Performance bound for blind CFO estimation in OFDM with real-valued constellations

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Abstract— **In this paper, we investigate the performance of the blind carrier frequency offset (CFO) estimation method for OFDM with real-valued constellations introduced in [1]. The method is based on minimizing the total off-diagonal power of the received pseudo-covariance matrix and the resulting solution has a simple closed-form expression.**

To assess the large sample performance, we derive the Cramér-Rao bound (CRB) for the blind CFO estimation problem. When deriving the CRB, the transmitted OFDM modulated signal is assumed to be a Gaussian process. Since real-valued constellations are used, the received signal is non-circular. As a result, the CRB has to be derived for non-circular Gaussian model. Simulation results highlight significant differences in performance between the complex circular and non-circular cases.

I. Introduction

Orthogonal Frequency Division Multiplexing (OFDM) is a powerful technique to handle impairments of wireless communication media such as multipath propagation. Its ability to turn frequency selective channels into a set of frequency flat ones leads to simplified receiver design. Hence, OFDM is a viable candidate for future 4G wireless communications standards. One of the main drawbacks of OFDM is its high sensitivity to carrier frequency offsets (CFO) caused by the oscillator inaccuracies and the Doppler shift due to mobility, giving rise to inter-carrier interference (ICI) [1], [5], [6]. Therefore, frequency offset estimation must be accomplished with high fidelity.

In this paper, we investigate the performance of a CFO estimation and compensation method introduced in [1] for real-valued constellations. The method is blind in a sense that no knowledge of the channel or transmitted data is required. A cost function minimizing the total off-diagonal power of pseudo-covariance matrices of the received signal is derived. Off-diagonal terms are induced by ICI and should be minimized. Enforcing a diagonal structure leads to perfectly frequency synchronized OFDM transmission. A closed-form expression is found for minimizing the criterion. This leads to low complexity and accurate computational solution.

To assess the large sample performance, we derive the Cramér-Rao bound (CRB) for blind CFO estimators. We model the OFDM signal as a Gaussian process. The received signal is not circular (proper) Gaussian [4] since real-valued constellations are used. Hence the pseudo-covariance matrix is non-zero. As a result, the CRB has to be derived for non-circular Gaussian model [5], [7]. Simulation results for the presented estimator show good performance with respect to the CRB, especially at low signal-to-noise ratio (SNR). Different behavior of the CRB is observed for circular and non-circular cases.

The rest of the paper is organized as follows. The system model is briefly described in Section 2, and Section 3 presents the blind CFO estimation algorithm. The Cramér-Rao bound is derived in Section 4. Simulation results in the case of a time-invariant frequency selective channel are presented in Section 5 for different noise levels. Finally, Section 6 concludes this paper.

The following notation is used throughout the paper:

E	expectation operator;
Tr	trace;
^{T}	transpose;
^{H}	conjugate transpose;
*	elementwise conjugation;
$\mathbf{A}^{1/2}$	Hermitian square-root;
$\mathbf{A}^{-1}$	matrix inverse;
$\mathbf{A}_{ij}$	(i,j) element of the matrix $\mathbf{A}$;
$\|\|\|_F$	Frobenius norm;
$\otimes$, $\odot$	Kronecker product, Hadamard product;
$\mathbf{I}_N$	identity matrix of size $N \times N$;
$\mathbb{1}_N$	$N \times N$ matrix filled with ones;
vec	stacks columns of a matrix on top of each other;
Re, Im	real part, imaginary part;
$\mathbf{\Pi}_{\Delta}^{\perp}$	projection matrix to subspace orthogonal to $\mathbf{\Delta}$;
diag $\{a_1, \dots, a_N\}$	diagonal matrix with $[a_1, \dots, a_N]^T$ on the main diagonal.

II. SYSTEM MODEL

We use a general OFDM transmission model from [6]. The k-th modulated OFDM block is written as:

$$\mathbf{b}(k) = \mathbf{F}_N \mathbf{a}(k), \tag{1}$$

where $\mathbf{F}_N$ is the $N \times N$ inverse discrete Fourier transform (IDFT) matrix, N is the total number of subcarriers, and $\mathbf{a}(k)$ is the real-valued $N \times 1$ symbol vector.

The received OFDM $N \times 1$ signal block in time domain after cyclic prefix removal, including frequency offset, may be expressed as:

$$\mathbf{r}(k) \quad = \quad \mathbf{C}_\epsilon \mathbf{u}(k) + \mathbf{w}(k), \tag{2}$$

where $\mathbf{u}(k) \triangleq \widetilde{\mathbf{H}}\mathbf{b}(k)$, $\widetilde{\mathbf{H}}$ is the $N \times N$ circulant channel matrix and the $N \times N$ diagonal matrix

$$\mathbf{C}_\epsilon \triangleq \operatorname{diag}\left\{\exp\left(j\frac{2\pi k\epsilon}{N}\right), k = 0, \ldots, P-1\right\} \tag{3}$$

is used to introduce the frequency offset. The quantity $\epsilon \in [0, 1)$ is referred to as normalized frequency offset with respect to intercarrier spacing. The length of the cyclic prefix is L and the total OFDM block length is $P = N + L$.

The channel matrix $\widetilde{\mathbf{H}}$ is circulant due to cyclic prefix insertion and removal operations. The channel is assumed to be time-invariant and to have a maximum of L_h taps, hence it is frequency selective. The length of the cyclic prefix is $L \geq L_h$ in order to avoid inter-block interference. The complex noise term $\mathbf{w}$ in (2) is assumed to be circular complex Gaussian [4]. The signal and noise processes are assumed to be mutually independent, and i.i.d. over time index k.

Given an estimate $\hat{\epsilon}$ of the true value ϵ, CFO compensation may be performed at the receiver in time domain prior to the discrete Fourier transform:

$$\begin{aligned} \mathbf{v}_{\hat{\epsilon}}(k) &= \mathbf{F}_N^H \mathbf{C}_{\hat{\epsilon}}^* \mathbf{r}(k) && (4) \\ &= \mathbf{F}_N^H \mathbf{C}_{\hat{\epsilon}-\epsilon}^* \mathbf{u}(k) + \mathbf{C}_{\hat{\epsilon}}^* \mathbf{w}(k), && (5) \end{aligned}$$

where $\mathbf{C}_{\hat{\epsilon}}$ and $\mathbf{C}_{\hat{\epsilon}-\epsilon}$ have the structure defined in (3).

III. BLIND CFO ESTIMATION EXPLOITING NON-CIRCULARITY

In this section we present briefly the proposed carrier frequency offset (CFO) estimation method for OFDM with real-valued constellations. An extensive description may be found in [1].

Let us define the covariance and pseudo-covariance matrices of $\mathbf{u}(k)$ in (2), respectively $\mathbf{Q}$ and $\mathbf{P}$, as follows:

$$\mathbf{Q} \triangleq \mathrm{E}\left[\mathbf{u}(k)\mathbf{u}^H(k)\right], \quad \mathbf{P} \triangleq \mathrm{E}\left[\mathbf{u}(k)\mathbf{u}^T(k)\right]. \tag{6}$$

Let $\widetilde{\mathbf{P}}(\mu) = \mathrm{E}\left[\mathbf{v}_\mu(k)\mathbf{v}_\mu^T(k)\right]$, where $\mathbf{v}_\mu(k)$ denotes $\mathbf{v}_{\hat{\epsilon}}(k)$ in (4) evaluated at $\hat{\epsilon} = \mu$. Then $\widetilde{\mathbf{P}}(\mu)$ and $\mathbf{P}$ are related by:

$$\widetilde{\mathbf{P}}(\mu) = \mathbf{F}_N^H \mathbf{C}_{\mu-\epsilon}^* \mathbf{P}\, \mathbf{C}_{\mu-\epsilon}^* \mathbf{F}_N^H. \tag{7}$$

Null or perfectly compensated frequency offset ($\mu = \epsilon$) leads to a perfectly orthogonal transmission, and $\widetilde{\mathbf{P}}(\mu)$ becomes diagonal. Off-diagonal elements are introduced by inter-carrier interference and should be minimized.

The justification to consider pseudo-covariance instead of the commonly used covariance is that it vanishes for circular complex random variables such as the complex noise term $\mathbf{w}$, but the information on the frequency offset ϵ is retained. Consequently, the proposed method may be applied to real constellations.

The total off-diagonal power $\mathcal{J}(\mu)$ of $\widetilde{\mathbf{P}}(\mu)$ may be written as:

$$\mathcal{J}(\mu) = \left\|\widetilde{\mathbf{P}}(\mu) \odot (\mathbb{1}_N - \mathbf{I}_N)\right\|_F^2. \tag{8}$$

It can be shown that $\mathcal{J}(\epsilon) = 0$, and $\mathcal{J}(\mu) > \mathcal{J}(\epsilon)$ for $\mu \neq \epsilon$, $\mu \in [0, 1)$. Hence, in theory, CFO can be found by driving $\mathcal{J}(\mu)$ to zero. In practice, only an estimate $\widehat{\widetilde{\mathbf{P}}}(\mu)$ of $\widetilde{\mathbf{P}}(\mu)$ is available. Then, an estimate of CFO, $\hat{\epsilon}$, may found by:

$$\hat{\epsilon} = \arg\min_\mu \widehat{\mathcal{J}}(\mu), \quad \widehat{\mathcal{J}}(\mu) = \left\|\widehat{\widetilde{\mathbf{P}}}(\mu) \odot (\mathbb{1}_N - \mathbf{I}_N)\right\|_F^2. \tag{9}$$

As shown in [1], the cost function may be written as:

$$\widehat{\mathcal{J}}(\mu) = A + B\cos(2\pi\mu) + C\sin(2\pi\mu), \tag{10}$$

where $A, B, C \in \mathbb{R}$. Hence, closed-form minimization of $\widehat{\mathcal{J}}$ may be performed and the extremum points are given by:

$$\mu_k = \frac{1}{2\pi}\arctan\left\{\frac{\sqrt{3}\left(\widehat{\mathcal{J}}\left(\frac{1}{3}\right) - \widehat{\mathcal{J}}\left(\frac{2}{3}\right)\right)}{2\widehat{\mathcal{J}}(0) - \widehat{\mathcal{J}}\left(\frac{1}{3}\right) - \widehat{\mathcal{J}}\left(\frac{2}{3}\right)}\right\} + \frac{k}{2} + \frac{1}{4},\ k = 0, 1. \tag{11}$$

Finally the frequency offset estimate is found by choosing:

$$\hat{\epsilon} = \arg\min_{k=0,1} \widehat{\mathcal{J}}(\mu_k). \tag{12}$$

IV. CRAMÉR-RAO BOUND

To assess the large sample performance of the proposed method, we derive the Cramér-Rao bound (CRB), under the assumption that the transmitted symbol vector $\mathbf{b}(k)$ in (1) is Gaussian. CRB gives the minimum variance an unbiased estimator may achieve. Since we are interested in real symbol constellations, $\mathbf{b}(k)$ cannot be modeled as a complex circular random vector. Hence, a complete second order statistics includes both signal covariance and pseudo-covariance matrices [7].

A. CRB for multivariate complex Gaussian distributions

First, we extend the result on Cramér-Rao bounds in [3] to multivariate complex Gaussian distributions. Let us consider the following zero-mean complex random vector:

$$\mathbf{z} = \mathbf{x} + j\mathbf{y},$$

where $\mathbf{x}$ and $\mathbf{y}$ are zero-mean jointly Gaussian real-valued random vectors ($\mathrm{E}[\mathbf{z}] = \mathbf{0}$) of size $N \times 1$. Let $\tilde{\mathbf{z}} = \begin{bmatrix} \mathbf{z} \\ \mathbf{z}^* \end{bmatrix}$

and define $\mathbf{\Omega}$ as $\mathbf{\Omega} = \mathrm{E}\left[\tilde{\mathbf{z}}\tilde{\mathbf{z}}^H\right]$. Then,

$$\mathbf{\Omega} = \begin{bmatrix} \mathbf{\Gamma} & \mathbf{C} \\ \mathbf{C}^* & \mathbf{\Gamma}^* \end{bmatrix}, \text{ with } \mathbf{\Gamma} \triangleq \mathrm{E}\left[\mathbf{z}\mathbf{z}^H\right] \text{ and } \mathbf{C} \triangleq \mathrm{E}\left[\mathbf{z}\mathbf{z}^T\right]. \tag{13}$$

Let us assume that $\tilde{\mathbf{z}}$ depends on a $R \times 1$ vector of parameters $\boldsymbol{\theta}$. Then, by assuming that $\mathbf{\Omega}$ is of full rank, the probability density function (pdf) of $\tilde{\mathbf{z}}$ given $\boldsymbol{\theta}$ is written as [7]:

$$f(\tilde{\mathbf{z}}, \boldsymbol{\theta}) = \left(\pi^{-N}\right) \det(\mathbf{\Omega}) \exp\left(-\frac{1}{2}\tilde{\mathbf{z}}^H \mathbf{\Omega}^{-1} \tilde{\mathbf{z}}\right). \tag{14}$$

The corresponding log-likelihood function is:

$$\ln f(\tilde{\mathbf{z}}, \boldsymbol{\theta}) = -N \ln \pi - \frac{1}{2} \ln \det(\mathbf{\Omega}) - \frac{1}{2} \tilde{\mathbf{z}}^H \mathbf{\Omega}^{-1} \tilde{\mathbf{z}}. \tag{15}$$

After some algebra and following derivations in [3], the (i, j) element of the Fisher information matrix (FIM) for the parameter vector $\boldsymbol{\theta}$ is obtained as:

$$[\mathrm{FIM}_{ij}] = -\mathrm{E}\left[[\ln f(\tilde{\mathbf{z}}, \boldsymbol{\theta})]''_{ij}\right] = \frac{1}{2} \mathrm{Tr}\left\{\mathbf{\Omega}^{-1}\mathbf{\Omega}'_i \mathbf{\Omega}^{-1} \mathbf{\Omega}'_j\right\}, \tag{16}$$

where $2[\ln f(\tilde{\mathbf{z}}, \boldsymbol{\theta})]''_{ij} \triangleq \frac{\partial^2}{\partial\theta_i \partial\theta_j} \ln f(\tilde{\mathbf{z}}, \boldsymbol{\theta})$, $i, j = 1, \ldots, R$, and $\mathbf{\Omega}'_i \triangleq \frac{\partial \mathbf{\Omega}}{\partial \theta_i}$. We notice that for circular complex random distributions (i.e. $\mathbf{C} = \mathbf{0}$), equation (16) is equivalent to the result in [3].

Let us next consider K i.i.d. random vectors $\mathbf{z}(k)$ of size $N \times 1$, $k = 1, \ldots, K$, with the pdf given in (14). Now, with $\mathbf{z}_K^T = \left[\mathbf{z}^T(1), \mathbf{z}^T(2), \ldots, \mathbf{z}^T(K)\right]^T$ as data vector, it follows from (16) that the FIM for the parameter vector $\boldsymbol{\theta}$ becomes:

$$[\mathrm{FIM}_{ij}] = \frac{K}{2} \mathrm{Tr}\left\{\mathbf{\Omega}^{-1}\mathbf{\Omega}'_i \mathbf{\Omega}^{-1} \mathbf{\Omega}'_j\right\}. \tag{17}$$

B. CRB for blind CFO estimation

With the problem of blind frequency synchronization at hand, the signal model becomes the following:

$$\mathbf{z} = \mathbf{C}_\epsilon \mathbf{s} + \mathbf{w}, \tag{18}$$

where:

1) The frequency offset matrix $\mathbf{C}_\epsilon$ is defined as in (3), and ϵ acts as a deterministic unknown parameter to be estimated.
2) Effects of the unknown wireless channel and transmitted data symbols are both embedded in the random vector $\mathbf{s}$, which is assumed to be non-circular complex Gaussian.
3) The noise term $\mathbf{w}$ is assumed to be circular complex Gaussian.
4) Vectors $\mathbf{z}$, $\mathbf{s}$ and $\mathbf{w}$ are of size $N \times 1$.

Due to the non-circularity induced by $\mathbf{s}$, the random vector $\mathbf{z}$ is also complex non-circular Gaussian. First, we express covariance and pseudo-covariance matrices of $\mathbf{z}$ in terms of those of $\mathbf{s}$:

$$\begin{aligned} \mathbf{\Gamma} &\triangleq \mathrm{E}\left[\mathbf{z}\mathbf{z}^H\right] = \mathbf{C}_\epsilon \mathbf{Q} \mathbf{C}_\epsilon^H + \sigma \mathbf{I}_N, \quad & \mathbf{Q} &\triangleq \mathrm{E}\left[\mathbf{s}\mathbf{s}^H\right] \\ \mathbf{C} &\triangleq \mathrm{E}\left[\mathbf{z}\mathbf{z}^T\right] = \mathbf{C}_\epsilon \mathbf{P} \mathbf{C}_\epsilon^T, & \mathbf{P} &\triangleq \mathrm{E}\left[\mathbf{s}\mathbf{s}^T\right] \end{aligned} \tag{19}$$

In more compact matrix form:

$$\begin{aligned} \mathbf{\Omega} &\triangleq \begin{bmatrix} \mathbf{\Gamma} & \mathbf{C} \\ \mathbf{C}^* & \mathbf{\Gamma}^* \end{bmatrix} & (20) \\ &= \widetilde{\mathbf{C}}_\epsilon \widetilde{\mathbf{\Omega}} \widetilde{\mathbf{C}}_\epsilon^H + \sigma \mathbf{I}_{2N}, & (21) \end{aligned}$$

where $\mathbf{\Omega}$ is assumed to be full rank and

$$\widetilde{\mathbf{C}}_\epsilon \triangleq \begin{bmatrix} \mathbf{C}_\epsilon & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_\epsilon^* \end{bmatrix}, \quad \widetilde{\mathbf{\Omega}} \triangleq \begin{bmatrix} \mathbf{Q} & \mathbf{P} \\ \mathbf{P}^* & \mathbf{Q}^* \end{bmatrix}.$$

The entire statistics $\{\mathbf{P}, \mathbf{Q}\}$ only depend on a finite number M of real-valued unknown parameters stacked into the vector:

$$\boldsymbol{\rho} = [\rho_1, \ldots, \rho_M]^T. \tag{22}$$

The vector $\boldsymbol{\rho}$ is made from $\{\mathrm{Re}(\mathbf{P}_{ij}), \mathrm{Im}(\mathbf{P}_{ij}) \text{ for } j \geq i\}$, $\{\mathbf{Q}_{ii}\}$ and $\{\mathrm{Re}(\mathbf{Q}_{ij}), \mathrm{Im}(\mathbf{Q}_{ij}) \text{ for } j > i\}$. Hence, there exists a matrix $\mathbf{J}$ of size $4N^2 \times M$ such that:

$$\mathrm{vec}\left(\widetilde{\mathbf{\Omega}}\right) = \mathbf{J}\boldsymbol{\rho}. \tag{23}$$

Parameters can be stacked into the vector $\boldsymbol{\theta} = \left[\epsilon\ \boldsymbol{\rho}^T\ \sigma\right]^T$, where ϵ is the parameter of interest, whereas $\boldsymbol{\rho}$ and σ are nuisance parameters. The CRB matrix [3] has the following structure:

$$\begin{aligned} \mathrm{CRB} &= \left[\begin{array}{c|cc} \mathrm{E}\left[\tilde{\epsilon}^2\right] & \mathrm{E}\left[\tilde{\epsilon}\tilde{\boldsymbol{\rho}}^T\right] & \mathrm{E}\left[\tilde{\epsilon}\tilde{\sigma}\right] \\ \hline \mathrm{E}\left[\tilde{\boldsymbol{\rho}}\tilde{\epsilon}\right] & \mathrm{E}\left[\tilde{\boldsymbol{\rho}}\tilde{\boldsymbol{\rho}}^T\right] & \mathrm{E}\left[\tilde{\boldsymbol{\rho}}\tilde{\sigma}\right] \\ \mathrm{E}\left[\tilde{\sigma}\tilde{\epsilon}\right] & \mathrm{E}\left[\tilde{\sigma}\tilde{\boldsymbol{\rho}}^T\right] & \mathrm{E}\left[\tilde{\sigma}^2\right] \end{array}\right] \\ &\triangleq \mathrm{FIM}^{-1}, \end{aligned} \tag{24}$$

where $\tilde{\epsilon} = \hat{\epsilon} - \epsilon$, $\tilde{\sigma} = \hat{\sigma} - \sigma$, $\tilde{\boldsymbol{\rho}} = \hat{\boldsymbol{\rho}} - \boldsymbol{\rho}$ are estimation errors, respectively on ϵ, σ and $\boldsymbol{\rho}$.

We are interested in obtaining the CRB for ϵ only. Hence, we need to calculate the $(1, 1)$ element of the general CRB matrix shown above, that is the $(1, 1)$ term of the inverse of the Fisher information matrix. The CRB for a similar model has also been independently derived in [9]. The derivation is based on calculating explicitly the nine blocks of the Fisher information matrix, which would be very tedious in our case.

Following [8] and using (17), the (i, j) element of the FIM is given by:

$$\begin{aligned} [\mathrm{FIM}_{ij}] &= \frac{K}{2} \mathrm{Tr}\left\{\mathbf{\Omega}^{-1} \frac{\partial \mathbf{\Omega}}{\partial \theta_i} \mathbf{\Omega}^{-1} \frac{\partial \mathbf{\Omega}}{\partial \theta_j}\right\} \\ &= \frac{K}{2} \mathrm{vec}\left(\frac{\partial \mathbf{\Omega}}{\partial \theta_i}\right)^H \mathrm{vec}\left(\mathbf{\Omega}^{-1} \frac{\partial \mathbf{\Omega}}{\partial \theta_j} \mathbf{\Omega}^{-1}\right) \\ &= \frac{K}{2} \mathrm{vec}\left(\frac{\partial \mathbf{\Omega}}{\partial \theta_i}\right)^H \left(\mathbf{\Omega}^{-T} \otimes \mathbf{\Omega}^{-1}\right) \mathrm{vec}\left(\frac{\partial \mathbf{\Omega}}{\partial \theta_j}\right), \end{aligned} \tag{25}$$

where we used the following well-known identities [2]:

$$\begin{aligned} \mathrm{Tr}(\mathbf{X}\mathbf{Y}) &= \left(\mathrm{vec}\left(\mathbf{X}^H\right)\right)^H \mathrm{vec}(\mathbf{Y}) & (26) \\ \mathrm{vec}(\mathbf{X}\mathbf{Y}\mathbf{Z}) &= \left(\mathbf{Z}^T \otimes \mathbf{X}\right) \mathrm{vec}(\mathbf{Y}), & (27) \end{aligned}$$

which hold for any conformable matrices $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$. Equivalently, based on (25), the whole FIM may be expressed as:

$$\text{FIM} = \frac{K}{2}\left(\frac{\partial \boldsymbol{\omega}}{\partial \boldsymbol{\theta}^T}\right)^H \left(\boldsymbol{\Omega}^{-T} \otimes \boldsymbol{\Omega}^{-1}\right)\left(\frac{\partial \boldsymbol{\omega}}{\partial \boldsymbol{\theta}^T}\right), \tag{28}$$

where $()^{-T}$ denotes $\left(()^{-1}\right)^T$ and $\boldsymbol{\omega}$ is defined as:

$$\boldsymbol{\omega} \triangleq \text{vec}\left(\boldsymbol{\Omega}\right) = \left(\widetilde{\mathbf{C}}_\epsilon^* \otimes \widetilde{\mathbf{C}}_\epsilon\right)\text{vec}\left(\widetilde{\boldsymbol{\Omega}}\right) + \sigma\,\text{vec}\left(\mathbf{I}_{2N}\right). \tag{29}$$

The Kronecker product $\left(\boldsymbol{\Omega}^{-T} \otimes \boldsymbol{\Omega}^{-1}\right)$ can be written as:

$$\left(\boldsymbol{\Omega}^{-T} \otimes \boldsymbol{\Omega}^{-1}\right) = \left(\boldsymbol{\Omega}^{-T/2} \otimes \boldsymbol{\Omega}^{-1/2}\right)\left(\boldsymbol{\Omega}^{-T/2} \otimes \boldsymbol{\Omega}^{-1/2}\right)$$

and the matrix $\left(\boldsymbol{\Omega}^{-T/2} \otimes \boldsymbol{\Omega}^{-1/2}\right)\left(\frac{\partial \boldsymbol{\omega}}{\partial \boldsymbol{\theta}^T}\right)$ is partitioned into

$$\begin{aligned} & \left(\boldsymbol{\Omega}^{-T/2} \otimes \boldsymbol{\Omega}^{-1/2}\right)\left[\frac{\partial \boldsymbol{\omega}}{\partial \epsilon}\Big|\frac{\partial \boldsymbol{\omega}}{\partial \boldsymbol{\rho}^T}\frac{\partial \boldsymbol{\omega}}{\partial \sigma}\right] \\ \triangleq \;& \left[\mathbf{g}_{4N^2\times 1} \mid \boldsymbol{\Delta}_{4N^2\times(M+1)}\right]. \end{aligned} \tag{30}$$

Using the matrix form in (30), the FIM from equation (28) can be also given as:

$$\text{FIM} = \frac{K}{2}\begin{bmatrix} \mathbf{g}^H \\ \boldsymbol{\Delta}^H \end{bmatrix}\begin{bmatrix} \mathbf{g} & \boldsymbol{\Delta} \end{bmatrix}. \tag{31}$$

Since CRB $=$ FIM^{-1}, applying a result on the inverse of partitioned matrices [3] onto (31), the $(1,1)$ element of the CRB matrix in (24) corresponding to the CFO, namely CRB (ϵ), is found to be:

$$\begin{aligned} \text{CRB}\left(\epsilon\right) &= \frac{2}{K}\left(\mathbf{g}^H\mathbf{g} - \mathbf{g}^H\boldsymbol{\Delta}\left(\boldsymbol{\Delta}^H\boldsymbol{\Delta}\right)^{-1}\boldsymbol{\Delta}^H\mathbf{g}\right)^{-1} \\ &= \frac{2}{K}\left(\mathbf{g}^H\boldsymbol{\Pi}_{\boldsymbol{\Delta}}^{\perp}\mathbf{g}\right)^{-1}, \end{aligned} \tag{32}$$

where the vector $\mathbf{g}$ was defined in equation (30) as $\mathbf{g} = \left(\boldsymbol{\Omega}^{-T/2} \otimes \boldsymbol{\Omega}^{-1/2}\right)\frac{\partial \boldsymbol{\omega}}{\partial \epsilon}$ with

$$\frac{\partial \boldsymbol{\omega}}{\partial \epsilon} = \text{vec}\left(\frac{\partial \boldsymbol{\Omega}}{\partial \epsilon}\right) = \text{vec}\left(\widetilde{\mathbf{D}}_\epsilon\widetilde{\boldsymbol{\Omega}}\widetilde{\mathbf{C}}_\epsilon^H + \widetilde{\mathbf{C}}_\epsilon\widetilde{\boldsymbol{\Omega}}\widetilde{\mathbf{D}}_\epsilon^H\right) \tag{33}$$

$$\widetilde{\mathbf{D}}_\epsilon \triangleq \frac{\partial}{\partial \epsilon}\widetilde{\mathbf{C}}_\epsilon. \tag{34}$$

Now, let us partition the matrix $\boldsymbol{\Delta}$ defined in (30) as:

$$\left(\boldsymbol{\Omega}^{-T/2} \otimes \boldsymbol{\Omega}^{-1/2}\right)\left[\frac{\partial \boldsymbol{\omega}}{\partial \boldsymbol{\rho}^T}\Big|\frac{\partial \boldsymbol{\omega}}{\partial \sigma}\right] \triangleq \left[\mathbf{V}_{4N^2\times M} \,|\, \mathbf{u}_{4N^2\times 1}\right] \tag{35}$$

and calculate the terms $\mathbf{V}$ and $\mathbf{u}$. By definition:

$$\mathbf{V} \triangleq \left(\boldsymbol{\Omega}^{-T/2} \otimes \boldsymbol{\Omega}^{-1/2}\right)\frac{\partial \boldsymbol{\omega}}{\partial \boldsymbol{\rho}^T}. \tag{36}$$

Differentiating the expression of $\boldsymbol{\omega}$ in (29) wrt. $\boldsymbol{\rho}$, we obtain:

$$\begin{aligned} \frac{\partial \boldsymbol{\omega}}{\partial \boldsymbol{\rho}^T} &= \left(\widetilde{\mathbf{C}}_\epsilon^* \otimes \widetilde{\mathbf{C}}_\epsilon\right)\frac{\partial}{\partial \boldsymbol{\rho}^T}\left(\text{vec}\left(\widetilde{\boldsymbol{\Omega}}\right)\right) \\ &= \left(\widetilde{\mathbf{C}}_\epsilon^* \otimes \widetilde{\mathbf{C}}_\epsilon\right)\mathbf{J}, \end{aligned} \tag{37}$$

because vec$\left(\widetilde{\boldsymbol{\Omega}}\right) = \mathbf{J}\boldsymbol{\rho}$ from (23). Then, the vector $\mathbf{u}$ is expressed as:

$$\begin{aligned} \mathbf{u} &= \left(\boldsymbol{\Omega}^{-T/2} \otimes \boldsymbol{\Omega}^{-1/2}\right)\frac{\partial \boldsymbol{\omega}}{\partial \sigma} \\ &= \left(\boldsymbol{\Omega}^{-T/2} \otimes \boldsymbol{\Omega}^{-1/2}\right)\text{vec}\left(\mathbf{I}_{2N}\right) \qquad (38) \\ &= \text{vec}\left(\boldsymbol{\Omega}^{-1}\right), \qquad (39) \end{aligned}$$

since $\frac{\partial \boldsymbol{\omega}}{\partial \sigma} = \text{vec}\left(\mathbf{I}_{2N}\right)$ and (39) follows from (38) using (27).

Finally, as we have derived closed-form expressions for $\mathbf{g}$ and $\boldsymbol{\Delta}$, the entire FIM may be also obtained using (31).

C. Application to blind synchronization in OFDM

In theory, the OFDM signal has a discrete probability distribution with $\mathcal{Q}^N$ states, $\mathcal{Q}$ being the symbol modulation order and N the number of subcarriers. However, as N is often large, this distribution is not tractable in practice. Therefore, the Gaussian approximation has been widely used in the literature related to OFDM. This approximation is valid component-wise for a large number of subcarriers, as each signal component is a sum of N i.i.d. random variables.

Hence, we approximate the vector $\mathbf{s}(k) = \widetilde{\mathbf{H}}\mathbf{F}_N\mathbf{a}(k)$ by a multivariate Gaussian vector. Real symbol constellations induce naturally non-circularity. Then, the OFDM transmission with imperfect frequency synchronization in (2) falls within the model presented in (18) and discussed previously. Therefore, the CRB for blind frequency offset estimation in OFDM may be approximated by:

$$\text{CRB}\left(\epsilon\right) = \frac{2}{K}\left(\mathbf{g}^H\boldsymbol{\Pi}_{\Delta}^{\perp}\mathbf{g}\right)^{-1}, \tag{40}$$

where K is the number of considered data blocks and the quantities $\mathbf{g}$, $\boldsymbol{\Pi}_{\Delta}^{\perp}$, are as presented in Section IV-B.

However, as the component-wise Gaussian approximation seems reasonable, the assumption on the joint Gaussianity of the components is more questionable, and this may lead to slight differences with the exact CRB. Also, the above finite sample CRB cannot be interpreted as such. In addition, the asymptotic CRB was studied in [9]. In the next section, numerical evaluations will give insight into the behavior of the CRB with respect to some system parameters.

V. Simulations

This section presents the simulation results, using the blind carrier offset estimation algorithm presented in Section III. The mean square error (MSE) of the estimator is plot against the previously calculated CRB. The OFDM system parameters are chosen as follows: the carrier frequency is $f_0 = 2.4$ GHz, the number of subcarriers is set to $N = 64$ and the available bandwidth is taken equal to $B = 0.5$ MHz. The length L of the cyclic prefix is 4. The subcarrier symbol rate is of 7.8 KHz. The employed symbol modulation is BPSK. The SNR, if not stated otherwise, is equal to 5 dB. The normalized frequency offset is $\epsilon = 0.43$.

The wireless channel is considered to be deterministic but unknown to the receiver. That not only affects the transmission and the proposed algorithm, but also the CRB. The channel impulse response chosen for our simulations has four transmission paths and is the following:

$$\mathbf{h} = \begin{bmatrix} 0.0731 - 0.8702j \\ 0.3613 - 0.4503j \\ -0.1098 + 0.4476j \\ -0.0270 - 0.0942j \end{bmatrix}.$$

Plots of the mean square error versus number of observed blocks and SNR are depicted in Figure 1. Results are close to the CRB in the low SNR region. The graph of the MSE does not show any important dependence of the performance on the noise level. In theory, the pseudo-covariance of the noise vanishes on average. Hence, the performance of the algorithm should not depend on the SNR if noise is circularly symmetric. However, in practice the sample estimates of the pseudo-covariance matrix experience perturbations because of noise which degrades the performance at low SNR.

The proposed method is biased for finite sample size (gap to the CRB in Figure 1), but can be shown to be asymptotically unbiased. However, even though the CRB is not attained, the residual estimation error is small (less than 2% on average after 200 blocks), and hence the proposed method performs accurately almost regardless of the SNR. Possible improvement of the estimate may be obtained using this as an initial estimate and refining it with e.g. the method in [5]. Note that no pilot or virtual carriers (VC) are exploited here. VC usually improve significantly the performance of blind CFO estimation [5], [6], as they induce a low rank signal model, which enables the use of high resolution subspace techniques.

In contrast with real symbol constellations considered throughout this paper, complex constellations such as QPSK, 8PSK or 16QAM induce circular complex random signals. As the CRB was derived for a general complex multivariate Gaussian model, it is interesting to compare together the bounds obtained for circular and non-circular cases. We also compute the CRB when the channel is known to the receiver. This is done as in Section IV, but without viewing anymore the channel as a nuisance parameter. The curves presented in Figure 2 highlight a significant difference in performance between the circular and non-circular case, which is in par with the results in [9]. Much lower CRBs are observed whenever the signal contains non-circularity. Moreover, for finite sample size, the Cramér-Rao bound tends to zero as the SNR tends to infinity for non-circular signals, which does not hold true for the circular case. This implies that blind CFO estimation algorithms may be significantly improved by using a complete second order statistics, in place of the covariance matrix only, when real constellations are in use.

VI. Conclusions

In this paper, we investigated the performance of the carrier frequency offset estimation method for OFDM with real-valued constellations introduced in [1]. To assess the large sample performance, we derived the Cramér-Rao bound for the general multivariate Gaussian model, as real symbol constellations generate non-circular signals. Unlike the approach in [9], the method we applied does not require to calculate and invert the entire Fisher information matrix. We directly derived a closed form expression for the element of the CRB matrix corresponding to CFO. Numerical results show a significant difference in the behavior of the CRB between circular and non-circular cases.

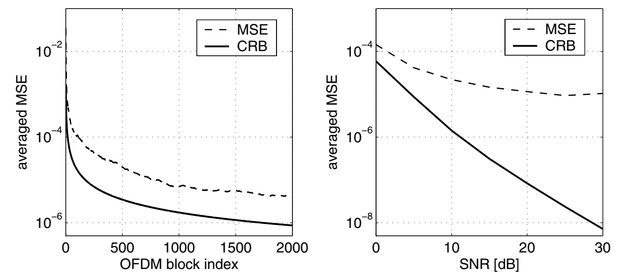

Fig. 1. On the left: MSE vs. number of observed blocks at SNR=5dB. On the right: MSE vs. SNR. ($K = 200$ blocks; ensemble average over 100 realizations; $\epsilon = 0.43$).

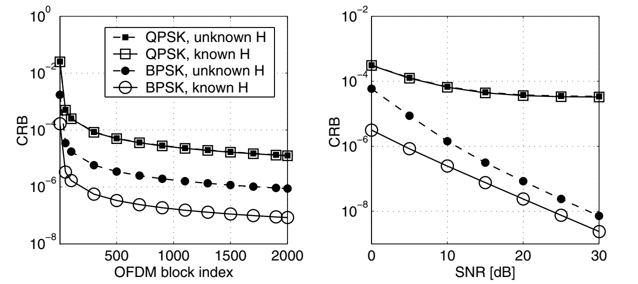

Fig. 2. On the left: CRB vs. number of observed blocks at SNR=5dB. On the right: CRB vs. SNR. ($K = 200$, $\epsilon = 0.43$). BPSK or QPSK modulation; known or unknown channel to the receiver.

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