Impact of Double-Rayleigh Fading on System Performance

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***Abstract*— Double-Rayleigh amplitude distribution, induced by double scattering in the radio channel, occurs in a variety of propagation scenarios, including propagation via diffracting wedges. It is unknown, however, how double scattering affects communication link performance. In this paper, an expression for symbol error probability for some common communication schemes over double-Rayleigh fading wireless channels is derived. It is shown that double scattering results in severe degradation in symbol error probability. Interestingly, our analysis suggests that for the double-Rayleigh channel there exists no meaningful diversity order for the SNR values of practical interest.**

I. Introduction

Rayleigh distribution is the classical amplitude distribution of a narrow-band fading wireless channel. Recently, it has been shown that the presence of a keyhole in the signal path will lead to *double-Rayleigh* distributed amplitude [1], [2]. The propagation phenomenon leading to the double-Rayleigh distributed amplitude is called double scattering. A generalization of the pure double scattering case, called the multiple scattering model, has been proposed in [3], [4]. The multiple scattering phenomenon has been predicted by simulations in [1], and also observed in measurements [3]–[5]. In this paper we shall focus on the special case of double scattering. Examples of propagation scenarios, where the double scattering phenomenon is likely to occur include:

- (Fig. 1a) When two rings of scatterers are separated by a large distance, and all propagation paths of the channel travel via the same narrow "pipe" [3]. The Multiple-Input Multiple-Output (MIMO) channel case was considered in [6].
- (Fig. 1b) Propagation in amplify-and-forward wireless relay networks. The amplitude distribution of the received signal will be a product of two Rayleigh random processes if at least two of the nodes (tx,A-F,rx) in the network are moving, hence giving rise to the double-Rayleigh amplitude distribution. We assume, for simplicity, a noiseless analogue repeater node (A-F) with fixed gain, or slow power control. (This is a somewhat contrived assumption, the more realistic case of noisy repeater is studied in [7].)
- (Fig. 1c) Propagation via diffracting wedges, such as street corners in urban micro cells. If both the transmitter and receiver are moving, the street corner effectively functions as a multiplier (keyhole) for the two Rayleigh processes [1].

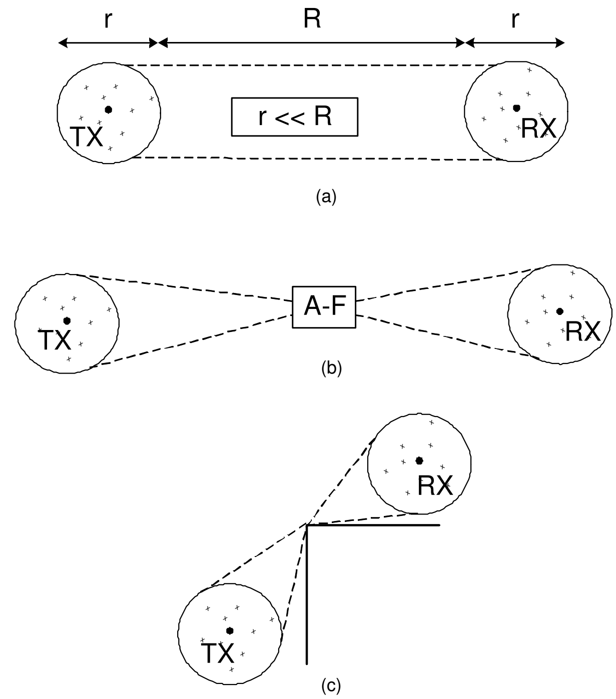

Fig. 1. Three propagation scenarios with double-Rayleigh fading. (a) keyhole created by two rings of scatterers separated by large distance [6]; (b) amplify-and-forward relay; (c) propagation via diffracting street corner [1].

If the double scattering phenomenon is found to occur commonly in nature, this may have some interesting implications. It is commonly thought that the canonical frequency-flat Rayleigh fading amplitude presents the worst-case fading model for a radio channel [8]. In Figs. 2 and 3 the density and distribution functions of Rayleigh and double-Rayleigh random variables are plotted. It can be seen that the amount of fading of the double-Rayleigh random variable is significantly larger than that of the Rayleigh random variable. This motivates us to pose the question: What is the impact of double scattering on radio link performance? After all, if the effect was found to be negligible, a large amount of effort could be saved in radio channel measurements, transceiver design, and so on.

In this paper we study the error performance of communication over double-Rayleigh fading radio links. We derive analytically the symbol error probability (SEP) of some common modulation schemes, and evaluate the performance

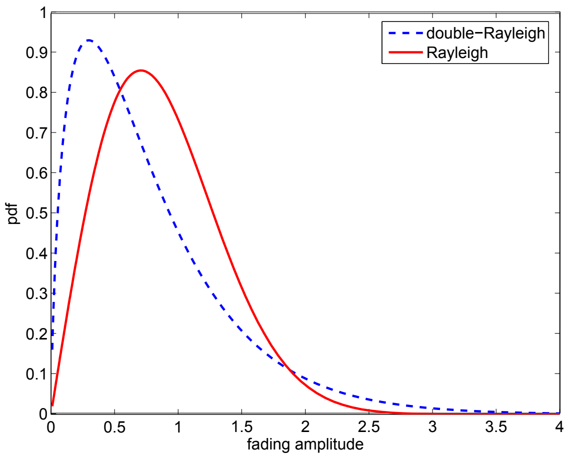

Fig. 2. Probability density functions of Rayleigh and double-Rayleigh channel amplitude. Mean power is normalized to unity.

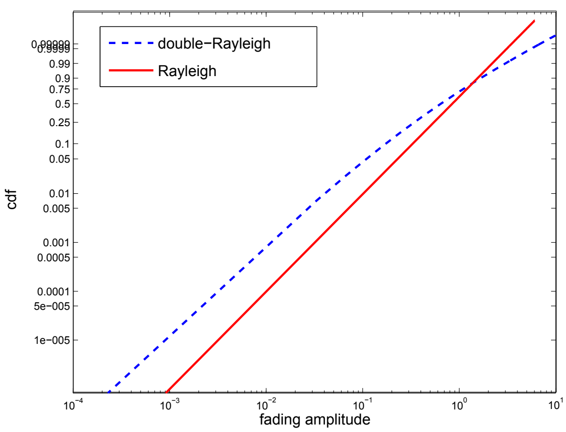

Fig. 3. Cumulative distribution functions of Rayleigh and double-Rayleigh channel amplitude. Mean power is normalized to unity.

degradation compared to the classical (single) Rayleigh fading channels. We also show that diversity order is not a meaningful channel measure for the double-Rayleigh channel. This is because the SEP for double-Rayleigh fading converges to its limiting high-SNR slope very slowly. In this sense the high-SNR behavior of the double-Rayleigh channel differs from other standard fading models.

The paper is organized as follows. In Section II we present the signal model. Section III gives the main result: SEP expression for some common modulation schemes over double-Rayleigh fading channels. The proof of the main result can be found in Section IV. The diversity order of double-Rayleigh fading channels is discussed in Section V, while numerical examples are presented in Section VI. Section VII concludes the paper.

II. Channel Model

The general form of the impulse response of a multiple scattering radio channel is [3]

$$H = C + H_1 + H_2H_3 + H_4H_5H_6 + \dots\,, \qquad (1)$$

TABLE I

The value of m for various modulation schemes [8].

modulation method	m
coherent amplitude-shift keying (ASK)	0.5
coherent frequency-shift keying (FSK)	1
coherent binary phase-shift keying (BPSK)	2
coherent quadrature phase-shift keying (QPSK)	2
minimum shift keying (MSK)	1.7

where the C is the deterministic line-of-sight component with constant magnitude. The H_is are assumed independent circularly symmetric Gaussian zero-mean complex random variables. For physical explanation behind the model, see [4]. Conventionally, only the first-order scattering is assumed, i.e. $H_i = 0$, for all $i > 1$; this results in the conventional Rayleigh fading signal magnitude (or Rice, if $|C| > 0$). In this paper we restrict to non-line-of-sight double scattering, and consequently assume that $C = 0$ and $H_i = 0$ for all $i \neq 2, 3$. From these assumptions it follows that $R = |H| = |H_2H_3|$ will have the double-Rayleigh distribution. We further assume that H has unit variance, so that $E[R^2] = 1$, which is the usual channel power normalization. Our main contribution is to derive an expression for the symbol error probability of some common communication schemes operating over the double-Rayleigh fading channel.

III. Main Result

The instantaneous SNR at the receiver is $\bar{\gamma}\beta$, where $\beta = R^2$, $E[\beta] = 1$, and $\bar{\gamma}$ is the average SNR. The SEP averaged over the fading SNR is [9]

$$P_E = \int_0^\infty P_E(\beta)p(\beta)\,\mathrm{d}\beta\,, \qquad (2)$$

where $p(\beta)$ denotes the pdf of the SNR, and $P_E(\beta)$ is the symbol error probability for fixed β. For example, for the classical (single) Rayleigh amplitude distribution $p(\beta) = e^{-\beta}$, whereas for the double-Rayleigh distribution we have $p(\beta) = 2K_0\left(2\sqrt{\beta}\right)$, where $K_0(x)$ is the modified Bessel function of the second kind [1].

Throughout the paper we assume that the symbol error probability conditioned on β is given by

$$P_E(\beta) = \mathrm{Q}(\sqrt{m\bar{\gamma}\beta})\,, \qquad (3)$$

where $\mathrm{Q}(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty \exp(-0.5t^2)\,\mathrm{d}t$. In (3) it implicitly assumed that channel noise is additive Gaussian. While it is possible to consider more complex forms of error probability, (3) encompasses a large class of practical modulation schemes, while affording reasonably simple analysis. In Table I, the value of m is listed for some common modulation schemes.

Assuming (3) holds, the average SEP is given by

$$P_E = 2\int_0^\infty \mathrm{Q}(\sqrt{m\bar{\gamma}\beta})K_0(2\sqrt{\beta})\,\mathrm{d}\beta\,. \qquad (4)$$

We do not know a closed-form solution to this integral. We shall show in Section IV that (4) is given in an easily computable series form as

$$P_E = \frac{1}{2}\sum_{n=1}^{\infty}\frac{2^n}{(m\bar{\gamma})^n}\frac{(2n-1)!!}{(2n)!!(n-1)!} \times \left[\frac{1}{n} - \ln\left(\frac{2}{m\bar{\gamma}}\right) - \psi\left(n+\frac{1}{2}\right) + 2\psi(n)\right] . \quad (5)$$

where $k!!$ is the double factorial and $\psi(x)$ is the digamma function. The series form in (5) provides some insight on the high-SNR performance of modulation schemes over double-Rayleigh fading channel; this will be discussed in Section V.

From practical numerical computation viewpoint, we remark that for average SNR above 0 dB about five terms is sufficient for practical usage, and for SNR above 10 dB only the first term is required. For SNR below 0 dB the convergence of the series becomes slower, and more terms are required. In practice this does not pose problems, since the only special function in (5), the digamma function, can be efficiently evaluated using finite sums for the positive integer and positive-integer-plus-one-half arguments appearing in (5).

IV. Proof of (5)

In this section we prove the main result. A reader wishing to skip details may proceed directly to Section V.

A. Main idea

Using the series presentations [10, §8.447.3]

$$K_0(x) = -\ln\left(\frac{x}{2}\right) I_0(x) + \sum_{k=0}^{\infty}\frac{x^{2k}}{2^{2k}(k!)^2}\,\psi(k+1)\,, \quad (6)$$

with the series expansion for the modified Bessel function

$$I_0(x) = \sum_{k=0}^{\infty}\frac{x^{2k}}{2^{2k}(k!)^2}\,, \quad (7)$$

we can write

$$K_0(x) = \sum_{k=0}^{\infty}\frac{x^{2k}}{2^{2k}(k!)^2}\left[-\ln\left(\frac{x}{2}\right) + \psi(k+1)\right] . \quad (8)$$

Using (8), we can write (4) as

$$\begin{aligned} P_E &= 2\sum_{k=0}^{\infty}\frac{1}{(k!)^2}\Bigg[-\frac{1}{2}\underbrace{\int_0^{\infty}\beta^k\ln(\beta)\,\mathrm{Q}\left(\sqrt{a\beta}\right)\mathrm{d}\beta}_{I_1} \\ &\qquad +\psi(k+1)\underbrace{\int_0^{\infty}\beta^k\,\mathrm{Q}\left(\sqrt{a\beta}\right)\mathrm{d}\beta}_{I_2}\Bigg] \\ &= 2\sum_{k=0}^{\infty}\frac{1}{(k!)^2}\left[-\frac{1}{2}I_1 + \psi(k+1)I_2\right] \quad (9)\end{aligned}$$

where we denoted $a = m\bar{\gamma}$. In the following subsections we shall evaluate the integrals I_1 and I_2 separately, starting with the easier one.

B. Evaluating I_2

The integral I_2 is

$$I_2 = \int_0^{\infty}\beta^k\,\mathrm{Q}(\sqrt{a\beta})\,\mathrm{d}\beta\,. \quad (10)$$

Using the well-known relation

$$\mathrm{Q}(x) = \frac{1}{2}\,\mathrm{erfc}(\frac{x}{\sqrt{2}}) \quad (11)$$

and change of variable $y = \sqrt{a\beta}$ we must integrate

$$I_2 = \frac{2^{k+1}}{a^{k+1}}\int_0^{\infty}y^{2k+1}\,\mathrm{erfc}(y)dy\,. \quad (12)$$

This can be evaluated in closed-form with [10, §6.281]

$$\int_0^{\infty}x^{2n-1}\,\mathrm{erfc}(x)dx = \frac{\Gamma(n+\frac{1}{2})}{2n\sqrt{\pi}}\,. \quad (13)$$

Therefore, we have

$$I_2 = \frac{2^k\Gamma\left(k+\frac{3}{2}\right)}{a^{k+1}(k+1)\sqrt{\pi}}\,. \quad (14)$$

In the sequel, we find it convenient to express I_2 as

$$I_2 = \frac{2^k k!\,(2k+1)!!}{a^{k+1}(2k+2)!!}\,, \quad (15)$$

where we used

$$\Gamma\left(k+\frac{3}{2}\right) = \frac{(2k+1)!!\sqrt{\pi}}{2^{k+1}} \quad (16)$$

and

$$(2k+2)!! = 2^{k+1}(k+1)!\,, \quad (17)$$

with $t!!$ denoting the double factorial.

C. Evaluating I_1

Next we evaluate

$$I_1 = \int_0^{\infty}\beta^k\ln(\beta)\,\mathrm{Q}\left(\sqrt{a\beta}\right)\mathrm{d}\beta\,. \quad (18)$$

The Craig presentation of the Q function for $x > 0$ is [11]

$$\mathrm{Q}(x) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin^2(\theta)}}\,\mathrm{d}\theta\,. \quad (19)$$

Plugging (19) into I_1 we need to evaluate [denote $b = 2\sin^2(\theta)$]

$$I_1 = \frac{1}{\pi}\int_0^{\infty}\int_0^{\frac{\pi}{2}}\beta^k\ln(\beta)e^{-\frac{a\beta}{b}}\,\mathrm{d}\beta\,\mathrm{d}\theta\,.$$

Using [10, §4.352.1]

$$\int_0^{\infty}\beta^k\ln(\beta)e^{-\frac{a\beta}{b}}\,d\beta = \frac{b^{k+1}}{a^{k+1}}k!\left[\psi(k+1) - \ln\left(\frac{a}{b}\right)\right]$$

leaves us

$$I_1 = \frac{2^{k+1}k!}{\pi a^{k+1}}\Bigg\{\left[\psi(k+1)+\ln\left(\frac{2}{a}\right)\right]\underbrace{\int_0^{\frac{\pi}{2}}(\sin\theta)^{2k+2}\,\mathrm{d}\theta}_{A}$$
$$+2\underbrace{\int_0^{\frac{\pi}{2}}\ln(\sin\theta)(\sin\theta)^{2k+2}\,\mathrm{d}\theta}_{B}\Bigg\}. \quad (20)$$

The first integral in (20) is given by [10, §3.621.3]

$$A = \int_0^{\frac{\pi}{2}}(\sin\theta)^{2k+2}\,\mathrm{d}\theta = \frac{\pi}{2}\frac{(2k+1)!!}{(2k+2)!!}. \quad (21)$$

The second integral in (20) is [10, §4.387.4]

$$B = \int_0^{\frac{\pi}{2}}\ln(\sin\theta)(\sin\theta)^{2k+2}\,\mathrm{d}\theta \quad (22)$$
$$= \frac{\pi}{2}\frac{(2k+1)!!}{(2k+2)!!}\left[\sum_{n=1}^{2k+2}\frac{(-1)^{n+1}}{n}-\ln 2\right]. \quad (23)$$

Noting that

$$\sum_{n=1}^{2k+2}\frac{(-1)^{k+1}}{k} = \ln 2+\frac{1}{2}\left[\psi\left(k+\frac{3}{2}\right)-\psi(k+2)\right], \quad (24)$$

with $\psi(x)$ denoting the digamma function, we can write B as

$$B = \frac{\pi}{4}\frac{(2k+1)!!}{(2k+2)!!}\left[\psi\left(k+\frac{3}{2}\right)-\psi(k+2)\right]. \quad (25)$$

By further noting that

$$\psi(k+1)-\psi(k+2) = -\frac{1}{k+1} \quad (26)$$

and inserting (21) and (25) in (20) we have

$$I_1 = \frac{2^k k!\,(2k+1)!!}{a^{k+1}(2k+2)!!} \times\left[\ln\left(\frac{2}{a}\right)+\psi\left(k+\frac{3}{2}\right)-\frac{1}{k+1}\right]. \quad (27)$$

D. Combining I_1 and I_2

Recalling that $a = m\bar{\gamma}$, and combining (15) and (27) with (9), and changing of index $n = k+1$, (5) follows.

V. Diversity Order

Diversity order d of a communication method is the high-SNR slope of the symbol error probability (SEP) determined from a SEP-SNR graph plotted in log-log scale. Mathematically, we can define

$$d = -\lim_{\bar{\gamma}\to\infty}\frac{\mathrm{d}\ln(P_E)}{\mathrm{d}\ln(\bar{\gamma})}, \quad (28)$$

where P_E is the symbol error probability defined in (2). For Rayleigh channel the bit error probability can be approximated at high SNR by $P_{E,\text{rayl}} \approx \frac{1}{2m\bar{\gamma}}$. This results in $d_{\text{rayl}} = 1$, which is independent of the average SNR and modulation scheme, i.e., it is a property of the channel fading statistics alone [12].

For the double-Rayleigh channel we can approximate SEP at high SNR by taking only the first term of (5):

$$P_E \approx \frac{1}{2m\bar{\gamma}}\left[1-\ln\left(\frac{2}{m\bar{\gamma}}\right)-\psi\left(\frac{3}{2}\right)-2\mu\right], \quad (29)$$

where $\mu \approx 0.5772$ is the Euler-Mascheroni constant. Note that (29) is actually a lower bound on SEP. Approximating (29) further we have a simple relation for high $\bar{\gamma}$

$$P_E \approx \frac{\ln(\bar{\gamma})}{2m\bar{\gamma}}. \quad (30)$$

From (30) we note that the slope of the symbol error probability decays as $\sim\bar{\gamma}^{-1}\ln(\bar{\gamma})$ rather than $\sim\bar{\gamma}^{-1}$ as is the case with the single-Rayleigh channel. Hence we would expect the diversity order of the double-Rayleigh channel to be smaller than unity, which is the Rayleigh channel's diversity order.

From its definition in (28) and (29), with constants subsumed in $q \approx -0.884$, we can compute the diversity order for the double-Rayleigh channel as

$$d = \lim_{\bar{\gamma}\to\infty}\left(1-\frac{1}{\ln(\bar{\gamma})+\ln(m)+q}\right) \quad (31)$$
$$= 1, \quad (32)$$

which is, in principle, the same as that of the Rayleigh channel. One should, however, note that the term inside the limit in (31) converges to (32) extremely slowly as illustrated in Fig. 4. It can be seen that, due to slow convergence to unity, the slope of the SEP curve is not a constant independent of the average SNR over the range shown in Fig. 4. Therefore, for practical average SNR values there does not exist a constant slope for the SEP curve. This behavior is in sharp contrast to most other standard fading models, including Rayleigh and Nakagami-m channels, for which the diversity order is a meaningful channel measure even for relatively low average

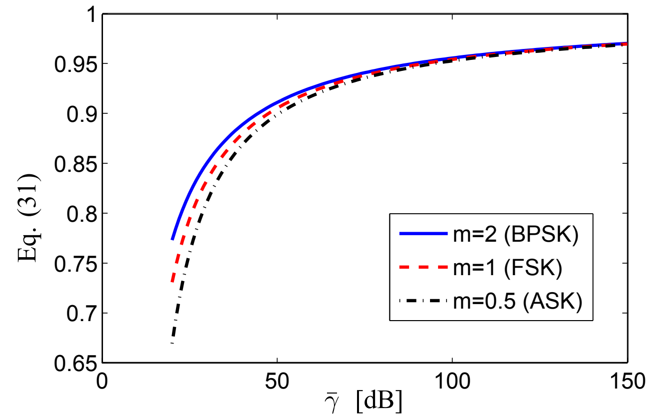

Fig. 4. Convergence of (31) for different m.

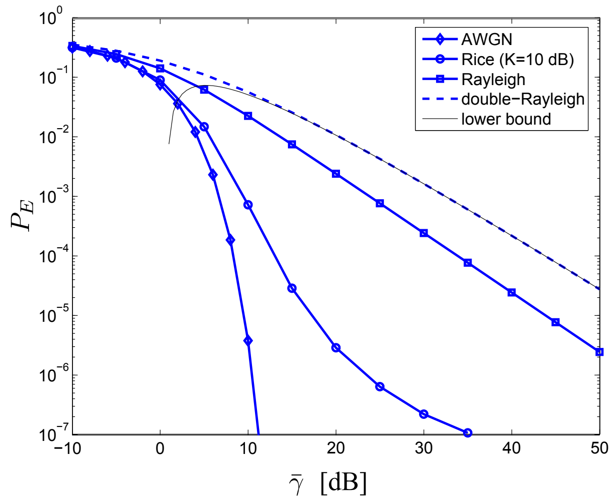

Fig. 5. The bit error probability of BPSK over AWGN, Rice, Rayleigh and double-Rayleigh channels, including the lower bound (29).

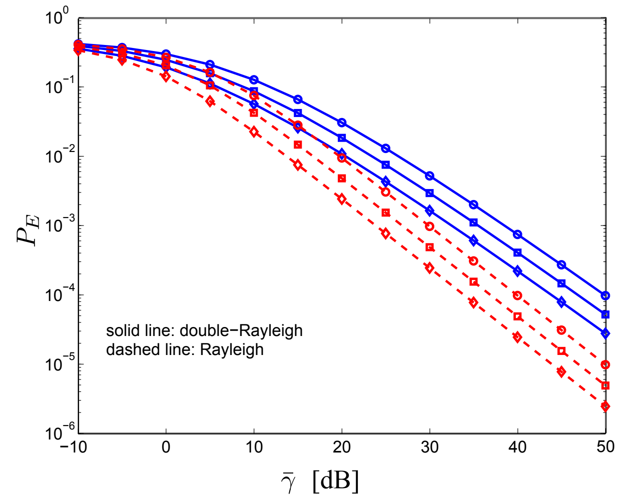

Fig. 6. The bit error probability of three modulation schemes over Rayleigh and double-Rayleigh channels. Circle markers denote ASK ($m = 0.5$), square markers FSK ($m = 1$), and diamond markers BPSK ($m = 2$)

SNR [12]. This pathological feature signals problems in the practical applicability of many high-SNR analysis methods to the double-Rayleigh channel. For example, in the framework introduced in [12] it is assumed that near origin the density of SNR can be accurately approximated by a polynomial term of form $p(\beta) \sim a\beta^t$. The constants a and t then define the diversity order and coding gain of the modulation scheme at high SNR. However, for the double-Rayleigh channel the SNR pdf for β small is of form $p(\beta) \sim -2\ln(2\sqrt{\beta})$. In other words, the SNR pdf is *not* polynomially smooth near the origin, which means that the high-SNR technique presented in [12] cannot be directly applied to the double-Rayleigh channel.

VI. Numerical Examples

In Fig. 5, we plot SEP of coherently detected BPSK for AWGN, Rice, Rayleigh and double-Rayleigh channels. In comparison to Rayleigh fading channel, we note that at SEP=10^{-3} there is about eight dB degradation due to double-Rayleigh fading. Furthermore, the high-SNR slope is clearly worse than that of the Rayleigh channel. The lower bound (29) is useful for about $\bar{\gamma} > 10$ dB.

In Fig. 6 we show the bit error probability for ASK, FSK and BPSK modulation schemes over Rayleigh and double-Rayleigh channels. For SEP below 10^{-3} the best scheme (BPSK) for the double-Rayleigh channel is worse than the worst scheme (ASK) for the Rayleigh channel.

VII. Conclusion

We have derived an expression for the symbol error probability of come common modulation schemes operating over a double-Rayleigh fading channel. It was shown that double-Rayleigh fading results in considerably degraded error performance compared to single Rayleigh fading. Furthermore, due to pathological behavior of the double-Rayleigh SNR distribution, diversity order is not a meaningful channel measure for SNR levels of practical interest.

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References

[1] V. Erceg, S. J. Fortune, J. Ling, A. Rustako, and R. A. Valenzuela, “Comparisons of a computer-based propagation prediction tool with experimental data collected in urban microcellular environments,” *IEEE J. Select. Areas Commun.*, vol. 15, no. 4, pp. 677–684, May 1997.

[2] D. Chizhik, G. J. Foschini, and R. A. Valenzuela, “Capacities of multi-element transmit and receive antennas: Correlations and keyholes,” *El. Lett.*, vol. 36, no. 13, pp. 1099 –1100, Jun. 2000.

[3] J. B. Andersen and I. Z. Kovács, “Power distributions revisited,” in *Proc. COST273 3rd Management Committee Meeting*, Jan. 17-18, 2002, Guildford, UK, TD(02)004.

[4] J. B. Andersen, “Statistical distributions in mobile communications using multiple scattering,” in *Proc. 27th URSI General Assembly*, Maastricht, Netherlands, Aug. 2002.

[5] I. Z. Kovács, P. Eggers, K. Olesen, and L. Petersen, “Investigations of outdoor-to-indoor mobile-to-mobile radio communication channels,” in *Proc. IEEE 56th Vehicular Technology Conference (VTC-Fall)*, vol. 1, Sept. 2002, pp. 430– 434.

[6] D. Gesbert, H. Bölcskei, D. A. Gore, and A. J. Paulraj, “Outdoor MIMO wireless channels: Models and performance prediction,” *IEEE Trans. Commun.*, vol. 50, no. 12, pp. 1926–1934, Dec. 2002.

[7] M. O. Hasna and M.-S. Alouini, “A performance study of dual-hop transmissions with fixed gain relays,” *IEEE Trans. Wireless Commun.*, vol. 3, no. 6, pp. 1963–1968, Nov. 2004.

[8] G. D. Durgin, *Space-Time Wireless Channels*. Prentice-Hall, 2003.

[9] J. G. Proakis, *Digital Communications*, 3rd ed. McGraw-Hill, Inc., 1995.

[10] I. Gradshteyn and I. Ryzhik, *Table of integrals, series, and products*, 4th ed. Academic Press, Inc., 1980.

[11] J. W. Craig, “A new, simple and exact result for calculating the probability of error for two-dimensional signal constellations,” in *Proc. MILCOM 91*, vol. 2, McLean, VA, USA, Nov. 1991, pp. 571–575.

[12] Z. Wang and G. B. Giannakis, “A simple and general parameterization quantifying performance in fading channels,” *IEEE Trans. Commun.*, vol. 51, no. 8, pp. 1389–1398, Aug. 2003.