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A FAMILY OF C^0 FINITE ELEMENTS FOR KIRCHHOFF PLATES I: ERROR ANALYSIS*

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Abstract. A new finite element formulation for the Kirchhoff plate model is presented. The method is a displacement formulation with the deflection and the rotation vector as unknowns, and it is based on ideas stemming from a stabilized method for the Reissner–Mindlin model [R. Stenberg, in *Asymptotic Methods for Elastic Structures*, P. Ciarlet, L. Trabucho, and J. M. Viano, eds., de Gruyter, Berlin, 1995] and a method to treat a free boundary [P. Destuynder and T. Nevers, *RAIRO Modél. Math. Anal. Numér.*, 22 (1988), pp. 217–242]. Optimal a priori and a posteriori error estimates are derived.

Key words. finite elements, Kirchhoff plate model, free boundary, a priori error analysis, a posteriori error analysis

AMS subject classifications. 65N30, 74K20, 74S05

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1. Introduction.

A conforming finite element method for the Kirchhoff plate-bending problem requires a C^1-continuity and hence leads to methods that are rarely used in practice. Instead, either a nonconforming method is used or the model is abandoned in favor of the Reissner–Mindlin model. For the latter, there exist several families of methods that have rigorously been shown to be free from locking and optimally convergent.

A natural idea is to consider the Kirchhoff model as the limit of the Reissner–Mindlin model when the plate thickness approaches zero and to use a good Reissner–Mindlin element with the thickness (after a scaling, see below) representing the parameter when penalizing the Kirchhoff constraint. In this approach, there are two obstacles. First, for a free boundary, this leads to a method which is not consistent. This inconsistency significantly reduces the convergence rate of the method. In the literature, this point is often ignored since mostly the clamped case is considered. A remedy to this was developed by Destuynder and Nevers, who showed that the consistency is obtained by adding a term penalizing the tangential Kirchhoff condition along the free boundary [7]. Even if this modification has been done, there remains a second drawback. In order for the solution to the penalized formulation to be close to the exact solution, the penalty parameter should be large. This, however, leads to an ill-conditioned discrete system.

The free boundary inconsistency of the limit problem is closely related to the strong boundary layer of the Reissner–Mindlin plate problem with free boundaries. For Reissner–Mindlin plates, the presence of free boundaries significantly reduces the regularity of the solution and hence decreases the convergence rate of finite element approximations [1, 10, 5]. In [5, 2], the regularity of the solution has been improved by modifying the boundary conditions for free boundaries. These modifications imitate the boundary conditions of the Kirchhoff model as well as couple the variational

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spaces for the deflection and the rotation through the tangential Kirchhoff constraint along free boundaries. Adopting the modified boundary conditions on the discrete level it has been proved in [5, 2] that a set of finite element methods maintain their optimal order of convergence in the free boundary case. However, it can be seen as a drawback that all of these methods follow the mixed formulation with the shear force as an additional unknown. For positive values of the thickness parameter t, as usual, the corresponding displacement formulations can be achieved by condensing the shear force from the formulation. For the limit case $t = 0$, however, this possibility is excluded due to the nominator t^2 of the factor penalizing the Kirchhoff condition. For this reason, applying these methods for Kirchhoff plates requires a mixed formulation with the additional shear force degrees of freedom.

Our aim in the present paper is to present a family of Kirchhoff plate-bending elements which follows the displacement formulation and for which the convergence rate is optimal even in the presence of free boundaries. The method is a formulation combining the ideas from the stabilized method for Reissner–Mindlin plates presented in [13] and the treatment of the free boundary presented in [7]. Altough the method resembles the one with the linked interpolation technique in [2] for Reissner–Mindlin plates, it has been independently derived for the Kirchhoff plate problem with free boundaries. The family includes “simple low-order” elements, and it is well-conditioned. In the second part [3] of this paper, we give the results of numerical tests and a more detailed and constructive motivation for the method (cf. [4] as well).

The paper is organized as follows. In the next section, we describe the plate-bending problem, and in section 3, we introduce the new family of finite elements. In section 4, an a priori error analysis is derived. This analysis leads to optimal results, with respect both to the regularity of the solution and to the polynomial degree used. In section 5, an a posteriori error analysis is performed. We derive a local error indicator which is shown to be both reliable and efficient.

2. The Kirchhoff plate-bending problem. We consider the problem of bending of an isotropic linearly elastic plate and assume that the undeformed plate midsurface is described by a given convex polygonal domain $\Omega \subset \mathbb{R}^2$. The plate is considered to be clamped on the part $\Gamma_{\rm C}$ of its boundary $\partial\Omega$, simply supported on the part $\Gamma_{\rm S} \subset \partial\Omega$, and free on $\Gamma_{\rm F} \subset \partial\Omega$. The deflection and transversal load are denoted by w and g, respectively.

In what follows, we indicate with $\mathcal{V}$ the set of all corner points in $\Gamma_{\rm F}$. Moreover, $\boldsymbol{n}$ and $\boldsymbol{s}$ represent the unit outward normal and the unit counterclockwise tangent to the boundary, respectively. Finally, for points $x \in \mathcal{V}$, we introduce the following notation. We indicate with $\boldsymbol{n}_1$ and $\boldsymbol{s}_1$ the unit vectors corresponding, respectively, to $\boldsymbol{n}$ and $\boldsymbol{s}$ on one of the two edges forming the boundary angle at x; with $\boldsymbol{n}_2$ and $\boldsymbol{s}_2$ we indicate the ones corresponding to the other edge. Note that which of the two edges correspond to the subscript 1 or 2 is not relevant.

The classical Kirchhoff plate-bending model is then given by the biharmonic partial differential equation

$$\mathsf{D}\Delta^2 w = g \qquad \text{in } \Omega, \tag{2.1}$$

the boundary conditions

$$\begin{aligned}
&w = 0, && \frac{\partial w}{\partial \boldsymbol{n}} = 0 && \text{on } \Gamma_{\rm C},\\
&w = 0, && \boldsymbol{n}\cdot\boldsymbol{M}\boldsymbol{n} = 0 && \text{on } \Gamma_{\rm S},\\
&\boldsymbol{n}\cdot\boldsymbol{M}\boldsymbol{n} = 0, && \frac{\partial}{\partial \boldsymbol{s}}\big(\boldsymbol{s}\cdot\boldsymbol{M}\boldsymbol{n}\big) + (\mathbf{div}\,\boldsymbol{M})\cdot\boldsymbol{n} = 0 && \text{on } \Gamma_{\rm F},
\end{aligned} \tag{2.2}$$

and the corner conditions

$$\big(\boldsymbol{s}_1 \cdot \boldsymbol{M}\boldsymbol{n}_1\big)(x) = \big(\boldsymbol{s}_2 \cdot \boldsymbol{M}\boldsymbol{n}_2\big)(x) \qquad \forall x \in \mathcal{V}. \tag{2.3}$$

Here

$$\mathsf{D} = \frac{\mathsf{E}t^3}{12(1-\nu^2)} \tag{2.4}$$

is the bending rigidity, with E, ν being the Young modulus and the Poisson ratio for the material, respectively. Note that for the shear modulus G it holds that

$$\mathsf{G} = \frac{\mathsf{E}}{2(1+\nu)}. \tag{2.5}$$

The moment tensor is given by

$$\boldsymbol{M}(\nabla w) = \mathsf{D}\big((1-\nu)\boldsymbol{\varepsilon}(\nabla w) + \nu \operatorname{div}(\nabla w)\boldsymbol{I}\big), \tag{2.6}$$

with the symmetric gradient $\boldsymbol{\varepsilon}$, and the shear force by

$$\boldsymbol{Q} = -\mathbf{div}\, \boldsymbol{M}. \tag{2.7}$$

Note that the independence of the Poisson ratio ν in the differential equation (2.1) is a consequence of cancellations when substituting (2.6) and (2.7) into the equilibrium equation

$$-\operatorname{div} \boldsymbol{Q} = g. \tag{2.8}$$

For the analysis below, it will be convenient to perform a scaling of the problem by assuming that the load is given by $g = \mathsf{G}t^3 f$, with f fixed. Then the differential equation (2.1) becomes independent of the plate thickness:

$$\frac{1}{6(1-\nu)} \Delta^2 w = f \qquad \text{in } \Omega. \tag{2.9}$$

Furthermore, we use the following scaled moment tensor $\boldsymbol{m}$:

$$\boldsymbol{M}(\nabla w) = \mathsf{G}t^3 \boldsymbol{m}(\nabla w), \tag{2.10}$$

and the shear force $\boldsymbol{q}$ is defined by

$$\boldsymbol{Q} = \mathsf{G}t^3 \boldsymbol{q}. \tag{2.11}$$

The unknowns in our finite element method will be the approximations to the deflection and its gradient, the rotation $\boldsymbol{\beta} = \nabla w$. With this as a new unknown, our problem can be written as the system of partial differential equations

$$\nabla w - \boldsymbol{\beta} = \mathbf{0}, \tag{2.12}$$

$$-\operatorname{div} \boldsymbol{q} = f, \tag{2.13}$$

$$\boldsymbol{L\beta} + \boldsymbol{q} = \mathbf{0} \qquad \text{in } \Omega, \tag{2.14}$$

the boundary conditions

$$w = 0, \ \boldsymbol{\beta} = \mathbf{0} \qquad \text{on } \Gamma_{\mathrm{C}}, \tag{2.15}$$

$$w = 0,\ \boldsymbol{\beta} \cdot \boldsymbol{s} = 0,\ \boldsymbol{n} \cdot \boldsymbol{m}(\boldsymbol{\beta})\boldsymbol{n} = 0 \qquad \text{on } \Gamma_{\mathrm{S}}, \tag{2.16}$$

$$\frac{\partial w}{\partial \boldsymbol{s}} - \boldsymbol{\beta} \cdot \boldsymbol{s} = 0,\ \boldsymbol{n} \cdot \boldsymbol{m}(\boldsymbol{\beta})\boldsymbol{n} = 0,\ \frac{\partial}{\partial \boldsymbol{s}}\big(\boldsymbol{s} \cdot \boldsymbol{m}(\boldsymbol{\beta})\boldsymbol{n}\big) - \boldsymbol{q} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma_{\mathrm{F}}, \tag{2.17}$$

and the corner conditions

$$\big(\boldsymbol{s}_1 \cdot \boldsymbol{m}(\boldsymbol{\beta})\boldsymbol{n}_1\big)(x) = \big(\boldsymbol{s}_2 \cdot \boldsymbol{m}(\boldsymbol{\beta})\boldsymbol{n}_2\big)(x) \qquad \forall x \in \mathcal{V}. \tag{2.18}$$

The operator $\boldsymbol{L}$ is defined as

$$\boldsymbol{L}\boldsymbol{\beta} = \mathbf{div}\, \boldsymbol{m}(\boldsymbol{\beta}), \tag{2.19}$$

and the scaled bending moment is considered as a function of the rotation:

$$\boldsymbol{m}(\boldsymbol{\beta}) = \frac{1}{6}\left(\boldsymbol{\varepsilon}(\boldsymbol{\beta}) + \frac{\nu}{1-\nu} \operatorname{div} \boldsymbol{\beta}\, \boldsymbol{I}\right). \tag{2.20}$$

In what follows, we will often write $\boldsymbol{m}$ instead of $\boldsymbol{m}(\boldsymbol{\beta})$. We further denote

$$a(\boldsymbol{\beta}, \boldsymbol{\eta}) = (\boldsymbol{m}(\boldsymbol{\beta}), \boldsymbol{\varepsilon}(\boldsymbol{\eta})). \tag{2.21}$$

In order to neglect plate rigid movements and the related technicalities, we will in what follows assume that the one-dimensional measure of Γ_{C} is positive.

3. The finite element formulation. In this section, we will introduce our finite element method. Even if our method is stable for all choices of finite element spaces, we will, for simplicity, present it for triangular elements and for the polynomial degrees that yield an optimal convergence rate. Hence, let a regular family of triangular meshes on Ω be given. For the integer $k \geq 1$, we then define the discrete spaces

$$W_h = \{v \in W \ \mid\ v_{|K} \in P_{k+1}(K) \ \ \forall K \in \mathcal{C}_h\}, \tag{3.1}$$

$$\boldsymbol{V}_h = \{\boldsymbol{\eta} \in \boldsymbol{V} \ \mid\ \boldsymbol{\eta}_{|K} \in [P_k(K)]^2 \ \ \forall K \in \mathcal{C}_h\}, \tag{3.2}$$

with

$$W = \left\{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_{\mathrm{C}} \cup \Gamma_{\mathrm{S}}\right\}, \tag{3.3}$$

$$\boldsymbol{V} = \left\{\boldsymbol{\eta} \in [H^1(\Omega)]^2 \mid \boldsymbol{\eta} = \boldsymbol{0} \text{ on } \Gamma_{\mathrm{C}}\,,\ \boldsymbol{\eta} \cdot \boldsymbol{s} = 0 \text{ on } \Gamma_{\mathrm{S}}\right\}. \tag{3.4}$$

Here $\mathcal{C}_h$ represents the set of all triangles K of the mesh, and $P_k(K)$ is the space of polynomials of degree k on K. In what follows, we will indicate with h_K the diameter of each element K, while h will indicate the maximum size of all of the elements in the mesh. Furthermore, we will indicate with E a general edge of the triangulation and with h_E the length of E. The set of all edges lying on the free boundary Γ_{F} we denote by $\mathcal{F}_h$.

Before introducing the method, we state the following result which trivially follows from classical scaling arguments and the coercivity of the form a.

LEMMA 3.1. *There exist positive constants C_I and C_I' such that*

$$C_I \sum_{K \in \mathcal{C}_h} h_K^2 \|\boldsymbol{L}\boldsymbol{\phi}\|_{0,K}^2 \leq a(\boldsymbol{\phi}, \boldsymbol{\phi}) \qquad \forall \boldsymbol{\phi} \in \boldsymbol{V}_h, \tag{3.5}$$

$$C_I' \sum_{E \in \mathcal{F}_h} h_E \, \|m_{ns}(\boldsymbol{\phi})\|_{0,E}^2 \leq a(\boldsymbol{\phi}, \boldsymbol{\phi}) \qquad \forall \boldsymbol{\phi} \in \boldsymbol{V}_h, \tag{3.6}$$

where the operator $m_{ns}(\boldsymbol{\phi}) = \boldsymbol{s} \cdot \boldsymbol{m}(\boldsymbol{\phi})\boldsymbol{n}$*, with* $\boldsymbol{n}, \boldsymbol{s}$*, being the unit outward normal and the unit counterclockwise tangent to the edge* E*, respectively, and with* $\boldsymbol{m}$ *defined in* (2.20)*.*

Let two real numbers γ and α be assigned: $\gamma > 2/C'_I$ and $0 < \alpha < C_I/4$. Then the discrete problem reads as follows.

Method 3.1. Find $(w_h, \boldsymbol{\beta}_h) \in W_h \times \boldsymbol{V}_h$, such that

$$\mathcal{A}_h(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta}) = (f, v) \qquad \forall (v, \boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h, \tag{3.7}$$

where the form $\mathcal{A}_h$ is defined as

$$\mathcal{A}_h(z, \boldsymbol{\phi}; v, \boldsymbol{\eta}) = \mathcal{B}_h(z, \boldsymbol{\phi}; v, \boldsymbol{\eta}) + \mathcal{D}_h(z, \boldsymbol{\phi}; v, \boldsymbol{\eta}), \tag{3.8}$$

with

$$\begin{aligned} \mathcal{B}_h(z, \boldsymbol{\phi}; v, \boldsymbol{\eta}) = a(\boldsymbol{\phi}, \boldsymbol{\eta}) - \sum_{K \in \mathcal{C}_h} \alpha h_K^2 (\boldsymbol{L}\boldsymbol{\phi}, \boldsymbol{L}\boldsymbol{\eta})_K \\ + \sum_{K \in \mathcal{C}_h} \frac{1}{\alpha h_K^2} (\nabla z - \boldsymbol{\phi} - \alpha h_K^2 \boldsymbol{L}\boldsymbol{\phi}, \nabla v - \boldsymbol{\eta} - \alpha h_K^2 \boldsymbol{L}\boldsymbol{\eta})_K \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} \mathcal{D}_h(z, \boldsymbol{\phi}; v, \boldsymbol{\eta}) = \langle m_{ns}(\boldsymbol{\phi}), [\nabla v - \boldsymbol{\eta}] \cdot \boldsymbol{s} \rangle_{\Gamma_{\rm F}} + \langle [\nabla z - \boldsymbol{\phi}] \cdot \boldsymbol{s}, m_{ns}(\boldsymbol{\eta}) \rangle_{\Gamma_{\rm F}} \\ + \sum_{E \in \mathcal{F}_h} \frac{\gamma}{h_E} \langle [\nabla z - \boldsymbol{\phi}] \cdot \boldsymbol{s}, [\nabla v - \boldsymbol{\eta}] \cdot \boldsymbol{s} \rangle_E \end{aligned} \tag{3.10}$$

for all $(z, \boldsymbol{\phi}), (v, \boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h$. Here $\langle \cdot, \cdot \rangle_{\Gamma_{\rm F}}$ and $\langle \cdot, \cdot \rangle_E$ denote the L^2-inner products on $\Gamma_{\rm F}$ and E, respectively.

The bilinear form $\mathcal{B}_h$ constitutes the Reissner–Mindlin method of [13] with the thickness t set equal to zero, while the additional form $\mathcal{D}_h$ is introduced in order to avoid the convergence deterioration in the presence of free boundaries.

Furthermore, we introduce the discrete shear force

$$\boldsymbol{q}_{h|K} = \frac{1}{\alpha h_K^2} (\nabla w_h - \boldsymbol{\beta}_h - \alpha h_K^2 \boldsymbol{L}\boldsymbol{\beta}_h)_{|K} \quad \forall K \in \mathcal{C}_h. \tag{3.11}$$

We note that, due to (2.14) and (2.12), it holds that

$$\boldsymbol{q}_{|K} = \frac{1}{\alpha h_K^2} (\nabla w - \boldsymbol{\beta} - \alpha h_K^2 \boldsymbol{L}\boldsymbol{\beta})_{|K} \quad \forall K \in \mathcal{C}_h, \tag{3.12}$$

and hence it follows that the definition (3.11) is consistent with the exact shear force.

For simplicity, in the rest of this section we assume that the deflection w belongs to $H^3(\Omega)$; this is a very reasonable assumption, as discussed at the end of this section. Note as well that, with some additional technical work involving the appropriate Sobolev spaces and their duals, such an assumption could probably be avoided. The following result states the consistency of the method.

THEOREM 3.2. *The solution* $(w, \boldsymbol{\beta})$ *of the problem* (2.14)–(2.18) *satisfies*

$$\mathcal{A}_h(w, \boldsymbol{\beta}; v, \boldsymbol{\eta}) = (f, v) \qquad \forall (v, \boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h. \tag{3.13}$$

Proof. The definition of the bilinear forms in Method 3.1, recalling (2.14) and the expression (3.12), give

$$\begin{aligned}\mathcal{B}_h(w,\boldsymbol{\beta};v,\boldsymbol{\eta}) &= a(\boldsymbol{\beta},\boldsymbol{\eta}) - \sum_{K\in\mathcal{C}_h} \alpha h_K^2(\boldsymbol{L\beta},\boldsymbol{L\eta})_K \\ &\quad + \sum_{K\in\mathcal{C}_h} \frac{1}{\alpha h_K^2}(\nabla w - \boldsymbol{\beta} - \alpha h_K^2 \boldsymbol{L\beta}, \nabla v - \boldsymbol{\eta} - \alpha h_K^2 \boldsymbol{L\eta})_K \\ &= a(\boldsymbol{\beta},\boldsymbol{\eta}) + \sum_{K\in\mathcal{C}_h} \alpha h_K^2(\boldsymbol{q},\boldsymbol{L\eta})_K + \sum_{K\in\mathcal{C}_h}(\boldsymbol{q},\nabla v - \boldsymbol{\eta} - \alpha h_K^2\boldsymbol{L\eta})_K \\ &= a(\boldsymbol{\beta},\boldsymbol{\eta}) + (\boldsymbol{q},\nabla v - \boldsymbol{\eta}). \end{aligned} \tag{3.14}$$

First, by the definition (2.21), then integrating by parts on each triangle, and finally using the regularity of the functions involved, and the boundary conditions (2.15), (2.16) on Γ_{C}, Γ_{S}, respectively, we get

$$\begin{aligned} a(\boldsymbol{\beta},\boldsymbol{\eta}) + (\boldsymbol{q},\nabla v - \boldsymbol{\eta}) &= (\boldsymbol{m}(\boldsymbol{\beta}),\boldsymbol{\varepsilon}(\boldsymbol{\eta})) + (\boldsymbol{q},\nabla v - \boldsymbol{\eta}) \\ &= -(\boldsymbol{L\beta} + \boldsymbol{q},\boldsymbol{\eta}) + \langle \boldsymbol{m}(\boldsymbol{\beta})\cdot\boldsymbol{n},\boldsymbol{\eta}\rangle_{\Gamma_{\mathrm{F}}} - (\operatorname{div}\boldsymbol{q},v) + \langle \boldsymbol{q}\cdot\boldsymbol{n},v\rangle_{\Gamma_{\mathrm{F}}}. \end{aligned} \tag{3.15}$$

Recalling (2.14) and (2.13), the identity above becomes

$$a(\boldsymbol{\beta},\boldsymbol{\eta}) + (\boldsymbol{q},\nabla v - \boldsymbol{\eta}) = (f,v) + \langle \boldsymbol{m}(\boldsymbol{\beta})\cdot\boldsymbol{n},\boldsymbol{\eta}\rangle_{\Gamma_{\mathrm{F}}} + \langle \boldsymbol{q}\cdot\boldsymbol{n},v\rangle_{\Gamma_{\mathrm{F}}}, \tag{3.16}$$

while using the boundary conditions of (2.17) on Γ_{F} and integration by parts along the boundary finally leads to

$$a(\boldsymbol{\beta},\boldsymbol{\eta}) + (\boldsymbol{q},\nabla v - \boldsymbol{\eta}) = (f,v) - \langle m_{ns}(\boldsymbol{\beta}), [\nabla v - \boldsymbol{\eta}]\cdot\boldsymbol{s}\rangle_{\Gamma_{\mathrm{F}}}. \tag{3.17}$$

Due to (2.17), we have

$$\begin{aligned}\mathcal{D}_h(w,\boldsymbol{\beta};v,\boldsymbol{\eta}) &= \langle m_{ns}(\boldsymbol{\beta}),[\nabla v - \boldsymbol{\eta}]\cdot\boldsymbol{s}\rangle_{\Gamma_{\mathrm{F}}} + \langle [\nabla w - \boldsymbol{\beta}]\cdot\boldsymbol{s}, m_{ns}(\boldsymbol{\eta})\rangle_{\Gamma_{\mathrm{F}}} \\ &\quad + \sum_{E\in\mathcal{F}_h}\frac{\gamma}{h_E}\langle [\nabla w - \boldsymbol{\beta}]\cdot\boldsymbol{s},[\nabla v - \boldsymbol{\eta}]\cdot\boldsymbol{s}\rangle_E \\ &= \langle m_{ns}(\boldsymbol{\beta}),[\nabla v - \boldsymbol{\eta}]\cdot\boldsymbol{s}\rangle_{\Gamma_{\mathrm{F}}}. \end{aligned} \tag{3.18}$$

The result now directly follows from (3.14), (3.17), and (3.18). □

Remark 3.1. If the Reissner–Mindlin method of [13] without the additional form $\mathcal{D}_h$ is employed by setting $t = 0$, then in the presence of a free boundary we obtain

$$\mathcal{B}_h(w,\boldsymbol{\beta};v,\boldsymbol{\eta}) = (f,v) + \langle m_{ns}(\boldsymbol{\beta}),[\nabla v - \boldsymbol{\eta}]\cdot\boldsymbol{s}\rangle_{\Gamma_{\mathrm{F}}} \qquad \forall (v,\boldsymbol{\eta})\in W_h\times\boldsymbol{V}_h. \tag{3.19}$$

Therefore, this would lead to an inconsistent method. We return to this in Remark 4.1 below.

4. Stability and a priori error estimates. For $(v,\boldsymbol{\eta})\in W_h\times\boldsymbol{V}_h$, we introduce the following mesh-dependent norms:

$$|(v,\boldsymbol{\eta})|_h^2 = \sum_{K\in\mathcal{C}_h} h_K^{-2}\|\nabla v - \boldsymbol{\eta}\|_{0,K}^2, \tag{4.1}$$

$$\|v\|_{2,h}^2 = \|v\|_1^2 + \sum_{K\in\mathcal{C}_h}|v|_{2,K}^2 + \sum_{E\in\mathcal{I}_h} h_E^{-1}\left\| \left[\!\left[\frac{\partial v}{\partial \boldsymbol{n}} \right]\!\right] \right\|_{0,E}^2 + \sum_{E\subset\Gamma_{\mathrm{C}}} h_E^{-1}\left\|\frac{\partial v}{\partial\boldsymbol{n}}\right\|_{0,E}^2, \tag{4.2}$$

$$|||(v,\boldsymbol{\eta})|||_h = \|\boldsymbol{\eta}\|_1 + \|v\|_{2,h} + |(v,\boldsymbol{\eta})|_h, \tag{4.3}$$

where $[\![\cdot]\!]$ represents the jump operator and $\mathcal{I}_h$ denotes the edges lying in the interior of the domain Ω.

In [12], the following lemma is proved.

LEMMA 4.1. *There exists a positive constant C such that*

$$\|v\|_{2,h} \le C\big(\|\boldsymbol{\eta}\|_1 + \|v\|_1 + |(v,\boldsymbol{\eta})|_h\big) \qquad \forall (v,\boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h. \tag{4.4}$$

Using the Poincaré inequality and the previous lemma, the following equivalence easily follows.

LEMMA 4.2. *There exists a positive constant C such that*

$$C|\!|\!|(v,\boldsymbol{\eta})|\!|\!|_h \le \|\boldsymbol{\eta}\|_1 + |(v,\boldsymbol{\eta})|_h \le |\!|\!|(v,\boldsymbol{\eta})|\!|\!|_h \quad \forall (v,\boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h. \tag{4.5}$$

We now have the following stability estimate.

THEOREM 4.3. *Let $0 < \alpha < C_I/4$ and $\gamma > 2/C_I'$. Then there exists a positive constant C such that*

$$\mathcal{A}_h(v,\boldsymbol{\eta}; v,\boldsymbol{\eta}) \ge C|\!|\!|(v,\boldsymbol{\eta})|\!|\!|_h^2 \quad \forall (v,\boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h. \tag{4.6}$$

Proof. Using the first inverse estimate of Lemma 3.1 we get

$$\begin{aligned}
&\mathcal{B}_h(v,\boldsymbol{\eta}; v,\boldsymbol{\eta}) \\
&\quad = a(\boldsymbol{\eta},\boldsymbol{\eta}) - \sum_{K\in\mathcal{C}_h} \alpha h_K^2 \|\boldsymbol{L}\boldsymbol{\eta}\|_{0,K}^2 + \sum_{K\in\mathcal{C}_h} \frac{1}{\alpha h_K^2} \|\nabla v - \boldsymbol{\eta} - \alpha h_K^2 \boldsymbol{L}\boldsymbol{\eta}\|_{0,K}^2 \\
&\quad \ge \left(1 - \frac{\alpha}{C_I}\right) a(\boldsymbol{\eta},\boldsymbol{\eta}) + \sum_{K\in\mathcal{C}_h} \frac{1}{\alpha h_K^2} \|\nabla v - \boldsymbol{\eta} - \alpha h_K^2 \boldsymbol{L}\boldsymbol{\eta}\|_{0,K}^2. \qquad (4.7)
\end{aligned}$$

Next, using locally the arithmetic-geometric mean inequality with the constant γ/h_E then the second inverse inequality of Lemma 3.1, we get

$$\begin{aligned}
&\mathcal{D}_h(v,\boldsymbol{\eta}; v,\boldsymbol{\eta}) \\
&\quad = \sum_{E\in\mathcal{F}_h} \Big(2\langle m_{ns}(\boldsymbol{\eta}), [\nabla v - \boldsymbol{\eta}]\cdot \boldsymbol{s}\rangle_E + \frac{\gamma}{h_E}\|[\nabla v - \boldsymbol{\eta}]\cdot \boldsymbol{s}\|_{0,E}^2\Big) \\
&\quad \ge \sum_{E\in\mathcal{F}_h} \Big(- \frac{\gamma}{h_E}\|[\nabla v - \boldsymbol{\eta}]\cdot \boldsymbol{s}\|_{0,E}^2 - \gamma^{-1} h_E \, \|m_{ns}(\boldsymbol{\eta})\|_{0,E}^2 + \frac{\gamma}{h_E}\|[\nabla v - \boldsymbol{\eta}]\cdot \boldsymbol{s}\|_{0,E}^2\Big) \\
&\quad = -\sum_{E\in\mathcal{F}_h} \gamma^{-1} h_E \, \|m_{ns}(\boldsymbol{\eta})\|_{0,E}^2 \\
(4.8)\ &\ge -\frac{\gamma^{-1}}{C_I'}\, a(\boldsymbol{\eta},\boldsymbol{\eta}) \ge -\frac{1}{2}\, a(\boldsymbol{\eta},\boldsymbol{\eta}).
\end{aligned}$$

Joining (4.7) with (4.8) and using Korn's inequality we then obtain

$$\begin{aligned}
&\mathcal{B}_h(v,\boldsymbol{\eta}; v,\boldsymbol{\eta}) + \mathcal{D}_h(v,\boldsymbol{\eta}; v,\boldsymbol{\eta}) \\
&\quad \ge \left(\frac{1}{2} - \frac{\alpha}{C_I}\right) a(\boldsymbol{\eta},\boldsymbol{\eta}) + \sum_{K\in\mathcal{C}_h} \frac{1}{\alpha h_K^2} \|\nabla v - \boldsymbol{\eta} - \alpha h_K^2 \boldsymbol{L}\boldsymbol{\eta}\|_{0,K}^2 \\
&\quad \ge C\Big(\|\boldsymbol{\eta}\|_1^2 + \sum_{K\in\mathcal{C}_h} \frac{1}{\alpha h_K^2} \|\nabla v - \boldsymbol{\eta} - \alpha h_K^2 \boldsymbol{L}\boldsymbol{\eta}\|_{0,K}^2\Big). \qquad (4.9)
\end{aligned}$$

From the triangle inequality, again the inverse estimate of Lemma 3.1, and the boundedness of the bilinear form a, it follows that

$$
\begin{aligned}
&\sum_{K\in\mathcal{C}_h}\frac{1}{\alpha h_K^2}\|\nabla v-\boldsymbol{\eta}\|_{0,K}^2\\
&\quad\le 2\left(\sum_{K\in\mathcal{C}_h}\frac{1}{\alpha h_K^2}\|\nabla v-\boldsymbol{\eta}-\alpha h_K^2\boldsymbol{L\eta}\|_{0,K}^2+\sum_{K\in\mathcal{C}_h}\frac{1}{\alpha h_K^2}\|\alpha h_K^2\boldsymbol{L\eta}\|_{0,K}^2\right)\\
&\quad\le 2\left(\sum_{K\in\mathcal{C}_h}\frac{1}{\alpha h_K^2}\|\nabla v-\boldsymbol{\eta}-\alpha h_K^2\boldsymbol{L\eta}\|_{0,K}^2+\sum_{K\in\mathcal{C}_h}\alpha h_K^2\|\boldsymbol{L\eta}\|_{0,K}^2\right)\\
&\quad\le C\left(\sum_{K\in\mathcal{C}_h}\frac{1}{\alpha h_K^2}\|\nabla v-\boldsymbol{\eta}-\alpha h_K^2\boldsymbol{L\eta}\|_{0,K}^2+a(\boldsymbol{\eta},\boldsymbol{\eta})\right)\\
&\quad\le C\left(\sum_{K\in\mathcal{C}_h}\frac{1}{\alpha h_K^2}\|\nabla v-\boldsymbol{\eta}-\alpha h_K^2\boldsymbol{L\eta}\|_{0,K}^2+\|\boldsymbol{\eta}\|_1^2\right),
\end{aligned}
\tag{4.10}
$$

which combined with (4.9) gives

$$
\mathcal{A}_h(v,\boldsymbol{\eta};v,\boldsymbol{\eta})\ge C\big(\|\boldsymbol{\eta}\|_1^2+|(v,\boldsymbol{\eta})|_h^2\big). \tag{4.11}
$$

The result then follows from the norm equivalence of Lemma 4.2. □

We can now derive the error estimates for the method. We note that the assumptions of the theorem are supposed to be valid for the further results below as well and hence are not repeated in what follows.

THEOREM 4.4. *Let $0<\alpha<C_I/4$ and $\gamma>2/C_I'$. Let $(w,\boldsymbol{\beta})$ be the exact solution of the problem, and let $(w_h,\boldsymbol{\beta}_h)$ be the approximate solution obtained with Method 3.1. Suppose that $w\in H^{s+2}(\Omega)$, with $1\le s\le k$. Then it holds that*

$$
|||(w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h)|||_h\le Ch^s\|w\|_{s+2}. \tag{4.12}
$$

Proof. Step 1. Let $(w_I,\boldsymbol{\beta}_I)\in W_h\times\boldsymbol{V}_h$ be the usual Lagrange interpolants to w and $\boldsymbol{\beta}$, respectively. Using first the stability result of Theorem 4.3 and then the consistency result of Theorem 3.2, one has the existence of a pair

$$
(v,\boldsymbol{\eta})\in W_h\times\boldsymbol{V}_h,\qquad |||(v,\boldsymbol{\eta})|||_h\le C \tag{4.13}
$$

such that

$$
\begin{aligned}
|||(w_h-w_I,\boldsymbol{\beta}_h-\boldsymbol{\beta}_I)|||_h&\le\mathcal{A}_h(w_h-w_I,\boldsymbol{\beta}_h-\boldsymbol{\beta}_I;v,\boldsymbol{\eta})\\
&=\mathcal{A}_h(w-w_I,\boldsymbol{\beta}-\boldsymbol{\beta}_I;v,\boldsymbol{\eta}),
\end{aligned}
\tag{4.14}
$$

where we recall that $\mathcal{A}_h=\mathcal{B}_h+\mathcal{D}_h$.

Step 2. For the $\mathcal{B}_h$-part, we have

$$
\begin{aligned}
\mathcal{B}_h(w-w_I,\boldsymbol{\beta}-\boldsymbol{\beta}_I;v,\boldsymbol{\eta})&=a(\boldsymbol{\beta}-\boldsymbol{\beta}_I,\boldsymbol{\eta})-\sum_{K\in\mathcal{C}_h}\alpha h_K^2(\boldsymbol{L}(\boldsymbol{\beta}-\boldsymbol{\beta}_I),\boldsymbol{L\eta})_K\\
&\quad+\sum_{K\in\mathcal{C}_h}\frac{1}{\alpha h_K^2}(\nabla(w-w_I)-(\boldsymbol{\beta}-\boldsymbol{\beta}_I)-\alpha h_K^2\boldsymbol{L}(\boldsymbol{\beta}-\boldsymbol{\beta}_I),\nabla v-\boldsymbol{\eta}-\alpha h_K^2\boldsymbol{L\eta})_K.
\end{aligned}
\tag{4.15}
$$

Due to the first inverse inequality of Lemma 3.1, we get

$$\left(\sum_{K\in\mathcal{C}_h} h_K^2\|\boldsymbol{L}\boldsymbol{\eta}\|_{0,K}^2\right)^{1/2} \leq C|||(v,\boldsymbol{\eta})|||_h \tag{4.16}$$

and

$$\left(\sum_{K\in\mathcal{C}_h} \frac{1}{\alpha h_K^2}\|\nabla v-\boldsymbol{\eta}-\alpha h_K^2\boldsymbol{L}\boldsymbol{\eta}\|_{0,K}^2\right)^{1/2} \leq C|||(v,\boldsymbol{\eta})|||_h. \tag{4.17}$$

Using these bounds in (4.15) and recalling (4.13), we obtain

$$\begin{aligned}&\mathcal{B}_h(w-w_I,\boldsymbol{\beta}-\boldsymbol{\beta}_I;v,\boldsymbol{\eta})\\ &\quad\leq C\left(|||(w-w_I,\boldsymbol{\beta}-\boldsymbol{\beta}_I)|||_h+\left(\sum_{K\in\mathcal{C}_h} h_K^2|\boldsymbol{\beta}-\boldsymbol{\beta}_I|_{2,K}^2\right)^{1/2}\right).\end{aligned} \tag{4.18}$$

Substituting the definition of the norm (4.3) in (4.18), using the triangle inequality, and finally applying the classical interpolation estimates, it easily follows that

$$\mathcal{B}_h(w-w_I,\boldsymbol{\beta}-\boldsymbol{\beta}_I;v,\boldsymbol{\eta})\leq Ch^s\big(\|w\|_{s+2}+\|\boldsymbol{\beta}\|_{s+1}\big). \tag{4.19}$$

Step 3. For the $\mathcal{D}_h$-part in (4.14), we have, by the definition (3.10),

$$\begin{aligned}\mathcal{D}_h(w-w_I,\boldsymbol{\beta}-\boldsymbol{\beta}_I;v,\boldsymbol{\eta}) &= \langle m_{ns}(\boldsymbol{\beta}-\boldsymbol{\beta}_I),[\nabla v-\boldsymbol{\eta}]\cdot\boldsymbol{s}\rangle_{\Gamma_{\mathrm{F}}}\\ &\quad+\langle[\nabla(w-w_I)-(\boldsymbol{\beta}-\boldsymbol{\beta}_I)]\cdot\boldsymbol{s},m_{ns}(\boldsymbol{\eta})\rangle_{\Gamma_{\mathrm{F}}}\\ &\quad+\sum_{E\in\mathcal{F}_h}\frac{\gamma}{h_E}\langle[\nabla(w-w_I)-(\boldsymbol{\beta}-\boldsymbol{\beta}_I)]\cdot\boldsymbol{s},[\nabla v-\boldsymbol{\eta}]\cdot\boldsymbol{s}\rangle_E\\ &=: T_1+T_2+T_3.\end{aligned} \tag{4.20}$$

Scaling arguments give

$$\|[\nabla v-\boldsymbol{\eta}]\cdot\boldsymbol{s}\|_{0,E}^2\leq\|\nabla v-\boldsymbol{\eta}\|_{0,E}^2\leq Ch_{K(E)}^{-1}\|\nabla v-\boldsymbol{\eta}\|_{0,K(E)}^2 \tag{4.21}$$

for all $E\in\mathcal{F}_h$, where $K(E)$ is the triangle with E as an edge. The l^2-Cauchy–Schwarz inequality, the bound (4.21), and the norm definition (4.3) now give

$$\begin{aligned}T_1&\leq\left(\sum_{E\in\mathcal{F}_h} h_{K(E)}\|m_{ns}(\boldsymbol{\beta}-\boldsymbol{\beta}_I)\|_{0,E}^2\right)^{1/2}\left(\sum_{E\in\mathcal{F}_h} h_{K(E)}^{-1}\|[\nabla v-\boldsymbol{\eta}]\cdot\boldsymbol{s}\|_{0,E}^2\right)^{1/2}\\ &\leq C\left(\sum_{E\in\mathcal{F}_h} h_{K(E)}\|m_{ns}(\boldsymbol{\beta}-\boldsymbol{\beta}_I)\|_{0,E}^2\right)^{1/2}|||(v,\boldsymbol{\eta})|||_h.\end{aligned} \tag{4.22}$$

Recalling the bound (4.13), classical polynomial interpolation properties give

$$T_1\leq C\left(\sum_{E\in\mathcal{F}_h} h_{K(E)}\|m_{ns}(\boldsymbol{\beta}-\boldsymbol{\beta}_I)\|_{0,E}^2\right)^{1/2}\leq Ch^s\|\boldsymbol{\beta}\|_{s+1}. \tag{4.23}$$

Again, by scaling we have

$$\|m_{ns}(\boldsymbol{\eta})\|_{0,E}^2 \leq h_{K(E)}^{-1} |\boldsymbol{\eta}|_{1,K(E)}^2 \qquad \forall E \in \mathcal{F}_h. \tag{4.24}$$

The l^2-Cauchy–Schwarz inequality, this bound, and the norm definition (4.3) give

$$T_2 \leq \left(\sum_{E \in \mathcal{F}_h} h_{K(E)}^{-1} \|\nabla(w - w_I) - (\boldsymbol{\beta} - \boldsymbol{\beta}_I)\|_{0,E}^2 \right)^{1/2} \left(\sum_{E \in \mathcal{F}_h} h_{K(E)} \|m_{ns}(\boldsymbol{\eta})\|_{0,E}^2 \right)^{1/2}$$

$$\leq C \left(\sum_{E \in \mathcal{F}_h} h_{K(E)}^{-1} \|\nabla(w - w_I) - (\boldsymbol{\beta} - \boldsymbol{\beta}_I)\|_{0,E}^2 \right)^{1/2} |||(v, \boldsymbol{\eta})|||_h. \tag{4.25}$$

Recalling the bound (4.13), classical polynomial interpolation estimates give

$$\begin{aligned} T_2 &\leq C \left(\sum_{E \in \mathcal{F}_h} h_{K(E)}^{-1} \|\nabla(w - w_I) - (\boldsymbol{\beta} - \boldsymbol{\beta}_I)\|_{0,E}^2 \right)^{1/2} \\ &\leq C h^s \big(\|\boldsymbol{\beta}\|_{s+1} + \|w\|_{s+2} \big). \end{aligned} \tag{4.26}$$

The bound for T_3 follows by combining the same techniques used for T_1 and T_2; we get

$$T_3 \leq C h^s \big(\|\boldsymbol{\beta}\|_{s+1} + \|w\|_{s+2} \big). \tag{4.27}$$

Now, joining all of the bounds (4.14), (4.19), (4.20), (4.23), (4.26), and (4.27) we obtain

$$|||(w_h - w_I, \boldsymbol{\beta}_h - \boldsymbol{\beta}_I)|||_h \leq C h^s \big(\|\boldsymbol{\beta}\|_{s+1} + \|w\|_{s+2} \big). \tag{4.28}$$

The triangle inequality and the classical polynomial interpolation estimates (recalling that $\boldsymbol{\beta} = \nabla w$) then yield

$$|||(w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)|||_h \leq C h^s \big(\|\boldsymbol{\beta}\|_{s+1} + \|w\|_{s+2} \big) \leq C h^s \|w\|_{s+2}. \tag{4.29}$$

Note that the result holds for real values of the regularity parameter s since the interpolation results used above are valid for real values of s. $\square$

Remark 4.1. As noted in Remark 3.1, the limiting Reissner–Mindlin method (i.e., without the additional correction $\mathcal{D}_h$) is inconsistent. Regardless of the solution regularity and the polynomial degree k, the inconsistency term can be bounded only with the order $O(h^{1/2})$. As is well known (see, for example, [10]), the inconsistency error is a lower bound for the error of finite element methods. As a consequence, the numerical scheme will not converge with a rate better than $h^{1/2}$ if $\Gamma_{\mathrm{F}} \neq \emptyset$. This observation is also confirmed by the numerical tests shown in [3]. See [6] for other numerical tests regarding this issue. Note further that this boundary inconsistency term is connected not only to the formulation in [13] but is common to any other Kirchhoff method which follows a "Reissner–Mindlin limit" approach.

For the shear force, the practical norm to use is the discrete negative norm

$$\|\boldsymbol{r}\|_{-1,h} = \left(\sum_{K \in \mathcal{C}_h} h_K^2 \|\boldsymbol{r}\|_{0,K}^2 \right)^{1/2}. \tag{4.30}$$

Since we assume that $w \in H^{s+2}(\Omega)$, with $s \geq 1$, we have $\boldsymbol{q} \in [L^2(\Omega)]^2$, and from the estimates above the lemma immediately follows.

LEMMA 4.5. *It holds that*

$$\|\boldsymbol{q} - \boldsymbol{q}_h\|_{-1,h} \leq Ch^s \|w\|_{s+2}. \tag{4.31}$$

From this follows a norm estimate in the dual to the space

$$\boldsymbol{V}_* = \left\{ \boldsymbol{\eta} \in [H^1(\Omega)]^2 \mid \boldsymbol{\eta} = \boldsymbol{0} \text{ on } \Gamma_{\mathrm{C}},\ \boldsymbol{\eta} \cdot \boldsymbol{s} = 0 \text{ on } \Gamma_{\mathrm{F}} \cup \Gamma_{\mathrm{S}} \right\}, \tag{4.32}$$

i.e., in the norm

$$\|\boldsymbol{r}\|_{-1,*} = \sup_{\boldsymbol{\eta} \in \boldsymbol{V}_*} \frac{\langle \boldsymbol{r}, \boldsymbol{\eta} \rangle}{\|\boldsymbol{\eta}\|_1}. \tag{4.33}$$

We have the following result.

LEMMA 4.6. *It holds that*

$$\|\boldsymbol{q} - \boldsymbol{q}_h\|_{-1,*} \leq Ch^s \|w\|_{s+2}. \tag{4.34}$$

Proof. The proof is essentially an application of the "Pitkäranta–Verfürth trick" (see [11, 14]). By the definition of the norm $\|\cdot\|_{-1,*}$ there exists a function $\boldsymbol{\eta} \in \boldsymbol{V}_*$ such that

$$\|\boldsymbol{q} - \boldsymbol{q}_h\|_{-1,*} \leq (\boldsymbol{q} - \boldsymbol{q}_h, \boldsymbol{\eta}), \qquad \|\boldsymbol{\eta}\|_1 \leq C. \tag{4.35}$$

Using a Clément-type interpolant we can find a piecewise linear function $\boldsymbol{\eta}_I \in \boldsymbol{V}_*$ such that it holds that

$$h_K^{s-1} \|\boldsymbol{\eta} - \boldsymbol{\eta}_I\|_{s,K} \leq C \|\boldsymbol{\eta}\|_{1,K} \leq C', \quad s = 0, 1, \tag{4.36}$$

for all $K \in \mathcal{C}_h$. Using the Cauchy–Schwarz inequality, the bound (4.36) with $s = 0$, and the definition (4.30), it follows that

$$\begin{aligned} (\boldsymbol{q} - \boldsymbol{q}_h, \boldsymbol{\eta}) &= (\boldsymbol{q} - \boldsymbol{q}_h, \boldsymbol{\eta} - \boldsymbol{\eta}_I) + (\boldsymbol{q} - \boldsymbol{q}_h, \boldsymbol{\eta}_I) \\ &\leq C \|\boldsymbol{q} - \boldsymbol{q}_h\|_{-1,h} + (\boldsymbol{q} - \boldsymbol{q}_h, \boldsymbol{\eta}_I). \end{aligned} \tag{4.37}$$

Note that $\boldsymbol{\eta}_I$ is in both $\boldsymbol{V}_h$ and $\boldsymbol{V}_*$; moreover, $\boldsymbol{L}\boldsymbol{\eta}_I = \boldsymbol{0}$ on each element K of $\mathcal{C}_h$. As a consequence, using (3.7), (3.11), (3.12), and Theorem 3.2, it follows that

$$\begin{aligned} (\boldsymbol{q} - \boldsymbol{q}_h, \boldsymbol{\eta}_I) &= a(\boldsymbol{\beta} - \boldsymbol{\beta}_h, \boldsymbol{\eta}_I) + \langle [\nabla w_h - \boldsymbol{\beta}_h)] \cdot \boldsymbol{s}, M_{ns}(\boldsymbol{\eta}_I) \rangle_{\Gamma_{\mathrm{F}}} \\ &=: T_1 + T_2. \end{aligned} \tag{4.38}$$

Due to the continuity of the bilinear form and using bound (4.36) with $s = 1$, it immediately follows that

$$T_1 \leq C \|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_1 \leq C |||(w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)|||_h. \tag{4.39}$$

Using first the Cauchy–Schwarz inequality, then the Agmon inequality, and finally the bound (4.36) with $s = 1$, Lemma 3.1, and the definition (4.3), we get

$$\begin{aligned} T_2 &\leq \left(\sum_{E \in \mathcal{F}_h} h_E^{-1} \|\nabla w_h - \boldsymbol{\beta}_h)\|_{0,E}^2 \right)^{1/2} \left(\sum_{E \in \mathcal{F}_h} h_E \|M_{ns}(\boldsymbol{\eta}_I)\|_{0,E}^2 \right)^{1/2} \\ &\leq \left(\sum_{K \in \mathcal{C}_h} h_K^{-2} \|\nabla w_h - \boldsymbol{\beta}_h)\|_{0,K}^2 \right)^{1/2} \|\boldsymbol{\eta}_I\|_1 \\ &\leq C |||(w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)|||_h, \end{aligned} \tag{4.40}$$

where in the last inequality we implicitly used the relation $\nabla w - \boldsymbol{\beta} = \mathbf{0}$. Combining (4.35), (4.37) with (4.38), (4.39), and (4.40), it follows that

$$\|\boldsymbol{q} - \boldsymbol{q}_h\|_{-1,*} \leq C\big(\|\boldsymbol{q} - \boldsymbol{q}_h\|_{-1,h} + |||(w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)|||_h\big). \tag{4.41}$$

Joining (4.41) and (4.31) and using Theorem 4.4 the proposition immediately follows. $\square$

The regularity of the solution to the Kirchhoff plate problems for convex polygonal domains, with all three main types of boundary conditions, is very case-dependent. We refer, for example, to the work [9], in which a rather complete study is accomplished. Note that if $f \in H^{-1}(\Omega)$, in most cases of interest, the regularity condition $w \in H^3(\Omega)$ is indeed achieved.

Note further that with classical duality arguments and technical calculations it is possible to derive the error bound

$$\|w - w_h\|_1 \leq Ch^{s+1}\|w\|_{s+2}, \tag{4.42}$$

if the regularity estimate

$$\|w\|_3 \leq C\|f\|_{-1} \tag{4.43}$$

holds. Moreover, if $k \geq 2$ and the regularity estimate

$$\|w\|_4 \leq C\|f\|_0 \tag{4.44}$$

is satisfied, then it holds that

$$\|w - w_h\|_0 \leq Ch^{s+2}\|w\|_{s+2}. \tag{4.45}$$

5. A posteriori error estimates. In this section, we prove the reliability and the efficiency for an a posteriori error estimator for our method. To this end, we introduce

$$\tilde{\eta}_K^2 := h_K^4\|f + \operatorname{div} \boldsymbol{q}_h\|_{0,K}^2 + h_K^{-2}\|\nabla w_h - \boldsymbol{\beta}_h\|_{0,K}^2, \tag{5.1}$$

$$\eta_E^2 := h_E^3\|[\![\boldsymbol{q}_h \cdot \boldsymbol{n}]\!]\|_{0,E}^2 + h_E\|[\![\boldsymbol{m}(\boldsymbol{\beta}_h)\boldsymbol{n}]\!]\|_{0,E}^2, \tag{5.2}$$

$$\eta_{S,E}^2 := h_E\|m_{nn}(\boldsymbol{\beta}_h)\|_{0,E}^2, \tag{5.3}$$

$$\eta_{F,E}^2 := h_E\|m_{nn}(\boldsymbol{\beta}_h)\|_{0,E}^2 + h_E^3\left\|\frac{\partial}{\partial s}m_{ns}(\boldsymbol{\beta}_h) - \boldsymbol{q}_h \cdot \boldsymbol{n}\right\|_{0,E}^2, \tag{5.4}$$

where h_E denotes the length of the edge E and $[\![\cdot]\!]$ represents the jump operator (which is assumed to be equal to the function value on boundary edges). Further, for a triangle $K \in \mathcal{C}_h$ we denote the sets of edges lying in the interior of Ω, on Γ_{S}, and on Γ_{F}, by $I(K)$, $S(K)$, and $F(K)$, respectively. By $\mathcal{S}_h$ we denote the set of all edges on Γ_{S} and by $\mathcal{I}_h$ the ones lying in the interior of the domain.

Given any element $K \in \mathcal{C}_h$, let the local error indicator be

$$\eta_K := \left(\tilde{\eta}_K^2 + \frac{1}{2}\sum_{E \in I(K)} \eta_E^2 + \sum_{E \in S(K)} \eta_{S,E}^2 + \sum_{E \in F(K)} \eta_{F,E}^2\right)^{1/2}. \tag{5.5}$$

Finally, the global error indicator is defined as

$$\eta := \left(\sum_{K \in \mathcal{C}_h} \eta_K^2\right)^{1/2}. \tag{5.6}$$

Remark 5.1. It is worth noting that, by the definition (3.11),

$$(\boldsymbol{q}_h + \boldsymbol{L}\boldsymbol{\beta}_h)_{|K} = \frac{1}{\alpha h_K^2}(\nabla w_h - \boldsymbol{\beta}_h)_{|K} \qquad \forall K \in \mathcal{C}_h, \tag{5.7}$$

which is the reason why there appear no terms of the kind $\|\boldsymbol{q}_h + \boldsymbol{L}\boldsymbol{\beta}_h\|_{0,K}$ in the error estimator. We note as well that scaling arguments give

$$\sum_{E\in\mathcal{F}_h} h_E^{-1}\|\nabla w_h - \boldsymbol{\beta}_h\|_{0,E}^2 \le C \sum_{K\in\mathcal{C}_h} h_K^{-2}\|\nabla w_h - \boldsymbol{\beta}_h\|_{0,K}^2, \tag{5.8}$$

which is the reason why there appear no boundary terms of the kind $\|\nabla w_h - \boldsymbol{\beta}_h\|_{0,E}$.

5.1. Upper bound. In order to derive the reliability of the method we need the following saturation assumption.

Assumption 5.1. Given a mesh $\mathcal{C}_h$, let $\mathcal{C}_{h/2}$ be the mesh obtained by splitting each triangle $K \in \mathcal{C}_h$ into four triangles connecting the edge midpoints. Let $(w_{h/2}, \boldsymbol{\beta}_{h/2})$ be the discrete solution corresponding to the mesh $\mathcal{C}_{h/2}$. We assume that there exists a constant ρ, $0 < \rho < 1$, such that

$$\begin{aligned} &|||(w - w_{h/2}, \boldsymbol{\beta} - \boldsymbol{\beta}_{h/2})|||_{h/2} + \|\boldsymbol{q} - \boldsymbol{q}_{h/2}\|_{-1,*} \\ &\qquad \le \rho\big(|||(w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)|||_h + \|\boldsymbol{q} - \boldsymbol{q}_h\|_{-1,*}\big), \end{aligned} \tag{5.9}$$

where by $|||\cdot|||_{h/2}$ we indicate the mesh-dependent norm with respect to the new mesh $\mathcal{C}_{h/2}$.

In what follows, we will need the following result.

LEMMA 5.1. *Let, for $v \in W_{h/2}$, the local seminorm be*

$$|v|_{2,h/2,K} = \left(\sum_{K'\in\mathcal{C}_{h/2},\ K'\subset K} |v|_{2,K'}^2\right)^{1/2}. \tag{5.10}$$

Then there is a positive constant C such that for all $v \in W_{h/2}$ there exists $v_I \in W_h$ with the bound

$$\|v - v_I\|_{0,K} + h_K^{1/2}\|v - v_I\|_{0,\partial K} \le C h_K^2 |v|_{2,h/2,K} \qquad \forall K \in \mathcal{C}_h. \tag{5.11}$$

Moreover, v_I interpolates v at all of the vertices of the triangulation $\mathcal{C}_{h/2}$.

Proof. We choose v_I as the only function in $H^1(\Omega)$ such that

$$\begin{aligned} &v_{I|K} \in P_2(K) \qquad \forall K \in \mathcal{C}_h, \\ &v_I(x) = v(x) \qquad \forall x \in \mathcal{V}_{h/2}, \end{aligned} \tag{5.12}$$

where $\mathcal{V}_{h/2}$ represents the set of all of the vertices of $\mathcal{C}_{h/2}$. Note that it is trivial to check that $v_I \in W_h$ for all $k \ge 1$. Observing that

$$|v|_{2,h/2,K} + \sum_{x\in\mathcal{V}_{h/2}\cap K} |v(x)|, \qquad v \in W_{h/2}, K \in \mathcal{C}_h, \tag{5.13}$$

is indeed a norm on the finite-dimensional space of the functions $v \in W_{h/2}$ restricted to K, the result follows applying the classical scaling argument. $\square$

For simplicity, in what follows we will treat the case $\Gamma_{\mathrm{S}} = \emptyset$, the general case following with identical arguments as the ones that follow. We have the following preliminary result.

THEOREM 5.2. *It holds that*

$$|||(w_{h/2} - w_h, \boldsymbol{\beta}_{h/2} - \boldsymbol{\beta}_h)|||_{h/2} \leq C\eta. \tag{5.14}$$

Proof. Step 1. Due to the stability of the discrete formulation, proved in Theorem 4.3, there exists a couple $(v, \boldsymbol{\eta}) \in W_{h/2} \times \boldsymbol{V}_{h/2}$ such that

$$|||(v, \boldsymbol{\eta})|||_{h/2} \leq C \tag{5.15}$$

and

$$|||(w_{h/2} - w_h, \boldsymbol{\beta}_{h/2} - \boldsymbol{\beta}_h)|||_{h/2} \leq \mathcal{A}_{h/2}(w_{h/2} - w_h, \boldsymbol{\beta}_{h/2} - \boldsymbol{\beta}_h; v, \boldsymbol{\eta}). \tag{5.16}$$

Furthermore, we have

$$\mathcal{A}_{h/2}(w_{h/2}, \boldsymbol{\beta}_{h/2}; v, \boldsymbol{\eta}) = (f, v). \tag{5.17}$$

Step 2. Simple calculations and the definition (3.11) give

$$\begin{aligned}
\mathcal{B}_{h/2}(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta}) &= a(\boldsymbol{\beta}_h, \boldsymbol{\eta}) - \sum_{K \in \mathcal{C}_{h/2}} \alpha h_K^2 (\boldsymbol{L}\boldsymbol{\beta}_h, \boldsymbol{L}\boldsymbol{\eta})_K \\
&\quad + \sum_{K \in \mathcal{C}_{h/2}} \frac{1}{\alpha h_K^2} (\nabla w_h - \boldsymbol{\beta}_h - \alpha h_K^2 \boldsymbol{L}\boldsymbol{\beta}_h, \nabla v - \boldsymbol{\eta} - \alpha h_K^2 \boldsymbol{L}\boldsymbol{\eta})_K \\
&= a(\boldsymbol{\beta}_h, \boldsymbol{\eta}) - \sum_{K \in \mathcal{C}_{h/2}} (\nabla w_h - \boldsymbol{\beta}_h, \boldsymbol{L}\boldsymbol{\eta})_K + \sum_{K \in \mathcal{C}_{h/2}} (\boldsymbol{q}_h, \nabla v - \boldsymbol{\eta})_K \\
&\quad + R_1(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta}) \\
&= \mathcal{B}_h(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta}) + R_1(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta}),
\end{aligned} \tag{5.18}$$

where $\boldsymbol{q}_h$ is defined as in (3.11), i.e., based on the coarser mesh, and

$$\begin{aligned}
R_1(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta}) &= \sum_{K \in \mathcal{C}_{h/2}} \frac{1}{\alpha h_K^2} (\nabla w_h - \boldsymbol{\beta}_h, \nabla v - \boldsymbol{\eta})_K \\
&\quad - \sum_{K \in \mathcal{C}_h} \frac{1}{\alpha h_K^2} (\nabla w_h - \boldsymbol{\beta}_h, \nabla v - \boldsymbol{\eta})_K.
\end{aligned} \tag{5.19}$$

The last term on the right-hand side is well defined since $\nabla v - \boldsymbol{\eta}$ is piecewise L^2-regular.

Let now $\mathcal{F}_{h/2}$ indicate the set of all edges of $\mathcal{C}_{h/2}$ lying on Γ_{F}. Adding and subtracting the difference between the two forms, it then follows that

$$\mathcal{D}_{h/2}(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta}) = \mathcal{D}_h(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta}) + R_2(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta}), \tag{5.20}$$

where

$$\begin{aligned}
R_2(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta}) &= \sum_{E \in \mathcal{F}_{h/2}} \frac{\gamma}{h_E} \langle [\nabla w_h - \boldsymbol{\beta}_h] \cdot \boldsymbol{s}, [\nabla v - \boldsymbol{\eta}] \cdot \boldsymbol{s} \rangle_E \\
&\quad - \sum_{E \in \mathcal{F}_h} \frac{\gamma}{h_E} \langle [\nabla w_h - \boldsymbol{\beta}_h] \cdot \boldsymbol{s}, [\nabla v - \boldsymbol{\eta}] \cdot \boldsymbol{s} \rangle_E
\end{aligned} \tag{5.21}$$

and where the first member on the right-hand side is indeed well defined due to the piecewise regularity of $(v, \boldsymbol{\eta})$. We will denote

$$R(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta}) = R_1(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta}) + R_2(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta}). \tag{5.22}$$

Joining (5.17)–(5.21) then yields

$$\mathcal{A}_{h/2}(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta}) = \mathcal{A}_h(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta}) + R(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta}). \tag{5.23}$$

Step 3. Let $v_I \in W_h$ be the interpolant defined in Lemma 5.1, and let $\boldsymbol{\eta}_I \in \boldsymbol{V}_h$ be the piecewise linear interpolant to $\boldsymbol{\eta}$. First, we have

$$\mathcal{A}_h(w_h, \boldsymbol{\beta}_h; v_I, \boldsymbol{\eta}_I) = (f, v_I). \tag{5.24}$$

This, together with (5.17) and (5.23), gives

$$\begin{aligned}
&\mathcal{A}_{h/2}(w_{h/2} - w_h, \boldsymbol{\beta}_{h/2} - \boldsymbol{\beta}_h; v, \boldsymbol{\eta}) \\
&\quad = \mathcal{A}_{h/2}(w_{h/2}, \boldsymbol{\beta}_{h/2}; v, \boldsymbol{\eta}) - \mathcal{A}_{h/2}(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta}) \\
&\quad = \mathcal{A}_{h/2}(w_{h/2}, \boldsymbol{\beta}_{h/2}; v, \boldsymbol{\eta}) - \mathcal{A}_h(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta}) - R(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta}) \\
&\quad = (f, v - v_I) - \mathcal{A}_h(w_h, \boldsymbol{\beta}_h; v - v_I, \boldsymbol{\eta} - \boldsymbol{\eta}_I) - R(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta}).
\end{aligned} \tag{5.25}$$

Step 4. Next, we bound the last terms above. Recalling that $\mathcal{C}_{h/2}$ is a subdivision of $\mathcal{C}_h$, the Cauchy–Schwarz inequality, (4.3), and (5.15) give

$$\begin{aligned}
|R_1(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta})| &\leq 2 \left| \sum_{K \in \mathcal{C}_{h/2}} \frac{1}{\alpha h_K^2} (\nabla w_h - \boldsymbol{\beta}_h, \nabla v - \boldsymbol{\eta})_K \right| \\
&\leq 2 \left(\sum_{K \in \mathcal{C}_{h/2}} \frac{1}{h_K^2} \|\nabla w_h - \boldsymbol{\beta}_h\|_{0,K}^2 \right)^{1/2} \left(\sum_{K \in \mathcal{C}_{h/2}} \frac{1}{h_K^2} \|\nabla v - \boldsymbol{\eta}\|_{0,K}^2 \right)^{1/2} \\
&\leq C \left(\sum_{K \in \mathcal{C}_{h/2}} \frac{1}{h_K^2} \|\nabla w_h - \boldsymbol{\beta}_h\|_{0,K}^2 \right)^{1/2}.
\end{aligned} \tag{5.26}$$

Using scaling and arguments similar to those already adopted in (5.26) it can be checked that

$$|R_2(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta})| \leq C \left(\sum_{K \in \mathcal{C}_{h/2}} \frac{1}{h_K^2} \|\nabla w_h - \boldsymbol{\beta}_h\|_{0,K}^2 \right)^{1/2}. \tag{5.27}$$

Combining (5.26) and (5.27) we get

$$|R(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta})| \leq |R_1(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta})| + |R_2(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta})| \leq C\eta. \tag{5.28}$$

Step 5. Next, we expand, substitute the expression (3.11) for $\boldsymbol{q}_h$, and regroup the terms:

$$
\begin{aligned}
&(f, v - v_I) - \mathcal{A}_h(w_h, \boldsymbol{\beta}_h; v - v_I, \boldsymbol{\eta} - \boldsymbol{\eta}_I) \\
&= (f, v - v_I) - \Big\{ a(\boldsymbol{\beta}_h, \boldsymbol{\eta} - \boldsymbol{\eta}_I) - \sum_{K \in \mathcal{C}_h} \alpha h_K^2 \big(\boldsymbol{L}\boldsymbol{\beta}_h, \boldsymbol{L}(\boldsymbol{\eta} - \boldsymbol{\eta}_I)\big)_K \\
&\qquad + \sum_{K \in \mathcal{C}_h} \frac{1}{\alpha h_K^2} \big(\nabla w_h - \boldsymbol{\beta}_h - \alpha h_K^2 \boldsymbol{L}\boldsymbol{\beta}_h, \nabla(v - v_I) - (\boldsymbol{\eta} - \boldsymbol{\eta}_I) - \alpha h_K^2 \boldsymbol{L}(\boldsymbol{\eta} - \boldsymbol{\eta}_I)\big)_K \\
&\qquad + \big\langle m_{ns}(\boldsymbol{\beta}_h), [\nabla(v - v_I) - (\boldsymbol{\eta} - \boldsymbol{\eta}_I)] \cdot \boldsymbol{s} \big\rangle_{\Gamma_{\mathrm{F}}} + \big\langle [\nabla w_h - \boldsymbol{\beta}_h] \cdot \boldsymbol{s}, m_{ns}(\boldsymbol{\eta} - \boldsymbol{\eta}_I) \big\rangle_{\Gamma_{\mathrm{F}}} \\
&\qquad + \sum_{E \in \mathcal{F}_h} \frac{\gamma}{h_E} \big\langle [\nabla w_h - \boldsymbol{\beta}_h] \cdot \boldsymbol{s}, [\nabla(v - v_I) - (\boldsymbol{\eta} - \boldsymbol{\eta}_I)] \cdot \boldsymbol{s} \big\rangle_E \Big\} \\
&= (f, v - v_I) - \Big\{ a(\boldsymbol{\beta}_h, \boldsymbol{\eta} - \boldsymbol{\eta}_I) - \sum_{K \in \mathcal{C}_h} \alpha h_K^2 \big(\boldsymbol{L}\boldsymbol{\beta}_h + \boldsymbol{q}_h, \boldsymbol{L}(\boldsymbol{\eta} - \boldsymbol{\eta}_I)\big)_K \\
&\qquad + \big(\boldsymbol{q}_h, \nabla(v - v_I) - (\boldsymbol{\eta} - \boldsymbol{\eta}_I)\big) \\
&\qquad + \big\langle m_{ns}(\boldsymbol{\beta}_h), [\nabla(v - v_I) - (\boldsymbol{\eta} - \boldsymbol{\eta}_I)] \cdot \boldsymbol{s} \big\rangle_{\Gamma_{\mathrm{F}}} + \big\langle [\nabla w_h - \boldsymbol{\beta}_h] \cdot \boldsymbol{s}, m_{ns}(\boldsymbol{\eta} - \boldsymbol{\eta}_I) \big\rangle_{\Gamma_{\mathrm{F}}} \\
&\qquad + \sum_{E \in \mathcal{F}_h} \frac{\gamma}{h_E} \big\langle [\nabla w_h - \boldsymbol{\beta}_h] \cdot \boldsymbol{s}, [\nabla(v - v_I) - (\boldsymbol{\eta} - \boldsymbol{\eta}_I)] \cdot \boldsymbol{s} \big\rangle_E \Big\} \\
&= \Big\{ (f, v - v_I) - \big(\boldsymbol{q}_h, \nabla(v - v_I)\big) - \big\langle m_{ns}(\boldsymbol{\beta}_h), [\nabla(v - v_I)] \cdot \boldsymbol{s} \big\rangle_{\Gamma_{\mathrm{F}}} \\
&\qquad - \sum_{E \in \mathcal{F}_h} \frac{\gamma}{h_E} \big\langle [\nabla w_h - \boldsymbol{\beta}_h] \cdot \boldsymbol{s}, [\nabla(v - v_I)] \cdot \boldsymbol{s} \big\rangle_E \Big\} \\
&\qquad - \Big\{ a(\boldsymbol{\beta}_h, \boldsymbol{\eta} - \boldsymbol{\eta}_I) - \sum_{K \in \mathcal{C}_h} \alpha h_K^2 \big(\boldsymbol{L}\boldsymbol{\beta}_h + \boldsymbol{q}_h, \boldsymbol{L}(\boldsymbol{\eta} - \boldsymbol{\eta}_I)\big)_K - \big(\boldsymbol{q}_h, \boldsymbol{\eta} - \boldsymbol{\eta}_I\big) \\
&\qquad - \big\langle m_{ns}(\boldsymbol{\beta}_h), [\boldsymbol{\eta} - \boldsymbol{\eta}_I] \cdot \boldsymbol{s} \big\rangle_{\Gamma_{\mathrm{F}}} + \big\langle [\nabla w_h - \boldsymbol{\beta}_h] \cdot \boldsymbol{s}, m_{ns}(\boldsymbol{\eta} - \boldsymbol{\eta}_I) \big\rangle_{\Gamma_{\mathrm{F}}} \\
&\qquad - \sum_{E \in \mathcal{F}_h} \frac{\gamma}{h_E} \big\langle [\nabla w_h - \boldsymbol{\beta}_h] \cdot \boldsymbol{s}, [\boldsymbol{\eta} - \boldsymbol{\eta}_I] \cdot \boldsymbol{s} \big\rangle_E \Big\} \\
&=: A - B.
\end{aligned}
\tag{5.29}
$$

Step 6. In the part A above, integration by parts and using the fact that $v(x) = v_I(x)$ at the corner points $x \in \mathcal{V}$ yields

$$
\begin{aligned}
&(f, v - v_I) - \big(\boldsymbol{q}_h, \nabla(v - v_I)\big) - \big\langle m_{ns}(\boldsymbol{\beta}_h), [\nabla(v - v_I)] \cdot \boldsymbol{s} \big\rangle_{\Gamma_{\mathrm{F}}} \\
&\quad = (f + \operatorname{div} \boldsymbol{q}_h, v - v_I) + \left\langle \frac{\partial}{\partial \boldsymbol{s}} m_{ns}(\boldsymbol{\beta}_h) - \boldsymbol{q}_h \cdot \boldsymbol{n}, v - v_I \right\rangle_{\Gamma_{\mathrm{F}}}.
\end{aligned}
\tag{5.30}
$$

The separate terms are then estimated as follows, using the Cauchy–Schwarz inequality and Lemma 5.1:

$$
\begin{aligned}
\bigl|(f+\operatorname{div}\boldsymbol{q}_h, v-v_I)\bigr| &= \Bigl|\sum_{K\in\mathcal{C}_h}(f_h+\operatorname{div}\boldsymbol{q}_h, v-v_I)_K\Bigr| \\
&\le \Bigl(\sum_{K\in\mathcal{C}_h} h_K^4\|f+\operatorname{div}\boldsymbol{q}_h\|_{0,K}^2\Bigr)^{1/2}\Bigl(\sum_{K\in\mathcal{C}_h} h_K^{-4}\|v-v_I\|_{0,K}^2\Bigr)^{1/2} \\
&\le C\Bigl(\sum_{K\in\mathcal{C}_h} h_K^4\|f+\operatorname{div}\boldsymbol{q}_h\|_{0,K}\Bigr)^{1/2}\Bigl(\sum_{K\in\mathcal{C}_h} |v|_{2,h/2,K}^2\Bigr)^{1/2} \\
&\le C\Bigl(\sum_{K\in\mathcal{C}_h}\tilde\eta_K^2\Bigr)^{1/2} \qquad (5.31)
\end{aligned}
$$

and

$$
\begin{aligned}
&\Bigl|\Bigl\langle \frac{\partial}{\partial \boldsymbol{s}} m_{ns}(\boldsymbol{\beta}_h)-\boldsymbol{q}_h\cdot\boldsymbol{n}, v-v_I\Bigr\rangle_{\Gamma_{\mathrm{F}}}\Bigr| = \Bigl|\sum_{E\in\mathcal{F}_h}\Bigl\langle \frac{\partial}{\partial \boldsymbol{s}} m_{ns}(\boldsymbol{\beta}_h)-\boldsymbol{q}_h\cdot\boldsymbol{n}, v-v_I\Bigr\rangle_E\Bigr| \\
&\le \Bigl(\sum_{E\in\mathcal{F}_h} h_E^3\Bigl\|\frac{\partial}{\partial \boldsymbol{s}} m_{ns}(\boldsymbol{\beta}_h)-\boldsymbol{q}_h\cdot\boldsymbol{n}\Bigr\|_{0,E}^2\Bigr)^{1/2}\Bigl(\sum_{E\in\mathcal{F}_h} h_E^{-3}\|v-v_I\|_{0,E}^2\Bigr)^{1/2} \\
&\le C\Bigl(\sum_{E\in\mathcal{F}_h}\eta_{F,E}^2\Bigr)^{1/2}\Bigl(\sum_{K\in\mathcal{C}_h}|v|_{2,h/2,K}^2\Bigr)^{1/2} \\
&\le C\Bigl(\sum_{E\in\mathcal{F}_h}\eta_{F,E}^2\Bigr)^{1/2}. \qquad (5.32)
\end{aligned}
$$

The last term in A is readily estimated by scaling estimates and Lemma 5.1:

$$
\begin{aligned}
&\Bigl|\sum_{E\in\mathcal{F}_h}\frac{\gamma}{h_E}\bigl\langle[\nabla w_h-\boldsymbol{\beta}_h]\cdot\boldsymbol{s}, [\nabla(v-v_I)]\cdot\boldsymbol{s}\bigr\rangle_E\Bigr| \\
&\le \Bigl(\sum_{E\in\mathcal{F}_h} h_E^{-1}\|\nabla w_h-\boldsymbol{\beta}_h\|_{0,E}^2\Bigr)^{1/2}\Bigl(\sum_{E\in\mathcal{F}_h} h_E^{-1}\|\nabla(v-v_I)\|_{0,E}^2\Bigr)^{1/2} \\
&\le C\Bigl(\sum_{K\in\mathcal{C}_h} h_K^{-2}\|\nabla w_h-\boldsymbol{\beta}_h\|_{0,K}^2\Bigr)^{1/2}\Bigl(\sum_{E\in\mathcal{F}_h} h_E^{-3}\|v-v_I\|_{0,E}^2\Bigr)^{1/2} \\
&\le C\Bigl(\sum_{K\in\mathcal{C}_h}\tilde\eta_K^2\Bigr)^{1/2}\Bigl(\sum_{K\in\mathcal{C}_h}|v|_{2,h/2,K}^2\Bigr)^{1/2} \le C\Bigl(\sum_{K\in\mathcal{C}_h}\tilde\eta_K^2\Bigr)^{1/2}. \qquad (5.33)
\end{aligned}
$$

Collecting (5.30)–(5.33) we obtain

$$
\bigl|A\bigr| \le C\eta. \qquad (5.34)
$$

Step 7. We will now estimate the term B. The following terms are directly estimated as the similar terms above:

$$\left|\left\langle[\nabla w_h - \boldsymbol{\beta}_h]\cdot\boldsymbol{s}, m_{ns}(\boldsymbol{\eta}-\boldsymbol{\eta}_I)\right\rangle_{\Gamma_{\mathrm{F}}}\right| + \left|\sum_{E\in\mathcal{F}_h}\frac{\gamma}{h_E}\left\langle[\nabla w_h - \boldsymbol{\beta}_h]\cdot\boldsymbol{s}, [\boldsymbol{\eta}-\boldsymbol{\eta}_I]\cdot\boldsymbol{s}\right\rangle_E\right|$$
$$\leq C\eta. \tag{5.35}$$

Since $\boldsymbol{\eta}_I$ is piecewise linear, it holds that $\boldsymbol{L}\boldsymbol{\eta}_{I|K} = \mathbf{0}$. The inverse estimate then gives

$$\left|\sum_{K\in\mathcal{C}_h}\alpha h_K^2\big(\boldsymbol{L}\boldsymbol{\beta}_h + \boldsymbol{q}_h, \boldsymbol{L}(\boldsymbol{\eta}-\boldsymbol{\eta}_I)\big)_K\right| = \left|\sum_{K\in\mathcal{C}_h}\alpha h_K^2\big(\boldsymbol{L}\boldsymbol{\beta}_h + \boldsymbol{q}_h, \boldsymbol{L}\boldsymbol{\eta}\big)_K\right|$$
$$\leq C\left(\sum_{K\in\mathcal{C}_h}\alpha h_K^2\|\boldsymbol{L}\boldsymbol{\beta}_h + \boldsymbol{q}_h\|_{0,K}^2\right)^{1/2}\|\boldsymbol{\eta}\|_1$$
$$\leq C\eta, \tag{5.36}$$

where we in the last step used (5.7). The final step in estimating the term B is to integrate by parts, use the Cauchy–Schwarz inequality, interpolation estimates, and again (5.7):

$$\begin{aligned}
&\left|a(\boldsymbol{\beta}_h, \boldsymbol{\eta}-\boldsymbol{\eta}_I) - \big(\boldsymbol{q}_h, \boldsymbol{\eta}-\boldsymbol{\eta}_I\big) - \left\langle m_{ns}(\boldsymbol{\beta}_h), [\boldsymbol{\eta}-\boldsymbol{\eta}_I]\cdot\boldsymbol{s}\right\rangle_{\Gamma_{\mathrm{F}}}\right| \\
&\quad = \Bigg| -\sum_{K\in\mathcal{C}_h}(\boldsymbol{L}\boldsymbol{\beta}_h + \boldsymbol{q}_h, \boldsymbol{\eta}-\boldsymbol{\eta}_I) + \sum_{E\in\mathcal{I}_h}\langle[\![\boldsymbol{m}(\boldsymbol{\beta}_h)\boldsymbol{n}]\!], \boldsymbol{\eta}-\boldsymbol{\eta}_I\rangle_E \\
&\qquad + \langle m_{nn}(\boldsymbol{\beta}_h), [\boldsymbol{\eta}-\boldsymbol{\eta}_I]\cdot\boldsymbol{n}\rangle_{\Gamma_{\mathrm{S}}\cup\Gamma_{\mathrm{F}}}\Bigg| \\
&\quad \leq \sum_{K\in\mathcal{C}_h}\|\boldsymbol{L}\boldsymbol{\beta}_h + \boldsymbol{q}_h\|_{0,K}\|\boldsymbol{\eta}-\boldsymbol{\eta}_I\|_{0,K} + \sum_{E\in\mathcal{I}_h}\|[\![\boldsymbol{m}(\boldsymbol{\beta}_h)\boldsymbol{n}]\!]\|_{0,E}\|\boldsymbol{\eta}-\boldsymbol{\eta}_I\|_{0,E} \\
&\qquad + \sum_{E\in\mathcal{S}_h\cup\mathcal{F}_h}\|m_{nn}(\boldsymbol{\beta}_h)\|_{0,E}\|\boldsymbol{\eta}-\boldsymbol{\eta}_I\|_{0,E}
\end{aligned}$$
$$\leq C\eta. \tag{5.37}$$

Collecting (5.35)–(5.37) we obtain

$$\big|B\big| \leq C\eta. \tag{5.38}$$

Step 8. The asserted estimate now follows from (5.16), (5.25), (5.28), (5.29), (5.34), and (5.38). ☐

We also have the following lemma for the shear force.

LEMMA 5.3. *It holds that*

$$\|\boldsymbol{q}_{h/2} - \boldsymbol{q}_h\|_{-1,*} \leq C\big(|||(w_{h/2} - w_h, \boldsymbol{\beta}_{h/2} - \boldsymbol{\beta}_h)|||_{h/2} + \eta\big). \tag{5.39}$$

Proof. We start by observing that, referring to the definition (3.11) and its "$h/2$" counterpart, $\boldsymbol{q}_h$ and $\boldsymbol{q}_{h/2}$ are defined on different meshes and therefore with different h_K^2 coefficients. However, recalling that the size ratio between the two meshes is bounded, it is easy to check that an opportune splitting and the triangle inequality give

$$\|\boldsymbol{q}_{h/2}-\boldsymbol{q}_h\|^2_{-1,h} \le C\left(\sum_{K\in\mathcal{C}_{h/2}} \|\nabla(w_{h/2}-w_h)-(\boldsymbol{\beta}_{h/2}-\boldsymbol{\beta}_h)\|^2_{0,K}\right.$$

$$\left. + \sum_{K\in\mathcal{C}_h}\|\nabla w_h-\boldsymbol{\beta}_h\|^2_{0,K} + \sum_{K\in\mathcal{C}_{h/2}} h_K^2\|\boldsymbol{L}\boldsymbol{\beta}_{h/2}-\boldsymbol{L}\boldsymbol{\beta}_h\|^2_{0,K}\right). \tag{5.40}$$

The first and the last term in (5.40) can be bounded in terms of the $|||\cdot|||_{h/2}$ norm, simply using the definition (4.3) and the inverse inequality

$$h_K^2\|\boldsymbol{L}\boldsymbol{\beta}_{h/2}-\boldsymbol{L}\boldsymbol{\beta}_h\|^2_{0,K} \le C\|\boldsymbol{\beta}_{h/2}-\boldsymbol{\beta}_h\|^2_{1,K}. \tag{5.41}$$

Therefore, recalling the definition (5.1), we get

$$\|\boldsymbol{q}_{h/2}-\boldsymbol{q}_h\|_{-1,h} \le C\big(|||(w_{h/2}-w_h,\boldsymbol{\beta}_{h/2}-\boldsymbol{\beta}_h)|||_{h/2}+\eta\big). \tag{5.42}$$

The transition from the $\|\boldsymbol{q}_{h/2}-\boldsymbol{q}_h\|_{-1,h}$ norm to the $\|\boldsymbol{q}_{h/2}-\boldsymbol{q}_h\|_{-1,*}$ norm is accomplished by using the "Pitkäranta–Verfürth trick" with steps almost identical to those used in Lemma 4.5, which are therefore omitted. □

Joining Theorem 5.2 and Lemma 5.3 gives the following a posteriori upper bound for the method.

THEOREM 5.4. *It holds that*

$$|||(w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h)|||_h + \|\boldsymbol{q}-\boldsymbol{q}_h\|_{-1,*} \;\le\; C\eta. \tag{5.43}$$

Proof. Theorem 5.2 combined with Lemma 5.3 trivially gives

$$|||(w_{h/2}-w_h,\boldsymbol{\beta}_{h/2}-\boldsymbol{\beta}_h)|||_{h/2} + \|\boldsymbol{q}_{h/2}-\boldsymbol{q}_h\|_{-1,*} \le C\eta. \tag{5.44}$$

From the saturation assumption it follows that

$$\begin{aligned} &|||(w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h)|||_{h/2} + \|\boldsymbol{q}-\boldsymbol{q}_h\|_{-1,*}\\ &\quad\le \frac{1}{1-\rho}\big(|||(w_{h/2}-w_h,\boldsymbol{\beta}_{h/2}-\boldsymbol{\beta}_h)|||_{h/2} + \|\boldsymbol{q}_{h/2}-\boldsymbol{q}_h\|_{-1,*}\big), \end{aligned} \tag{5.45}$$

and hence the assertion follows from (5.44). □

5.2. Lower bound. In this section, we prove the efficiency of the error estimator. Given any edge E of the triangulation, we define ω_E as the set of all of the triangles $K\in\mathcal{C}_h$ that have E as an edge. Given any $K\in\mathcal{C}_h$, we define ω_K as the set of all of the triangles in $\mathcal{C}_h$ that share an edge with K. We then have the following lemma [8].

LEMMA 5.5. *Given any edge E of the triangulation $\mathcal{C}_h$, let $P_k(E)$ be the space of polynomials of degree at most k on E. There exists a linear operator*

$$\Pi_E \,:\, P_k(E) \longrightarrow H^2_0(\omega_E) \tag{5.46}$$

such that for all $p_k\in P_k(E)$ it holds that

$$C_1\|p_k\|^2_{0,E} \le \langle p_k,\Pi_E(p_k)\rangle_E \le \|p_k\|^2_{0,E}, \tag{5.47}$$

$$\|\Pi_E(p_k)\|_{0,\omega_E} \le C_2 h_E^{1/2}\|p_k\|_{0,E}, \tag{5.48}$$

where the positive constants C_i above depend only on k and the minimum angle of the triangles in $\mathcal{C}_h$.

Next, we define a local counterpart of the negative norm defined in (4.33) for the shear force.

$$\|\boldsymbol{r}\|_{-1,*,\omega_K} = \sup_{\substack{\boldsymbol{\eta}\in V_* \\ \boldsymbol{\eta}=\boldsymbol{0} \text{ in } \Omega\setminus\omega_K}} \frac{\langle \boldsymbol{r},\boldsymbol{\eta}\rangle}{\|\boldsymbol{\eta}\|_1}. \tag{5.49}$$

We then have the following reliability result.

THEOREM 5.6. *It holds that*

$$\eta_K \ \le\ C\big(|||(w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h)|||_{h,\omega_K} + \|\boldsymbol{q}-\boldsymbol{q}_h\|_{-1,*,\omega_K} + h_K^2\|f-f_h\|_{0,\omega_K}\big), \tag{5.50}$$

where f_h is some approximation of the load f. Here $|||\cdot|||_{h,\omega_K}$ and $\|\cdot\|_{0,\omega_K}$ represent, respectively, the standard restrictions of the norms $|||\cdot|||_h$ and $\|\cdot\|_0$ to the domain ω_K.

Proof. The proof of the theorem consists of bounding separately all of the addenda of η_K in (5.5).

Step 1. We first bound the terms of $\tilde{\eta}_K^2$ in (5.1). Considering the right-hand side of (5.50), the triangle inequality immediately shows that it is sufficient to bound the term $h_K^2\|f_h + \operatorname{div}\boldsymbol{q}_h\|_{0,K}$.

Given any $K\in\mathcal{C}_h$, let b_K indicate the standard third-order polynomial bubble function on K, scaled such that $\|b_K\|_{L^\infty(K)}=1$. Given $K\in\mathcal{C}_h$, let now $\varphi_K\in H_0^2(K)$ be defined as

$$\varphi_K = (f_h + \operatorname{div}\boldsymbol{q}_h)\, b_K^2. \tag{5.51}$$

The standard scaling arguments then easily show that

$$\|f_h + \operatorname{div}\boldsymbol{q}_h\|_{0,K}^2 \le C(f_h+\operatorname{div}\boldsymbol{q}_h,\varphi_K)_K, \tag{5.52}$$

$$\|\varphi_K\|_{0,K} \le C\|f_h+\operatorname{div}\boldsymbol{q}_h\|_{0,K}. \tag{5.53}$$

For the first term in $\tilde{\eta}_K^2$, the equilibrium equation (2.13) and integration by parts give

$$\begin{aligned} h_K^2\|f_h+\operatorname{div}\boldsymbol{q}_h\|_{0,K}^2 &\le Ch_K^2(f_h+\operatorname{div}\boldsymbol{q}_h,\varphi_K)_K \\ &= Ch_K^2\big((f+\operatorname{div}\boldsymbol{q}_h,\varphi_K)_K + (f_h-f,\varphi_K)_K\big) \\ &= Ch_K^2\big((-\operatorname{div}\boldsymbol{q}+\operatorname{div}\boldsymbol{q}_h,\varphi_K)_K + (f_h-f,\varphi_K)_K\big) \\ &= Ch_K^2\big((\boldsymbol{q}_h-\boldsymbol{q},\nabla\varphi_K)_K + (f_h-f,\varphi_K)_K\big). \end{aligned} \tag{5.54}$$

We note, in particular, that $\nabla\varphi_K\in \boldsymbol{V}_*$ and $\nabla\varphi_K=\boldsymbol{0}$ in $\Omega\setminus K$. Therefore, the duality inequality and the Cauchy–Schwarz inequality followed by the inverse inequality and the bound (5.53) lead to the estimate

$$\begin{aligned} &Ch_K^2\big((\boldsymbol{q}_h-\boldsymbol{q},\nabla\varphi_K)_K + (f_h-f,\varphi_K)\big) \\ &\quad\le C\|\boldsymbol{q}-\boldsymbol{q}_h\|_{-1,*,K}\, h_K^2\|\nabla\varphi_K\|_{1,K} + Ch_K^2\|f-f_h\|_{0,K}\|\varphi_K\|_{0,K} \\ &\quad\le C\big(\|\boldsymbol{q}-\boldsymbol{q}_h\|_{-1,*,K} + h_K^2\|f-f_h\|_{0,K}\big)\|f_h+\operatorname{div}\boldsymbol{q}_h\|_{0,K}. \end{aligned} \tag{5.55}$$

Combining now (5.54) with (5.55) gives

$$h_K^2\|f_h+\operatorname{div}\boldsymbol{q}_h\|_{0,K} \le C\big(\|\boldsymbol{q}-\boldsymbol{q}_h\|_{-1,*,K} + h_K^2\|f-f_h\|_{0,K}\big). \tag{5.56}$$

The second term of $\tilde{\eta}_K^2$ in (5.1) can be directly bounded by using the Kirchhoff condition (2.12) with the definitions (4.1)–(4.3):

$$
\begin{aligned}
h_K^{-1}\|\nabla w_h - \boldsymbol{\beta}_h\|_{0,K} &= h_K^{-1}\|\nabla(w - w_h) - (\boldsymbol{\beta} - \boldsymbol{\beta}_h)\|_{0,K}^2 \\
&\le |||(w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)|||_{h,K}. \qquad (5.57)
\end{aligned}
$$

Step 2. We next bound the terms of η_E^2 in (5.2). Given now $E \in I(K)$, an edge of the element K lying in the interior of Ω, let

$$
\boldsymbol{\varphi}_E = \Pi_E([\![\boldsymbol{m}(\boldsymbol{\beta}_h)\boldsymbol{n}]\!]), \qquad (5.58)
$$

where, with a little abuse of notation, the operator Π_E is intended as applied on each single component. Then, from (5.47) with integration by parts, it follows that

$$
\begin{aligned}
h_E^{1/2}\|[\![\boldsymbol{m}(\boldsymbol{\beta}_h)\boldsymbol{n}]\!]\|_{0,E}^2 &\le C h_E^{1/2}\langle[\![\boldsymbol{m}(\boldsymbol{\beta}_h)\boldsymbol{n}]\!], \boldsymbol{\varphi}_E\rangle_E \\
&= C h_E^{1/2}\big((\boldsymbol{L}\boldsymbol{\beta}_h, \boldsymbol{\varphi}_E)_{\omega_E} + (\boldsymbol{m}(\boldsymbol{\beta}_h), \nabla\boldsymbol{\varphi}_E)_{\omega_E}\big), \qquad (5.59)
\end{aligned}
$$

where we recall that ω_E was defined at the start of this section. Integration by parts and the equation (2.14) immediately lead to the identity

$$
(\boldsymbol{m}(\boldsymbol{\beta}), \nabla\boldsymbol{\varphi}_E)_{\omega_E} = -(\boldsymbol{L}\boldsymbol{\beta}, \boldsymbol{\varphi}_E)_{\omega_E} = (\boldsymbol{q}, \boldsymbol{\varphi}_E)_{\omega_E}, \qquad (5.60)
$$

which, applied to (5.59), gives

$$
\begin{aligned}
&h_E^{1/2}\|[\![\boldsymbol{m}(\boldsymbol{\beta}_h)\boldsymbol{n}]\!]\|_{0,E}^2 \\
&\quad \le C h_E^{1/2}\big((\boldsymbol{L}\boldsymbol{\beta}_h + \boldsymbol{q}, \boldsymbol{\varphi}_E)_{\omega_E} + (\boldsymbol{m}(\boldsymbol{\beta}_h) - \boldsymbol{m}(\boldsymbol{\beta}), \nabla\boldsymbol{\varphi}_E)_{\omega_E}\big) \\
&\quad = C h_E^{1/2}\big((\boldsymbol{L}\boldsymbol{\beta}_h + \boldsymbol{q}_h, \boldsymbol{\varphi}_E)_{\omega_E} + (\boldsymbol{q} - \boldsymbol{q}_h, \boldsymbol{\varphi}_E)_{\omega_E} \\
&\qquad + (\boldsymbol{m}(\boldsymbol{\beta}_h) - \boldsymbol{m}(\boldsymbol{\beta}), \nabla\boldsymbol{\varphi}_E)_{\omega_E}\big). \qquad (5.61)
\end{aligned}
$$

Next, we bound the three terms on the right-hand side of (5.61). For the first term, the identity (5.7), the Cauchy–Schwarz inequality, the definition (5.58), and the bound (5.48) give

$$
\begin{aligned}
&h_E^{1/2}(\boldsymbol{L}\boldsymbol{\beta}_h + \boldsymbol{q}_h, \boldsymbol{\varphi}_E)_{\omega_E} \\
&\quad \le C\left(\sum_{K \subset \omega_E} h_K^{-2}\|\nabla w_h - \boldsymbol{\beta}_h\|_{0,K}^2\right)^{1/2} \|[\![\boldsymbol{m}(\boldsymbol{\beta}_h)\boldsymbol{n}]\!]\|_{0,E} \\
&\quad \le C\, |||(w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)|||_{h,\omega_E}\|[\![\boldsymbol{m}(\boldsymbol{\beta}_h)\boldsymbol{n}]\!]\|_{0,E}. \qquad (5.62)
\end{aligned}
$$

For the second term on the right-hand side of (5.61), we note that $\boldsymbol{\varphi}_E \in \boldsymbol{V}_*$ and $\boldsymbol{\varphi}_E = \boldsymbol{0}$ in $\Omega\backslash\omega_E$. Therefore, the duality inequality and the definition (5.58) combined with the bound (5.48) give

$$
\begin{aligned}
h_E^{1/2}(\boldsymbol{q} - \boldsymbol{q}_h, \boldsymbol{\varphi}_E)_{\omega_E} &\le h_E^{1/2}\|\boldsymbol{q} - \boldsymbol{q}_h\|_{-1,*,\omega_E}\|\boldsymbol{\varphi}_E\|_{1,\omega_E} \\
&\le C\|\boldsymbol{q} - \boldsymbol{q}_h\|_{-1,*,\omega_E}\|[\![\boldsymbol{m}(\boldsymbol{\beta}_h)\boldsymbol{n}]\!]\|_{0,E}. \qquad (5.63)
\end{aligned}
$$

For the third term of (5.61), the Cauchy–Schwarz inequality, then the inverse inequality, and finally (5.58) combined with the bound (5.48) lead to the estimate

$$
\begin{aligned}
&h_E^{1/2}(\boldsymbol{m}(\boldsymbol{\beta}_h) - \boldsymbol{m}(\boldsymbol{\beta}), \nabla\boldsymbol{\varphi}_E)_{\omega_E} \le C\|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_{1,\omega_E}\, h_K^{-1/2}\|\boldsymbol{\varphi}_E\|_{0,\omega_E} \\
&\quad \le C\|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_{1,\omega_E}\|[\![\boldsymbol{m}(\boldsymbol{\beta}_h)\boldsymbol{n}]\!]\|_{0,E}. \qquad (5.64)
\end{aligned}
$$

Now, by combining (5.62), (5.63), and (5.64) with (5.61) it follows that

$$h_E^{1/2}\|[\![\boldsymbol{m}(\boldsymbol{\beta}_h)\boldsymbol{n}]\!]\|_{0,E} \leq C\big(|||(w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h)|||_{h,\omega_E} + \|\boldsymbol{q}-\boldsymbol{q}_h\|_{-1,*,\omega_E}\big). \tag{5.65}$$

The remaining term of η_E^2 is bounded with similar arguments; with the notation

$$\varphi_E = \Pi_E([\![\boldsymbol{q}_h\cdot\boldsymbol{n}]\!]), \tag{5.66}$$

the identity

$$-(\operatorname{div}\boldsymbol{q},\varphi_E)_{\omega_E} = (\boldsymbol{q},\nabla\varphi_E)_{\omega_E} \tag{5.67}$$

with (5.54) implies

$$\begin{aligned} h_E^{1/2}\|[\![\boldsymbol{q}\cdot\boldsymbol{n}]\!]\|_{0,E}^2 &\leq Ch_E^{1/2}\langle[\![\boldsymbol{q}\cdot\boldsymbol{n}]\!],\varphi_E\rangle_E \\ &\leq Ch_E^{1/2}\big((f-f_h,\varphi_E)_{\omega_E} + (\boldsymbol{q}_h-\boldsymbol{q},\nabla\varphi_E)_{\omega_E}\big). \end{aligned} \tag{5.68}$$

Finally, we note that $\nabla\varphi_E \in \boldsymbol{V}_*$ and $\nabla\varphi_E = \boldsymbol{0}$ in $\Omega\backslash\omega_E$. Therefore,

$$h_E^{3/2}\|[\![\boldsymbol{q}_h\cdot\boldsymbol{n}]\!]\|_{0,E} \leq C\big(\|\boldsymbol{q}-\boldsymbol{q}_h\|_{-1,*,\omega_E} + h_K^2\|f-f_h\|_{0,\omega_E}\big). \tag{5.69}$$

Step 3. Third, we bound the only term of $\eta_{S,E}^2$ in (5.3) which appears in $\eta_{F,E}^2$ as well. Given now a triangulation edge E in $S(K)\cup F(K)$, let

$$\varphi_E = \Pi_E(m_{nn}(\boldsymbol{\beta}_h)). \tag{5.70}$$

Due to (5.47) and (2.19), integration by parts gives (here $\boldsymbol{\nabla}$ denotes the tensor-valued gradient applied to a vector-valued function)

$$\begin{aligned} h_E^{1/2}\|m_{nn}(\boldsymbol{\beta}_h)\|_{0,E}^2 &\leq h_E^{1/2}\langle m_{nn}(\boldsymbol{\beta}_h-\boldsymbol{\beta}),\varphi_E\rangle_E \\ &= h_E^{1/2}\langle \boldsymbol{m}_n(\boldsymbol{\beta}_h-\boldsymbol{\beta}),\varphi_E\boldsymbol{n}\rangle_E \\ &= h_E^{1/2}\big((\boldsymbol{m}(\boldsymbol{\beta}_h-\boldsymbol{\beta}),\boldsymbol{\nabla}(\varphi_E\boldsymbol{n}))_{\omega_E} + (\boldsymbol{L}(\boldsymbol{\beta}_h-\boldsymbol{\beta}),\varphi_E\boldsymbol{n})_{\omega_E}\big), \end{aligned} \tag{5.71}$$

where $\boldsymbol{n}$ is, as usual, the chosen normal unit vector to E. For the first term, using the Cauchy–Schwarz inequality, then the inverse inequality, and finally the bound (5.48), we easily get

$$\begin{aligned} h_E^{1/2}(\boldsymbol{m}(\boldsymbol{\beta}_h-\boldsymbol{\beta}),\boldsymbol{\nabla}(\varphi_E\boldsymbol{n}))_{\omega_E} &\leq h_E^{1/2}\|\boldsymbol{\beta}-\boldsymbol{\beta}_h\|_{1,\omega_E}\|\boldsymbol{\nabla}(\varphi_E\boldsymbol{n})\|_{0,\omega_E} \\ &\leq C\|\boldsymbol{\beta}-\boldsymbol{\beta}_h\|_{1,\omega_E}\|m_{nn}(\boldsymbol{\beta}_h)\|_{0,E}. \end{aligned} \tag{5.72}$$

For the second term in (5.71), recalling (2.14) we have

$$\begin{aligned} &h_E^{1/2}(\boldsymbol{L}(\boldsymbol{\beta}_h-\boldsymbol{\beta}),\varphi_E\boldsymbol{n})_{\omega_E} \\ &\quad = h_E^{1/2}(\boldsymbol{L}\boldsymbol{\beta}_h+\boldsymbol{q}_h,\varphi_E\boldsymbol{n})_{\omega_E} + h_E^{1/2}(\boldsymbol{q}-\boldsymbol{q}_h,\varphi_E\boldsymbol{n})_{\omega_E}. \end{aligned} \tag{5.73}$$

Observing now that $\varphi_E\boldsymbol{n} \in \boldsymbol{V}_*$ and $\varphi_E\boldsymbol{n} = \boldsymbol{0}$ in $\Omega\backslash\omega_E$, the two terms on the right-hand side of (5.73) can be bounded with the same arguments used above, respectively, in (5.62) and (5.63). Omitting the details, we therefore get

$$\begin{aligned} h_E^{1/2}(\boldsymbol{L}(\boldsymbol{\beta}_h-\boldsymbol{\beta}),\varphi_E\boldsymbol{n})_{\omega_E} &\leq C\big(|||(w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h)|||_{h,\omega_E} \\ &\quad + \|\boldsymbol{q}-\boldsymbol{q}_h\|_{-1,*,\omega_E}\big)\|m_{nn}(\boldsymbol{\beta}_h)\|_{0,E}. \end{aligned} \tag{5.74}$$

From (5.71), (5.72), and (5.74) we get

$$h_E^{1/2}\|m_{nn}(\boldsymbol{\beta}_h)\|_{0,E} \le C\big(|||(w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h)|||_{h,\omega_E} + \|\boldsymbol{q}-\boldsymbol{q}_h\|_{-1,*,\omega_E}\big). \tag{5.75}$$

Step 4. Finally, we bound the last term of $\eta^2_{F,E}$ in (5.4). Given now a triangulation edge E in $F(K)$, let

$$\varphi_E = \Pi_E\left(\frac{\partial}{\partial s} m_{ns}(\boldsymbol{\beta}_h) - \boldsymbol{q}_h\cdot\boldsymbol{n}\right). \tag{5.76}$$

Using (5.47) and recalling (2.17), we obtain

$$\begin{aligned} &h_E^{3/2}\left\|\frac{\partial}{\partial s} m_{ns}(\boldsymbol{\beta}_h) - \boldsymbol{q}_h\cdot\boldsymbol{n}\right\|^2_{0,E}\\ &\quad\le h_E^{3/2}\left(\left\langle\frac{\partial}{\partial s} m_{ns}(\boldsymbol{\beta}_h-\boldsymbol{\beta}),\varphi_E\right\rangle_E + \langle[\boldsymbol{q}-\boldsymbol{q}_h]\cdot\boldsymbol{n},\varphi_E\rangle_E\right). \end{aligned} \tag{5.77}$$

For the first term, integration by parts on the edge and simple algebra give

$$\begin{aligned} h_E^{3/2}\left\langle\frac{\partial}{\partial s} m_{ns}(\boldsymbol{\beta}_h-\boldsymbol{\beta}),\varphi_E\right\rangle_E &= h_E^{3/2}\langle m_{ns}(\boldsymbol{\beta}-\boldsymbol{\beta}_h),\nabla\varphi_E\cdot\boldsymbol{s}\rangle_E\\ &= h_E^{3/2}\big(\langle\boldsymbol{m}(\boldsymbol{\beta}-\boldsymbol{\beta}_h)\boldsymbol{n},\nabla\varphi_E\rangle_E - \langle m_{nn}(\boldsymbol{\beta}-\boldsymbol{\beta}_h),\nabla\varphi_E\cdot\boldsymbol{n}\rangle_E\big). \end{aligned} \tag{5.78}$$

Using again integration by parts, the first term in (5.78) can be written as

$$\begin{aligned} &h_E^{3/2}\langle\boldsymbol{m}(\boldsymbol{\beta}-\boldsymbol{\beta}_h)\boldsymbol{n},\nabla\varphi_E\rangle_E\\ &\quad= h_E^{3/2}\big(\boldsymbol{L}(\boldsymbol{\beta}-\boldsymbol{\beta}_h),\nabla\varphi_E)_{\omega_E} + (\boldsymbol{m}(\boldsymbol{\beta}-\boldsymbol{\beta}_h),\boldsymbol{\nabla}\nabla\varphi_E)_{\omega_E}\big). \end{aligned} \tag{5.79}$$

The second term in (5.77), again due to integration by parts and recalling (2.13), is instead equivalent to

$$\begin{aligned} h_E^{3/2}\langle[\boldsymbol{q}-\boldsymbol{q}_h]\cdot\boldsymbol{n},\varphi_E\rangle_E &= h_E^{3/2}\big(\boldsymbol{q}-\boldsymbol{q}_h,\nabla\varphi_E)_{\omega_E}\\ &\quad-(f_h+\operatorname{div}\boldsymbol{q}_h,\varphi_E)_{\omega_E} - (f-f_h,\varphi_E)_{\omega_E}\big). \end{aligned} \tag{5.80}$$

For the first term, due to (2.14) and (3.11), we now have

$$\begin{aligned} &h_E^{3/2}(\boldsymbol{q}-\boldsymbol{q}_h,\nabla\varphi_E)_{\omega_E}\\ &\quad= h_E^{3/2}\big(\boldsymbol{L}(\boldsymbol{\beta}_h-\boldsymbol{\beta}),\nabla\varphi_E)_{\omega_E} - \frac{1}{\alpha h^2_{\omega_E}}\big(\nabla w_h-\boldsymbol{\beta}_h,\nabla\varphi_E\big)_{\omega_E}, \end{aligned} \tag{5.81}$$

where h_{ω_E} is the size of the triangle ω_E. Combining all of the identities from (5.77) to (5.81), it follows that

$$\begin{aligned} &h_E^{3/2}\left\|\frac{\partial}{\partial s} m_{ns}(\boldsymbol{\beta}_h) - \boldsymbol{q}_h\cdot\boldsymbol{n}\right\|^2_{0,E}\\ &\quad\le h_E^{3/2}\Bigg((\boldsymbol{m}(\boldsymbol{\beta}-\boldsymbol{\beta}_h),\boldsymbol{\nabla}\nabla\varphi_E)_{\omega_E} - (m_{nn}(\boldsymbol{\beta}-\boldsymbol{\beta}_h),\nabla\varphi_E\cdot\boldsymbol{n}\rangle_E\\ &\qquad-\frac{1}{\alpha h^2_{\omega_E}}(\nabla w_h-\boldsymbol{\beta}_h,\nabla\varphi_E)_{\omega_E} - (f_h+\operatorname{div}\boldsymbol{q}_h,\varphi_E)_{\omega_E}\\ &\qquad-(f-f_h,\varphi_E)_{\omega_E}\Bigg). \end{aligned} \tag{5.82}$$

For the second term on the right-hand side of (5.82), recalling (2.17), using the Cauchy–Schwarz inequality and the bound (5.75), we have

$$\begin{aligned} h_E^{3/2}\langle m_{nn}(\boldsymbol{\beta}-\boldsymbol{\beta}_h), \nabla\varphi_E\cdot\boldsymbol{n}\rangle_E &\le h_E^{1/2}\|m_{nn}(\boldsymbol{\beta}_h)\|_{0,E}\, h_E\|\nabla\varphi_E\|_{0,E} \\ &\le C\big(|||(w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h)|||_{h,\omega_E} + \|\boldsymbol{q}-\boldsymbol{q}_h\|_{-1,*,\omega_E}\big) h_E\|\nabla\varphi_E\|_{0,E}, \end{aligned} \tag{5.83}$$

which, using the inverse inequality and the bound (5.48), gives

$$\begin{aligned} h_E^{3/2}\langle m_{nn}(\boldsymbol{\beta}-\boldsymbol{\beta}_h), \nabla\varphi_E\cdot\boldsymbol{n}\rangle_E &\le C\big(|||(w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h)|||_{h,\omega_E} \\ &\quad + \|\boldsymbol{q}-\boldsymbol{q}_h\|_{-1,*,\omega_E}\big)\left\|\frac{\partial}{\partial s}m_{ns}(\boldsymbol{\beta}_h) - \boldsymbol{q}_h\cdot\boldsymbol{n}\right\|_{0,E}. \end{aligned} \tag{5.84}$$

The remaining terms on the right-hand side of (5.82) can all be bounded using the Cauchy–Schwarz inequality, the inverse inequality, and the bounds (5.56), (5.48) as already shown for the similar previous cases. Without showing all of the details, we finally get

$$\begin{aligned} &h_E^{3/2}\left\|\frac{\partial}{\partial s}m_{ns}(\boldsymbol{\beta}_h) - \boldsymbol{q}_h\cdot\boldsymbol{n}\right\|_{0,E}^2 \\ &\quad\le C\big(|||(w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h)|||_{h,\omega_E} + \|\boldsymbol{q}-\boldsymbol{q}_h\|_{-1,*,\omega_E} \\ &\qquad + h_K^2\|f-f_h\|_{0,K}\big)\left\|\frac{\partial}{\partial s}m_{ns}(\boldsymbol{\beta}_h) - \boldsymbol{q}_h\cdot\boldsymbol{n}\right\|_{0,E} \end{aligned} \tag{5.85}$$

or, trivially,

$$\begin{aligned} &h_E^{3/2}\left\|\frac{\partial}{\partial s}m_{ns}(\boldsymbol{\beta}_h) - \boldsymbol{q}_h\cdot\boldsymbol{n}\right\|_{0,E} \\ &\quad\le C\big(|||(w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h)|||_{h,\omega_E} + \|\boldsymbol{q}-\boldsymbol{q}_h\|_{-1,*,\omega_E} + h_K^2\|f-f_h\|_{0,K}\big). \end{aligned} \tag{5.86}$$

Recalling now the definitions for η_K in (5.1) and the local negative norm in (5.49), the proposition is proved. □

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