Backward Wave Region and Negative Material Parameters of a Structure Formed by Lattices of Wires and Split-Ring Resonators

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***Abstract*—A structure formed by combined lattices of infinitely long wires and split-ring resonators is studied. A dispersion equation is derived and then used to calculate the effective permittivity and permeability in the frequency band where the lattice can be homogenized. The backward wave region in which both the effective permittivity and permeability are negative is analyzed. Some open and controversial questions are discussed. It is shown that previous experimental results confirming the existence of backward waves in such a structure can be in deed explained in terms of negative material parameters. However, these parameters are not quasi-static and thus the known analytical formulas for the effective material parameters of this structure, which have been widely used and discussed in the literature, were not correct, and it was the reason of some objections to the authors of that experiment.**

***Index Terms*—Analytical formula, backward wave, homogenization, left-handed media, negative permeability, negative permittivity, negative refraction, split ring resonators, Veselago media.**

I. INTRODUCTION

METAMATERIALS with negative permittivity and negative permeability, which were first suggested in [1], have attracted much attention recently. A metamaterial that has simultaneously negative permittivity ϵ and negative permeability μ within a certain frequency band at microwave frequencies has been introduced recently [2]. This structure consists of two lattices: a lattice of infinitely long parallel wires, and a lattice of relatively small (compared to the wavelength λ in the host medium) particles which are called *split-ring resonators* (SRRs). In [3] and [4] two analytical models of SRR (similar to each other) have been developed for the resonant permeability at microwave frequencies. Lattices of wires at low frequencies (when the lattice period d is smaller than $\lambda/2$) were considered as homogenous dielectric media long time ago in [5] and were studied again recently in the low-frequency region [6], [7]. At these low frequencies the negative permittivity is due to the lattice of wires according to the models of [5]–[7] (for waves propagating normally to the wires with the electric field polarized along these wires). These results were combined in [2], [10]–[12] to form a simple model predicting simultaneously negative ϵ and μ within the resonant band of a SRR. In [10] this prediction has been qualitatively confirmed by numerical simulations using the MAFIA code. Dispersion curves obtained numerically contain the passband within the SRR resonant band (due to the presence of the SRR lattice). This passband can also be predicted by the analytical model. However, the numerical dispersion data obtained in [10] have not been used to extract the material parameters. The experimental observation of the negative refraction of a wave in such a structure is reported in [8]. The phenomenon of the negative refraction was predicted in [1] for media with $\epsilon < 0$ and $\mu < 0$, and according to this theory they correspond to the backward wave region (where the Poynting vector of the eigenwave is opposite to the wave vector).

Does the experimental observation of [8] mean that the structure suggested in [2] can be described through ϵ and μ which are both negative within the SRR resonant band? Based on [8] one can only assert that the negative refraction region necessarily exists within this frequency band that can be explained in terms of backward-waves. Backward waves in a lattice correspond to negative dispersion, i.e., the group velocity (the derivative of the eigenfrequency with respect to the wavenumber) is in the opposite direction of the phase velocity. Negative dispersion for a lattice is quite common in high frequency bands ($kd > \pi$, where $k = 2\pi/\lambda$ is the wavenumber in the host medium). However, at low frequencies ($kd < \pi$) negative dispersion is an abnormal phenomenon. In [2], [8], and [10] the lattice at low frequencies was treated as a continuous medium and the concept of the negative dispersion is equivalent to the concept of negative material parameters. However, is it possible to describe the structure formed by lattices of wires and SRRs (studied in [2] and [8]) in terms of ϵ and μ within the resonant band of SRRs? This question remains open since the analytical model [2] with which these material parameters were introduced is incomplete and can be wrong.

In [8], the negative refraction was observed only for waves whose electric field is parallel to the wires and the wave vector is perpendicular to the wires. Thus, the permittivity ϵ considered in [6] and [2] are the xx component of the permittivity tensor (we assume the wires are along the x axis) and the permeability

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μ considered there are the transversal component of the permeability tensor $\mu_t = \mu_{yy} = \mu_{zz}$ (there are two orthogonal SRRs in each unit cell of the structure studied in [8]). If one considers only the propagation in the transversal plane (i.e., the $y-z$ plane in our case), one can neglect the spatial dispersion in the wire lattice and consider it as a medium with negative permittivity [5]. In the case of perfectly conducting wires one has [5]

$$\epsilon = 1 - \frac{\omega_p^2}{\omega^2} \tag{1}$$

where ω_p is the analog of the plasma frequency. In [4], [6], and [12], the following frequency dependencies of the effective medium parameters were suggested for their structure when both the SRRs and the wires are made of real metal

$$\epsilon = 1 - \frac{\omega_p^2 - \omega_e^2}{\omega^2 - \omega_e^2 - j\gamma\omega} \tag{2}$$

$$\mu = 1 - \frac{\omega_{mp}^2 - \omega_m^2}{\omega^2 - \omega_m^2 - j\gamma\omega} \tag{3}$$

where ω_{mp} is the analog of the resonant frequency of a magnetic plasma, ω_e is the electronic resonant frequency and ω_m is the magnetic resonant frequency (i.e., the resonant frequency of the magnetic polarizaibility for an individual SRR). These formulas correspond to the conventional quasi-static model for the permittivity and permeability, i.e., the electromagnetic interaction between the SSRs and wires is neglected. In the present paper, we take into account this interaction. We consider the lossless case [corresponding to $\gamma = 0$, $\omega_e = 0$ in (2) and (3)].

In the present paper, we also discuss the following three questions:

- How crucial is the electromagnetic interaction between the SRRs and the wires when calculating the material parameters of the whole structure?
- How to find the frequency region within the resonant band of SRR scatterers in which the homogenization of the whole structure is allowed?
- Which is the correspondence between two bands: backward-wave region and the band in which the permittivity and permeability are both negative?

The first question has been considered briefly in [13]. It was indicated that the structure considered in [8] was fortunately built so that each SRR is located exactly at the center of two adjacent wires and thus there is no quasi-static interaction between the wire lattice and the SRR lattice (i.e., the magnetic fields produced by the two adjacent equivalent line currents cancel out at the center where the SRR is located). If one considers ϵ and μ as quasistatic parameters (as was done in [13]), the absence of the quasi-static interaction should lead to the following result: the effective permittivity of the structure is identical to the effective permittivity of the lattice of wires and the effective permeability of the structure is identical to the effective permeability of the lattice of SRRs. However, we will show in the present paper that this is not correct since the electromagnetic interaction between the wires and the SRRs in such a structure is not quasistatic (or local) and will dramatically influence the effective permittivity.

For the second question, one must be very careful in the homogenization of the complex structure studied in [2], [8] and [10]. In fact, the results of [8] can not be interpreted quantitatively in terms of the permittivity and permeability used in [2], [4], [6], [8], and [10]. This has been revealed in [14]. In the present paper, we develop an analytical model for a structure similar to the one studied in [8] (i.e., formed by combined lattices of infinitely long wires and split-ring resonators). The model allows the structure to be homogenized and its valid frequency domain to be identified. A self-consistent dispersion equation is derived and then used to calculate correctly the effective permittivity and permeability in the frequency band where the lattice can be homogenized. For the third question, the low-frequency backward wave region is analyzed and it is found that both effective permittivity and permeability are negative in it.

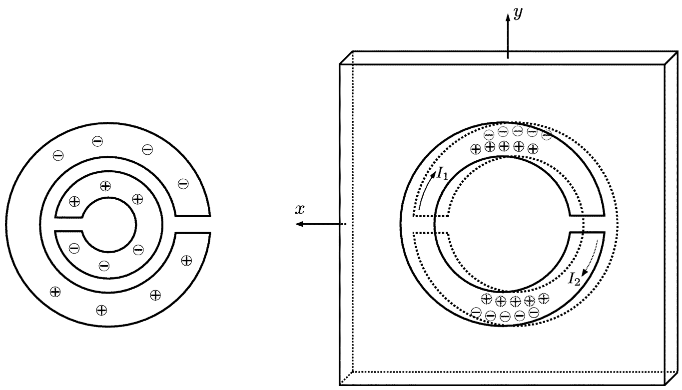
Our results have shown that the homogenization is allowed only over part of the resonant band of the SRR scatterers and the homogenization is forbidden in a subband inside the SRR resonant band (even though the frequencies of this subband are low and the spatial dispersion exists there).

II. SRR With Identical Rings

The SRR particle considered in [3] and [4] is a pair of two coplanar broken rings. Since the two loops are not identical the analytical model for this particle is rather cumbersome (SRR models more complete than those suggested in [3] and [4] have been developed in [15] and [16]). It is not correct that the SRR particle can be described simply as a resonant magnetic scatterer [16]. The structure considered in [8] (if homogenized) has to be described through three material parameters: ϵ, μ and κ (the magnetoelectric coupling parameter). It was shown in [16] that a SRR is actually a bianisotropic particle and the role of bianisotropy is destructive for negative refraction.

The bianisotropy is not the only disadvantage of this SRR particle. Another disadvantage is its resonant electric polarizability (i.e., the polarization produced by the electric field), which was also ignored in [3] and [4]. Electric polarizability resonance occurs at frequencies very close to the resonant frequency of the magnetic polarizability [15]. If the SRRs are made of **lossy** metal, the resonant electric polarizability will lead to a dramatic increase in the resistive loss. The resonant electric polarizability also makes the analytical modeling of the whole structure very complicated.

A modified SRR which does not possess bianisotropy was proposed in [16]. This SRR also consists of two loops but they are identical and parallel to each other (located on both sides of a dielectric plate in practice). Fig. 1 shows two kinds of SRR. The left one is the SRR considered in [3] and [4]. The right is the SRR introduced in [16] and the one considered in the present paper. It has been mentioned in [16] that the magnetic resonant frequency of their SRR is lower than that of the SRR considered in [3] and [4] (for the same size). This is because the mutual capacitance C_{mut} between the two parallel broken rings is now the capacitance of a conventional parallel-plate capacitor and is significantly higher than the mutual capacitance of two coplanar split rings considered in [4]. This fact is illustrated in Fig. 1(b):



(a) (b)

Fig. 1. (a) coplanar SRR and (b) parallel SRR. The concept of the mutual capacitance is illustrated by indicating the charges in the polarized rings.

if the upper half of ring 1 is charged positively the negative induced charges appear in the upper half of ring 2. The same situation happens for the SRR shown in Fig. 1(a), but not so effectively since the strips are coplanar and weakly interacted.

There is one more advantage that was not mentioned in [16] for the SRR of two parallel broken rings. The resonances of the electric and magnetic polarizabilities of this particle do not overlap in frequency. Actually, particle suggested in [16] is a special case of the bi-helix particle introduced (with the aim to create novel low-reflective shields) and studied in [17]. Unlike the SRR considered in [16] this bi-helix particle contains four stems orthogonal to the loop planes. Note that the theory of [17] remains valid even if the length of these stems becomes zero.

Both rings 1 and 2 [see Fig. 1(b)] have the same radius r and area $S = \pi r^2$. The impedance Z_0 for each broken ring can be calculated by

$$Z_0 = \frac{1}{Y_{\rm ring}} + \frac{1}{Y_{\rm split}}$$

where $Y_{\rm ring}$ is the admittance for the corresponding closed ring and $Y_{\rm split}$ is the admittance for the split (associated with the capacitance between the two broken ends). The magnetization arises due to the magnetic field orthogonal to the ring plane (i.e., the xy plane in Fig. 1). Resonant electric polarization is caused by the y component of the external local electric field $\mathbf{E}^{\rm loc}$. The x component of $\mathbf{E}^{\rm loc}$ has no influence over the resonant polarization and can be neglected [18]. Thus, voltages (electromotive forces) $\mathcal{E}^{\rm H}$ and $\mathcal{E}^{\rm E}$ will be induced in each loop by the external local electric and magnetic fields, respectively, [17], [18]

$$\mathcal{E}^{\rm H} = -j\omega\mu_0 S H_z^{\rm loc} \tag{4}$$

and

$$\mathcal{E}^{\rm E} = \frac{4rJ_1'(kr)Z_0}{j\eta A_1(kr)}\left(1 + \frac{j}{\pi\eta(Y_{\rm ring} + Y_{\rm split})}\right)E_y^{\rm loc} \tag{5}$$

where J_1' is the derivative (with respect to the argument) of the Bessel function, and A_1 is one of the so-called King's coefficients known in the theory of loop antennas (see, e.g., [18]). Then the induced currents $I_{1,2}$ (due to the changing of the charges at the tips of the split arms) at the split gaps of rings 1 and 2 satisfy the following equations [17]

$$I_1 Z_0 + I_2 Z_{\rm mut} = \mathcal{E}^{\rm E} + \mathcal{E}^{\rm H} \tag{6}$$
$$I_2 Z_0 + I_1 Z_{\rm mut} = \mathcal{E}^{\rm E} + \mathcal{E}^{\rm H} \tag{7}$$

where $Z_{\rm mut}$ is the mutual impedance of the two broken rings.

It is clear from (6) and (7) that there are two eigenmodes of currents in the SRR. The first mode (excited by the local magnetic field) corresponds to $I_1 = I_2$ when the electric dipole moment of the SRR is zero and the magnetic dipole moment $\mathbf{m} = m\mathbf{z}_0$ is twice the magnetic moment of a single ring (see Fig. 1). The second mode (excited by the local electric field) corresponds to $I_1 = -I_2$ when the magnetic moment is zero and the electric dipole moment $\mathbf{p} = p\mathbf{y}_0$ of the SRR is twice the electric dipole moment of a single ring. The electric resonant frequency (at which the electric polarizability resonates) is always higher than the magnetic resonant frequency [17]. If the distance h between the two parallel broken rings is very small, The relative difference of these two resonant frequencies may exceed 50% [17], and we assume that the electric polarizability of the SSR is negligible within the frequency band of the magnetic resonance. Note that h is also the thickness of the dielectric layer between the two parallel broken rings in Fig. 1, the rings are assumed to be perfectly conducting with infinitesimal thickness.

Therefore, unlike the SSR considered in [3] and [4], the SSR suggested in [16] is appropriate for creating artificial magnetic resonance (no resonant electric properties within the frequency band of interest). This is the reason why we choose the SSR shown in the right part of Fig. 1 to study in the present paper.

An analytical model for an individual SRR particle is simpler than the one considered in [17] due to the absence of the stems. The model used in this section is quasistatic since it refers to an isolated particle of small size (with respect to the wavelength). Assume that the SRR shown in Fig. 1 is excited by magnetic field $H_z^{\rm loc}$. Also assume that the dielectric plate separating the two parallel rings has the same permittivity as the background medium (then we can avoid the influence of the dielectric plate which can be very strong if there is a mismatch in the permittivity). If the nonuniformity of the azimuthal current distribution in both rings and SRRs can be neglected, the magnetic polarizability can be written as

$$a_{mm} = \frac{m}{H_z^{\rm loc}} = \frac{2I\mu_0 S}{H_z^{\rm loc}} \tag{8}$$

where $I = I_1 = I_2$. From (6) and (4) it follows that

$$I = \frac{\mathcal{E}^{\rm H}}{Z_0 + Z_{\rm mut}} = \frac{-j\omega S\mu_0 H_z^{\rm loc}}{Z_0 + Z_{\rm mut}}. \tag{9}$$

Calculating the total impedance of the loop by taking into account the mutual coupling of the loops (as it was done in [15] and [16] for SRR of coplanar rings), we obtain

$$Z_{\rm tot} = Z_0 + Z_{\rm mut} = j\omega(L + L_{\rm mut}) + \frac{1}{j\omega C_{\rm tot}} + R_r.$$

Here, R_r is the radiation resistance of the whole particle, L is the ring inductance

$$L = \mu_0 r \left(\log \frac{32r}{w} - 2\right)$$

where w is the width of the strip from which the ring is made of, and $L_{\rm mut}$ is the mutual inductance of the two parallel coaxial rings:

$$L_{\rm mut} = \mu_0 r \left[\left(1 + \frac{3\xi^2}{4}\right) \log \frac{4}{\xi} - 2\right]$$

where $\xi = h/2r$. The total capacitance $C_{\rm tot}$ attributed to the split can be calculated (taking into account the capacitive mutual coupling; cf. [15] and [16]) as half of the mutual capacitance formed by the two parallel rings

$$C_{\rm tot} = \frac{C_{\rm mut}}{2} = \frac{\epsilon_0 \epsilon w \pi r}{2h}.$$

In this formula, the capacitance of the split is neglected since it is small as compared to $C_{\rm mut}$.

From (8) and (9) we obtain

$$a_{mm} = \frac{2\mu_0^2 S^2}{(L + L_{\rm mut})\left(\frac{\omega_0^2}{\omega^2} - 1\right) - j\frac{R_r}{\omega}} \qquad (10)$$

where

$$\omega_0^2 = \frac{1}{C_{\rm tot}(L + L_{\rm mut})}. \qquad (11)$$

In a similar way one can show that the electric polarizability resonates at the frequency ω_1 (see also [17])

$$\omega_1^2 = \frac{1}{C_{\rm tot}(L - L_{\rm mut})}$$

and $\omega_0 < \omega_1$.

We will also use the following result of (10)

$$\mathrm{Re}\left(\frac{1}{a_{mm}}\right) = \frac{(L + L_{\rm mut})\left(\frac{\omega_0^2}{\omega^2} - 1\right)}{2\mu_0^2 S^2}. \qquad (12)$$

The radiation resistance R_r can be found from the following condition [21]–[23]

$$\mathrm{Im}\left(\frac{1}{a_{mm}}\right) = \frac{k^3}{6\pi\mu_0}. \qquad (13)$$

In the dispersion equation for a lattice, R_r cancels out and does not influence the result.

III. The Structure

The structure we study in the present paper is similar to the one studied experimentally in [8], however, instead of the coplanar SRRs we use the parallel SRRs (as described in SectionII).

When the wave propagates along the z axis the electric field excites the x–directed current I_{n_y,n_z} in the wire numbered (n_y, n_z) (for the reference wire we have $n_y = n_z = 0$). The magnetic field excites those SRRs which are parallel to the x-z plane. Their magnetic moments are parallel to the y axis. Then the lattice can be considered as a set of two-dimensional (2-D) grids parallel to the x-y plane and orthogonal to the propagation direction. Each grid contains magnetic and electric polarizations. The magnetic moments as well as the currents are tangential to the grid plane, and each grid can be considered as a sheet of surface magnetic moment $\mathbf{M} = M\mathbf{y}_0$ and a surface electric current $\mathbf{J} = J\mathbf{x}_0$ or surface electric polarization $P = J/j\omega$ ($\mathbf{x}_0$, $\mathbf{y}_0$ are unit vectors of Cartesian axes). Similar situation holds for the wave propagation along the y axis. Then one has $\mathbf{M} = M\mathbf{z}_0$ and $\mathbf{J} = J\mathbf{x}_0$, and the electric and magnetic polarizations for each 2-D grid (parallel to the x-z plane) is again tangential to the grid and orthogonal to the propagation direction. The wire lattice and the SRR lattice have the same periods along the y and z axes and are denoted as b and d, respectively. The period of the SRR lattice along the x axis is denoted as a. Fig. 1 shows the two orthogonal sets of SRRs separated with each other by $a/2$ along the x axis. This separation plays no role in our model since we do not consider the electromagnetic interaction between these two orthogonal sets of magnetic dipoles. When the wave propagates along the z axis or y axis, one of the two sets of SRRs is not excited and the interaction is completely absent.

In the case $b = d$ such a structure behaves (within the frequency band where the homogenization is possible) like a uniaxial magneto-dielectric medium with relative axial permittivity ϵ_{xx} (mainly due to the presence of the wires) and relative transversal permeability $\mu_{yy} = \mu_{zz} = \mu_t$ (mainly due to the SRR particles). This indicates that in order to find the effective material parameters of the whole structure we can consider only the case of the normal propagation (along the z or y axis).

Note that the transversal permittivity and the axial permeability of the structure are equal to those of the background medium. For simplicity we assume this background medium is vacuum.

We will see that the SRRs strongly interact with the wires at each frequency. Their interaction is not quasi-static and influences the propagation constant starting from zero frequency. In this way it influences the material parameters of the whole structure.

IV. Dispersion Equation

Let the wave propagate along the z axis with propagation factor β (to be determined). Consider the whole structure as a set of parallel 2-D grids which are parallel to the $x-y$ plane and denote the surface magnetic moments $M(n_z)$ and the surface currents $J(n_z)$ at those grids numbered n_z. Then we choose an arbitrary SRR in the grid with $n_z = 0$ as the reference particle and an arbitrarily chosen wire (in the same grid) as the reference wire.

When we evaluate the magnetic moment m of the reference SRR (which is related to the surface magnetic moment M by $m = Mab$), we take into account its electromagnetic interaction with all the other SRRs following the work of [23] where a simple model of 3-D dipole lattice was suggested. As to the influence of the wires to the reference SRR, we can replace each grid of wires with a sheet of current $J(n_z)$ because of the absence of the a quasi-static interaction between SRRs and wires. This gives the well-known plane-wave approximation of the electromagnetic interaction in lattices (see [23] and the references cited there).

When we evaluate the current I of the reference wire (which is related to the surface electric current by $I = Jb$), we take into account its interaction with all the other wires following the work of [19] where a simple model of doubly-periodic wire lattice was suggested. The influence of the SRR lattice on the reference wire is taken into account under the plane-wave approximation and the reciprocity principle is satisfied.

Each sheet of electric or magnetic polarization produces a plane wave [20]. Since $M(n_z)$ and $J(n_z)$ satisfy

$$M(n_z) = Me^{-jn_z\beta d} \tag{14}$$

$$J(n_z) = Je^{-jn_z\beta d} \tag{15}$$

we can write the following relations for the x–component of the electric field (produced by all the sheets of magnetic moment M and acting on the reference wire) and the y–component of the magnetic field (produced by all the sheets of current J and acting on the reference SRR)

$$E_x^M = \sum_{n_z=-\infty}^{\infty} \mathrm{sign}(n_z)\frac{j\omega M}{2}e^{-jn_z\beta d - jk|n_z|d} \tag{16}$$

$$H_y^J = \sum_{n_z=-\infty}^{\infty} \mathrm{sign}(n_z)\frac{J}{2}e^{-jn_z\beta d - jk|n_z|d}. \tag{17}$$

Both series can be analytically carried out and we easily obtain

$$E_x^M = -\frac{\omega M}{2}\frac{\sin\beta d}{\cos kd - \cos\beta d} \tag{18}$$

$$H_y^J = \frac{jJ}{2}\frac{\sin\beta d}{\cos kd - \cos\beta d}. \tag{19}$$

The local electric field acting on the reference wire is the sum of E_x^M and the contribution of the wires

$$E_x^{\mathrm{loc}} = E_x^M + C_w I = E_x^M + C_w bJ \tag{20}$$

where C_w (the interaction factor of the wire lattice) was determined in [19]

$$C_w = \frac{-j\eta}{2b}\left[\frac{\sin kd}{\cos kd - \cos\beta d} + \frac{kb}{\pi}\left(\log\frac{kb}{4\pi} + \gamma\right) + j\frac{kb}{2}\right]. \tag{21}$$

Here η is the wave impedance of the host material and $\gamma = 0.5772$ is the Euler constant.

The local magnetic field acting on the reference SRR is the sum of H_y^M and the contribution of the SRR particles

$$H_y^{\mathrm{loc}} = H_y^J + C_d m = H_y^J + C_d abM \tag{22}$$

where C_d (the interaction factor of the lattice of magnetic dipoles) is given by [23]

$$C_d = \frac{\omega q}{2ab\eta} + j\frac{k^3}{6\pi\mu_0} + \frac{\omega}{2ab\eta}\frac{\sin kd}{\cos kd - \cos\beta d}. \tag{23}$$

A similar relation has been given in [23] for a lattice of electric dipoles (the only difference as compared to (23) is the factor η^2). Here q denotes the real part of the dimensionless interaction factor of a 2-D grid of dipoles with periods a, b. In [23], the closed-form expression for q is given for the case $a = b$

$$q_0 = \frac{1}{2}\left(\frac{\cos kas}{kas} - \sin kas\right)$$

where the number s is approximately equal to $1/1.4380 = 0.6954$. Relation (23) is very accurate for the case $d \gg a$, and in the case $d = a$ its error is still quite small [23].

The responses of the reference SRR and the reference wire to the local fields can be written as

$$m = a_{mm} H^{\mathrm{loc}} \tag{24}$$

$$I = \frac{E^{\mathrm{loc}}}{Z_w} \tag{25}$$

where Z_w is given by [19]

$$Z_w = \frac{k\eta}{4}\left[1 - \frac{2j}{\pi}\left(\log\frac{kr_0}{2} + \gamma\right)\right] \tag{26}$$

where r_0 is the effective radius of the wire ($r_0 = w/4$ if made from a strip with width w).

To obtain the dispersion equation we substitute (20), (22), (21), (23), (18), and (19) into (24) and (25). Since $m = Mab$ and $I = Jb$ we obtain the following system of equations

$$M\left(\frac{1}{a_{mm}} - A - j\frac{k^3}{6\pi\mu_0}\right) = j\frac{BJ}{ab} \tag{27}$$

$$J(Z_w - C_w) = -\omega BM \tag{28}$$

where

$$A = \frac{\omega}{2ab\eta}\left(\frac{\sin kd}{\cos kd - \cos\beta d} + q\right)$$

and

$$B = \frac{1}{2}\frac{\sin\beta d}{\cos kd - \cos\beta d}.$$

The parameter B describes the interaction between the currents in the wires and the magnetic moments of the SRRs. B is not a quasi-static parameter even at low frequencies since it does not approach zero at zero frequency. Its presence in the dispersion equation strongly influences the result for the propagation constant β at all frequencies.

Relations (13) and (26) lead to the cancellation of the imaginary part at the left-hand side of (27) and the real part at the left-hand side of (28). Thus, system of (27) and (28) gives the following real-valued dispersion equation:

$$(\mathrm{Im}(Z_w) - \mathrm{Im}(C_w))\left[\mathrm{Re}\left(\frac{1}{a_{mm}}\right) - A\right] = -\frac{\omega^2 B^2}{ab}.$$

It can be rewritten as the following quadratic equation with respect to $\cos\beta d$:

$$\left[\frac{2ab}{\omega\eta}\mathrm{Re}\left(\frac{1}{a_{mm}}\right) - q\right](\cos kd - \cos\beta d)\sin kd - \left(1 + \frac{kb}{\pi}\log\frac{b}{2\pi r_0}\right)\sin^2 kd - \cos^2\beta d + 1 = 0. \tag{29}$$

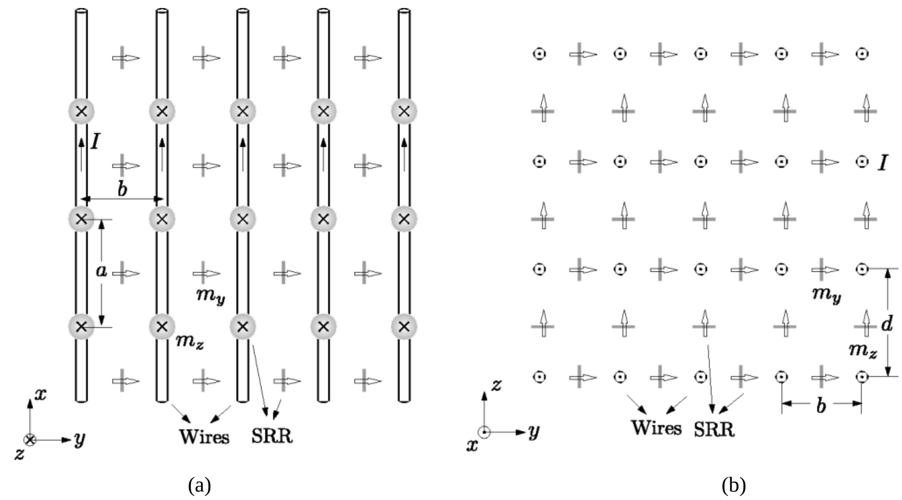


Fig. 2. (a) Front view of the lattice of SRRs (shown as disks; their magnetic dipoles are indicated with arrows) and straight wires. (b) Top view: wave propagation is in the y-z plane.

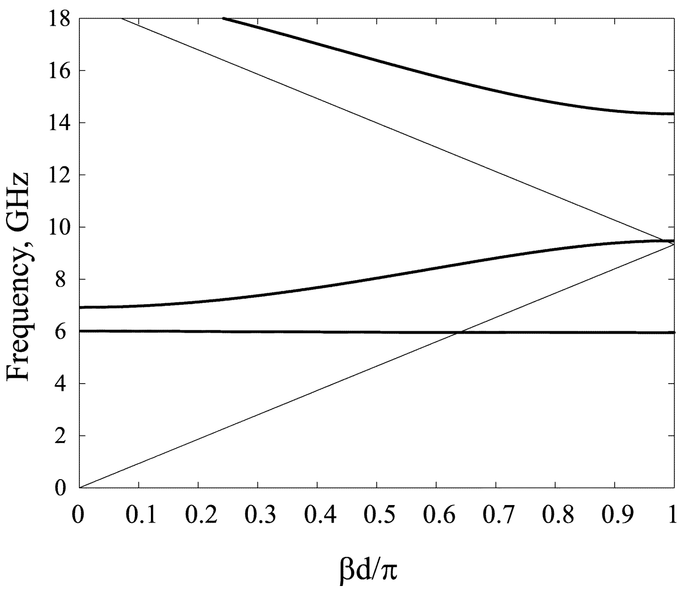


Fig. 3. Dispersion plot of the structure for $d = 2a = 2b = 16$ mm (thick lines). Thin lines coincide with the dispersion plot of host medium and correspond to the wave polarization $\mathbf{E} = E_y\mathbf{y}_0$, $\mathbf{H} = H_x\mathbf{x}_0$.

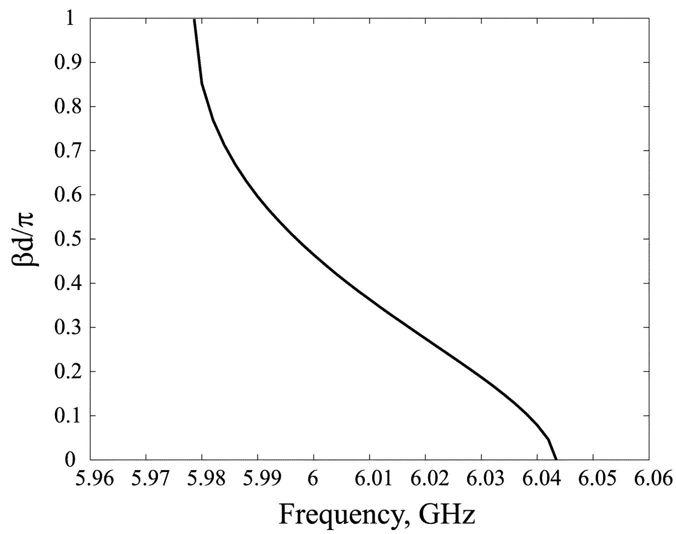


Fig. 4. Normalized propagation factor of the backward wave versus frequency for $d = a = b = 8$ mm.

There are two roots $\beta_{1,2}$ for the dispersion (29) at each frequency. One of them is exactly equal to k (the wavenumber in the host medium). This root corresponds to the wave with polarization $\mathbf{E} = E_y\mathbf{y}_0$ and $\mathbf{H} = H_x\mathbf{x}_0$. This wave excites neither wires nor SRRs and does not interact with the structure. Another root corresponds to the wave with polarization $\mathbf{E} = E_x\mathbf{x}_0$ and $\mathbf{H} = H_y\mathbf{y}_0$. This is the interacting wave which is of interest. In our dispersion curves we keep both solutions of (29).

V. Dispersion Curves

As numerical examples we choose the following parameters for the structure shown in Fig. 2: the size of SRR particle (outer diameter of the rings) is $D = 3.8$ mm, the width of the strip (forming the rings) is $w = 1$ mm, the radius of wire cross section is $r_0 = 0.2$ mm, the distance between the rings (which is chosen so that the resonance of a_{mm} is at 6 GHz) is $h = 0.84$ mm. Lattice periods $a = b = 8$ mm and $d = 16$ mm are chosen in our first example. In the second example, we choose $a = b = d = 8$ mm. In the third example, we choose $a = b = d = 4$ mm.

Fig. 3 gives the dispersion curve in a form commonly used in the literature of photonic crystals (see, e.g., [25]). It represents the dependence of the eigenfrequencies on the normalized propagation factor $\beta d/\pi$ over the first Brillouin zone ($0 < \beta < \pi/d$) for the case when $d = 2a = 2b = 16$ mm. Straight lines correspond to noninteracting waves. Curved lines correspond to interacting waves. The only difference of this curve as compared to the well-known plot for the wire medium (see, e.g., [19] and [24]) is the miniband at about 6 GHz, in which the group velocity is opposite with respect to $\mathbf{z}_0$ (our choice of β as that belonging to the first Brillouin zone fixes the positive direction of the phase velocity along the z axis).

In this narrow frequency band, wave propagation is prohibited in the lattice of wires. Therefore, the miniband is due to the presence of SRRs and the resonant magnetization of the SRR lattice.

The resonant passband becomes wider if the period d decreases. From Fig. 4 one can see the dependence of the prop-

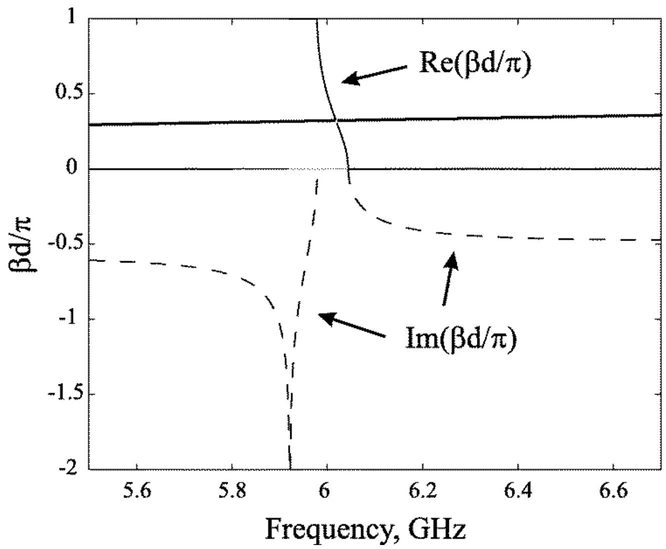


Fig. 5. Real (thin solid line) and imaginary (dashed line) parts of normalized propagation factor versus frequency for $d = a = b = 8$ mm. Thick line corresponds to the propagation of wave with polarization $\mathbf{E} = E_y\mathbf{y}_0$, $\mathbf{H} = H_x\mathbf{x}_0$ in host material.

agation factor on the frequency in the vicinity of the SRR resonance for the case $a = b = d = 8$ mm.

The backward-wave region corresponds to the frequencies 5.980–6.045 GHz, whereas the resonant frequency of $\mathrm{Re}(a_{mm})$ given by (11) is 6.000 GHz. $\mathrm{Re}(a_{mm})$ becomes negative at 6.000 GHz. Thus, within the backward-wave band $\mathrm{Re}(a_{mm})$ is mainly negative. The group velocity of the backward wave is relatively small (it approximately equals $5.3 \cdot 10^{-3}c$, where c is the speed of light).

In order to understand whether it is possible to homogenize the structure at the frequencies when the backward-wave region exists, we studied β for both propagating and decaying modes.

Fig. 5 shows the frequency dependence of both the real and imaginary parts of the normalized propagation factor $\beta d/\pi$ for the case $a = b = d = 8$ mm. It is clear that outside the SRR resonant band the eigenmodes of the structure are the same as those of the wire medium [19]. The thick straight line corresponds to the noninteracting wave.

From Fig. 5 one can see that the eigenmodes within the frequency band 5.92–5.98 GHz are complex. The lower limit of the backward wave region (5.980–6.045 GHz) is the upper limit of the complex-mode band. Complex modes cannot exist in continuous media. These modes are known for electromagnetic crystals with different geometries (see, e.g., [19], [22]). These are decaying modes though the real part of the propagation factor is $\mathrm{Re}\beta = \pi/d$. The existence of this real part of β reflects the fact that the directions of the currents in the wires are alternating along the propagation axis (two adjacent currents have opposite directions and this can be interpreted as the phase shift π between them due to the real part of the complex propagation factor).

Therefore, the homogenization is possible within one (the upper) half of the SRR resonant band but impossible within another (the lower) half of the SRR resonant band (though for these frequencies the structure periods are much smaller than the wavelength in the background medium).

VI. Homogenization

Let us try to consider the structure (in the case $a = b = d$) as a uniaxial magnetodielectric medium. Then the interacting wave (propagating along the z axis with $\mathbf{E} = E_x\mathbf{x}_0, \mathbf{H} = H_y\mathbf{y}_0$) also satisfies the following constitutive equations:

$$\begin{aligned} D_x &= \epsilon_0 E_x + P_x^{\mathrm{bulk}} = \epsilon_0\epsilon_{xx}E_x \\ B_y &= \mu_0 H_y + M_y^{\mathrm{bulk}} = \mu_0\mu_t H_y. \end{aligned}$$

Define the following ratio:

$$\alpha = \eta\frac{P_x^{\mathrm{bulk}}}{M_y^{\mathrm{bulk}}} = \frac{1}{\eta}\frac{(\epsilon_{xx}-1)}{(\mu_t-1)}\frac{E_x}{H_y} \tag{30}$$

where E_x and H_y are the field components averaged over the cubic cell $a \times a \times a$. P_x^{bulk} and M_y^{bulk} are the bulk electric and magnetic polarizations related with the surface current J and surface magnetic polarization

$$P_x^{\mathrm{bulk}} = \frac{J}{j\omega d} \qquad M_y^{\mathrm{bulk}} = \frac{M}{d}.$$

From Maxwell's equations we easily obtain

$$\frac{E_x}{H_y} = \eta\sqrt{\frac{\mu_t}{\epsilon_{xx}}}. \tag{31}$$

Substituting (21) and (26) into (28), we obtain

$$\begin{aligned} \alpha(\omega) &= \frac{\eta J}{j\omega M} = \frac{\eta B}{j(Z_w - C_w)} \\ &= -\frac{\pi\sin\beta d}{\pi\sin kd + kb\log\frac{b}{2\pi r_0}(\cos kd - \cos\beta d)}. \end{aligned} \tag{32}$$

From (30) and (31) it follows that

$$\alpha = \frac{(\epsilon_{xx}-1)}{(\mu_t-1)}\sqrt{\frac{\mu_t}{\epsilon_{xx}}}. \tag{33}$$

In the above equation, $\beta(\omega)$ is already known from the dispersion curve and $k = \omega\sqrt{\epsilon_0\mu_0} = \omega/c$.

Equating the propagation factor β to the value $\omega\sqrt{\epsilon_0\mu_0\epsilon_{xx}\mu_t}$, we obtain

$$\mu_t = \frac{\beta^2}{k^2\epsilon_{xx}}. \tag{34}$$

Substituting this expression for μ_t into (33), we obtain

$$\epsilon_{xx}(\omega) = \frac{\frac{c^2\beta^2(\omega)}{\omega^2} + \frac{c\beta}{\omega\alpha(\omega)}}{1 + \frac{\beta(\omega)}{k\alpha(\omega)}}. \tag{35}$$

After ϵ_{xx} is found, we then evaluate μ_t through [cf. (35) and (34)]

$$\mu_t(\omega) = \frac{1 + \frac{c\beta(\omega)}{\omega\alpha(\omega)}}{1 + \frac{\omega}{c\beta(\omega)\alpha(\omega)}}. \tag{36}$$

We have taken into account the nonlocal interaction in the structure in (35) and (36) though the effective permittivity and permeability are introduced as the parameters relating $\mathbf{B}$, $\mathbf{D}$ with $\mathbf{E}$, $\mathbf{H}$ at the same point. Therefore, unlike (2) and (3), our material parameters are not quasi-static.

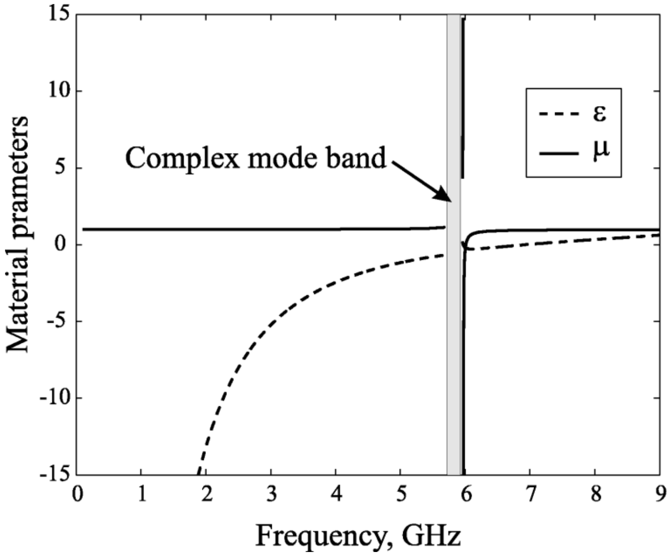


Fig. 6. Axial permittivity (dashed line) and the transversal permeability (solid line) for the case $a = b = d = 8$ mm.

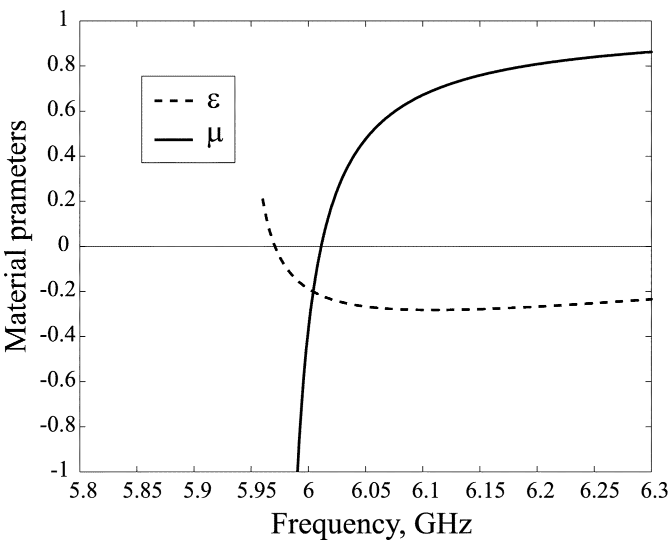


Fig. 7. Axial permittivity (dashed line) and the transversal permeability (solid line) for the case $a = b = d = 8$ mm within the SRR resonant band.

Frequency dependencies of both ϵ_{xx} and μ_t are shown in Fig. 6 for the case $a = b = d = 8$ mm. Outside the resonant band of the SRR particles the frequency dependence of the permittivity repeats the known result for wire media (treated as artificial dielectric media) [5]. The permeability is practically equal to the unity outside the resonant band of SRRs. Within the complex-mode band the homogenization is forbidden [21], [22] and this frequency region is removed in this figure (both ϵ_{xx} and μ_t calculated through (35) and (34) are complex within this band).

Let us consider the resonant frequency behavior of the material parameters in details. Fig. 7 shows the same curves as those in Fig. 6, however, in another scale starting from the upper limit of the complex-mode band. From Fig. 7 one sees that the permittivity and permeability are both negative between 5.970 and 6.020 GHz. From Fig. 4 it follows that the backward wave propagates between 5.980 and 6.045 GHz. Thus, the backward-wave region almost coincides with the region where both the permittivity and permeability are negative. Note that this coincidence is only approximate in our model. Probably this small difference between the band of backward waves and the band of negative ϵ and μ results from approximations that are inherent to this model.

Also we have indicated in Fig. 7 the point at which permittivity and permeability are equal (at about 6.005 GHz). At this frequency, the medium is impedance-matched with the free space (this is useful for some applications) and the values of ϵ_{xx} and μ_t are not very high (the homogenization is then allowed).

As a main result, one can see from Fig. 7 that the permittivity does not follow (within the resonant band) the law (even qualitatively) suggested in [2], [8], and [10]. It is the frequency dependence of the permittivity is nonmonotonous (from Fig. 7 one sees that ϵ_{xx} decreases over 5.96–6.09 and increases after 6.090 GHz as the frequency increases).

In the theory of continuous media, one can prove that both the permittivity and permeability must grow as the frequency increases in the lossless case [26]. In our case, the permeability grows everywhere as the frequency increases (until the first spatial resonance of the lattice, i.e., $kd = \pi$ when it loses the physical meaning). Thus, the frequency behavior of the permeability is normal. However, the permittivity grows as the frequency increases only at the frequencies when the magnetization of SRR is small and the interaction of the SRRs and wires is negligible. Within the band of the backward wave the permittivity decreases as the frequency increases. Therefore, the homogenization procedure we have developed is not completely consistent with the theory of [26]. The reason for this disagreement is that the material parameters considered in [26] are quasi-static (i.e., the polarization of the medium at a given point is determined by the field at this point) while our model takes into account the nonlocal interaction of the SRR lattice and the wire lattice. We found that the visible difference between the quasi-static model and our model is within the SRR resonant band. However, the influence of the nonlocal interaction is revealed in the permittivity of the wire lattice disturbed by the presence of the SRRs.

The lattice of infinite wires is spatially dispersive at all frequencies since the wires are longer than any possible wavelength. When the wave propagates strictly in the plane orthogonal to the axis of the wires one can still neglect the spatial dispersion since all parameters are independent of the x coordinate. Thus, the problem is 2-D and possible to be homogenized [5]. However, if there is a lattice of scatterers with which the wires interact, the situation becomes quite different (even for propagation orthogonal to the wires). Here the problem is not 2-D and the wire current is influenced by all the SRR particles positioned along its infinite length. It results in the abnormal frequency behavior of the effective permittivity of the structure.

VII. Conclusion

In the present paper, we have developed an analytical model for a structure similar to the one for which the negative refraction at microwave frequencies was first observed (formed by combined lattices of infinitely long wires and split-ring resonators) [8]. We have derived a self-consistent dispersion equation and studied the dispersion properties of the lattice. The explicit dispersion equation clearly confirms the existence of the narrow passband within the resonant band of the split-ring resonators. In this passband, the group and phase velocities of the propagating wave are in opposite directions (i.e., a backward wave). Negative refraction can be explained in terms of backward waves without introducing the concept of negative material parameters. However, in the present case the homogenization turns out to be possible in the frequency range of low-frequency backward waves. The obtained dispersion curves have been used to calculate correctly the effective permittivity and permeability in the frequency band where the structure can be homogenized. It is interesting to see that the dispersion curves agree well with the Veselago theory which predicts backward waves when both permittivity and permeability are negative. Outside the resonant band of the SRR particles, the effective permittivity of the whole structure is the same as that of the wire lattice and the effective permeability is equal to 1. However, within the SRR resonant band, there is a subband where the homogenization is forbidden since the complex mode satisfies the dispersion equation at these frequencies. We found that the frequency region in which both ϵ and μ are negative coincides approximately with the backward wave band. In this region, the frequency dependence of the effective permittivity is abnormal. We interpret this as the result of the low-frequency spatial dispersion which is inherent for the wire medium in the presence of the resonant scatterers. The approximate coincidence of the backward-wave band and the band of negative material parameters confirms, in general, the concept of the structure under consideration as a uniaxial variant of Veselago media. However, the coincidence of two bands is not exact, and this question needs to be clarified in a more accurate way than above. We think that the most constructive approach is to take into account the resistive losses of the metal, which can be a crucial factor for the homogenization model. If the presence of losses increases this small difference, the concept of the uniaxial Veselago medium for the structure from wires and SRRs must be revised. This study is planned for the future.

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