http://doi.acm.org/10.1145/1089444.1089478

Probability of Successful Transmission in a Random Slotted-Aloha Wireless Multihop Network Employing Constant Transmission Power

Henri Koskinen and Jorma Virtamo
Helsinki University of Technology (TKK), Networking Laboratory
P.O. Box 3000, FIN-02015 TKK, Finland
{Henri.Koskinen, Jorma.Virtamo}@tkk.fi

ABSTRACT

In [3], it was shown that the optimal throughput scaling in wireless multihop networks is achieved using slotted Aloha, and quantitative performance results for this protocol were derived under the assumption of exponentially distributed transmission powers. In this paper, we extend the analysis of this MAC scheme: assuming that all nodes employ some common constant power, we evaluate the probability of successful transmission in a random time slot. When interfering nodes are assumed to be randomly located, this temporal probability is a random variable with its own distribution. We develop numerical approximations for evaluating both the mean and the tail probability of this distribution; as far as we are aware, the distribution itself has not been studied before. The accuracy of our approximations can be improved indefinitely, with the cost of added numerical computations. We validate the approximations against simulation results.

Categories and Subject Descriptors

C.2.1 [**Computer-communication networks**]: Network architecture and design—*wireless communication, distributed networks*; C.2.2 [**Computer-communication networks**]: Network protocols—*ALOHA*; C.4 [**Performance of systems**]: *Performance attributes*; G.3 [**Probability and statistics**]: *distribution functions, stochastic processes*

General Terms

Performance, Theory

Keywords

Ad hoc networks, interference, sensor networks, slotted Aloha, stochastic geometry, wireless multihop networks

1. INTRODUCTION

Wireless multihop networks are known to pose several design problems, one of them being medium access control (MAC). Although not so much of an issue in wired networks, the solution used in wireless networks should allow the spatial reuse of the shared medium. Moreover, the time-varying network topology and the lack of centralized control in multihop networks render the use of coordinated MAC schemes difficult, making random access seem like the preferred choice.

Probably the simplest random-access protocol is Aloha [1], in which network nodes transmit any time they desire, and conflicts due to simultaneous transmissions destructively interfering are deduced from missing acknowledgements. Retransmissions are randomly delayed to avoid repeated collisions. The efficiency of this scheme is improved if transmissions are only allowed to occupy synchronized time slots; this is referred to as the slotted Aloha [2].

Another approach to accessing the medium, known as Carrier Sense Multiple Access (CSMA), is that network nodes determine whether or not to transmit by "listening" to any possible ongoing transmissions [8]. Although perfectly viable as such in wired networks, the implementation in wireless networks requires additional procedures around the receiver to overcome problems concomitant with the spatial aspect, such as so-called hidden and exposed terminals. CSMA is the basis for the medium access protocols used in WLANs, while slotted Aloha is used e.g. on the Random Access CHannel of GSM [5].

The information-theoretic transport capacity of these networks is an important quantity when assessing any MAC option, giving an upper bound for the achievable throughput no matter what the used solution. In [6], Gupta and Kumar derive bounds and scaling laws for the throughput of wireless multihop networks under different network models. In particular, for a network of nodes located randomly with spatial density λ that all employ the same transmission power, the upper bound for total throughput is shown to be $\Theta(\sqrt{\lambda})$, i.e. $\Theta(1/\sqrt{\lambda})$ per node, when a one-hop transmission is assumed successfully received (in [6] referred to as the Physical Model) whenever a signal-to-noise-and-interference ratio (SINR) threshold is exceeded at reception. ($f(\lambda) = \Theta(g(\lambda))$ denotes that $f(\lambda) = O(g(\lambda))$ and $g(\lambda) = O(f(\lambda))$.)

The main reference of this paper [3] presents an analysis of random slotted-Aloha networks under the above physical model. Accounting for all interferences in an exact way, unlike in previous studies of Aloha, it is shown that the above capacity bound is reached by applying slotted Aloha. Furthermore, the only parameter of the protocol, the medium access probability, can be optimized a priori, depending only on the required SINR threshold and, more importantly, not on the spatial density of nodes. This allows for a decentralized implementation, provided that nodes have some local information on the location of other nodes. In contrast, the authors point out that for optimal spatial reuse and hence optimal throughput scaling in a network using CSMA, the carrier sense range must be adapted to the node density. In other words, the range within which one transmission should be prohibited from another depends on the density of nodes. This is an impediment to the decentralized implementation of CSMA in wireless multihop networks.

The quantitative analysis in [3] assumes the transmission powers of the network nodes to be i.i.d. random variables with exponential distribution. To be precise, although the authors in [3] claim their protocol to reach *the* optimal capacity bound $\Theta(\sqrt{\lambda})$ cited above, this bound was only shown to hold with a given fixed power employed by all nodes in [6]. The same bound remained a conjecture for arbitrary networks, i.e., for networks where everything, including the individual transmission powers, can be arbitrarily selected and optimized. The aim of the present paper is to extend the analysis of the Aloha protocol proposed in [3]. In particular, we wish to determine the probability of successful transmission (or, more aptly, reception) in a random time slot – one of the most fundamental performance quantities in the network – when all the nodes employ a common transmission power. When taken as an average over all configurations of interfering nodes around the receiver, this probability can be evaluated by deriving the distribution of near-by interference exactly and utilizing the Laplace transform of the remaining interference in an approximation adapted from one presented by Bahadur and Rao in [4]. The use of this approximation is motivated by the observation that the remaining interference has, in a limit, a Gaussian distribution. We also address the distribution of the temporal (over time slots) probability of successful reception. This probability is a random variable since interfering nodes are randomly located around the recipient; its distribution (over different recipients) has not been studied before to our knowledge. We develop a recursive method for assessing the tail probability of this distribution, where the main idea is to divide the neighborhood of the receiver into concentric zones, with the interference from every node in a given zone approximated to be equal, and to average out the effect of different configurations beyond some maximum distance. All our results are approximations that, with sufficient computation effort, can be used to evaluate all the studied quantities to an arbitrary level of accuracy. We validate these approximations against simulation results.

The structure of this paper is the following. The next section gives a review of the essential assumptions and results in [3], and our assumptions and problem statements are presented in Section 3. The first performance quantity, namely, the probability of successful reception averaged over all recipients is evaluated in Section 4, whereas the distribution of this probability over different recipients is addressed in Section 5. Section 6 concludes the paper with a summary and some discussion. For completeness, the Bahadur-Rao approximation, applied to evaluating exceedance probabilities both above and below the mean in Section 4, is presented in the Appendix.

2. BACKGROUND

This section reviews the network modelling assumptions and some of the results in [3] relevant for this paper. Readers familiar with [3] may skip this section.

2.1 Network model

The network studied in [3] is infinite, with nodes located at the points of a Poisson point process $\Phi = \{X_i\}$ with intensity λ on the plane $\mathbb{R}^2$.

The medium access control in the network is arranged in the spirit of slotted Aloha: the operation of the network is divided into time slots, and each node is allowed to transmit in any time slot with a fixed medium access probability p. For any time slot, the states of the nodes are denoted by the indicator variables $\{e_i\}$: if $e_i = 1$, node i is allowed to transmit, and $e_i = 0$ means that node i refrains from transmitting, thus making it a potential receiver. The variables e_i are independent (among both the nodes and the time slots) Bernoulli-distributed random variables with parameter p.

In the general case considered in [3], the transmission powers $\{\mathcal{P}_i\}$ for nodes i with $e_i = 1$ are also random, assumed i.i.d. among both nodes and time slots. The quantitative results were based on exponentially distributed powers.

To determine the effect of interferences between concurrent transmissions, the model includes the attenuation function $L(x, y)$ that gives the path loss in power for a signal propagated from point $x \in \mathbb{R}^2$ to point y. Most of the quantitative results in [3] assume the commonly used power-law attenuation function that only depends on the distance $||x - y||$:

$$L(x, y) = l(||x - y||) = C||x - y||^{-\alpha} \quad \text{with} \quad C > 0,\, \alpha > 2. \tag{1}$$

The condition for the successful reception of a transmission is as given by the Physical Model in [6]: in any time slot, given a potential receiver node i at point $X_i = x_i$ and a transmitting node j at $X_j = x_j$, the receiver can successfully decode the transmission from node j – producing what is chosen as the unit throughput from j to i over this time slot – if and only if the Signal-to-Interference-and-Noise Ratio (SINR) at reception exceeds some common threshold T, i.e.

$$\frac{\mathcal{P}_j l(||x_j - x_i||)}{N_0 + I} \geq T, \tag{2}$$

where N_0 is a time and location independent power of the background noise on the frequency channel utilized by the network, and I denotes the interference power sum $\sum_{k \neq i,j} e_k \mathcal{P}_k l(||X_k - x_i||)$.

2.2 Existing results

Assume that all the fixed network parameters and the distance $||x_j - x_i||$ between the transmitter and the receiver are given. Then we may ask what is the probability, accounting for both the random locations $\{X_k\}$ and medium access states $\{e_k\}$ of all other nodes, that (2) holds in a random time slot? In other words, if a random configuration $\{X_k\}$ is observed in a random time slot, what is the probability that (2) holds, given the distance $d = ||x_j - x_i||$? (Indeed, by the stationarity of the homogeneous Poisson process, this probability does not depend on x_i, nor does it depend on x_j, by the independence of nodes existing in non-intersecting regions.) Let us write this probability as $\Pr_{\{X_k\},\{e_k\}}[(2) \text{ holds} \,|\, ||x_j - x_i|| = d]$.

Because we are interested in any random configuration for only one time slot, we may limit our attention in any configuration to the nodes that transmit in that time slot. By the properties of the Poisson process, we may then write the interference power sum in (2) as $I = \sum_k \mathcal{P}_k l(||Y_k - x_i||)$, the shot noise of a Poisson process $\{Y_k\}$ with intensity λp at x_i. Because of the stationarity, we may without loss of generality assume that the point x_i is at the origin, i.e. let $I = I_{\Phi(\lambda p)} = \sum_k \mathcal{P}_k l(||Y_k||)$.

As pointed out in [3], the Laplace transform of a general Poisson shot noise $I_{\Phi(\lambda)}$ with i.i.d. transmission powers $\mathcal{P}_k \sim \mathcal{P}$, calculated at the origin, is

$$I^*_{\Phi(\lambda)}(s) = \exp\left(-\lambda \int_{\mathbb{R}^2} 1 - \mathrm{E}_{\mathcal{P}}\left[\exp(-s\mathcal{P} \cdot l(||x||))\right] dx\right). \quad (3)$$

The generalization of the above to the interference at the origin from transmitters in some arbitrary domain simply amounts to integrating over that domain instead of $\mathbb{R}^2$.

The basis of the quantitative analysis of the above network model in [3] was that for transmission powers $\{\mathcal{P}_k\}$ i.i.d. exponentially distributed with mean $1/\mu$,
$\Pr_{\{X_k\},\{e_k\}}[(2) \text{ holds} \,|\, d] = I^*_{\Phi(\lambda p)}(\mu T/l(d)) \cdot N_0^*(\mu T/l(d))$
where $N_0^*(s)$ is the Laplace transform of the background noise. As a corollary, for exponential $\{\mathcal{P}_k\}$, $N_0 \equiv 0$ and the power-law attenuation function (1), we have

$$\begin{aligned}
&\Pr_{\{X_k\},\{e_k\}}[(2) \text{ holds} \,|\, ||x_j - x_i|| = d] = I^*_{\Phi(\lambda p)}\left(\frac{\mu T}{l(d)}\right) \\
&= \exp\left(-\lambda p \int_{\mathbb{R}^2} 1 - \frac{\mu}{\mu + [\mu T/l(d)] l(||x||)} dx\right) \\
&= \exp\left(-2\pi\lambda p \int_0^\infty \frac{r}{1 + r^\alpha/(Td^\alpha)} dr\right) \\
&= \exp\left(-\frac{2\pi\lambda p (T^{1/\alpha} d)^2}{\alpha} \int_0^\infty \frac{r^{2/\alpha - 1}}{1 + r} dr\right) \\
&= \exp\left(-\frac{2\pi\lambda p (T^{1/\alpha} d)^2}{\alpha} \Gamma(2/\alpha)\Gamma(1 - 2/\alpha)\right), \quad (4)
\end{aligned}$$

where $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ is the Gamma function. The authors also presented the following scaling result: for a general distribution of the transmission powers, $N_0 \equiv 0$ and the power-law attenuation function (1), $\Pr_{\{X_k\},\{e_k\}}[(2) \text{ holds} \,|\, d]$ only depends on λ, p, T and d through the product $dT^{1/\alpha}\sqrt{\lambda p}$.

Also in [3], an optimization criterion, referred to as the mean density of progress, was proposed for selecting the medium access probability p with given SINR threshold T. The density of progress is measured in bits per second crossing a meter of the line perpendicular to the direction of transfer in the planar network. For later reference, Table 1 shows thus optimized values of p in the case $N_0 = 0$ when $\alpha = 3$ and the transmission powers are assumed exponentially distributed, with various values of the SINR threshold T.

3. NETWORK MODEL AND PROBLEM STATEMENTS

We study the slotted-Aloha network model presented in [3] and described in Section 2.1, assuming that all nodes

Table 1: Optimized values of the medium access probability p (as given in [3]; partly determined visually* from figures therein) when $N_0 = 0$, $\alpha = 3$, and the transmission powers of the nodes are assumed i.i.d. exponentially distributed.

T [dB]	0	4	8	10	13	15
p_{opt}	0.17*	0.11*	0.07*	0.052	0.034	0.026

are stationary. In particular, we depart the somewhat unrealistic assumption of exponentially distributed and hence unbounded transmission powers, and assume instead that all nodes employ some common, constant transmission power: $\mathcal{P}_k \equiv P$. Note that this makes quite a difference: for instance, the ratio of two transmission powers has infinite expectation in the exponential case.

Throughout this paper, we also assume the power-law attenuation function (1), and as with most results in [3], we assume that the power of the background noise $N_0 = 0$. By the stationarity of the homogeneous Poisson process, without loss of generality we select the location x_i of the receiving node i as the origin.

Under these assumptions, the condition (2) for successful reception over distance $d = ||x_j - x_i|| = ||x_j||$ reduces to the one for the Signal-to-Interference Ratio (SIR) and can be written in the following equivalent forms:

$$\begin{aligned}
&\frac{PCd^{-\alpha}}{I} \geq T \quad \Leftrightarrow \quad I \leq PC(T^{1/\alpha}d)^{-\alpha} \\
\Leftrightarrow \quad &\frac{I}{PC(T^{1/\alpha}d)^{-\alpha}} = \sum_k e_k \left(\frac{||X_k||}{T^{1/\alpha}d}\right)^{-\alpha} \leq 1, \quad (5)
\end{aligned}$$

where $I = \sum_k e_k PC||X_k||^{-\alpha}$ is the interference power sum at the recipient. In this paper, we are interested in the probability that this condition holds given d, i.e., in the same quantity that played key role in the analysis in [3]. However, as will next be explained, we take a deeper look at this quantity than was done in [3].

The next section is devoted to evaluating the same probability as determined in [3], namely, the probability that (5) holds for a random configuration $\{X_k\}$ in a random time slot, given d. We will write this probability as
$\Pr_{\{X_k\},\{e_k\}}[(5) \text{ holds} \,|\, d]$.

We may also study the probability that (5) holds in a random time slot for a given configuration of surrounding nodes $\{X_k\} = \{x_k\}$ representing – and completely characterizing – the interference environment of one receiving node in the network; we write this conditional probability as $\Pr_{\{e_k\}}[(5) \text{ holds} \,|\, d;\, \{X_k\} = \{x_k\}]$. This probability is different for different configurations $\{x_k\}$, but it is fully determined once given $\{x_k\}$. It is thus a function of the random node locations $\{X_k\}$ and therefore itself a random variable with a probability distribution over $\{X_k\}$. This distribution describes how different nodes in the network are in different positions with regard to the success of communication; we will study this distribution in Section 5. In fact, the probability $\Pr_{\{X_k\},\{e_k\}}[(5) \text{ holds} \,|\, d]$ discussed in the next section can be seen to be the expected value of this distribution over $\{X_k\}$.

4. PROBABILITY OF SUCCESSFUL RECEPTION FROM DISTANCE d: EXPECTED VALUE OVER $\{X_k\}$

We begin with $\Pr_{\{X_k\},\{e_k\}}[(5) \text{ holds} \,|\, d]$, the probability that (5) holds for a random configuration $\{X_k\}$ in a random time slot, given d. As already mentioned in Section 2.2, we may in this case treat I in (5) as the shot noise of a Poisson process $\{Y_k\}$ with intensity λp at the origin, i.e. let $I = I_{\Phi(\lambda p)} = \sum_k P \cdot l(||Y_k||)$.

Let P_r denote the power received from a transmitter at distance r, i.e. $P_r = P \cdot l(r) = PCr^{-\alpha} = P_{T^{1/\alpha}d}\left(\frac{r}{T^{1/\alpha}d}\right)^{-\alpha}$. By (3), the Laplace transform of I is now

$$I^*_{\Phi(\lambda p)}(s) = \exp\left[\lambda p 2\pi \int_0^\infty \left(e^{-sP_{T^{1/\alpha}d}\left(\frac{r}{T^{1/\alpha}d}\right)^{-\alpha}} - 1\right) r\,dr\right]$$

$$\Rightarrow I^*_{\Phi(\lambda p)}\left(\frac{s}{P_{T^{1/\alpha}d}}\right) =$$

$$\exp\left[\lambda p 2\pi (T^{1/\alpha}d)^2 \int_0^\infty \left(e^{-st^{-\alpha}} - 1\right) t\,dt\right], \qquad (6)$$

where the last expression can also be seen as the Laplace transform of $I/P_{T^{1/\alpha}d}$. Note that it is precisely $I/P_{T^{1/\alpha}d}$ whose distribution we are interested in, since (5) is also equivalent to $I/P_{T^{1/\alpha}d} \leq 1$. Consistent with the scaling result mentioned in Section 2.2, we see that this Laplace transform – and hence the probability of interest to us – indeed only depends on the product $dT^{1/\alpha}\sqrt{\lambda p}$.

A closer look reveals that $I/P_{T^{1/\alpha}d}$ has infinite expectation and variance. This is a side effect caused by the assumed power-law attenuation function (1) which has a singularity at zero distance. This shortcoming is implicitly dealt with by the decomposition of the interference that is about to follow. Actually, the Laplace transform of $I/P_{T^{1/\alpha}d}$ could also be utilized directly in approximating the probability that (5) holds, but as we will demonstrate by numerical studies in Section 4.3, this method turns out to work poorly.

For more accurate approximations, we will treat the total interference I as the sum of two parts. The distribution of one part is approximated using its Laplace transform as implied above, whereas the distribution of the other part is calculated exactly. The key observation allowing this division is that for (5) to hold, there may be at most m active transmitters at distances y satisfying

$$\frac{PCd^{-\alpha}}{(m+1)PCy^{-\alpha}} < T \quad \Leftrightarrow \quad y < [(m+1)T]^{1/\alpha} d \stackrel{def}{=} r_m,$$

i.e., $m+1$ active transmitters alone at distance r_m would still satisfy the condition (5), but moving them any closer would violate this condition. Now, we will partition the total interference into that originating from two zones, i.e. $I_{\Phi(\lambda p)} = I = I_{\text{in}} + I_{\text{out}}$ where I_{in} denotes the interference originating from distances up to r_m and I_{out} denotes that from distances beyond r_m (see Figure 1). Because of the above limitation to up to m active transmitters within r_m, we may determine the distribution of I_{in} exactly, as follows.

4.1 Distribution of I_{in}

By the properties of the Poisson process, given the number of nodes in the inner zone, their locations in that zone are i.i.d. uniformly distributed. Thus, the distribution of the interference I_1 from a single node in this zone can be easily

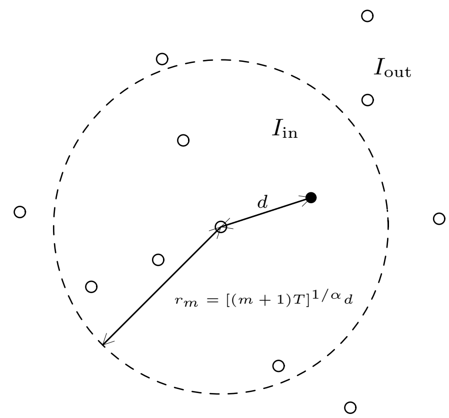


Figure 1: Division of the interference into that originating from two zones, for some $(m+1)T > 1$

determined: denoting by R the distance of the node from the origin and noting that $I_1 = P_R = P_{T^{1/\alpha}d}\left(\frac{R}{T^{1/\alpha}d}\right)^{-\alpha} \stackrel{def}{=} P_{T^{1/\alpha}d}Q$ where $P_{T^{1/\alpha}d}$ is constant, we obtain the probability density of Q as follows:

$$\Pr(R \leq r) = \frac{\pi r^2}{\pi r_m^2} = \Pr\left[\left(\frac{R}{T^{1/\alpha}d}\right)^{-\alpha} \geq \left(\frac{r}{T^{1/\alpha}d}\right)^{-\alpha}\right]$$

$$\Leftrightarrow \quad \Pr(Q \leq q) = 1 - \frac{(T^{1/\alpha}d)^2 q^{-2/\alpha}}{r_m^2} = 1 - \frac{q^{-2/\alpha}}{(m+1)^{2/\alpha}}$$

$$\Rightarrow \quad f_Q(q) = \begin{cases} \frac{2}{\alpha(m+1)^{2/\alpha}} q^{-(2+\alpha)/\alpha}, & q > \frac{1}{m+1}, \\ 0, & q \leq \frac{1}{m+1}. \end{cases}$$

Note that this distribution has infinite expectation. The summed interference from i nodes in the inner zone is then $P_{T^{1/\alpha}d} \cdot \sum_{j=1}^i Q_j$, where the probability density of $\sum_{j=1}^i Q_j$ is obtained as the convolution of i instances of the above density $f_Q(q)$, which we denote by $f_Q^{*i}(q)$. Given the condition that there are at most m active transmitters within r_m, the conditional distribution of I_{in} is then obtained by conditioning on i with the truncated Poisson distribution with parameter $\lambda p \pi r_m^2$:

$$f_{I_{\text{in}}/P_{T^{1/\alpha}d}}(q) = \frac{\sum_{i=0}^m \frac{(\lambda p \pi r_m^2)^i}{i!} f_Q^{*i}(q)}{\sum_{k=0}^m \frac{(\lambda p \pi r_m^2)^k}{k!}}.$$

As an example, with $m = 2$ it is possible to calculate this conditional distribution analytically. The above numerator now becomes

$$\sum_{i=0}^m \frac{(\lambda p \pi r_m^2)^i}{i!} f_Q^{*i}(q)$$

$$= 1 \cdot \delta(q) + \lambda p \pi r_m^2 f_Q(q) + (\lambda p \pi r_m^2)^2/2[f_Q * f_Q](q)$$

$$= \begin{cases} \delta(q), & q \leq \frac{1}{m+1}, \\ 2\lambda p\pi(T^{1/\alpha}d)^2/\alpha \cdot q^{-(2+\alpha)/\alpha}, & \frac{1}{m+1} < q \leq \frac{2}{m+1}, \\ 2\lambda p\pi(T^{1/\alpha}d)^2/\alpha \cdot q^{-(2+\alpha)/\alpha} & \\ +(\lambda p \pi r_m^2)^2/2[f_Q * f_Q](q), & q > \frac{2}{m+1}, \end{cases} \qquad (7)$$

where $\delta(\cdot)$ is the dirac delta function representing the distribution of the deterministic value 0. The convolution is obtained as

$$[f_Q * f_Q](q) = \int_{-\infty}^{\infty} f_Q(t) f_Q(q-t) dt = \frac{2}{\alpha(m+1)^{4/\alpha}} g(q),$$

where $g(q)$ equals

$$\Big\{ q^{-(2+\alpha)/\alpha} \left[(m+1)^{2/\alpha} - (q-(m+1)^{-1})^{-2/\alpha} \right] + \frac{2}{\alpha+4} \left[(q-(m+1)^{-1})^{-(\alpha+4)/\alpha} - (m+1)^{(\alpha+4)/\alpha} \right] \Big\}.$$

4.2 Distribution of I_{out}

Let us next turn to approximating the distribution of I_{out}. Applying (3) again yields the Laplace transform

$$I^*_{\text{out}}(s) = \exp\left[\lambda p 2\pi \int_{r_m}^{\infty} \left(e^{-sP_{r_m}\left(\frac{r}{r_m}\right)^{-\alpha}} - 1 \right) r\, dr \right]$$

$$\Rightarrow \quad I^*_{\text{out}}\left(\frac{s}{P_{r_m}}\right) = I^*_{\text{out}}\left(\frac{s}{P_{T^{1/\alpha}d}/(m+1)}\right) = \exp\left[\lambda p 2\pi r_m^2 \int_1^{\infty} \left(e^{-st^{-\alpha}} - 1 \right) t\, dt \right],$$

where the last expression can also be seen as the Laplace transform of $J \stackrel{def}{=} I_{\text{out}}/P_{r_m} = (m+1) I_{\text{out}}/P_{T^{1/\alpha}d}$. In the last integral, all the variables are dimensionless. For $\alpha > 2$ the integral is convergent and can easily be evaluated for any s. For numerical computations, however, it is advantageous to make a change of variables $u = t^{-(\alpha-2)}$, which yields the alternative form

$$J^*(s) = \exp\left[\frac{\lambda p 2\pi r_m^2}{\alpha-2} \int_0^1 \frac{e^{-su^{\alpha/(\alpha-2)}} - 1}{u^{\alpha/(\alpha-2)}} du \right].$$

For the logarithmic moment generating function $\varphi(\beta) = \log \mathrm{E}\left[e^{\beta J}\right] = \log J^*(-\beta)$ of J, we then have

$$\begin{aligned} \varphi(\beta) &= 2\lambda p\pi r_m^2 \int_1^{\infty} (e^{\beta t^{-\alpha}} - 1) t\, dt \\ &= \frac{2\lambda p\pi r_m^2}{\alpha-2} \int_0^1 \frac{e^{\beta u^{\alpha/(\alpha-2)}} - 1}{u^{\alpha/(\alpha-2)}} du, \\ \varphi'(\beta) &= \frac{2\lambda p\pi r_m^2}{\alpha-2} \int_0^1 e^{\beta u^{\alpha/(\alpha-2)}} du, \\ \varphi''(\beta) &= \frac{2\lambda p\pi r_m^2}{\alpha-2} \int_0^1 u^{\alpha/(\alpha-2)} e^{\beta u^{\alpha/(\alpha-2)}} du, \end{aligned} \tag{8}$$

whereby the mean and variance of I_{out} are

$$\mathrm{E}[I_{\text{out}}] = P_{r_m} \varphi'(0) = PC \frac{2\lambda p\pi}{\alpha-2} r_m^{2-\alpha},$$

$$\mathrm{Var}[I_{\text{out}}] = P_{r_m}^2 \varphi''(0) = (PC)^2 \frac{2\lambda p\pi}{2\alpha-2} r_m^{2-2\alpha}. \tag{9}$$

Now, consider generating a random realization of I_{out}, taking only nodes within some maximum distance into account for conceptual simplicity. This can be done by drawing the Poisson-distributed number of interfering nodes, placing these nodes independently and uniformly at random on the considered domain, no closer than r_m nor further than the maximum distance, and calculating I_{out} as the sum of the individual interference powers. Thus, I_{out} is the sum of i.i.d. random variables, and hence, by the Central limit theorem (see e.g. [7]), should obey a distribution that tends to the Gaussian as the node density tends to infinity. More precisely, provided that the density is so large that there is likely to be many nodes at the smaller distances with nearly equal contributions to I_{out}, the distribution of I_{out} – and hence that of the scaled quantity J – should be close to Gaussian.

Recall that given α, the distribution of J is fully characterized by the product $\lambda p r_m^2$. In fact, the quantity $\lambda p\pi r_m^2$ is the expected number of transmitting nodes within distance r_m from an arbitrary reference point. This quantity also determines how close to Gaussian the distribution of I_{out} – and hence J – is: if this number is small, then the total interference is likely to be dominated by few terms, resulting in a distribution far from the Gaussian. Accordingly, the larger the value, the better the Gaussian approximation. Figure 2 shows how the Gaussian approximation agrees with the simulated distribution of J with different values of $\lambda p\pi r_m^2$. To cater for the simulation, interference from distances beyond $k \cdot r_m$, with k chosen to be some large number, was neglected; this changes the results in (8) so that the upper limit ∞ in any of the integrals now becomes k and the lower limit 0 becomes $k^{-(\alpha-2)}$. One can see in the figure that the accuracy of the approximation indeed improves as $\lambda p\pi r_m^2$ increases, but the fit at the tails of the distribution remains poor. This can be remedied by adopting an approximation from large deviations theory (see [4, Section 6]) and applying it also below the mean; this approximation is introduced in the Appendix. This Bahadur-Rao (BR) approximation significantly improves the fit in the tails, while it coincides with the Gaussian approximation at the mean of the distribution: it is also shown in Figure 2.

4.3 Adding up I_{in} and I_{out}

We may now combine the means to evaluate the distributions of I_{in} and I_{out} to obtain an approximation for the

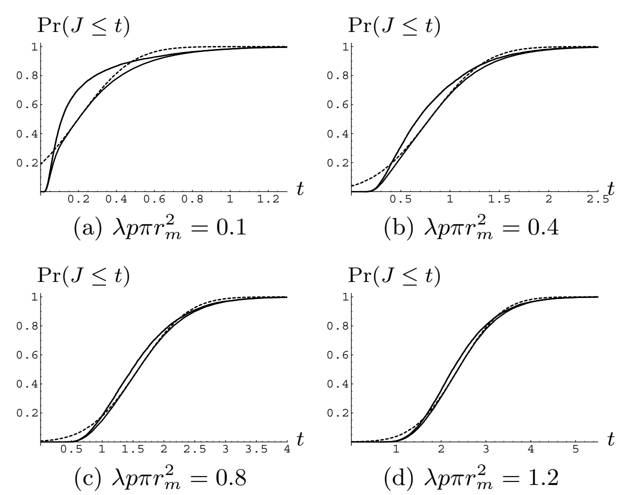


Figure 2: Cumulative distribution of $J = I_{\text{out}}/P_{r_m}$ with different values of $\lambda p\pi r_m^2$ when $\alpha = 3$. The simulation domain has been defined using $k = 100$. Upper line: simulated distribution; dashed line: Gaussian approximation; lower line: Bahadur-Rao approximation.

probability that (5) holds: this can be written as

$$
\begin{aligned}
&\Pr\left[(I_{\mathrm{in}}+I_{\mathrm{out}})/P_{T^{1/\alpha}d} \le 1\right]\\
=&\Pr(\text{At most } m \text{ active transmitters within } r_m)\\
&\times \int_0^1 f_{I_{\mathrm{in}}/P_{T^{1/\alpha}d}}(q) \Pr\left[I_{\mathrm{out}}/P_{T^{1/\alpha}d} \le 1-q\right] dq\\
=&e^{-\lambda p\pi r_m^2} \sum_{k=0}^{m} \frac{(\lambda p\pi r_m^2)^k}{k!} \int_0^1 \frac{\sum_{i=0}^{m} \frac{(\lambda p\pi r_m^2)^i}{i!} f_Q^{*i}(q)}{\sum_{k=0}^{m} \frac{(\lambda p\pi r_m^2)^k}{k!}}\\
&\times \Pr[(m+1)I_{\mathrm{out}}/P_{T^{1/\alpha}d} \le (m+1)(1-q)]dq\\
=&e^{-\lambda p\pi r_m^2} \int_0^1 \sum_{i=0}^{m} \frac{(\lambda p\pi r_m^2)^i}{i!} f_Q^{*i}(q) \Pr[J \le (m+1)(1-q)]dq.
\end{aligned} \tag{10}
$$

In the case of the example with $m = 2$, plugging in (7) gives this in the form

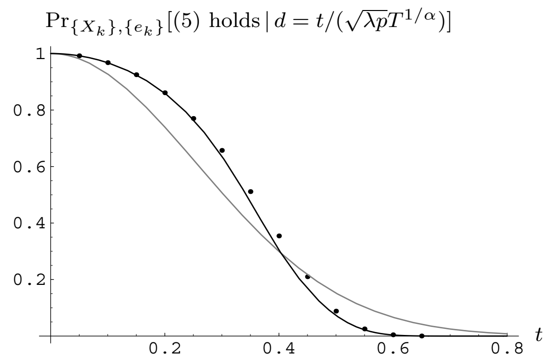
$$
\begin{aligned}
&e^{-\lambda p\pi[(m+1)T]^{2/\alpha}d^2} \Big\{ \Pr[J \le (m+1)]\\
&+\frac{2\lambda p\pi(T^{1/\alpha}d)^2}{\alpha} \int_{\frac{1}{m+1}}^{1} q^{-(2+\alpha)/\alpha} \Pr[J \le (m+1)(1-q)]dq\\
&+\frac{\left[\lambda p\pi(T^{1/\alpha}d)^2\right]^2}{\alpha} \int_{\frac{2}{m+1}}^{1} g(q) \Pr[J \le (m+1)(1-q)]dq \Big\},
\end{aligned} \tag{11}
$$

where the dependence only on the product $\sqrt{\lambda p}T^{1/\alpha}d$ (because it also applies to the distribution of J) is more apparent than in (10).

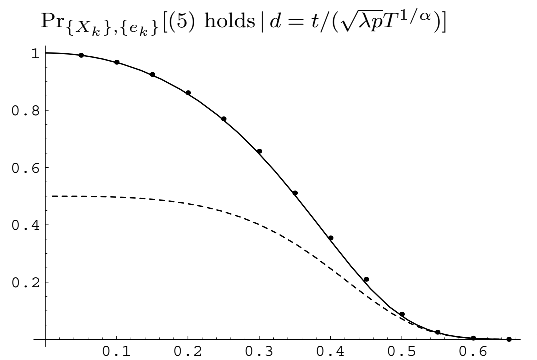
The accuracy of different approximations for $\Pr_{\{X_k\},\{e_k\}}[(5) \text{ holds} \,|\, d]$ is demonstrated in Figure 3. As mentioned earlier, the Laplace transform (6) of $I/P_{T^{1/\alpha}d}$ can also be used directly, by applying the BR approximation; this method has also been included in the figure and can be seen to result in a very poor approximation. On the other hand, using (10) with $m = 2$ already proves to be notably accurate and gives a significant improvement from using $m = 0$.

Note that we can improve the approximation to any level of accuracy by choosing sufficiently large m. The gain from increasing m is twofold. First, through increasing $\lambda p\pi r_m^2$, it allows approximating the distribution of J more accurately, as shown by Figure 2. Second, it decreases the share of I_{out} in the total interference, thus mitigating the effect of the remaining inaccuracy. The cost of increasing m is the added numerical labor in computing further convolutions $f_Q^{*i}(q)$. (Note that (11) already involves several nested numerical integrations, most of them contained in applying the BR approximation.)

One may also note in Figure 3(a) how the probability behaves differently under the assumption of exponentially distributed transmission powers. In particular, the probability makes a sharper transition with increasing distance d in our case. This is because the Signal-to-Interference ratio at reception varies less due to the lack of randomness in the transmission powers.



(a) $m = 0$



(b) $m = 2$

Figure 3: The probability of (5) holding when $\alpha = 3$, determined using (10) and applying the BR approximation with different values of m (solid lines), and by simulation (points). For comparison, the corresponding probability in the case of exponentially distributed transmission powers as given by (4) (gray line) and the result of applying the BR approximation directly to the Laplace transform (6) of $I/P_{T^{1/\alpha}d}$ (dashed line) are also shown. The interference from distances beyond $100T^{1/\alpha}d$ has been neglected in all cases (this results in the upper limit 100^α in the last integral in (4)).

5. PROBABILITY OF SUCCESSFUL RECEPTION FROM DISTANCE d: DISTRIBUTION OVER $\{X_k\}$

We now turn to evaluating $\Pr_{\{e_k\}}[(5) \text{ holds} \,|\, d;\, \{X_k\} = \{x_k\}]$, the probability that (5) holds in a random time slot, given the distance d and the configuration of surrounding nodes $\{X_k\} = \{x_k\}$. As we mentioned earlier, this probability is a function of $\{X_k\}$ and therefore itself a random variable. For brevity, we will use the notation $\Pr_{\{e_k\}}[(5) \text{ holds} \,|\, d;\, \{X_k\}] = \Pi(\{X_k\})$. In this section, we are interested in the distribution of $\Pi(\{X_k\})$.

With α and T fixed, the last form in (5) can be seen to be a condition for the transmission indicators $\{e_k\}$ and the distances of the other nodes from the recipient, relative to the distance $T^{1/\alpha}d$. Thus, with $\{X_k\}$ fixed, $\Pi(\{X_k\})$ only depends on the medium access probability p. Under the assumption that $\{X_k\}$ is a realization of a Poisson process, the distribution of $\Pi(\{X_k\})$ then depends only on p and the average number of nodes within distance $T^{1/\alpha}d$, equal to $\lambda\pi(T^{1/\alpha}d)^2$. This should be contrasted with the scaling result referred to earlier, according to which the averaged probability $\Pr_{\{X_k\},\{e_k\}}[(5) \text{ holds} \,|\, d]$, i.e. the mean of the distribution of $\Pi(\{X_k\})$, only depends on the product $\lambda p\pi(T^{1/\alpha}d)^2$.

In what follows, we will concentrate on the tail probability $\Pr_{\{X_k\}}[\Pi(\{X_k\}) > \hat{P}]$. Let $I(\mathcal{S})$ denote the random interference originating from the set $\mathcal{S} \subseteq \mathbb{R}^2$ and observed by the recipient at the origin in any time slot, with the convention $I = I(\mathbb{R}^2)$. Also, let $\mathcal{B}_r$ denote the disk with radius r centered at the origin, i.e. $\mathcal{B}_r = \{x : ||x|| \leq r\}$, and denote its complement with $\bar{\mathcal{B}}_r$. With this notation, given the locations of nodes in $\mathcal{B}_r$ and hence the probability distribution of $I(\mathcal{B}_r)$ in any time slot, the conditional tail probability of $\Pi(\{X_k\})$ can be written as

$$\Pr_{\{X_k \in \bar{\mathcal{B}}_r\}} \left\{ \Pi(\{X_k\}) > \hat{P} \,|\, F_{I(\mathcal{B}_r)}(i) \right\}, \tag{12}$$

where we have denoted the distribution of $I(\mathcal{B}_r)$ by its cumulative distribution function $F_{I(\mathcal{B}_r)}(i)$.

5.1 Recursive approximation for the tail probability of $\Pi(\{X_k\})$

We will now derive a method to approximately evaluate this conditional probability. Let us partition the exterior of $\mathcal{B}_r$ as $\bar{\mathcal{B}}_r = (\bar{\mathcal{B}}_r \bigcap \mathcal{B}_{\hat{r}}) \bigcup \bar{\mathcal{B}}_{\hat{r}}$, i.e. into an annulus with inner radius r and some outer radius $\hat{r} > r$, and the rest (see Figure 4). Next we make the approximation that the interference from every transmitting node in the annulus $\bar{\mathcal{B}}_r \bigcap \mathcal{B}_{\hat{r}}$ (drawn as black points in the figure) is equal to $P_{\tilde{r}}$, for some $r \leq \tilde{r} \leq \hat{r}$. Then, given the number of nodes N in the annulus, the interference originating from the annulus in each time slot is $I(\bar{\mathcal{B}}_r \bigcap \mathcal{B}_{\hat{r}}) \,|\, N = X \cdot P_{\tilde{r}}$ with $X \sim \mathrm{Bin}(N, p)$, and we have $I(\mathcal{B}_{\hat{r}}) \,|\, N = I(\mathcal{B}_r) + I(\bar{\mathcal{B}}_r \bigcap \mathcal{B}_{\hat{r}}) \,|\, N$. Since this allows computing the distribution of $I(\mathcal{B}_{\hat{r}}) \,|\, N$, conditioning on N – which in our model is Poisson-distributed with mean $\lambda\pi(\hat{r}^2 - r^2)$ – now leads to the following recursion for the conditional probability (12):

$$\Pr_{\{X_k \in \bar{\mathcal{B}}_r\}} \left\{ \Pi(\{X_k\}) > \hat{P} \,|\, F_{I(\mathcal{B}_r)}(i) \right\} =$$
$$\sum_{n=0}^{\infty} \Pr(N = n) \cdot \Pr_{\{X_k \in \bar{\mathcal{B}}_{\hat{r}}\}} \left\{ \Pi(\{X_k\}) > \hat{P} \,|\, F_{I(\mathcal{B}_{\hat{r}})|N}(i|n) \right\}.$$

In fact, by starting with $r = 0$ and $F_{I(\mathcal{B}_0)}(0) = 1$, this recursion can be used to evaluate the tail probability $\Pr_{\{X_k\}}[\Pi(\{X_k\}) > \hat{P}]$ to an arbitrary level of accuracy, by partitioning the plane into sufficiently thin annuli.

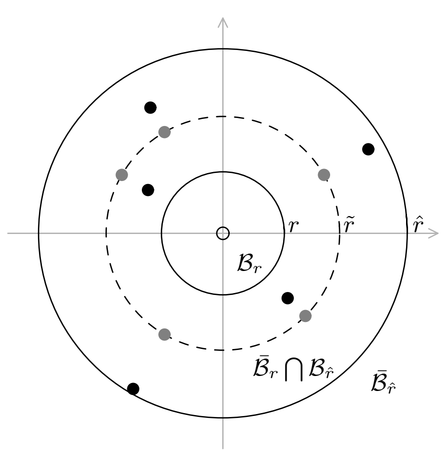


Figure 4: Schematic representation of the partitioning of $\bar{\mathcal{B}}_r$ and the approximation made in the annulus $\bar{\mathcal{B}}_r \bigcap \mathcal{B}_{\hat{r}}$

Of course, this recursion in itself is infinite, through the infinite sum on the one hand and through the partitioning of $\mathbb{R}^2$ into an infinite number of annuli on the other. Proper pruning and termination conditions are therefore needed. The first and obvious termination condition is that (12) equals 0 for such a distribution of $I(\mathcal{B}_r)$ for which $\Pr[I(\mathcal{B}_r) \leq P_{T^{1/\alpha}d}] \leq \hat{P}$. Since a high enough value of N gives $I(\mathcal{B}_{\hat{r}}) \,|\, N$ such a distribution, we only need to add new terms to the sum as long as the conditional probability $\Pr_{\{X_k \in \bar{\mathcal{B}}_{\hat{r}}\}} \left\{ \Pi(\{X_k\}) > \hat{P} \,|\, F_{I(\mathcal{B}_{\hat{r}})|N}(i|n) \right\}$ differs from zero by this termination condition.

As for dealing with the infinite plane, we may, for some $r_{\max}$, ignore how different configurations of nodes in $\bar{\mathcal{B}}_{r_{\max}}$ result in different distributions of $I(\bar{\mathcal{B}}_{r_{\max}})$, and instead use the distribution averaged over all possible configurations, as if the configuration of nodes producing the interference was different in every time slot. This amounts to approximating the distribution of interference from transmitters in $\bar{\mathcal{B}}_{r_{\max}}$, located according to a Poisson process with intensity λp, by utilizing its Laplace transform exactly as done in the previous section. With such an approximation for the distribution of $I(\bar{\mathcal{B}}_{r_{\max}})$, the final level of recursion simply gives

$$\Pr_{\{X_k \in \bar{\mathcal{B}}_{r_{\max}}\}} \left\{ \Pi(\{X_k\}) > \hat{P} \,|\, F_{I(\mathcal{B}_{r_{\max}})}(i) \right\}$$
$$= \begin{cases} 1, & \Pr\left[I(\bar{\mathcal{B}}_{r_{\max}}) + I(\mathcal{B}_{r_{\max}}) \leq P_{T^{1/\alpha}d}\right] > \hat{P}, \\ 0, & \Pr\left[I(\bar{\mathcal{B}}_{r_{\max}}) + I(\mathcal{B}_{r_{\max}}) \leq P_{T^{1/\alpha}d}\right] \leq \hat{P}, \end{cases}$$

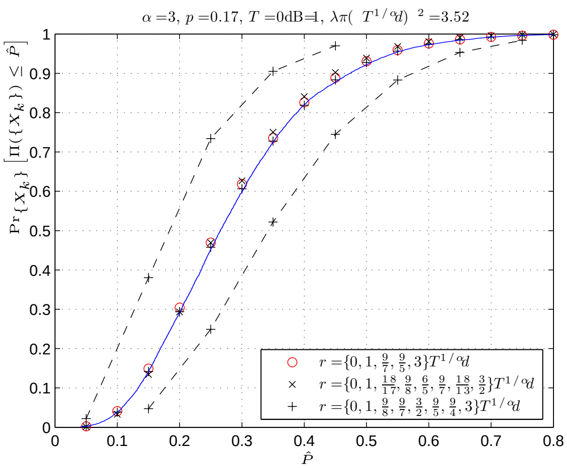
where the probability is calculated by conditioning on $I(\mathcal{B}_{r_{\max}})$, which has a discrete distribution with a finite number of values.

Because of the scaling result that applies to the distribution of $\Pi(\{X_k\})$, the parameter that completely characterizes the above recursion for evaluating the tail probability $\Pr_{\{X_k\}}[\Pi(\{X_k\}) > \hat{P}]$ is an increasing sequence $\{r/(T^{1/\alpha}d)\}$ of distances r, starting with zero and ending with $r_{\max}$, given relative to $T^{1/\alpha}d$. These are the outer radii of the nested annuli to consider at the successive levels of recursion. Because no other node may transmit within distance $T^{1/\alpha}d$ for (5) to hold, it is sensible to choose the first two distances as $\{r/(T^{1/\alpha}d)\} = \{0, 1\}$. Our method of choosing the remaining distances has been to fix $r_{\max}$ and the number of annuli to divide the distances $[T^{1/\alpha}d, r_{\max}]$, and select the annuli so that the expected interference from each annulus is equal, i.e. the integral $\int_r^{\hat{r}} \lambda p 2\pi t P C t^{-\alpha} dt$ is the same for each annulus.

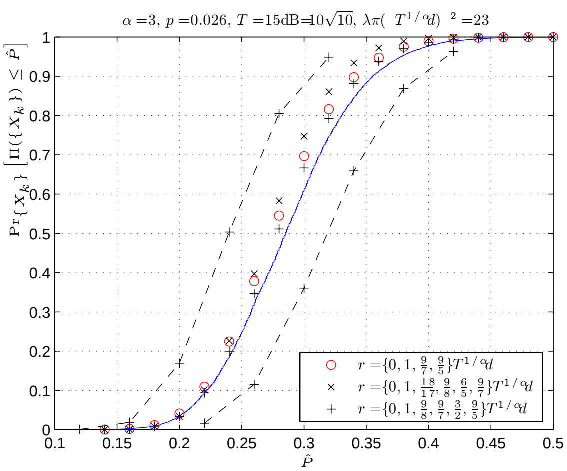
The choice of $\tilde{r}$ with which the interference from every node in an annulus with inner and outer radius r and $\hat{r}$, respectively, is taken to be $P_{\tilde{r}}$, determines the nature of our approximation: choosing $\tilde{r} = r$ naturally results in a conservative approximation, whereas setting $\tilde{r} = \hat{r}$ results in underestimating the interference. To aim at an approximation as accurate as possible, we may choose $P_{\tilde{r}}$ as the expected interference from a node placed uniformly at random in the annulus; this has been our choice in the demonstration that follows. The fact that this expected interference from the inmost, degenerate annulus is infinite does not affect the final result, since the condition (5) in any case prohibits all nodes in this annulus from transmitting.

5.2 Validation

In Figure 5 we compare the results given by this recursion with simulated distributions of $\Pi(\{X_k\})$. Each simulated sample represents the proportion of 1000 time slots in which (5) was satisfied in a given configuration of nodes, and 10000 random configurations were considered. The two subplots show how the accuracy of the recursion improves as the range covered by the annuli is increased and a larger number of annuli is used. This is particularly clear in the latter subplot, where the most accurate setting already required rather extensive computation time from the recursion, due to the high value of $\lambda\pi(T^{1/\alpha}d)^2$, i.e. wide ranges of numbers of nodes to consider in each annulus. For comparison, upper and lower bounds obtained by choosing $\tilde{r} = r$ and $\tilde{r} = \hat{r}$ have also been plotted; these points have been connected with dashed lines in the figure.



(a) Mean of simulation data: $\Pi(\{X_k\}) = 0.2866$



(b) Mean of simulation data: $\Pi(\{X_k\}) = 0.2874$

Figure 5: Empirical cumulative distribution functions of 10000 simulated estimates of $\Pi(\{X_k\})$ (solid curves) and $1 - \Pr_{\{X_k\}}\left[\Pi(\{X_k\}) > \hat{P}\right]$ as determined by the recursion using different radius sequences (see legends), with $\alpha = 3$ and $\lambda p\pi(T^{1/\alpha}d)^2 = 0.598$. To ease simulations, interference from distances beyond $10T^{1/\alpha}d$ was neglected in all cases.

The (T/p)-pairs selected for the two validation cases represent the extreme ends of reference values in Table 1, and $\lambda\pi d^2$ was chosen to make $\lambda p\pi(T^{1/\alpha}d)^2$ the same in both cases, implying the same means for the two distributions. The means obtained from the simulation data can be compared with Figure 3, considering that here $\sqrt{\lambda p}T^{1/\alpha}d \approx 0.44$. The slightly lower value predicted by the figure is due to the fact that here we have only taken interference from distances up to $10T^{1/\alpha}d$ into account, as opposed to $100T^{1/\alpha}d$ in Figure 3; substituting these distances in the place of r_m along with the assumed $\alpha = 3$ in (9), we see that we have here neglected $1/10$ of the expected interference from distances beyond $T^{1/\alpha}d$, whereas only $1/100$ was neglected in Figure 3.

The fact that the latter distribution with a higher value of $\lambda\pi(T^{1/\alpha}d)^2$ has a smaller variance can be explained by the fact that the number of nodes in any annulus with given inner and outer radii (relative to $T^{1/\alpha}d$) is Poisson-distributed with parameter proportional to $\lambda\pi(T^{1/\alpha}d)^2$, whereby a higher value implies a lower coefficient of variation for this number. Therefore, the number of nodes located within any distance interval has the smaller relative variance the greater the $\lambda\pi(T^{1/\alpha}d)^2$, resulting in a smaller variance for $\Pi(\{X_k\})$.

6. SUMMARY AND DISCUSSION

We analyzed the random planar wireless multihop network using slotted Aloha as proposed and modelled in [3]. As our contribution, we developed numerical approximations for evaluating the temporal probability of successful reception when all nodes in the network are assumed to employ a common constant transmission power. The accuracy of these approximations can be improved indefinitely, with the cost of added numerical computations. We demonstrated the accuracy of the approximations with simulation results.

We first focused on the probability of successful reception averaged over all configurations of other nodes surrounding the receiver, which was derived analytically in [3] assuming exponentially distributed transmission powers. This probability only depends on the distribution of a certain Poisson shot noise. The approximation was obtained by deriving the distribution of interference from the proximity of the receiver exactly and utilizing the Laplace transform of the remaining interference in an approximation adapted from [4].

We also addressed the distribution of the temporal probability of successful reception over different configurations of surrounding nodes and hence over different recipients. Dividing the neighborhood of the receiver into zones where each transmitter is assumed to produce equally strong interference and taking the effect of different configurations outside this neighborhood as an average, we obtained a recursion for evaluating the tail probability of this distribution.

As seen in Figure 3(a), under the assumption of one common transmission power the probability of successful reception has a stronger dependence on the distance of transmission than with random transmission powers. As a potential direction for future work, it might be interesting to utilize the results in this paper to see how this assumption changes the values of the medium access probability that maximize the mean density of progress.

7. ACKNOWLEDGMENTS

This work has been financially supported by the Finnish Defence Forces Technical Research Centre. Henri Koskinen has also been partly supported by a grant from the Nokia Foundation.

8. REFERENCES

[1] N. Abramson. The ALOHA system – another alternative for computer communications. In *Proceedings of the AFIPS Fall Joint Computer Conference*, volume 37, pages 281–285. AFIPS Press, 1970.

[2] N. Abramson. Packet switching with satellites. In *Proceedings of the AFIPS National Computer Conference*, volume 42, pages 695–702. AFIPS Press, 1973.

[3] F. Baccelli, B. Blaszczyszyn, and P. Mühlethaler. An Aloha protocol for multihop mobile wireless networks. In *Proceedings of 16th ITC Specialist Seminar*, pages 28–39, Aug. 2004.

[4] R. R. Bahadur and R. R. Rao. On deviations of the sample mean. *The Annals of Mathematical Statistics*, 31(4):1015–1027, Dec. 1960.

[5] A. Brand and H. Aghvami. *Multiple access protocols for mobile communications: GPRS, UMTS and beyond*. Wiley, Chichester, 2002.

[6] P. Gupta and P. R. Kumar. The capacity of wireless networks. *IEEE Transactions on Information Theory*, 46(2):388–404, Mar. 2000.

[7] A. F. Karr. *Probability*. Springer texts in statistics. Springer-Verlag, New York, 1993.

[8] L. Kleinrock and F. Tobagi. Packet switching in radio channels: Part I – Carrier Sense Multiple-Access modes and their throughput-delay characteristics. *IEEE Transactions on Communications*, 23(12):1400–1416, Dec. 1975.

APPENDIX

The Bahadur-Rao approximation

Assume a random variable X with probability density $f(x)$ and logarithmic moment generating function $\varphi(\beta)$. The tail probability of X can be written in terms of the distribution $f(x)$ twisted with parameter β:

$$\begin{aligned}\Pr(X > x) &= \int_x^\infty f(y)dy = \int_x^\infty e^{-\beta y+\varphi(\beta)} f_\beta(y)dy \\ &= e^{-\beta x+\varphi(\beta)} \int_x^\infty e^{-\beta(y-x)} f_\beta(y)dy.\end{aligned}$$

Because the value of the last integral does not exceed 1, this yields the bound $\Pr(X > x) \le e^{-\beta x+\varphi(\beta)}$. The tightest bound obtained by minimizing with respect to β, i.e. solving $\varphi'(\beta_x) = x$, is known as the Chernoff bound. This amounts to making the mean of the twisted distribution $f_{\beta_x}(\cdot)$ equal x.

If we further approximate the twisted distribution in the last integral by the normal one $\mathrm{N}(\varphi'(\beta_x) = x, \varphi''(\beta_x))$, we obtain

$$\begin{aligned}&\int_x^\infty e^{-\beta_x(y-x)} f_{\beta_x}(y)dy \\ \approx&\frac{1}{\sqrt{2\pi\varphi''(\beta_x)}} \int_x^\infty e^{-\beta_x(y-x)} e^{-\frac{1}{2}(y-x)^2/\varphi''(\beta_x)} dy \\ =&\frac{e^{\frac{1}{2}\beta_x^2\varphi''(\beta_x)}}{\sqrt{2\pi}} \int_{\beta_x\sqrt{\varphi''(\beta_x)}}^\infty e^{-\frac{1}{2}z^2} dz,\end{aligned}$$

which leads to a pretty accurate approximation for $\Pr(X > x)$ when $x \ge \mathrm{E}[X]$, i.e. $\beta_x \ge 0$, that naturally coincides with the one obtained by using the normal approximation for $f(\cdot)$ when $x = \mathrm{E}[X]$. This approximation is presented in [4]. On the other hand, we may repeat the above procedure for values of x below the mean, and combining the results, we have an approximation which is applicable everywhere:

$$\Pr(X > x) \approx \begin{cases} e^{-\beta_x x+\varphi(\beta_x)+\frac{1}{2}\beta_x^2\varphi''(\beta_x)} Q(\beta_x\sqrt{\varphi''(\beta_x)}), \\ \qquad\qquad x \ge \mathrm{E}[X] \Leftrightarrow \beta_x \ge 0, \\ 1 - e^{-\beta_x x+\varphi(\beta_x)+\frac{1}{2}\beta_x^2\varphi''(\beta_x)} [1 - Q(\beta_x\sqrt{\varphi''(\beta_x)})], \\ \qquad\qquad x \le \mathrm{E}[X] \Leftrightarrow \beta_x \le 0, \end{cases}$$

where $Q(x)$ is the tail probability of the standard normal distribution N(0,1). In this paper we use the Bahadur-Rao approximation in this form which is applicable also in the non-asymptotic regime, in contrast to the form usually referred to as the Bahadur-Rao approximation in the literature, which is based on using the asymptotic form $e^{-\frac{1}{2}x^2}/(\sqrt{2\pi}x)$ for $Q(x)$.