hat{q}_{-i}, \alpha_{-i})$, i.e., L maps $[q_{-i}^\lambda, \hat{q}_{-i}]$ into $[q_i^\lambda, \tilde{q}_i]$. Because π_i is growing on $[q_i^\lambda, \tilde{q}_i]$ it follows that for all $q_{-i}^0 \in I_i(\hat{q}_{-i})$ it is optimal to choose $q_i^1 = L(q_{-i}^0, \alpha_i)$. Thus, $q_{-i}^L \geq \hat{q}_{-i} > q_{-i}^\lambda$, and we have $q_i^\alpha > q_i^\lambda$. □

Proof of Lemma 5: Let us first observe that $\partial\pi_i(q^\lambda)/\partial q_i > 0$ means that π_i is growing at q^λ and hence the best response to q_{-i}^λ is greater than q_i^λ, i.e., $q_i^\lambda < R_i(q_{-i}^\lambda)$.

Let us now assume that $q_i^\lambda < R_i(q_{-i}^\lambda)$. By the definition of differential

$$
\partial\pi_i(q^\lambda)/\partial q_i = \lim_{\rho\to 0^+} \left[\pi_i\left((1-\rho)q_i^\lambda + \rho R_i(q_{-i}^\lambda), q_{-i}^\lambda\right) - \pi_i(q^\lambda)\right] / \left[\rho(R_i(q_{-i}^\lambda) - q_i^\lambda)\right],
$$

where $\rho \to 0^+$ means that ρ converges to zero from the positive side. When $\rho \in (0, 1]$ the concavity of π_i with respect to q_i yields

$$
\pi_i\left((1-\rho)q_i^\lambda + \rho R_i(q_{-i}^\lambda), q_{-i}^\lambda\right) \geq (1-\rho)\pi_i(q^\lambda) + \rho\pi_i\left(R_i(q_{-i}^\lambda), q_{-i}^\lambda\right).
$$

This inequality together with the above formula of the differential gives

$$
\partial\pi_i(q^\lambda)/\partial q_i \geq [\pi_i\left(R_i(q_{-i}^\lambda), q_{-i}^\lambda\right) - \pi_i(q^\lambda)]/[R_i(q_i^\lambda) - q_i^\lambda].
$$

By the strict concavity of π_i with respect q_i we know that the best response is unique and $\pi_i\left(R_i(q_{-i}^\lambda), q_{-i}^\lambda\right) > \pi_i(q^\lambda)$. Hence, $\partial\pi_i(q^\lambda)/\partial q_i > 0$, which concludes the proof. □

Proof of Lemma 6: The proof goes as in Lemma 3. Now we denote $q' = \left(f_{i,j}(q_j^0), f_{-i,j}(q_j^0)\right)$ and $q_{-i}^1 = f_{-i,j}(q_j^0)$. Again the necessary condition for optimality of sequence $q_i^1 = f_{i,j}(q_j^0)$, $q^k = q^\lambda$, $k \geq 2$, can be written as $S_1 + S_2 \leq 0$, where S_1 contains the terms that include q' and q_i^1, and S_2 contains the rest. As earlier, it can be shown that $S_2 \leq 0$.

Hence, we need to show that $S_1 \leq 0$. Now

$$S_1 = \begin{cases} [\partial \pi_i(q')/\partial q_i](q_i^1 - q_i') + \alpha_i \delta_i (q_i^1 - q_i')[\partial \pi_i(q^\lambda)/\partial q_{-i}] & \text{if } q_i^1 > f_{i,j}(q_j^0), \\ [\partial \pi_i(q')/\partial q_i](q_i^1 - q_i') & \text{otherwise.} \end{cases}$$

Let us assume that $q_i^1 > f_{i,j}(q_j^0)$. By the concavity of π_i with respect to q_i we know that $\partial \pi_i(q)/\partial q_i$ is decreasing with respect to q_i. Since $f_{i,j}$ is increasing, we have

$$\partial \pi_i \left(q'\right)/\partial q_i \leq \partial \pi_i \left(q_i^\lambda, f_{-i,j}(q_j)\right)/\partial q_i.$$

It follows from (A3) that

$$\partial \pi_i \left(q_i^\lambda, f_{-i,j}(q_j)\right)/\partial q_i \leq \partial \pi_i(q^\lambda)/\partial q_i,$$

and thus $\partial \pi_i \left(q'\right)/\partial q_i \leq \partial \pi_i \left(q^\lambda\right)/\partial q_i$. As $q_j \leq q_{j,i}^L$, we have $\partial \pi_i(q')/\partial q_i \geq 0$ and hence

$$[\partial \pi_i(q')/\partial q_i](q_i^1 - q_i') \leq [\partial \pi_i(q^\lambda)/\partial q_i](q_i^1 - q_i').$$

As in the proof of Lemma 3 it follows that $S_1 \leq 0$. We also have $S_1 \leq 0$ for $q_i^1 \leq f_{i,j}(q_j^0)$, because $\partial \pi_i(q')/\partial q_i \geq 0$. Thus, (11) holds for all feasible sequences. □

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