Subspace method for blind CFO estimation for OFDM systems with constant modulus constellations

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Abstract—In this paper, we propose a novel subspace based approach for blind carrier frequency offset estimation in OFDM. Correlation in squared spectrum of the channel is exploited. Low rank signal model is thereby obtained without virtual subcarriers. The proposed estimator accomplishes frequency synchronization with a single OFDM block. No extensive time averaging is needed, which makes the approach very attractive for time and frequency selective channels where the offset may be time varying. Superior performance is achieved over existing algorithms exploiting virtual carriers or constant modulus criterion.

I. INTRODUCTION

Orthogonal Frequency Division Multiplexing (OFDM) is a powerful technique to mitigate channel impairments in wireless communication such as multipath propagation, with simplified receiver design. OFDM is a viable candidate for future 4G wireless communications standards. One of the main drawbacks of OFDM is its high sensitivity to carrier frequency offsets (CFO) caused by the oscillator inaccuracies and the Doppler shift due to mobility. This gives rise to inter-carrier interference (ICI). Therefore, frequency offset compensation must be accomplished with high fidelity.

In this paper, we introduce a novel blind CFO estimation technique applicable to OFDM with constant modulus constellations (e.g. BPSK, QPSK or 8PSK). The method is blind since it does not require a priori knowledge of the transmitted data or the channel. The proposed frequency offset estimator needs only a single OFDM block to work with, unlike the majority of blind techniques [1], [2], [3] which do almost always require extensive time averaging.

The key idea is to exploit correlation among OFDM subcarriers and more specifically in squared amplitude spectrum of the channel. Novelty of the paper relies on the fact that low rank signal model is derived without any virtual subcarriers, which are commonly assumed to be available in existing subspace CFO estimation methods for OFDM [3], [4].

Simulation results demonstrate that the proposed method outperforms existing blind estimators based on virtual carriers [2], [3], [4], [5] or constant modulus criterion [6].

Performance is superior to some of the semi-blind [7], [8] and pilot-aided CFO estimation schemes [9].

The rest of the paper is organized as follows. The system model is briefly described next. Frequency domain channel correlation in OFDM is studied in Section 3. Then, Section 4 presents the blind CFO estimation algorithm. Finally, simulation results and performance comparison are presented in Section 5. Proofs and derivations may be found in the Appendix.

II. SYSTEM MODEL

We use a general OFDM transmission model from [3]. Let $\mathbf{a}(k) = [a_0(k), \ldots, a_{N-1}(k)]^T$ be the $N \times 1$ symbol vector at time instance k. We assume unit energy complex-valued symbol constellations are used, i.e. $|a_i(k)|^2 = 1$, $i = 0, \ldots, N-1$. The received OFDM $N \times 1$ signal block in time domain after cyclic prefix removal, including the frequency offset, is expressed as

$$\mathbf{z}(k) = \mathbf{C}_\epsilon \mathbf{F}^H \mathbf{D}_{\tilde{\mathbf{h}}}(k)\mathbf{a}(k) + \mathbf{w}(k), \tag{1}$$

where $\mathbf{F} = \left\{ \frac{1}{\sqrt{N}} \exp\left(-j\frac{2\pi kl}{N}\right) \right\}_{k,l=0,\ldots,N-1}$ is the $N \times N$ discrete Fourier transform (DFT) matrix, ^{H} denotes the Hermitian transpose and N is the total number of subcarriers. The diagonal matrix $\mathbf{C}_\epsilon$ introduces the frequency offset and is defined as

$$\mathbf{C}_\epsilon = \exp\left(j\frac{2\pi\epsilon L_{CP}}{N}\right) \cdot \text{diag}\left\{ 1, \ldots, \exp\left(j\frac{2\pi(N-1)\epsilon}{N}\right) \right\} \tag{2}$$

where L_{CP} is the length of the cyclic prefix ($L_{CP} < N$). The length of the whole OFDM block is $P = N + L_{CP}$. The quantity $\epsilon \in [0, 1)$ is referred to as normalized frequency offset (wrt. intercarrier spacing). The diagonal matrix $\mathbf{D}_{\tilde{\mathbf{h}}}(k)$ of size $N \times N$ in (1) contains the channel frequency response $\tilde{\mathbf{h}}(k) = [\tilde{h}_1, \ldots, \tilde{h}_N]^T$ at time instance k on its main diagonal. The complex noise term $\mathbf{w}$ is assumed to be zero-mean proper complex Gaussian. The signal and noise processes are assumed to be mutually independent, and i.i.d. over time.

Given an estimate μ of the true value ϵ, CFO compensation may be performed in time domain at the receiver prior

to DFT. The resulting $N \times 1$ vector $\mathbf{u}_\mu$ in frequency domain may be expressed as

$$\mathbf{u}_\mu(k) = \mathbf{F}\mathbf{C}_\mu^* \mathbf{z}(k), \tag{3}$$

where the matrix $\mathbf{C}_\mu$ has the same structure as in (2) and * denotes the complex conjugate.

III. FREQUENCY DOMAIN CORRELATION

A. Correlation in channel frequency response

The channel is assumed to be block fading and to have a maximum of L_h taps, hence it is frequency selective. The length of the cyclic prefix is set as $L_{CP} \geq L_h$ in order to avoid inter-block interference. A key idea in OFDM transmission is the frequency correlation of the channel among subcarriers induced by the DFT. Let $\mathbf{h}(k)$ be the $L_h \times 1$ channel impulse response in time domain corresponding to the $N \times 1$ channel frequency response vector $\bar{\mathbf{h}}(k)$. Since $L_h < N$, vectors $\mathbf{h}(k)$ and $\bar{\mathbf{h}}(k)$ are related by an N point DFT as $\bar{\mathbf{h}}(k) = \sqrt{N}\mathbf{F}_{\{:,1:L_h\}}\mathbf{h}(k)$, where the matrix $\mathbf{F}_{\{:,1:L_h\}}$ is made from the L_h first columns of the DFT matrix $\mathbf{F}$. Recalling that the $L_h \times 1$ vector $\mathbf{h}(k)$ may be obtained from $\bar{\mathbf{h}}(k)$ via Inverse Discrete Fourier (IDFT) transform as $\mathbf{h}(k) = \frac{1}{\sqrt{N}}\mathbf{F}_{\{:,1:L_h\}}^H \bar{\mathbf{h}}(k)$, the following relationship may be established:

$$\bar{\mathbf{h}}(k) = \mathbf{F}_{\{:,1:L_h\}}\mathbf{F}_{\{:,1:L_h\}}^H \bar{\mathbf{h}}(k) = \mathbf{A}\bar{\mathbf{h}}(k), \tag{4}$$

where the DFT/IDFT pair is denoted by $\mathbf{A} = \mathbf{F}_{\{:,1:L_h\}}\mathbf{F}_{\{:,1:L_h\}}^H$. Since the DFT matrix $\mathbf{F}$ is full rank, the rank of $\mathbf{F}_{\{:,1:L_h\}}$ is L_h, and consequently rank $\{\mathbf{A}\} = L_h$ [10].

B. Correlation in channel squared amplitude spectrum

In the following, we denote by $\odot$ the element-wise Hadamard product [10]. Channel spectrum may not be uniquely derived from the squared spectrum (ambiguity in phase). Therefore we resort to study and characterize correlation properties of squared amplitude spectrum $\bar{\mathbf{h}}(k) \odot \bar{\mathbf{h}}^*(k)$. First, by using the following theorem, we identify the vector subspace of channel squared amplitude spectrum.

Theorem 1: Let $\mathbf{x}$ be a $N \times 1$ vector such that $\mathbf{x} = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a $N \times N$ matrix of rank $L \leq N$. Then, the vector $\mathbf{x} \odot \mathbf{x}^*$ of squared amplitudes lies in the column space of $\mathbf{A} \odot \mathbf{A}^*$. Proof is given in the Appendix.

From (4), $\bar{\mathbf{h}}(k) = \mathbf{A}\bar{\mathbf{h}}(k)$, and according to the above theorem, the squared amplitude spectrum $\bar{\mathbf{h}}(k) \odot \bar{\mathbf{h}}^*(k)$ lies in the column space of $\mathbf{A} \odot \mathbf{A}^*$. Next, Theorem 2 below provides us with a basis for the column space of $\mathbf{A} \odot \mathbf{A}^*$.

Theorem 2: Let $\mathbf{A} = \mathbf{F}_{\{:,1:L_h\}}\mathbf{F}_{\{:,1:L_h\}}^H$ with $\mathbf{F}_{\{:,1:L_h\}} = [\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_{L_h}]$, where $\mathbf{f}_k$ is the k-th

column vector of DFT matrix $\mathbf{F}$. Then, let us contruct the basis $\mathcal{G} = \{\mathbf{g}_{-(L_h-1)}, \dots, \mathbf{g}_{(L_h-1)}\}$ from the $2L_h - 1$ vectors $\mathbf{g}_d$ defined as $\mathbf{g}_d \triangleq \sqrt{N}(\mathbf{f}_k \odot \mathbf{f}_l^*)$, $k, l = 1, \dots, L_h$, $d = k - l$, $d = -(L_h - 1), \dots, (L_h - 1)$. Then, $\mathcal{G}$ forms an orthonormal basis of the column space of $\mathbf{A} \odot \mathbf{A}^*$. Proof is given in the Appendix.

Therefore, the space of channel squared amplitude spectrum is of dimension $2L_h - 1$ We conclude that rank $\{\mathbf{A} \odot \mathbf{A}^*\} = 2L_h - 1$. Because $2L_h - 1 < N$ in practice, low rank model arises from correlation in channel squared amplitude spectrum. Based on this property, we propose next a subspace method for frequency synchronization in OFDM.

IV. SUBSPACE METHOD FOR BLIND CFO ESTIMATION

In the following, we introduce a blind CFO estimator which aims at restoring correlation in channel squared frequency response. Unlike the majority of blind CFO recovery techniques, the proposed estimator needs only a single OFDM block to operate with (i.e. no time averaging needs to be performed). Hence it allows finding an estimate for each block independently. This is obviously significant advantage in case of time-selective channels. Next, we provide description of the proposed algorithm. From now on, we drop the time index k, for simplicity of the notation.

Let us consider the noise-free case in equations (1-3) and compute the element-wise Hadamard product $\mathbf{u}_\mu \odot \mathbf{u}_\mu^*$ as

$$\mathbf{u}_\mu \odot \mathbf{u}_\mu^* = (\mathbf{M}_{\mu-\epsilon}\mathbf{D}_{\bar{\mathbf{h}}}\mathbf{a}) \odot (\mathbf{M}_{\mu-\epsilon}^*\mathbf{D}_{\bar{\mathbf{h}}}^*\mathbf{a}^*), \tag{5}$$

where we defined $\mathbf{M}_{\mu-\epsilon} = \mathbf{F}\mathbf{C}_\mu^*\mathbf{C}_\epsilon\mathbf{F}^H$. In case of perfect frequency synchronization, $\mu = \epsilon$ and $\mathbf{M}_{\mu-\epsilon} = \mathbf{I}_N$, where $\mathbf{I}_N$ is the $N \times N$ identity matrix. Then (5) becomes

$$\mathbf{u}_\epsilon \odot \mathbf{u}_\epsilon^* = (\mathbf{D}_{\bar{\mathbf{h}}}\mathbf{a}) \odot (\mathbf{D}_{\bar{\mathbf{h}}}^*\mathbf{a}^*) \tag{6}$$

$$= \bar{\mathbf{h}} \odot \bar{\mathbf{h}}^*, \tag{7}$$

since $\mathbf{D}_{\bar{\mathbf{h}}}$ is diagonal and $\mathbf{a} \odot \mathbf{a}^* = [|a_0|^2, \dots, |a_{N-1}|^2]^T = [1, \dots, 1]^T$ under the assumption of constant modulus complex-valued modulations ($|a_i| = 1$, $i = 0, \dots, N - 1$). Therefore $\mathbf{u}_\epsilon \odot \mathbf{u}_\epsilon^*$ is equal to the squared amplitude spectrum $\bar{\mathbf{h}} \odot \bar{\mathbf{h}}^*$. Consequently, it lies in the same subspace and inherits the same correlation properties as well. Frequency mismatch ($\mu \neq \epsilon$) leads to intercarrier interference (ICI) and alters the components of $\mathbf{u}_\mu \odot \mathbf{u}_\mu^*$, which does not lie in the correct subspace anymore, i.e. the one of $\bar{\mathbf{h}} \odot \bar{\mathbf{h}}^*$.

This leads to the idea of restoring the subspace structure induced by Fourier transforms in case of perfect synchronization. It is performed by maximizing the projection of $\mathbf{u}_\mu \odot \mathbf{u}_\mu^*$ in the subspace spanned by $\bar{\mathbf{h}} \odot \bar{\mathbf{h}}^*$, or equivalently, by minimizing the projection in the orthogonal subspace. Initial subspace is restored for $\mu = \epsilon$. The spanned subspaces are depicted in Figures 1. Since $\bar{\mathbf{h}} \odot \bar{\mathbf{h}}^*$ lies in the column space of $\mathbf{A} \odot \mathbf{A}^*$ and the rank of the latter is

$2L_h - 1 < N$, subspace technique may then be employed. Note that virtual subcarriers (i.e. carriers carrying no data) are not needed to ensure low rank model, since it naturally arises from correlation in channel spectrum.
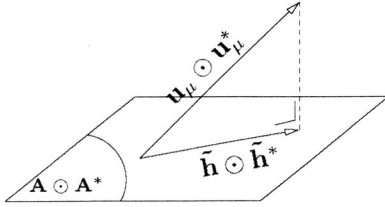

Fig. 1. Subspaces used in proposed blind CFO estimation. Projection of $\mathbf{u}_\mu \odot \mathbf{u}_\mu^*$ to subspace of $\tilde{\mathbf{h}} \odot \tilde{\mathbf{h}}^*$ is maximized, or equivalently projection to orthogonal subspace is minimized.

Let us define the squared norm of the projection of $\mathbf{u}_\mu \odot \mathbf{u}_\mu^*$ to the orthogonal subspace of $\mathbf{A} \odot \mathbf{A}^*$ as cost function $\mathcal{C}$,

$$\mathcal{C}(\mu) = \left\| \mathbf{\Pi}_{\mathbf{A} \odot \mathbf{A}^*}^{\perp} \left(\mathbf{u}_\mu \odot \mathbf{u}_\mu^* \right) \right\|^2, \tag{8}$$

where $\| \ \|^2$ is the squared Euclidean norm and $\mathbf{\Pi}_{\mathbf{A} \odot \mathbf{A}^*}^{\perp}$ denotes the projection matrix to subspace orthogonal to the columns of $\mathbf{A} \odot \mathbf{A}^*$. The cost function $\mathcal{C}(\mu)$ is periodic with period 1, because replacing μ by $\mu + 1$ only produces a shift by one of the OFDM subcarriers, and therefore the correlation features remain unchanged.

As the matrix $\mathbf{A}$ depends only on the DFT size N and the channel length L_h, it may be computed offline, as well as the projection matrix $\mathbf{\Pi}_{\mathbf{A} \odot \mathbf{A}^*}^{\perp}$. Furthermore, since Theorem 2 provides us with an orthonormal basis for the column subspace of $\mathbf{A} \odot \mathbf{A}^*$, we may express this projection matrix in closed form as

$$\mathbf{\Pi}_{\mathbf{A} \odot \mathbf{A}^*}^{\perp} = \mathbf{I} - \mathbf{G} \left(\mathbf{G}^H \mathbf{G} \right)^{-1} \mathbf{G}^H = \mathbf{I} - \mathbf{G} \mathbf{G}^H, \tag{9}$$

where the $N \times (2L_h - 1)$ matrix $\mathbf{G}$ is made out from the basis vectors of $\mathcal{G}$ stacked in matrix form. Note that having an orthonormal basis significantly reduces the computational cost (no matrix inversion is required).

Finally, an estimate $\hat{\epsilon}$ of the CFO is found by minimizing the projection of vector $\mathbf{u}_\mu \odot \mathbf{u}_\mu^*$ to orthogonal subspace of $\mathbf{A} \odot \mathbf{A}^*$ as

$$\hat{\epsilon} = \arg \min_{\mu \in [0,1)} \mathcal{C}(\mu). \tag{10}$$

Numerical solution to (10) may be found e.g. using a gradient descent method. Computational cost is not prohibitive due to a one dimensional search space with unique minimum in $[0, 1]$. The cost function to be minimized is shown in Figure 2. We compare in simulations the performance of the proposed subspace based constant modulus approach (denoted Proposed CM in simulation graphs) to the method introduced by Liu and Tureli in [4] exploiting virtual subcarriers (VSC). Since the VSC estimator would fail in a fully loaded OFDM system, we consider $N_v = 15$ virtual carriers for this specific method. In [6] Ghogho and Swami derived a criterion exploiting constant modulus (CM) symbols as well as a composite cost function involving both CM property and VSCs (VSC+CM). We compare our technique to those methods as well (Fig. 2,4 & 5).
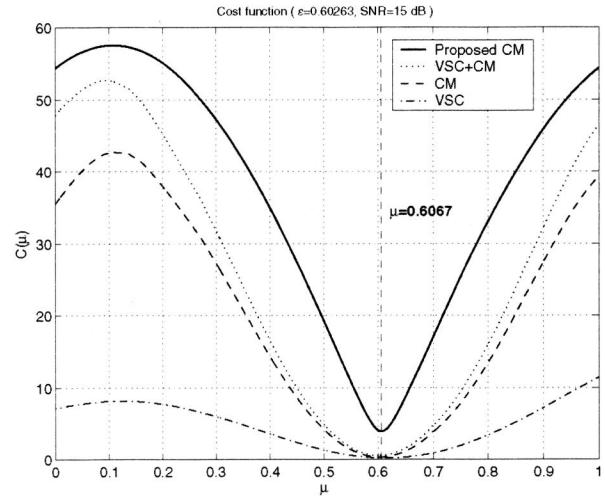

Fig. 2. Cost functions for proposed subspace based blind CFO estimation, VSC+CM based method, and CM and VSC only; $\epsilon = 0.60263$, SNR=15 dB.

V. NUMERICAL RESULTS

In this section, simulation results are reported. The OFDM system parameters are chosen as follows: the carrier frequency is $f_0 = 2.4$ GHz, the number of subcarriers is set to $N = 64$ and the available bandwidth is $B = 0.5$ MHz. The length L_{CP} of the cyclic prefix is 10. QPSK modulation is used.

The wireless channel is considered to have eight independent paths ($L_h = 8$) with unit variance Rayleigh i.i.d. distributed coefficients. Block fading is assumed, i.e. the channel stays constant within one OFDM block and varies from block to block. The normalized frequency offset is assumed to be uniformly distributed between $[0, 1]$, and to vary block-wise.

Numerical gradient descent method was used to solve the minimization problem in (10). Squared amplitude spectra before and after CFO correction are plotted in Figure 3 against the true one (i.e. with the true channel frequency response and perfect synchronization). The proposed algorithm restores correlation among subcarriers with high fidelity, without any knowledge of the frequency selective channel, under noise and severe frequency mismatch conditions (SNR = 15 dB and $\epsilon = 0.4496$ in Fig. 3).

A. Performance versus SNR

The Mean Square Error (MSE) is chosen as an error criterion for carrier offset estimation: MSE $= \mathrm{E} |\hat{\epsilon} - \epsilon|^2$. Plot of the MSE versus the signal-to-noise ratio (SNR) is depicted in Figure 4. Results are ensemble averaged on 2000
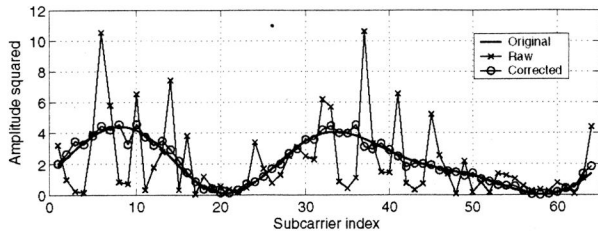
Fig. 3. Squared amplitude spectrum; QPSK modulation; $\epsilon = 0.4496$; SNR=15 dB. Channel amplitude spectrum is restored with high fidelity, under noise and severe frequency mismatch.

different channel and CFO realizations. Highly accurate tracking of time-varying CFO is achieved with a single OFDM block in time-frequency selective channels, in a fully blind manner, and for broad-range of SNRs (0 ot 50 dB). At 10^{-6} MSE, we obtained 12.3 dB gain over the VSC method [4], 2.1 dB over the CM method [6] and 1.6 dB over the composite VSC+CM method [6]. Thus, the constant modulus assumption is more effective than having virtual carriers only, and leads to more than 10 dB performance gain. Computational complexity compared to [6] is lower as well.

The high-resolution subspace technique based on ESPRIT algorithm proposed by Liu and Tureli in [4] requires virtual subcarriers to employ low rank signal model. Virtual subcarriers provide help especially for time and frequency synchronization problems, but at the expense of bandwidth efficiency, since those carriers do not carry any data. VSC based estimation would fail without virtual carriers (i.e. fully loaded OFDM system), while our subspace based approach does not need any. Due to DFT and IDFT operations, low rank model is feasible in OFDM transmission provided that the channel length L_h is such that $2L_h - 1 < N$. This is generally the case for a well designed OFDM system since $L \leq L_{CP} < (N+1)/2$.

Performance is also superior to other existing blind [3] or semi-blind techniques [7]. Numerous blind frequency synchronization methods have been proposed in the literature related to OFDM, but most of them need extensive time-averaging in order to get rid of the influence of both noise and data symbols. Hence they cannot perform well in estimating accurately CFO with a single OFDM block [3]. Results are also comparable to pilot-aided CFO estimation techniques such as in [9].

B. Sensitivity to unknown channel order

Previous results were obtained assuming the knowledge of channel length at the receiver. Delay spread may be estimated in practice, e.g. by investigating second-order cyclo-stationarity introduced by cyclic prefix. The actual channel length is set here to $L_h = 8$, and the assumed length at the receiver varies from 1 to 32, at SNR=15 dB. Results are presented in Figure 5.

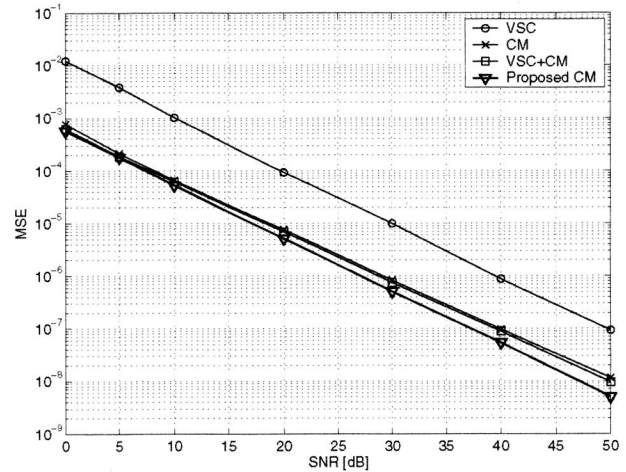
Fig. 4. MSE of CFO estimators vs. SNR; QPSK modulation; i.i.d. Rayleigh block fading channel of length $L_h = 8$; uniformly distributed CFO in $[0, 1]$ and varying block-wise; $N_v = 15$ virtual carriers considered for the VSC and VSC+CM estimators; SNR=15 dB; ensemble average (2000 blocks).
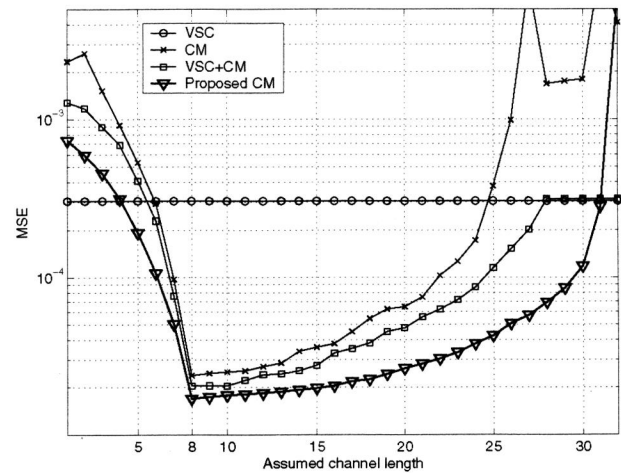

Fig. 5. MSE of CFO estimators vs. assumed channel length; the actual channel length is $L_h = 8$; SNR=15 dB; ensemble average (1000 blocks).

The CM based, VSC+CM and the proposed estimators need the channel length as input parameter. They all suffer from under-estimated delay spread, while they still perform well with (slightly) larger assumed channel order. The proposed algorithm is shown to work accurately over a broader range of values compared to the other ones. The length of the cyclic prefix ($L_{CP} = 10$, in our case) yields a reasonable upper limit in practice. Note that purely VSC based estimators are not affected by an unknown channel length, as the subspace structure is determined by the location of virtual carriers, assumed to be of prior knowledge.

VI. CONCLUSIONS

In this paper, we propose a novel subspace based algorithm for blind carrier frequency offset estimation in

OFDM. Exploiting correlation in squared amplitude spectrum of the channel is the key idea of the method. Low rank signal model is obtained without virtual subcarriers. Hence, bandwidth efficiency is high. The proposed estimator performs frequency synchronization with only one OFDM block. Hence, no extensive time averaging is needed. Highly accurate estimation is achieved in time-frequency selective channels. The proposed method outperforms other considered blind CFO estimators exploiting virtual carriers or constant modulus criterion, with lower complexity as well.

VII. Appendix

Proof of Theorem 1: Let $\mathbf{x} = [x_1, \ldots, x_N]^T$ and let $\mathbf{a}_1, \ldots, \mathbf{a}_N$ be the $N \times 1$ column vectors of $\mathbf{A}$. Then, we exploit the relationship $\mathbf{x} = \mathbf{A}\mathbf{x}$ and express the vector of squared amplitudes $\mathbf{x} \odot \mathbf{x}^*$ as

$$\mathbf{x} \odot \mathbf{x}^* = (\mathbf{A}\mathbf{x}) \odot (\mathbf{A}^*\mathbf{x}^*) \tag{11}$$

$$= \left(\sum_{k_1=1}^{N} x_{k_1} \mathbf{a}_{k_1} \right) \odot \left(\sum_{k_2=1}^{N} x_{k_2}^* \mathbf{a}_{k_2}^* \right) \tag{12}$$

$$= \sum_{k_1=1}^{N} \sum_{k_2=1}^{N} x_{k_1} x_{k_2}^* \left(\mathbf{a}_{k_1} \odot \mathbf{a}_{k_2}^* \right) \tag{13}$$

where (12) follows from (11) due to the distributivity of the Hadamard product. Since rank $\{\mathbf{A}\} = L$, there exist a basis $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_L\}$ of vectors such that $\mathbf{a}_i = \sum_{l=1}^{L} \alpha_{il}\mathbf{c}_l$, $\alpha_{il} \in \mathbb{C}$, $i = 1, \ldots, N$, $l = 1, \ldots, L$.

We now prove that the vector $\mathbf{x} \odot \mathbf{x}^*$ lies in the column space of $\mathbf{A} \odot \mathbf{A}^*$, that is each vector term $\mathbf{a}_{k_1} \odot \mathbf{a}_{k_2}^*$ in (13) lies in span $\{\mathbf{A} \odot \mathbf{A}^*\}$. First, for $k_1 = k_2 = k$, the term $\mathbf{a}_k \odot \mathbf{a}_k^*$ is by definition the kth column of $\mathbf{A} \odot \mathbf{A}^*$, and obviously lies in its column space. Then, the Hadamard product $\mathbf{a}_{k_1} \odot \mathbf{a}_{k_2}^*$ may be written using the basis $\mathcal{C}$ as

$$\mathbf{a}_{k_1} \odot \mathbf{a}_{k_2}^* = \sum_{l_1=1}^{L} \sum_{l_2=1}^{L} \alpha_{k_1 l_1} \alpha_{k_2 l_2}^* \left(\mathbf{c}_{l_1} \odot \mathbf{c}_{l_2}^* \right). \tag{14}$$

Finally for $k_1 \neq k_2$, $\mathbf{a}_{k_1} \odot \mathbf{a}_{k_2}^*$ is a linear combination of vectors $\mathbf{c}_{l_1} \odot \mathbf{c}_{l_2}^*$, as it is also for each column $\mathbf{a}_k \odot \mathbf{a}_k^*$ of $\mathbf{A} \odot \mathbf{A}^*$ (see (14)). Hence $\mathbf{a}_{k_1} \odot \mathbf{a}_{k_2}^*$ lies in the same vector space. ∎

Proof of Theorem 2: Given $\mathbf{A} = \mathbf{F}_{\{:,1:L_h\}}\mathbf{F}_{\{:,1:L_h\}}^H$, the k-th column vector of $\mathbf{A} \odot \mathbf{A}^*$, denoted by $\mathbf{b}_k$, may be written as

$$\mathbf{b}_k = \sum_{l_1=1}^{L_h} \sum_{l_2=1}^{L_h} F_{\{k,l_1\}} F_{\{k,l_2\}}^* \left(\mathbf{f}_{l_1} \odot \mathbf{f}_{l_2}^* \right), \tag{15}$$

where $\mathbf{f}_k$ and $F_{\{k,l\}}$ are respectively the k-th column vector and the (k, l) element of DFT matrix $\mathbf{F}$, $k, l = 1, \ldots, N$. First, one may check that the Hadamard product $\left(\mathbf{f}_{l_1} \odot \mathbf{f}_{l_2}^* \right)$

is of the form:

$$\left(\mathbf{f}_{l_1} \odot \mathbf{f}_{l_2}^* \right) = (1/N) \cdot \left[1, \omega_{l_1-l_2}, \omega_{l_1-l_2}^2, \ldots, \omega_{l_1-l_2}^{N-1} \right]^T, \tag{16}$$

where $\omega_{l_1-l_2}$, $l_1, l_2 = 1, \ldots, L_h$, is equal to $\omega_d = \exp\{-j2\pi d/N\}$ evaluated at $d = l_1 - l_2$.

Then, let us define the set of indices $\mathcal{L}_d = \{l_1, l_2 = 1, \ldots, L_h \mid l_1 - l_2 = d\}$, $d = -(L_h - 1), \ldots, (L_h - 1)$. Notice that any pairs of indices (l_1, l_2) and (l_1', l_2') picked from $\mathcal{L}_d$ will lead to the same product vector, i.e.

$$\left(\mathbf{f}_{l_1} \odot \mathbf{f}_{l_2}^* \right) \equiv \left(\mathbf{f}_{l_1'} \odot \mathbf{f}_{l_2'}^* \right), \quad \forall (l_1, l_2), (l_1', l_2') \in \mathcal{L}_d. \tag{17}$$

Let us define $d = l_1 - l_2$ and the $N \times 1$ vector $\mathbf{g}_d$ as

$$\mathbf{g}_d = (1/\sqrt{N}) \cdot \left[1, \omega_d, \omega_d^2, \ldots, \omega_d^{N-1} \right]^T. \tag{18}$$

One may now verify the orthonormality property between vectors $\mathbf{g}_d$ above, i.e.

$$\mathbf{g}_{d_1}^H \mathbf{g}_{d_2} = \begin{cases} 1 & \text{if } d_1 = d_2 \\ 0 & \text{otherwise} \end{cases}, \tag{19}$$

for $d_1, d_2 = -(L_h - 1), \ldots, (L_h - 1)$ and provided that $2L_h - 1 < N$.

Finally, we define the orthonormal basis $\mathcal{G}$ comprised of the $2L_h - 1$ vectors $\mathbf{g}_d$ as $\mathcal{G} = \{\mathbf{g}_{-(L_h-1)}, \ldots, \mathbf{g}_{(L_h-1)}\}$. Using the property in (17), we notice that any vector of the form $\left(\mathbf{f}_{l_1} \odot \mathbf{f}_{l_2}^* \right)$, $l_1, l_2 = 1, \ldots, L_h$, is one of the vectors of $\mathcal{G}$. Hence, any column vector $\mathbf{b}_k$ in (15), $k = 1, \ldots, N$, may be expressed in terms of vectors of $\mathcal{G}$. One concludes that $\mathcal{G}$ yields an orthonormal basis for the column space of $\mathbf{A} \odot \mathbf{A}^*$. ∎

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