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Determination of Generalized Permeability Function and Field Energy Density in Artificial Magnetics Using the Equivalent-Circuit Method

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***Abstract*—The equivalent-circuit model for artificial magnetic materials based on various arrangements of broken loops is generalized by taking into account losses in the substrate or matrix material. It is shown that a modification is needed to the known macroscopic permeability function in order to correctly describe these materials. Depending on the dominating loss mechanism (conductive losses in metal parts or dielectric losses in the substrate), the permeability function has different forms. The proposed circuit model and permeability function are experimentally validated. Furthermore, starting from the generalized circuit model, we derive an explicit expression for the electromagnetic field energy density in artificial magnetic media. This expression is valid at low frequencies and in the vicinity of the resonance also when dispersion and losses in the material are strong. The currently obtained results for the energy density are compared with the results obtained using different methods. As a practical application example, we calculate the quality factor of a microwave resonator made of an artificial magnetic material using the proposed equivalent-circuit method and compare the result with a formula derived for a special case by a different method known from the literature.**

***Index Terms*—Artificial magnetic materials, circuit model, energy density, permeability function.**

I. Introduction

ARTIFICIAL electromagnetic media with extraordinary properties (often called *metamaterials*) attract increasing attention in the microwave community. One of the widely studied subclasses of metamaterials are artificial magnetic materials operating in the microwave regime, e.g., [1]–[8]. For example, broken loops have been used as one of the building blocks to implement double-negative (DNG) media [9], [10], artificial magneto-dielectric substrates are today considered as one of the most promising ways to miniaturize microstrip antennas [11]–[17], and several suggestions have lately been proposed to use artificial magnetic resonators, e.g., in filter design (see, e.g., [18] and [19]).

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The extraordinary features of metamaterials call for careful analysis when studying the fundamental electromagnetic quantities in these materials. Recently, a lot of research has been devoted to the definition of field energy density in DNG media [20]–[23]. The authors of [20], [21], and [23] derived the energy density expression starting from the macroscopic media model and writing down the equation of motion for polarization (electric charge) or magnetization density in the medium. Furthermore, the complex Poynting theorem was used to search for expressions having the mathematically correct form to be identified as energy densities. Following the terminology presented in [23], we call this method the "electrodynamic method." In [22], one of the authors of this paper used another method. Starting from the material microstructure, an equivalent-circuit representation was derived for the unit cell constituting specific artificial dielectric and magnetic media. Lattices of thin wires and arrays of split-ring resonators were considered in [22]. The stored reactive energy and, furthermore, the field energy density, were calculated using the classical circuit theory. The authors of [23] later called this method the "equivalent-circuit method." Though these two methods apply to media with the same macroscopic permeabilities and permittivities, they are fundamentally very different, as will be clarified later in more detail. Moreover, the final expressions for the field energy density in artificial magnetic media given in [22] and [23] differ from each other. One of the motivations of this study is to clarify the reasons for this difference.

Here we concentrate only on artificial magnetic media and set two main goals for our work, which are as follows.

- To understand the differences and assumptions behind the electrodynamic and equivalent-circuit methods when deriving the field energy density expressions. We verify using a specific example (magnetic material unit cell) that, in the presence of nonnegligible losses, one should always calculate the stored energy at the microscopic level.
- To generalize the previously reported equivalent-circuit representation for artificial magnetic media [22].

The generalized circuit model takes into account losses in the matrix material. It is shown that this generalization forces a modification to the widely accepted permeability function used to macroscopically describe artificial magnetic media. The generalization has a significant importance, as it is shown that, in a practical situation, matrix losses strongly dominate over conductive losses. The proposed circuit representation and permeability function are experimentally validated. We measure the magnetic polarizability of a small magnetic material sample and compare the results with those given by the proposed analytical model and the previously used model. The results given by the proposed model agree very well with the measured results, whereas the old model leads to dramatic overestimation for the polarizability. Using the generalized circuit model, we derive an expression for the field energy density in artificial magnetic media. This expression is compared with the results obtained using the electrodynamic method in [23], and reasons for the differences are discussed. As a practical application example, we calculate the quality factor of a microwave resonator made of artificial magnetic material using the equivalent-circuit and electrodynamic methods. The differences between the final expressions are discussed.

II. Electrodynamic Method versus Equivalent-Circuit Method

It is well explained in reference books (e.g., [24] and [25]) that, for the definition of field energy density in a material having nonnegligible losses, one always needs to know the material microstructure. First of all, the reactive energy stored in any material sample is a quantity that can be measured. When the material is *lossless*, no information is needed about the material microstructure for this measurement. Indeed, we can measure the total power flux through the surface of the sample volume[1] and, since there is no power loss inside, we can use the Poynting theorem to determine the change in the stored energy. This is the reason why the field energy density in a dispersive media with negligible losses can be expressed through the frequency derivative of macroscopic material parameters. In the circuit theory, the same conclusion is true for circuits that contain only reactive elements. It is possible to find the stored reactive energy in the whole circuit knowing only the input impedance of a two-port [26].

Simple reasoning reveals that in the presence of nonnegligible material losses, the above described "black box" representation and direct measurement are not possible. Without knowing the material microstructure or the circuit topology, we do not know which portion of the input power is dissipated and which is stored in the reactive elements. Thus, the energy stored in lossy media cannot be *uniquely* defined by only utilizing the knowledge about the macroscopic behavior of the media [24], [25].

When defining the energy density in a certain material using the electrodynamic method, one first writes down the equation of motion for charge density or for magnetization in the medium using the macroscopic media model [21], [23]. We stress that this equation is the *macroscopic* equation of motion, containing the same physical information as the macroscopic permittivity and permeability. Further, the complex Poynting theorem is used to identify the mathematical form of the general macroscopic energy density expressions. Having the form of these expressions in mind, one searches for similar expressions in the equation of motion and defines them as energy densities. The problem of the electrodynamic method is the fact that it only utilizes the knowledge about the macroscopic behavior of the media, which, as explained above, in not enough. The aforementioned difficulty is avoided in the equivalent-circuit method [22]. Based on the known microscopic medium structure, one constructs the equivalent circuit for the unit cell of the medium. Careful analysis is needed to make sure that the circuit physically corresponds to the analyzed unit cell. After this check, the stored reactive energy and the corresponding field energy density can be uniquely calculated using the classical circuit theory.

[1]In practice, this measurement can be done, for example, by positioning a material sample inside a closed waveguide and measuring the S_{11} and S_{21} parameters.

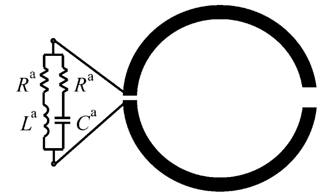

Fig. 1. Broken loop loaded with an infinitesimally small lumped circuit.

Next, we illustrate the difference between the electrodynamic and equivalent-circuit methods using a specific case of broken-loop composites. Consider the broken loop shown in Fig. 1. Following an example given in [25], we load the loop with an electrically infinitesimally small circuit consisting of lumped elements. Let us assume that the additional inductance and capacitance are chosen to have values $L^a = \tau R^a, C^a = \tau / R^a, \tau > 0$. A simple check reveals that, in this case, the input impedance of the load circuit is frequency independent and purely resistive: $Z_{\rm in} = R^a$. When the loop is electrically small, it can be represented as a resonant contour and the total loss resistance reads $R_{\rm tot} = R + R^a$, where R is the loss resistance due to the finite conductivity of the loop material. Next, we consider two different broken loops. The first loop is made of silver and loaded with the circuit shown in Fig. 1. The second loop has no loading circuit and is made of copper, thus, it has much higher conductive losses, as compared to the silver ring. If the load resistor for the silver loop R^a is chosen so that the total resistance $R_{\rm tot}$ equals R of the unloaded loop made of copper, the macroscopic descriptions of media formed by these two different loops are exactly the same. Thus, the electrodynamic method predicts the same value for the reactive energy stored in these two media. Inspection of Fig. 1 clearly indicates, however, that this is not the case. There is some additional energy stored in the load inductance and capacitance, which is *invisible* on the level of the macroscopic permeability description. Proper definition of the stored energy must be done at the microscopic level, which is possible with the equivalent-circuit method.

III. Equivalent-Circuit Method: Brief Review of Earlier Results

A commonly accepted permeability model as an effective medium description of dense (in terms of the wavelength) arrays of broken loops, split-ring resonators, and other similar structures reads

$$\mu(\omega) = \mu_0 \mu_r(\omega) = \mu_0 \left(1 + \frac{A\omega^2}{\omega_0^2 - \omega^2 + j\omega\Gamma}\right) \tag{1}$$

(see, e.g., [1], [2], [4], and [8].) In (1), A is the amplitude factor $(0 < A < 1), \omega_0$ is the undamped angular frequency of the zeroth pole pair (the resonant frequency of the array), and Γ

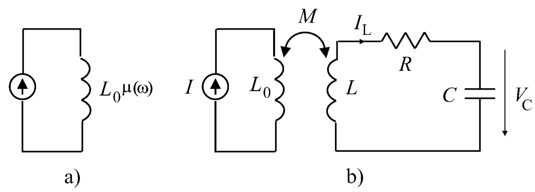

Fig. 2. (a) Magnetic material sample in the probe magnetic field of a tightly wound long solenoid. (b) Equivalent-circuit model (losses in the matrix material are not taken into account).

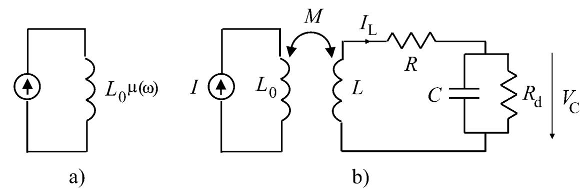

Fig. 3. (a) Magnetic material sample in the probe magnetic field of a tightly wound long solenoid. (b) Equivalent-circuit model taking into account losses in the matrix material.

is the loss factor. The model is obviously applicable only in the quasi-static regime since, in the limit $\omega \rightarrow \infty$, the permeability does not tend to μ_0. At extremely high frequencies, materials cannot be polarized due to inertia of electrons, thus, a physically sound high-frequency limit is μ_0 [24]. However, (1) gives correct results at low frequencies and in the vicinity of the resonance. This is the typical frequency range of interest, e.g., when utilizing artificial magneto-dielectric substrates in antenna miniaturization [13], [16], [17]. The other relevant restriction on the permeability function is the inequality [24]

$$\frac{\partial(\omega\mu(\omega))}{\partial\omega} > \mu_0 \tag{2}$$

valid in the frequency regions with negligible losses. Physically, the last restriction means that the stored energy density in a passive linear lossless medium must always be larger than the energy density of the same field in vacuum. Macroscopic model (1) violates restriction (2) at high frequencies, which is another manifestation of the quasi-static nature of the model. In the vicinity of the magnetic resonance, the effective permittivity of a dense array of split-ring resonators is weakly dispersive, and can be assumed to be constant.

In [22], the energy density in dispersive and lossy magnetic materials was introduced via a thought experiment. A small (in terms of the wavelength or the decay length in the material) sample of a magnetic material [described by (1)] was positioned in the magnetic field created by a tightly wound long solenoid having inductance L_0, Fig. 2(a). The inclusion changes the impedance of the solenoid to

$$Z(\omega) = j\omega L_0 \mu_r(\omega) = j\omega L_0 + \frac{j\omega^3 L_0 A}{\omega_0^2 - \omega^2 + j\omega\Gamma}. \tag{3}$$

The equivalent circuit with the same impedance was found to be that shown in Fig. 2(b), [22] with the impedance seen by the source

$$Z(\omega) = j\omega L_0 + \frac{j\omega^3 M^2/L}{\frac{1}{LC} - \omega^2 + j\omega\frac{R}{L}} \tag{4}$$

which is the same as (3) if

$$\frac{M^2}{LL_0} = A \quad \frac{1}{LC} = \omega_0^2 \quad \frac{R}{L} = \Gamma. \tag{5}$$

The aforementioned equivalent-circuit model is correct from the microscopic point-of-view since the modeled material is a collection of capacitively loaded loops magnetically coupled to the incident magnetic field. The electromagnetic field energy density in the material was found to be [22]

$$w_m = \frac{\mu_0}{2}\left(1 + \frac{A\omega^2\left(\omega_0^2 + \omega^2\right)}{\left(\omega_0^2 - \omega^2\right)^2 + \omega^2\Gamma^2}\right)|H|^2. \tag{6}$$

An important assumption in [22] and in this paper is that the current distribution is nearly uniform over the loop. This means that the electric dipole moment created by the excitation field is negligible as compared to the magnetic moment. The aforementioned assumption also sets the validity range for the equivalent-circuit method. The inclusions must be small compared to the wavelength so that higher order modes are not induced in the loops and the medium can be described using a single resonant circuit.

In [22], only losses due to nonideally conducting metal of loops were taken into account, and losses in the matrix material (substrate material on which metal loops are printed) were neglected. It will be shown below that neglecting the matrix losses can lead to severe overestimation of the achievable permeability values.

IV. Generalized Equivalent-Circuit Model and Permeability Function

Losses in the matrix material (typically a lossy dielectric laminate) can be modeled by an additional resistor in parallel with the capacitor. Indeed, if a capacitor is filled with a lossy dielectric material, the admittance reads

$$Y = j\omega C(\epsilon' - j\epsilon'') = j\omega C\epsilon' + \omega C\epsilon'' \tag{7}$$

where the latter expression denotes a loss conductance. Thus, the microscopically correct equivalent-circuit model is as shown in Fig. 3(b). The impedance seen by the source can be readily solved as follows:

$$Z = j\omega L_0 + \frac{j\omega^3 M^2/L + \omega^2 M^2/(LCR_d)}{\left(1 + \frac{R}{R_d}\right)\frac{1}{LC} - \omega^2 + j\omega\left(\frac{R}{L} + \frac{1}{CR_d}\right)}. \tag{8}$$

The macroscopic permeability function corresponding to this model reads

$$\mu(\omega) = \mu_0\left(1 + \frac{\omega^2 M^2/(LL_0) - j\omega M^2/(LL_0 CR_d)}{\left(1 + \frac{R}{R_d}\right)\frac{1}{LC} - \omega^2 + j\omega\left(\frac{R}{L} + \frac{1}{CR_d}\right)}\right). \tag{9}$$

Comparing (1) and (9), we immediately notice that (1) is an insufficient macroscopic model for the artificial material if the losses in the host matrix are not negligible. A proper macroscopic model correctly representing the composite from the microscopic point-of-view is

$$\mu(\omega) = \mu_0\mu_r(\omega) = \mu_0\left(1 + \frac{A\omega^2 - j\omega B}{\widetilde{\omega}_0^2 - \omega^2 + j\omega(\Gamma + \Gamma_d)}\right). \tag{10}$$

Equation (9) is the same as (10) if

$$\begin{aligned} \frac{M^2}{LL_0} &= A \\ \frac{M^2}{LL_0CR_d} &= B \\ \left(1 + \frac{R}{R_d}\right)\omega_0^2 &= \widetilde{\omega}_0^2 \\ \frac{R}{L} &= \Gamma \\ \frac{1}{CR_d} &= \Gamma_d. \end{aligned} \tag{11}$$

Above we have denoted $\omega_0^2 = 1/(LC)$. The macroscopic permeability function of different artificial magnetic materials can be conveniently estimated using (10), as several results are known in the literature for the effective circuit parameter values for different unit cells, e.g., [2], [6], and [8].

For the use of (10), it is important to know the physical nature of the equivalent loss resistor R_d. If losses in the matrix material are due to a finite conductivity of the dielectric material, the complex permittivity reads

$$\epsilon = \epsilon' - j\epsilon'' = \epsilon' - j\frac{\sigma}{\omega} \tag{12}$$

where σ is the conductivity of the matrix material. Thus, we see from (7) that the loss resistor is independent from the frequency and can be interpreted as a "true" resistor. Moreover, in this case, the permeability function is that given by (10). However, depending on the nature of the dielectric material, the loss mechanism can be very different from (12), and in other situations, the macroscopic permeability function needs other modifications. For example, let us assume that the permittivity obeys the Lorentzian type dispersion law

$$\epsilon = \epsilon'\left(1 + \frac{\Pi}{\omega_0'^2 - \omega^2 + j\omega\Lambda}\right) \tag{13}$$

where ω_0' is the angular frequency of the electric resonance, Π is the amplitude factor, and Λ is the loss factor. Moreover, we assume that the material is utilized well below the electric resonance, thus, $\omega \ll \omega_0'$. With this assumption, the permittivity becomes

$$\epsilon \approx \epsilon'(1 + \Pi) - j\omega\epsilon'\Pi\Lambda/\omega_0'^2. \tag{14}$$

We notice from (7) that, in this case, the equivalent loss resistor R_d becomes frequency dependent as follows:

$$R_d \propto \frac{1}{\omega^2} \tag{15}$$

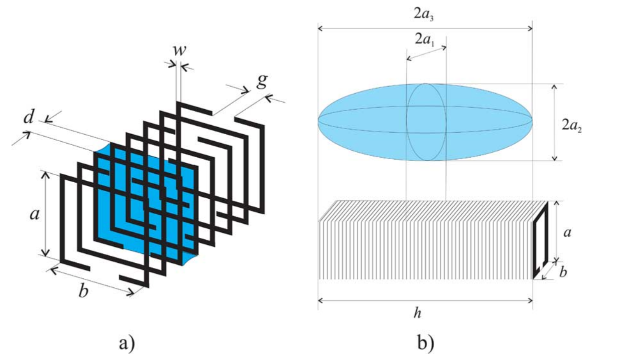

Fig. 4. (a) Schematic illustration of the metasolenoid. (b) Finite-size metasolenoid approximated as a magnetic ellipsoid.

and the permeability function takes the form

$$\mu(\omega) = \mu_0\mu_r(\omega) = \mu_0\left(1 + \frac{A\omega^2 - j\omega^3 B'}{\omega_0^2 - K\omega^2 + j\omega\left(\Gamma + \omega^2\Gamma_d'\right)}\right) \tag{16}$$

where K is a real-valued coefficient depending on the dielectric material. For other dispersion characteristics of the matrix material, the permeability function can have other forms.

V. Experimental Validation of the Proposed Circuit Model and Permeability Function

Here, we present experimental results that validate the generalized equivalent-circuit model and the corresponding macroscopic permeability function. The measurement campaign and the experimental results are described in detail in [8]. For the convenience of the reader, we briefly outline the main steps of the measurement procedure.

The measured artificial magnetic particle, metasolenoid, is schematically presented in Fig. 4(a). In that figure, a and b denote the cross-sectional dimensions of the broken loops, d is the separation between the loops, w is the strip width, and g is the width of the gap in the loops. In [8], the effective permeability of a medium densely filled with infinitely long metasolenoids was derived in the form

$$\mu_{\rm eff} = 1 - V_r\frac{j\omega\mu_0 S}{Z_{\rm tot}d} \tag{17}$$

where V_r is the volume filling ratio, $S = a\times b$ is the cross section area of the broken loop, and the total effective impedance was presented in the form

$$Z_{\rm tot} = j\omega L_{\rm eff} + \frac{1}{j\omega C_{\rm eff}} + R_{\rm eff}. \tag{18}$$

For the experimental validation, a finite-size metasolenoid was approximated as an ellipsoid cutoff from a magnetic medium described by (17) [see Fig. 4(b)], where h is the longitudinal length of the metasolenoid sample. The magnetic polarizability of the ellipsoid $\alpha_{\rm mm}$ and the effective permeability of the medium are related using the classical formula for the polarizability of an ellipsoid (e.g., [27]). On the other hand, permeability is defined using the equivalent circuit [see Fig. 3(b)]. Thus, the magnetic polarizablity of the measured sample contains all the relevant data for validating both the proposed equivalent circuit and the permeability function.

The field amplitude scattered by an electrically small material sample was measured by positioning the sample inside a standard rectangular waveguide. In addition to this, the scattered field amplitude was analytically calculated enabling the extraction of $\alpha_{\rm mm}$ from the measured results [8]. Though it is not explicitly mentioned in [8], substrate losses were taken into account when analytically calculating the magnetic polarizability of the sample. The authors used (18) to define the total impedance of the metasolenoid unit cell, however, complex permittivity was used when calculating the effective capacitance. Thus, the equivalent circuit of the unit cell used to analyze the measured sample is the proposed circuit shown in Fig. 3(b).[2] It can easily be verified using the data presented in [8] that the following expression for the effective impedance [derived using the circuit in Fig. 3(b)] exactly repeats the analytical estimation for the magnetic polarizability of the sample:

$$Z'_{\rm tot} = j\omega L_{\rm eff} + R_{\rm eff} + \frac{R_d}{1+j\omega C_{\rm eff}R_d} \tag{19}$$

where $R_d = \epsilon'/(\epsilon''\omega C_{\rm eff})$. The analytically calculated [$Z_{\rm tot}$ given by (19) is used in (17)] and measured magnetic polarizabilities are shown in Fig. 5. The measured and calculated key parameters are gathered in Table I. The polarizability and permeability values in Table I are the maximum values.

The measured results agree rather closely with the analytical calculations when the proposed model is used. The slight difference in the resonant frequencies, and the slightly lowered polarizability values in the measurement case are most likely caused by limitations in the accuracy of the manufacturing process. The implemented separation between the rings is probably slightly larger than the design value. This lowers the effective capacitance and is seen as a weakened magnetic response and a higher resonant frequency. Moreover, the measured $\mathrm{Im}\{\alpha_{\rm mm}\}$ clearly indicates that the effect of the lossy glue used to stack the rings is underestimated in the analytical calculations. In the analytical calculations, $\tan\delta = 0.002$ was used for the total loss tangent [8]. A loss-tangent value $\tan\delta = 0.0025$ would accurately produce the measured polarizability values, and the bandwidths (defined from the $\mathrm{Im}\{\alpha_{\rm mm}\}$ curve) would visually coincide.

If the matrix losses are neglected in the circuit model [$R_d \to \infty$ in (19)], the analytical calculations lead to dramatically overestimated polarizability and permeability values. It is, therefore, evident that the proposed generalization of the circuit model and the permeability function have a significant practical importance. Though the model has been validated using a specific example, we can conclude that matrix losses can strongly dominate over conductive losses in structures where the unit cells are closely spaced. This is physically well understandable since, in this case, the electric field is strongly concentrated inside the substrate.

[2]Please note that the unit cell of the metasolenoid contains two broken loops [8]. In the quasi-static regime the structure can be described using the *total* effective current (the sum of the currents in two adjacent loops) flowing in the unit cell. Thus, the use of the equivalent circuit in Fig. 3(b) is justified, just like for a "double split-ring particle" [3] in the fundamental mode.

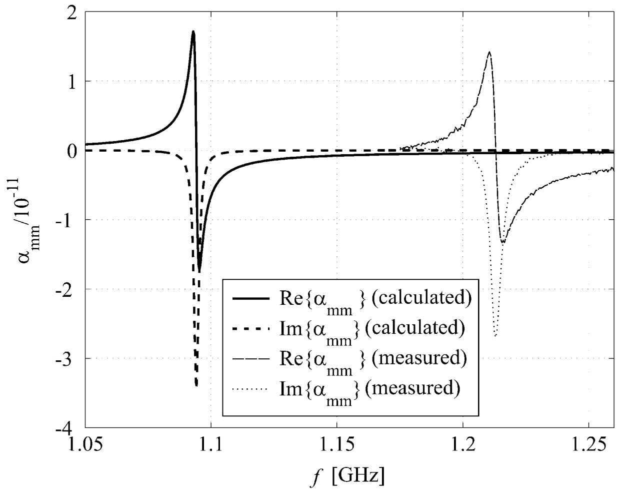

Fig. 5. Analytically calculated (proposed model) and measured magnetic polarizabilities.

TABLE I
ANALYTICALLY CALCULATED AND MEASURED PROPERTIES OF THE METASOLENOID SAMPLE

	$f_{\rm res}$ GHz	$\mathrm{Re}\{\alpha_{\rm mm}\}$ H·m^2	$\lvert\alpha_{\rm mm}\rvert$ H·m^2	$\mathrm{Re}\{\mu_{\rm eff}\}$
Analytical†	1.09	1.7×10^{-11}	3.3×10^{-11}	230
Measured	1.21	1.4×10^{-11}	2.7×10^{-11}	—
Analytical‡	1.09	19.9×10^{-11}	60.1×10^{-11}	4000

†proposed model, ‡old model

VI. ELECTROMAGNETIC FIELD ENERGY DENSITY

Following the approach introduced in [22], we will next generalize the expression for the energy density in artificial magnetics using the experimentally validated circuit model. In the time–harmonic regime, the total stored energy reads [notations are clear from Fig. 3(b)]

$$\begin{aligned} W &= \frac{1}{2}(L_0|I|^2 + L|I_L|^2 + C|V_C|^2) \\ &= \frac{1}{2}\left[L_0|I|^2 + |I_L|^2\left(L + \frac{1}{C\left(\omega^2+\dfrac{1}{C^2R_d^2}\right)}\right)\right] \end{aligned} \tag{20}$$

$$|I_L|^2 = \frac{\omega^2\dfrac{M^2}{L^2}\left(\omega^2+\dfrac{1}{C^2R_d^2}\right)}{\left[\left(1+\dfrac{R}{R_d}\right)\dfrac{1}{LC}-\omega^2\right]^2 + \omega^2\left(\dfrac{R}{L}+\dfrac{1}{CR_d}\right)^2}|I|^2. \tag{21}$$

Using the notations in (11), the stored energy can be written as

$$W = \frac{1}{2}L_0|I|^2\left(1+\frac{A\omega^2\left(\omega_0^2+\omega^2+\Gamma_d^2\right)}{\left(\widetilde{\omega}_0^2-\omega^2\right)^2+\omega^2(\Gamma+\Gamma_d)^2}\right). \tag{22}$$

The inductance per unit length of a tightly wound long solenoid is $L_0 = \mu_0 n^2 S$, where n is the number of turns per unit length and S is the cross-sectional area. The relation between the current I and magnetic field H inside the solenoid is $I = H/n$. Thus, the stored energy in one unit-length section of the solenoid reads

$$W = w_m S = \frac{1}{2}\mu_0 n^2 S \frac{|H|^2}{n^2}\left(1+\frac{A\omega^2\left(\omega_0^2+\omega^2+\Gamma_d^2\right)}{\left(\widetilde{\omega}_0^2-\omega^2\right)^2+\omega^2(\Gamma+\Gamma_d)^2}\right) \quad (23)$$

from which we obtain the expression for the electromagnetic field energy density in the artificial material sample

$$w_m = \frac{\mu_0}{2}\left(1+\frac{A\omega^2\left(\omega_0^2+\omega^2+\Gamma_d^2\right)}{\left(\widetilde{\omega}_0^2-\omega^2\right)^2+\omega^2(\Gamma+\Gamma_d)^2}\right)|H|^2. \quad (24)$$

We immediately notice that if there is no loss in the matrix material ($R_d \to \infty$ and $\Gamma_d \to 0$), then $\widetilde{\omega}_p^2 \to \omega_p$ and (24) reduces to (6).

A. Comparison with the Results Obtained Using the Electrodynamic Method

The above derived result differs from the result found in [23] and obtained using the electrodynamic method

$$w_m = \frac{\mu_0}{2}\left(1+\frac{A\omega^2\left[\omega_0^2\left(3\omega_0^2-\omega^2\right)+\omega^2\Gamma^2\right]}{\omega_0^2\left[\left(\omega_0^2-\omega^2\right)^2+\omega^2\Gamma^2\right]}\right)|H|^2. \quad (25)$$

The procedure and the underlying assumptions to obtain (25) have been briefly reviewed in Section II. The classical expression for the magnetic field energy density in media where absorption due to losses can be neglected reads [24], [25]

$$w_m = \frac{\mu_0}{2}\frac{\partial(\omega\mu_r(\omega))}{\partial\omega}|H|^2. \quad (26)$$

It is seen that in the presence of negligible losses [$\Gamma \to 0$ in (1)] the energy density result given by (25) is the same as the result predicted by the classical expression (26). However, (24) predicts a different result. The authors of [23] use this fact to state that the result obtained using the electrodynamic method is more inherently consistent than the result obtained using the equivalent-circuit method.

The equivalent-circuit method is known to give a perfectly inherently consistent result for the energy density in dielectrics obeying the general Lorentzian type dispersion law [22]. The general Lorentz model is a strictly causal model. This is, however, not the case with the modified Lorentz model (1). As is speculated already in [22], the reason for the difference in the results obtained using (24) and (26) in the small-loss limit is related to the physical limitations of the quasi-static permeability model (1). Thus, when (1) is used as the macroscopic medium description, (24) should be also used in the presence of vanishingly small losses. The electrodynamic method, though being inherently consistent with the classical expression, predicts nonphysical behavior at high frequencies. At high frequencies, the energy density given by the electrodynamic method is smaller than the energy density in vacuum (when there are no losses, this nonphysical behavior takes place at frequencies $\omega > \sqrt{3}\omega_0$, where the quasi-static model is still valid, but restriction (2) is violated) [23, Fig. 1(a) and (b)]. This behavior is avoided with the result obtained using the equivalent-circuit method since that approach is based on the microscopic description of the medium, which is always in harmony with the causality principle.

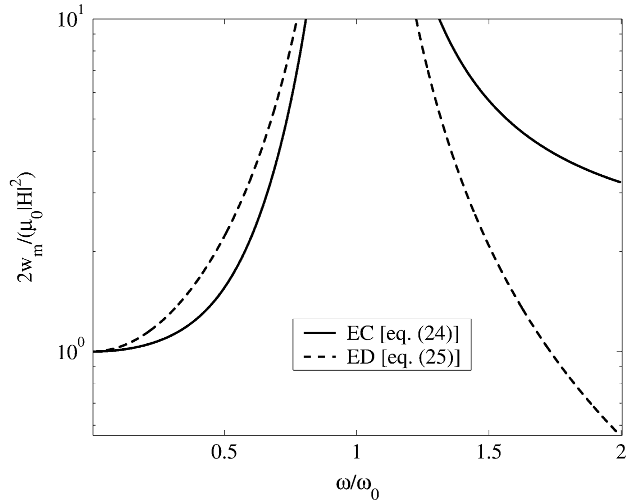

Fig. 6. Magnetic field energy density given by different expressions. ω_0 is the resonant frequency of the metasolenoid medium.

Fig. 6 depicts the normalized magnetic field energy density in a medium formed by the metasolenoids introduced in Section V (in Fig. 6, "EC" denotes the equivalent-circuit method and "ED" denotes the electrodynamic method). The amplitude factor $A = 1$ and the loss factors have been estimated using (11) and the data presented in [8]. In this particular example, the energy densities given by (6) and (24) are practically the same over the whole studied frequency range (the result given by (6) is not plotted in Fig. 6 since that curve visually coincides with that given by the equivalent-circuit method). This is due to the fact that large values of ω and ω_0 mask the effect of Γ and Γ_d in (6) and (24). The results given by the electrodynamic method and the classical expression (26) also visually coincide. However, as was mentioned above, the energy density expression given by the electrodynamic method predicts the same nonphysical behavior as the classical expression. The field energy density is smaller than the energy density in vacuum at frequencies $\omega > \sqrt{3}\omega_0$.

VII. Practical Application Example

Here, we present a specific application example for the established theory and study the differences between the equivalent-circuit method and the electrodynamic method using this example. We consider an electrically small microwave resonator made of an artificial magnetic material, and calculate the quality factor of the resonator using the two methods. The quality-factor estimation of artificial magnetic resonators has a strong practical importance as several suggestions have been recently made to use such resonators, e.g., in filter design (see, e.g., [18] and [19]).

To be able to compare the results from the two methods, we first consider (1) as the macroscopic medium description since substrate losses are not taken into account in [23]. After the comparison, we use (10) as the medium description, and calculate the quality factor for the case when the substrate losses are taken into account.

A. Quality Factor Using the Equivalent-Circuit Method

The power lost in the unit volume of a medium described by (1) can be derived based on the circuit presented in Fig. 2(b) and the result reads

$$P_L = \frac{\mu_0}{2} \frac{A\omega^4 \Gamma}{\left(\omega_0^2 - \omega^2\right)^2 + \omega^2\Gamma^2} |H|^2. \qquad (27)$$

Above we have assumed that radiation losses in the quasi-static regime are negligible compared to the ohmic losses. The energy stored in the same unit volume of the medium reads

$$W = \frac{\mu_0}{2} \frac{A\omega^2 \left(\omega_0^2 + \omega^2\right)}{\left(\omega_0^2 - \omega^2\right)^2 + \omega^2\Gamma^2} |H|^2 \qquad (28)$$

and for the quality factor we get

$$Q = \left.\frac{\omega W}{P_L}\right|_{\omega=\omega_0} = 2\frac{\omega_0}{\Gamma} \qquad (29)$$

which, as expected, is the quality factor of a series resonant circuit.

B. Quality Factor Using the Electrodynamic Method

The electrodynamic method also yields (27) when calculating the power loss (see [23, eqs. (26) and (27)]). However, the stored energy in this case reads

$$W = \frac{\mu_0}{2} \left(\frac{A\omega^2 \left[\omega_0^2 \left(3\omega_0^2 - \omega^2\right) + \omega^2\Gamma^2\right]}{\omega_0^2 \left[\left(\omega_0^2 - \omega^2\right)^2 + \omega^2\Gamma^2\right]} \right) |H|^2 \qquad (30)$$

leading to the following expression for the quality factor:

$$Q = \left.\frac{\omega W}{P_L}\right|_{\omega=\omega_0} = 2\frac{\omega_0}{\Gamma} + \frac{\Gamma}{\omega_0}. \qquad (31)$$

When comparing the quality factors obtained using the two methods, we notice that both of them behave in a physically sound manner when losses in the medium are low and further decrease. We remind here that in the low-loss limit (26) can also be used to determine the energy stored in the resonator. Further, if we use (27) to calculate the dissipated power [now Γ is small in (27)], the resulting quality factor coincides with (29). However, if losses become significant, (26) is not valid anymore, but the quality factor obtained using the electrodynamic method also behaves in a nonphysical manner. The quality factor increases with increasing strong losses. This nonphysical behavior is avoided with the result obtained using the equivalent-circuit method.

It is interesting to note that when the general (causal) Lorentz model [ω_0 replaces ω in the nominator of (1)] is used, the quality factor, obtained using the electrodynamic method, is given by (29) [23, eqs. (25) and (28)]. Thus, for instance, for artificial dielectrics that are modeled by the general Lorentz dispersion model, both macroscopic electrodynamic method and the microscopic circuit method give the same correct result for the quality factor. This shows that the problem of the electrodynamic method is indeed in the noncausal nature of the quasi-static model of broken loop composites.

C. Quality Factor Taking Into Account Substrate Losses

When substrate losses are taken into account, the equivalent-circuit method yields the following expression for the power dissipated in the medium:

$$P_L = \frac{\mu_0}{2} \frac{A\omega^2 \left(\omega^2 + \Gamma_d^2\right) \left(\Gamma + \dfrac{\omega_0^2 \Gamma_d}{\omega^2 + \Gamma_d^2}\right)}{\left(\widetilde{\omega}_0^2 - \omega^2\right)^2 + \omega^2\left(\Gamma^2 + \Gamma_d^2\right)} |H|^2. \qquad (32)$$

Furthermore, the stored energy reads

$$W = \frac{\mu_0}{2} \frac{A\omega^2 \left(\omega_0^2 + \omega^2 + \Gamma_d^2\right)}{\left(\widetilde{\omega}_0^2 - \omega^2\right)^2 + \omega^2\left(\Gamma^2 + \Gamma_d^2\right)} |H|^2 \qquad (33)$$

and for the quality factor we get

$$Q = \left.\frac{\omega W}{P_L}\right|_{\omega=\omega_0} = \frac{\widetilde{\omega}_0 \left(\omega_0^2 + \Gamma_d^2\right) \left(\widetilde{\omega}_0^2 + \omega_0^2 + \Gamma_d^2\right)}{\left(\widetilde{\omega}_0^2 + \Gamma_d^2\right) \left(\omega_0^2 \Gamma + \omega_0^2 \Gamma_d + \Gamma\Gamma_d^2\right)}. \qquad (34)$$

We immediately notice that if there is no loss in the matrix material, (34) reduces to (29).

VIII. Conclusion

In this paper, we have explained differences between recent approaches used to derive field energy density expressions for artificial lossy and dispersive magnetic media. The equivalent-circuit model of broken loops and other similar structures has been generalized to take into account losses in the dielectric matrix material. It has been shown that a modification is needed to the macroscopic permeability function commonly used to model these materials in the quasi-static regime. Moreover, depending on the nature of the dominating loss mechanism in the matrix material, the permeability function has different forms. The proposed circuit model and the modified permeability function have been experimentally validated, and it has been shown that, in a practical situation, matrix losses can dramatically dominate over conduction losses. Using the validated circuit model, we have derived an expression for the electromagnetic field energy density in artificial magnetic media. This expression is also valid when losses in the material cannot be neglected and when the medium is strongly dispersive. The results have been compared to recently reported alternative approaches. As a practical application example, we have calculated the quality factor of a microwave resonator made of an artificial magnetic material using the proposed method and a different method introduced in the literature.

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