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Abstract A local a posteriori error indicator for the well known Morley element for the Kirchhoff plate bending problem is presented. The error indicator is proven to be both reliable and efficient. The technique applied is general and it is shown to have also other applications.

Mathematics Subject Classification 74K20

1 Introduction

We consider the classical Kirchhoff plate bending problem. The natural variational space for this biharmonic problem is the second order Sobolev space. Thus, a conforming finite element approximation requires globally C^1-continuous elements which imply a high polynomial order. As a consequence, nonconforming elements are a widely adopted choice. A well known finite element for the Kirchhoff problem is the Morley element which uses just second order piecewise polynomial functions (see for example [9,14]).

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In the present paper, we derive a reliable and efficient a posteriori error estimator for the Morley element. Our analysis initially takes the steps from the pioneering work on a posteriori estimates for nonconforming elements [11]. In particular, the error is divided into a regular and irregular part using a new Helmholtz type decomposition.

On the other hand, as underlined for example in [5,12], a key property in this approach is the existence of a discrete space $\tilde{V}_h$, such that:

1. $\tilde{V}_h$ is contained in the adopted finite element space,
2. $\tilde{V}_h$ is contained in the variational space of the continuous formulation,
3. $\tilde{V}_h$ satisfies some minimal approximation properties.

In the case of the Morley element, the previous conditions do not hold. In the present work, this difficulty is dealt with simply making a different use of the exact and discrete variational identities. An approach similar to ours has turned out to be independently applied for second order problems in [7].

The paper is organized as follows. In Sect. 2 we briefly review the Kirchhoff plate bending problem and its Morley finite element approximation. The following, and the main, section is divided into three parts: In the first part we introduce some preliminaries, namely, two interpolation operators and a Helmholtz type decomposition, while in the following two subsections we prove, respectively, upper and lower error bounds for our local error indicator.

We finally observe that the principle applied here is general; it could be applied for example to obtain a posteriori error estimates for nonconforming elements without relying on the aforementioned space $\tilde{V}_h$ (see Remark 1).

For the convenience of the reader, a set of differential operators and the corresponding formula for integration by parts, widely used throughout the text, are recalled in the Appendix.

2 The Kirchhoff plate bending problem

We consider the bending problem of an isotropic linearly elastic plate. Let the undeformed plate midsurface be described by a given convex polygonal domain $\Omega \subset \mathbb{R}^2$. For simplicity, the plate is considered to be clamped on its boundary Γ. A transverse load $F = Gt^3 f$ is applied, where t is the thickness of the plate and G the shear modulus for the material.

2.1 The continuous variational formulation

Let the Sobolev space for the deflection be

$$W = H_0^2(\Omega)\,. \tag{2.1}$$

Let also the bilinear form for the problem be

$$a(u,v) = (\boldsymbol{E}\,\boldsymbol{\varepsilon}(\nabla u), \boldsymbol{\varepsilon}(\nabla v))_\Omega \quad \forall u, v \in W\,, \tag{2.2}$$

where the parentheses $(\cdot,\cdot)_\Omega$ above indicate the $L^2(\Omega)$ scalar product, and the fourth order positive definite elasticity tensor $\boldsymbol{E}$ is defined by

$$\boldsymbol{E}\,\boldsymbol{\sigma} = \frac{E}{12(1+\nu)}\Big(\boldsymbol{\sigma} + \frac{\nu}{1-\nu}\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I}\Big) \quad \forall \boldsymbol{\sigma} \in \mathbb{R}^{2\times 2}\,, \tag{2.3}$$

with E, ν the being Young modulus and the Poisson ratio for the material, respectively.

Then, following the Kirchhoff plate bending model, the deflection w of the plate can be found as the solution of the following variational problem:

Find $w \in W$ such that

$$a(w,v) = (f,v) \quad \forall v \in W\,. \tag{2.4}$$

2.2 The Morley finite element formulation

Let a regular family of triangular meshes $\{\mathcal{C}_h\}_h$ on Ω be given. In the sequel, we will indicate by h_K the diameter of each element K, while h will indicate the maximum size of all the elements in the mesh. Also, we will indicate with $\mathcal{E}_h$ the set of all the edges and with $\mathcal{E}'_h$ its subset comprising only the internal edges. Given any $e \in \mathcal{E}_h$, the scalar h_e will represent its length. Finally, to each edge $e \in \mathcal{E}_h$ we associate a normal unit vector $\boldsymbol{n}_e$ and a tangent unit vector $\boldsymbol{s}_e$, the latter given by a counter clockwise 90° rotation of $\boldsymbol{n}_e$; the choice of the particular normal is arbitrary, but is considered to be fixed once and for all.

In the sequel, we will also need the definition of jumps: Let K_+ and K_- be any two triangles with an edge e in common, such that the unit outward normal to K_- at e corresponds to $\boldsymbol{n}_e$. Furthermore, given a piecewise continuous scalar function v on Ω, call v^+ (respectively v^-) the trace $v|_{K_+}$ (respectively $v|_{K_-}$) on e. Then, the jump of v across e is a scalar function living on e, given by

$$[\![v]\!] = v^+ - v^-\,. \tag{2.5}$$

For a vector valued function also the jump is vector valued, defined as above component by component. Finally, the jump on boundary edges is simply given by the trace of the function on each edge.

We can now introduce the discrete Morley space

$$W_h = \left\{ v \in M_{2,h} \;\mid\; \int_e [\![\nabla v \cdot \boldsymbol{n}_e]\!] = 0 \quad \forall e \in \mathcal{E}_h \right\}, \tag{2.6}$$

where $M_{2,h}$ is the space of the second order piecewise polynomial functions on $\mathcal{C}_h$ which are continuous at the vertices of all the internal triangles and zero at all the triangle vertices on the boundary.

A set of degrees of freedom for this finite element space is given by the nodal values at the internal vertices of the triangulation plus the value of $\nabla v \cdot \boldsymbol{n}_e$ at the midpoints of the internal edges.

The finite element approximation of the problem (2.4) with the Morley element reads:

Method *Find $w_h \in W_h$ such that*

$$a_h(w_h, v_h) = (f, v_h) \quad \forall v_h \in W_h\,, \tag{2.7}$$

where

$$a_h(u_h, v_h) = \sum_{K \in \mathcal{C}_h} (\boldsymbol{E}\,\boldsymbol{\varepsilon}(\nabla u_h), \boldsymbol{\varepsilon}(\nabla v_h))_K \quad \forall u_h, v_h \in W_h\,. \tag{2.8}$$

The bilinear form a_h is definite positive on the space W_h, therefore there is a unique solution to the problem (2.7).

Let, here and in the sequel, C indicate a generic positive constant independent of h, possibly different at each occurrence. Introducing the discrete norm

$$\begin{aligned} |||v|||_h^2 = \sum_{K \in \mathcal{C}_h} |v|^2_{H^2(K)} + \sum_{e \in \mathcal{E}_h} h_e^{-3} \|[\![v]\!]\|^2_{L^2(e)} \\ + \sum_{e \in \mathcal{E}_h} h_e^{-1} \|[\![\nabla v \cdot \boldsymbol{n}_e]\!]\|^2_{L^2(e)} \end{aligned} \tag{2.9}$$

on $W_h + H^2$, the following a priori error estimate holds (see [13,16]).

Proposition 1 *Let w be the solution of the problem* (2.4) *and w_h the solution of the problem* (2.7). *Then it holds*

$$|||w - w_h|||_h \le Ch \left(|w|_{H^3(\Omega)} + h\|f\|_{L^2(\Omega)} \right). \tag{2.10}$$

3 A posteriori error estimates

In this section we derive reliable and efficient a posteriori error estimates for the Morley element. After some preliminaries, we will show the reliability and efficiency, up to a higher order load approximation term, of the error estimator

$$\eta = \Big(\sum_{K \in \mathcal{C}_h} \eta_K^2 \Big)^{1/2}, \tag{3.1}$$

where

$$\eta_K^2 = h_K^4 \|f_h\|^2_{L^2(K)} + \sum_{e\in\partial K} c_e h_e^{-3} \|[\![w_h]\!]\|^2_{L^2(e)} + \sum_{e\in\partial K} c_e h_e^{-1} \|[\![\nabla w_h \cdot \boldsymbol{n}_e]\!]\|^2_{L^2(e)} \tag{3.2}$$

and f_h is some approximation of f, while $c_e = 1/2$ if $e \in \mathcal{E}_h'$ and 1 otherwise.

Other a posteriori error estimates for Kirchhoff finite elements can be found for instance in [2,8].

Remark 1 As noted in the Introduction, the following a posteriori analysis does not rely on the existence of a subspace $\tilde{V}_h \subset W \cap W_h$ having some minimal approximation properties. The same idea can be generalized to other elements as well. One example is the nonparametric nonconforming quadrilateral element of [15] which does not satisfy such a property. In [12], the authors develop an a posteriori analysis for the element of [15], but are forced to add artificial bulb functions to the method in order to recover the existence of a space $\tilde{V}_h$. As the authors underline, a different proving technique should be found. Following the same path that follows, it is easy to check that reliable and efficient a posteriori error estimates can be obtained for the nonparametric element of [15] in a straightforward—at least in the affine mapped quadrilaterals case—manner, and without the additional bulb functions. For a similar approach to such elements see also [7].

3.1 Preliminaries

We start by introducing the following interpolant:

Definition 1 Given any $v \in H^2(\Omega)$, we indicate with v_I the only function in W_h such that

$$v_I(p) = v(p) \quad \text{for every vertex } p \text{ of the mesh } \mathcal{C}_h, \tag{3.3}$$

$$\int_e (\nabla v - \nabla v_I) \cdot \boldsymbol{n}_e = 0 \quad \forall e \in \mathcal{E}_h. \tag{3.4}$$

We note that it holds

$$\|v - v_I\|_{L^2(K)} \leq C h_K^2 |v|_{H^2(K)} \quad \forall K \in \mathcal{C}_h,\ v \in H^2(\Omega). \tag{3.5}$$

Moreover, a simple integration by parts along the edges gives

$$\int_e (\nabla v - \nabla v_I) \cdot \boldsymbol{s}_e = 0 \quad \forall e \in \mathcal{E}_h, \tag{3.6}$$

which will be also needed in the sequel.

Let now Π_C indicate the classical Clément interpolation operator from $H^1(\Omega)$ to the space of continuous piecewise linear functions (see for instance [3,4,10]). Given any $v \in H^1(\Omega)$, the following properties are well known:

$$\|v - \Pi_C(v)\|_{H^m(K)} \leq Ch_K^{1-m} \|v\|_{H^1(\tilde{K})} \quad \forall K \in \mathcal{C}_h, m = 0, 1, \tag{3.7}$$

$$\|v - \Pi_C(v)\|_{L^2(e)} \leq Ch_K^{1/2} \|v\|_{H^1(\tilde{K})} \quad \forall e \in \partial K, K \in \mathcal{C}_h, \tag{3.8}$$

where $\tilde{K}$ indicates the set of all the triangles of $\mathcal{C}_h$ with a nonempty intersection with $K \in \mathcal{C}_h$.

We also introduce the following operator: Given any edge $e \in \mathcal{E}_h$, let B_e indicate the globally continuous, piecewise second order polynomial function which is equal to 1 at the midpoint of e and zero at all the other vertices and edge midpoints of the mesh. Moreover, let V_B indicate the discrete space given by the span of all B_e, $e \in \mathcal{E}_h$. We then introduce the operator Π_B defined by

$$\Pi_B : H^1(\Omega) \to V_B \,, \qquad \int_e (v - \Pi_B(v)) = 0 \quad \forall e \in \mathcal{E}_h \,. \tag{3.9}$$

Using the definition (3.9), inverse inequalities and the Agmon inequality (see [1]), it is easy to check that Π_B satisfies the following property for all $v \in H^1(\Omega)$

$$\|\Pi_B(v)\|_{H^m(K)} \leq Ch_K^{1-m} \left(h_K^{-1} \|v\|_{L^2(K)} + |v|_{H^1(K)} \right) \quad \forall K \in \mathcal{C}_h. \tag{3.10}$$

We are now able to introduce our second interpolant:

Definition 2 Given any $v \in H^1(\Omega)$, we indicate with v_{II} the continuous piecewise polynomial function of second order given by

$$v_{II} = \Pi_C(v) + \Pi_B(v - \Pi_C(v)). \tag{3.11}$$

Using the properties (3.7), (3.8) and (3.10) we easily get

$$\|v - v_{II}\|_{H^m(K)} \leq Ch_K^{1-m} \|v\|_{H^1(\tilde{K})} \quad \forall K \in \mathcal{C}_h \,, m = 0, 1 \tag{3.12}$$

for all $v \in H^1(\Omega)$.

Moreover, directly from (3.9) and Definition 2, it follows

$$\int_e (v - v_{II}) = 0 \quad \forall e \in \mathcal{E}_h \,, v \in H^1(\Omega). \tag{3.13}$$

We finally need the following Helmholtz decomposition for second order tensors with components in $L^2(\Omega)$. Let in the sequel the space $\tilde{H}^m(\Omega)$, $m \in \mathbb{N}$, indicate the quotient space of $H^m(\Omega)$ where the seminorm $|\cdot|_{H^m(\Omega)}$ is null.

Moreover, let $L^2_0(\Omega)$ indicate as usual the space of functions in $L^2(\Omega)$ with zero average over Ω. The differential operators used below are defined in the Appendix.

Lemma 1 *Let $\boldsymbol{\sigma}$ be a second order tensor field in $L^2(\Omega;\mathbb{R}^{2\times 2})$. Then, there exist $\psi \in H^2_0(\Omega)$, $\rho \in L^2_0(\Omega)$ and $\boldsymbol{\phi} \in [\tilde{H}^1(\Omega)]^2$ such that*

$$\boldsymbol{\sigma} = \boldsymbol{E}\,\boldsymbol{\varepsilon}(\nabla\psi) + \boldsymbol{\rho} + \mathbf{Curl}\,\boldsymbol{\phi}\,, \tag{3.14}$$

where the second order tensor

$$\boldsymbol{\rho} = \begin{pmatrix} 0 & -\rho \\ \rho & 0 \end{pmatrix}. \tag{3.15}$$

Moreover,

$$\|\psi\|_{H^2(\Omega)} + \|\rho\|_{L^2(\Omega)} + \|\boldsymbol{\phi}\|_{H^1(\Omega)} \leq C\|\boldsymbol{\sigma}\|_{L^2(\Omega)}\,. \tag{3.16}$$

Proof The proof will be shown briefly. Let ψ be the solution of the following problem:

Find $\psi \in H^2_0(\Omega)$ such that

$$(\boldsymbol{E}\,\boldsymbol{\varepsilon}(\nabla\psi), \boldsymbol{\varepsilon}(\nabla v)) = (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\nabla v)) \quad \forall v \in H^2_0(\Omega)\,. \tag{3.17}$$

Note that the problem above has a unique solution due to the coercivity of the considered bilinear forms on the respective spaces. From (3.17) it immediately follows

$$\operatorname{div}\mathbf{div}\,(\boldsymbol{\sigma} - \boldsymbol{E}\,\boldsymbol{\varepsilon}(\nabla\psi)) = 0 \tag{3.18}$$

in the distributional sense. As a consequence of (3.18), there exists a scalar function $\rho \in L^2_0(\Omega)$ such that

$$\mathbf{div}\,(\boldsymbol{\sigma} - \boldsymbol{E}\,\boldsymbol{\varepsilon}(\nabla\psi)) = \mathbf{curl}\,\rho\,, \tag{3.19}$$

$$\|\rho\|_{L^2(\Omega)} \leq C\|\boldsymbol{\sigma}\|_{L^2(\Omega)} + \|\psi\|_{H^2(\Omega)}. \tag{3.20}$$

Now we observe that, by definition,

$$\mathbf{curl}\,\rho = \mathbf{div}\,\boldsymbol{\rho}, \tag{3.21}$$

which, recalling (3.19), implies

$$\mathbf{div}\,(\boldsymbol{\sigma} - \boldsymbol{E}\,\boldsymbol{\varepsilon}(\nabla\psi) - \boldsymbol{\rho}) = \mathbf{0}\,. \tag{3.22}$$

Identity (3.22) implies the existence of a vector function $\boldsymbol{\phi} \in [\tilde{H}^1(\Omega)]^2$ such that

$$\boldsymbol{\sigma} - \boldsymbol{E}\,\boldsymbol{\varepsilon}(\nabla\psi) - \boldsymbol{\rho} = \mathbf{Curl}\,\boldsymbol{\phi}, \tag{3.23}$$

$$\|\boldsymbol{\phi}\|_{H^1(\Omega)} \leq C\|\boldsymbol{\sigma} - \boldsymbol{E}\,\boldsymbol{\varepsilon}(\nabla\psi) - \boldsymbol{\rho}\|_{L^2(\Omega)}\,. \tag{3.24}$$

The second part of the proposition follows from the stability of the problem (3.17), and the bounds (3.20), (3.24). □

Remark 2 We note that, due to the boundary conditions required on ψ, we cannot obtain a similar result simply combining Lemma 3.1 in [6] with the classical Helmholtz decomposition.

3.2 Reliability

We have the following lower bound for the error estimator:

Theorem 1 *Let w be the solution of the problem* (2.4) *and w_h the solution of the problem* (2.7)*. Then it holds*

$$|||w - w_h|||_h \leq C \left(\sum_{K \in \mathcal{C}_h} \eta_K^2 + \sum_{K \in \mathcal{C}_h} h_K^4 \|f - f_h\|^2_{L^2(K)} \right)^{1/2} . \tag{3.25}$$

Proof Recalling that $w \in H^2_0(\Omega)$, it immediately follows

$$\begin{aligned} |||w - w_h|||_h^2 = & \sum_{K \in \mathcal{C}_h} |w - w_h|^2_{H^2(K)} + \sum_{e \in \mathcal{E}_h} h_e^{-3} \|[\![w_h]\!]\|^2_{L^2(e)} \\ & + \sum_{e \in \mathcal{E}_h} h_e^{-1} \|[\![\nabla w_h \cdot \boldsymbol{n}_e]\!]\|^2_{L^2(e)} . \end{aligned} \tag{3.26}$$

Therefore, due to the definition of η_K in (3.2) and the norm (2.9), what needs to be proved is

$$\sum_{K \in \mathcal{C}_h} |w - w_h|^2_{H^2(K)} \leq C \left(\sum_{K \in \mathcal{C}_h} \eta_K^2 + \sum_{K \in \mathcal{C}_h} h_K^4 \|f - f_h\|^2_{L^2(K)} \right) . \tag{3.27}$$

For convenience, we divide the proof of (3.27) into three steps.
Step 1. Let in the sequel e_h represent the error $w - w_h$. First due to the positive definiteness and symmetry of the fourth order tensor $\boldsymbol{E}$, then applying Lemma 1 to the tensor field $\boldsymbol{E}\,\boldsymbol{\varepsilon}(\nabla e_h)$, we have

$$\begin{aligned} \sum_{K \in \mathcal{C}_h} |e_h|^2_{H^2(K)} & \leq C a_h(e_h, e_h) \\ & = \sum_{K \in \mathcal{C}_h} (\boldsymbol{\varepsilon}(\nabla e_h), \boldsymbol{E}\,\boldsymbol{\varepsilon}(\nabla e_h))_K = T_1 + T_2 + T_3 , \end{aligned} \tag{3.28}$$

where

$$T_1 = \sum_{K\in\mathcal{C}_h} (\boldsymbol{\varepsilon}(\nabla e_h), \boldsymbol{E}\,\boldsymbol{\varepsilon}(\nabla\psi))_K\,, \tag{3.29}$$

$$T_2 = \sum_{K\in\mathcal{C}_h} (\boldsymbol{\varepsilon}(\nabla e_h), \boldsymbol{\rho})_K\,, \tag{3.30}$$

$$T_3 = \sum_{K\in\mathcal{C}_h} (\boldsymbol{\varepsilon}(\nabla e_h), \mathbf{Curl}\,\boldsymbol{\phi})_K\,. \tag{3.31}$$

We note that, recalling (3.16), it holds

$$\|\psi\|^2_{H^2(\Omega)} + \|\boldsymbol{\phi}\|^2_{H^1(\Omega)} \le C \sum_{K\in\mathcal{C}_h} |e_h|^2_{H^2(K)}\,. \tag{3.32}$$

Step 2. We now bound the three terms T_1, T_2, T_3 above. Due to the symmetry of $\boldsymbol{E}$, from (2.4) we get

$$T_1 = (f,\psi)_\Omega - \sum_{K\in\mathcal{C}_h} (\boldsymbol{E}\,\boldsymbol{\varepsilon}(\nabla w_h), \boldsymbol{\varepsilon}(\nabla\psi)))_K\,. \tag{3.33}$$

Let now $\psi_I \in W_h$ be the approximation of ψ defined in Definition 1. Recalling (2.7) and integrating by parts on each triangle, from (3.33) it follows

$$\begin{aligned} T_1 &= (f,\psi-\psi_I)_\Omega - \sum_{K\in\mathcal{C}_h} (\boldsymbol{E}\,\boldsymbol{\varepsilon}(\nabla w_h), \boldsymbol{\varepsilon}(\nabla(\psi-\psi_I)))_K \\ &= (f,\psi-\psi_I)_\Omega - \sum_{K\in\mathcal{C}_h}\sum_{e\in\partial K} (\boldsymbol{E}\,\boldsymbol{\varepsilon}(\nabla w_h)\boldsymbol{n}_K, \nabla(\psi-\psi_I)))_e\,, \end{aligned} \tag{3.34}$$

where, here and in the sequel, $\boldsymbol{n}_K$ indicates the outward unit normal to each edge of $K \in \mathcal{C}_h$.

Observing that $\boldsymbol{E}\,\boldsymbol{\varepsilon}(\nabla w_h)\boldsymbol{n}_K$ is constant on each edge, then the properties (3.4) and (3.6) applied to (3.34) imply

$$T_1 = (f,\psi-\psi_I)_\Omega = (f-f_h,\psi-\psi_I)_\Omega + (f_h,\psi-\psi_I)_\Omega\,. \tag{3.35}$$

Two Hölder inequalities and the interpolation property (3.5) therefore give

$$T_1 \le C\Big(\sum_{K\in\mathcal{C}_h} h_K^4\|f-f_h\|^2_{L^2(K)} + \sum_{K\in\mathcal{C}_h} h_K^4\|f_h\|^2_{L^2(K)}\Big)^{1/2}\|\psi\|_{H^2(\Omega)}\,. \tag{3.36}$$

Regarding the term T_2, it is sufficient to observe that, due to the symmetry of $\boldsymbol{\varepsilon}(\nabla e_h)$ and the definition of $\boldsymbol{\rho}$ in (3.15), it follows immediately

$$T_2 = \sum_{K \in \mathcal{C}_h} (\boldsymbol{\varepsilon}(\nabla e_h), \boldsymbol{\rho})_K = 0\,. \tag{3.37}$$

We now bound the term in (3.31). Recalling that $w \in H_0^2(\Omega)$ and the fact that $\operatorname{div} \mathbf{div}\, \mathbf{Curl}\, \boldsymbol{\phi} = 0$, integration by parts (see the Appendix) for the w part in T_3 gives

$$T_3 = \sum_{K \in \mathcal{C}_h} (\boldsymbol{\varepsilon}(\nabla w_h), \mathbf{Curl}\, \boldsymbol{\phi})_K\,. \tag{3.38}$$

We have

$$\begin{aligned} T_3 &= \sum_{K \in \mathcal{C}_h} (\boldsymbol{\varepsilon}(\nabla w_h), \mathbf{Curl}\, \boldsymbol{\phi})_K \\ &= \sum_{K \in \mathcal{C}_h} \big(\boldsymbol{\varepsilon}(\nabla w_h), \mathbf{Curl}\,(\boldsymbol{\phi} - \boldsymbol{\phi}_{\varPi})\big)_K + \sum_{K \in \mathcal{C}_h} \big(\boldsymbol{\varepsilon}(\nabla w_h), \mathbf{Curl}\, \boldsymbol{\phi}_{\varPi}\big)_K\,, \end{aligned} \tag{3.39}$$

where $\boldsymbol{\phi}_{\varPi}$ is the approximation of $\boldsymbol{\phi}$, component by component, introduced in Definition 2. Integrating by parts triangle by triangle and recalling (3.13), we have

$$\begin{aligned} &\sum_{K \in \mathcal{C}_h} \big(\boldsymbol{\varepsilon}(\nabla w_h), \mathbf{Curl}\,(\boldsymbol{\phi} - \boldsymbol{\phi}_{\varPi})\big)_K \\ &\quad = \sum_{K \in \mathcal{C}_h} \sum_{e \in \partial K} \big(\boldsymbol{\varepsilon}(\nabla w_h)\boldsymbol{s}_K, \boldsymbol{\phi} - \boldsymbol{\phi}_{\varPi}\big)_e = 0, \end{aligned} \tag{3.40}$$

where $\boldsymbol{s}_K$ represents the unit vector which is the counter clockwise rotation of $\boldsymbol{n}_K$ at each edge of $K \in \mathcal{C}_h$.

Again integrating by parts and observing that

$$\mathbf{Curl}\, \boldsymbol{\phi}_{\varPi} \boldsymbol{n}_K = -\nabla \boldsymbol{\phi}_{\varPi} \boldsymbol{s}_K \tag{3.41}$$

is continuous across edges, it follows

$$\begin{aligned} \sum_{K \in \mathcal{C}_h} \big(\boldsymbol{\varepsilon}(\nabla w_h), \mathbf{Curl}\, \boldsymbol{\phi}_{\varPi}\big)_K &= -\sum_{K \in \mathcal{C}_h} \sum_{e \in \partial K} \big(\nabla w_h, \nabla \boldsymbol{\phi}_{\varPi} \boldsymbol{s}_K\big)_e \\ &= -\sum_{e \in \mathcal{E}_h} \big([\![\nabla w_h]\!], \nabla \boldsymbol{\phi}_{\varPi} \boldsymbol{s}_K\big)_e\,. \end{aligned} \tag{3.42}$$

First Hölder inequalities, then the Agmon and the inverse inequality, and finally the property (3.12) with $m = 1$ give

$$
\begin{aligned}
&\sum_{e\in\mathcal{E}_h} \left([\![\nabla w_h]\!], \nabla \boldsymbol{\phi}_{II} \boldsymbol{s}_K \right)_e \\
&\quad \le \left(\sum_{e\in\mathcal{E}_h} h_e^{-1} \| [\![\nabla w_h]\!] \|_{L^2(e)}^2 \right)^{1/2} \left(\sum_{e\in\mathcal{E}_h} h_e \| \nabla \boldsymbol{\phi}_{II} \boldsymbol{s}_K \|_{L^2(e)}^2 \right)^{1/2} \\
&\quad \le C \left(\sum_{e\in\mathcal{E}_h} h_e^{-1} \| [\![\nabla w_h]\!] \|_{L^2(e)}^2 \right)^{1/2} \left(\sum_{K\in\mathcal{K}_h} \| \boldsymbol{\phi}_{II} \|_{H^1(K)}^2 \right)^{1/2} \\
&\quad \le C \left(\sum_{e\in\mathcal{E}_h} h_e^{-1} \| [\![\nabla w_h]\!] \|_{L^2(e)}^2 \right)^{1/2} \| \boldsymbol{\phi} \|_{H^1(\Omega)} \, .
\end{aligned} \tag{3.43}
$$

Combining the bound (3.43) with the identities (3.39), (3.40) and (3.42) grants

$$
\begin{aligned}
T_3 &\le C \left(\sum_{e\in\mathcal{E}_h} h_e^{-1} \| [\![\nabla w_h]\!] \|_{L^2(e)}^2 \right)^{1/2} \| \boldsymbol{\phi} \|_{H^1(\Omega)} \\
&\le C \left(\sum_{e\in\mathcal{E}_h} h_e^{-1} \| [\![\nabla w_h \cdot \boldsymbol{n}_e]\!] \|_{L^2(e)}^2 \right. \\
&\qquad \left. + \sum_{e\in\mathcal{E}_h} h_e^{-1} \| [\![\nabla w_h \cdot \boldsymbol{s}_e]\!] \|_{L^2(e)}^2 \right)^{1/2} \| \boldsymbol{\phi} \|_{H^1(\Omega)} .
\end{aligned} \tag{3.44}
$$

Observing that

$$
[\![\nabla w_h \cdot \boldsymbol{s}_e]\!] = \frac{\partial}{\partial s} [\![w_h]\!] \quad \forall e \in \mathcal{E}_h \, , \tag{3.45}
$$

where s represents the coordinate along the edge e, standard scaling arguments give

$$
\sum_{e\in\mathcal{E}_h} h_e^{-1} \| [\![\nabla w_h]\!] \cdot \boldsymbol{s}_e \|_{L^2(e)}^2 \le C \sum_{e\in\mathcal{E}_h} h_e^{-3} \| [\![w_h]\!] \|_{L^2(e)}^2 \, . \tag{3.46}
$$

Combining (3.44) with (3.46) finally gives

$$
\begin{aligned}
T_3 \le C &\left(\sum_{e\in\mathcal{E}_h} h_e^{-1} \| [\![\nabla w_h \cdot \boldsymbol{n}_e]\!] \|_{L^2(e)}^2 \right. \\
&\quad \left. + \sum_{e\in\mathcal{E}_h} h_e^{-3} \| [\![w_h]\!] \|_{L^2(e)}^2 \right)^{1/2} \| \boldsymbol{\phi} \|_{H^1(\Omega)} \, .
\end{aligned} \tag{3.47}
$$

Step 3. Combining (3.28) with (3.36), (3.37), (3.47) and recalling (3.32), it follows

$$\sum_{K\in\mathcal{C}_h} |e_h|^2_{H^2(K)} \leq C\Big(\sum_{K\in\mathcal{C}_h} h_K^4\|f-f_h\|^2_{L^2(K)} + \sum_{K\in\mathcal{C}_h}\eta_K^2\Big)^{1/2}\Big(\sum_{K\in\mathcal{C}_h}|e_h|_{H^2(K)}\Big)^{1/2}, \tag{3.48}$$

which implies (3.27). □

3.3 Efficiency

We have the following upper bound for the error estimator:

Theorem 2 *Let w be the solution of the problem* (2.4) *and w_h the solution of the problem* (2.7). *Then it holds*

$$\eta_K \leq |||w-w_h|||_{h,K} + h_K^2\|f-f_h\|_{L^2(K)}\,, \tag{3.49}$$

where $|||\cdot|||_{h,K}$ represents the local restriction of the norm $|||\cdot|||_h$ to the triangle K:

$$|||v|||^2_{h,K} = |v|^2_{H^2(K)} + \sum_{e\in\partial K} c_e h_e^{-3}\|[\![v]\!]\|^2_{L^2(e)} + \sum_{e\in\partial K} c_e h_e^{-1}\|[\![\nabla v\cdot \boldsymbol{n}_e]\!]\|^2_{L^2(e)}\,. \tag{3.50}$$

Proof As already observed, it holds

$$|||e_h|||^2_{h,K} = |e_h|^2_{H^2(K)} + \sum_{e\in\partial K} c_e h_e^{-3}\|[\![w_h]\!]\|^2_{L^2(e)} + \sum_{e\in\partial K} c_e h_e^{-1}\|[\![\nabla w_h\cdot \boldsymbol{n}_e]\!]\|^2_{L^2(e)}\,, \tag{3.51}$$

where we recall that $e_h = w - w_h$.

Therefore, due to the definition of η_K in (3.2), it is sufficient to prove that

$$h_K^2\|f_h\|_{L^2(K)} \leq C\left(|||e_h|||_{h,K} + h_K^2\|f-f_h\|_{L^2(K)}\right). \tag{3.52}$$

Let now K be any fixed triangle in $\mathcal{C}_h$. We indicate with b_K the standard third order polynomial bubble on K, scaled such that $\|b_K\|_{L^\infty(K)} = 1$. Moreover, let $\varphi_K \in H_0^2(K)$ be defined as

$$\varphi_K = f_h b_K^2. \tag{3.53}$$

Standard scaling arguments then easily show that

$$\|f_h\|^2_{L^2(K)} \leq C(f_h, \varphi_K)_K, \tag{3.54}$$

$$\|\varphi_K\|_{L^2(K)} \leq C\|f_h\|_{L^2(K)}. \tag{3.55}$$

Furthermore, noting that $\varphi_K \in H^2_0(K)$ and $\boldsymbol{E}\,\boldsymbol{\varepsilon}(\nabla w_h)$ is constant on K, integration by parts gives

$$(\boldsymbol{E}\,\boldsymbol{\varepsilon}(\nabla w_h), \boldsymbol{\varepsilon}(\nabla\varphi_K))_K = 0\,. \tag{3.56}$$

Applying the bound (3.54) and using (2.4), we get

$$\begin{aligned} h_K^2\|f_h\|^2_{L^2(K)} &\leq Ch_K^2(f_h, \varphi_K)_K \\ &= Ch_K^2\left((f, \varphi_K)_K + (f_h - f, \varphi_K)_K\right) \\ &= Ch_K^2\left((\boldsymbol{E}\,\boldsymbol{\varepsilon}(\nabla w), \boldsymbol{\varepsilon}(\nabla\varphi_K))_K + (f_h - f, \varphi_K)_K\right). \end{aligned} \tag{3.57}$$

First applying the identity (3.56), then the Hölder and inverse inequalities, and finally using the bound (3.55), it follows

$$\begin{aligned} h_K^2(\boldsymbol{E}\,\boldsymbol{\varepsilon}(\nabla w), \boldsymbol{\varepsilon}(\nabla\varphi_K))_K &= h_K^2(\boldsymbol{E}\,\boldsymbol{\varepsilon}(\nabla e_h), \boldsymbol{\varepsilon}(\nabla\varphi_K))_K \\ &\leq C|e_h|_{H^2(K)} h_K^2\|\boldsymbol{\varepsilon}(\nabla\varphi_K)\|_{L^2(K)} \\ &\leq C|e_h|_{H^2(K)}\|\varphi_K\|_{L^2(K)} \\ &\leq C|e_h|_{H^2(K)}\|f_h\|_{L^2(K)}\,. \end{aligned} \tag{3.58}$$

For the second term in (3.57), the Hölder inequality and the bound (3.55) give

$$h_K^2(f_h - f, \varphi_K)_K \leq Ch_K^2\|f - f_h\|_{L^2(K)}\|f_h\|_{L^2(K)}\,. \tag{3.59}$$

Combining (3.57) with (3.58) and (3.59) we get (3.52), and the proposition is proved. □

Appendix

Let v indicate a sufficiently regular scalar field $\Omega \to \mathbb{R}$. Analogously, let $\boldsymbol{\phi}$ and $\boldsymbol{\sigma}$ represent, respectively, a vector field $\Omega \to \mathbb{R}^2$ and a second order tensor field $\Omega \to \mathbb{R}^{2\times 2}$, both sufficiently regular. Finally, a subindex i after a comma will indicate a derivative with respect to the coordinate x_i, $i = 1, 2$.

We then have the following definitions for the differential operators:

$$\nabla v = \begin{pmatrix} v_{,1} \\ v_{,2} \end{pmatrix}, \quad \mathbf{curl}\, v = \begin{pmatrix} -v_{,2} \\ v_{,1} \end{pmatrix},$$
$$\nabla \boldsymbol{\phi} = \begin{pmatrix} \phi_{1,1} & \phi_{1,2} \\ \phi_{2,1} & \phi_{2,2} \end{pmatrix}, \quad \mathbf{Curl}\, \boldsymbol{\phi} = \begin{pmatrix} -\phi_{1,2} & \phi_{1,1} \\ -\phi_{2,2} & \phi_{2,1} \end{pmatrix},$$

$$\operatorname{div} \boldsymbol{\phi} = \phi_{1,1} + \phi_{2,2}, \quad \operatorname{rot} \boldsymbol{\phi} = \phi_{2,1} - \phi_{1,2},$$

$$\mathbf{div}\, \boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11,1} + \sigma_{12,2} \\ \sigma_{21,1} + \sigma_{22,2} \end{pmatrix}, \quad \mathbf{rot}\, \boldsymbol{\sigma} = \begin{pmatrix} \sigma_{12,1} - \sigma_{11,2} \\ \sigma_{22,1} - \sigma_{21,2} \end{pmatrix}.$$

Finally, the strain tensor is defined as the symmetric gradient,

$$\boldsymbol{\varepsilon}(\boldsymbol{\phi}) = \begin{pmatrix} \phi_{1,1} & \dfrac{\phi_{1,2} + \phi_{2,1}}{2} \\ \dfrac{\phi_{1,2} + \phi_{2,1}}{2} & \phi_{2,2} \end{pmatrix}.$$

The corresponding formula for integration by parts are, for a scalar v and a vector $\boldsymbol{\phi}$,

$$(\nabla v, \boldsymbol{\phi})_\Omega = -(v, \operatorname{div} \boldsymbol{\phi})_\Omega + (v, \boldsymbol{\phi} \cdot \boldsymbol{n})_{\partial\Omega},$$
$$(\mathbf{curl}\, v, \boldsymbol{\phi})_\Omega = -(v, \operatorname{rot} \boldsymbol{\phi})_\Omega + (v, \boldsymbol{\phi} \cdot \boldsymbol{s})_{\partial\Omega},$$

and for a vector $\boldsymbol{\phi}$ and a tensor $\boldsymbol{\sigma}$,

$$(\nabla \boldsymbol{\phi}, \boldsymbol{\sigma})_\Omega = -(\boldsymbol{\phi}, \mathbf{div}\, \boldsymbol{\sigma})_\Omega + (\boldsymbol{\phi}, \boldsymbol{\sigma} \boldsymbol{n})_{\partial\Omega},$$
$$(\mathbf{Curl}\, \boldsymbol{\phi}, \boldsymbol{\sigma})_\Omega = -(\boldsymbol{\phi}, \mathbf{rot}\, \boldsymbol{\sigma})_\Omega + (\boldsymbol{\phi}, \boldsymbol{\sigma} \boldsymbol{s})_{\partial\Omega}.$$

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