R. Korte and N. Shanmugalingam, Equivalence and self–improvement of *p*–fatness and Hardy's inequality, and association with uniform perfectness, arXiv:0709.2013v2 [math.FA].

# EQUIVALENCE AND SELF–IMPROVEMENT OF $p$–FATNESS AND HARDY'S INEQUALITY, AND ASSOCIATION WITH UNIFORM PERFECTNESS

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ABSTRACT. We present an easy proof that $p$–Hardy's inequality implies uniform $p$–fatness of the boundary when $p = n$. The proof works also in metric space setting and demonstrates the self–improving phenomenon of the $p$–fatness. We also explore the relationship between $p$–fatness, $p$–Hardy inequality, and the uniform perfectness for all $p \geq 1$, and demonstrate that in the Ahlfors $Q$–regular metric measure space setting with $p = Q$, these three properties are equivalent. When $p \neq 2$, our results are new even in the Euclidean setting.

## 1. INTRODUCTION

The purpose of this paper is to study the relation between $p$–Hardy's inequality

$$\int_\Omega \frac{|u(x)|^p}{\operatorname{dist}(x,\partial\Omega)^p}\,dx \leq C\int_\Omega |\nabla u(x)|^p\,dx$$

for all $u \in C_0^\infty(\Omega)$ and $C$ independent of $u$, the uniform perfectness of $\partial\Omega$, and the uniform $p$–fatness of $X \setminus \Omega$ in the metric space setting. By $p$–fatness we mean a capacitary version of the measure thickness condition. Rather surprisingly, these analytic, metric and geometric conditions turn out to be equivalent in certain situations. We also consider self–improving phenomena related to these conditions. Our results are new even in the Euclidean setting, when $p \neq 2$.

The fact that when $p = n$, a domain satisfies $p$–Hardy's inequality if and only if the complement is uniformly $p$–fat, was first proved by Ancona [1] in $\mathbb{R}^2$. Later, these results were generalized for all $n = p > 1$ by Lewis [14]. Sugawa proved in [19] that for $n = p = 2$ these conditions are equivalent to the uniform perfectness of the complement in the Euclidean plane. See also Buckley–Koskela [3] for studies relevant to Orlicz–Sobolev spaces.

In metric spaces, for all $p > 1$, it has been shown that uniform $p$–fatness of the complement of a domain implies that the domain supports $p$–Hardy's inequality under some conditions, see [2]. See also [9] for similar results involving a measure thickness condition. In [13], the equivalence of the $p$–fatness and a pointwise Hardy's inequality has been studied. In this paper, we prove that if a metric space is Ahlfors $Q$–regular and satisfies a weak

2000 *Mathematics Subject Classification.* 31B15, 46E35.

The first author is supported by the Finnish Academy of Science and Letters, Vilho, Yrjö and Kalle Väisälä Foundation, and the second author is partially supported by the NSF grant DMS-0355027.

$(1,Q)$–Poincaré inequality, then the support of a $Q$–Hardy inequality on a domain implies uniform $Q$–fatness of the complement of the domain. Our proof is rather transparent and it is based on estimating the Hausdorff–content of the boundary.

We will also prove a self–improvement property for both uniform $Q$–fatness and $Q$–Hardy's inequality in the setting of Ahlfors $Q$–regular metric measure spaces. That is, if a set satisfies $Q$–Hardy's inequality or is uniformly $Q$–fat, then there exists $q < Q$ such that the set satisfies $q$–Hardy's inequality or is uniformly $q$–fat, respectively. The self–improving property of Hardy's inequality has been studied in [12] and that of uniform $p$–fatness in [2] and in [16]. Our approach gives a more elementary proof of self–improvement of uniform $p$–fatness when $p = Q$.

## 2. Preliminaries

We assume that $X = (X, d, \mu)$ is a metric measure space equipped with a metric $d$ and a Borel regular outer measure $\mu$ such that $0 < \mu(B) < \infty$ for all balls $B = B(x, r) = \{y \in X : d(x, y) < r\}$. The measure $\mu$ is said to be *doubling* if there exists a constant $c_D \geq 1$, called the *doubling constant*, such that

$$\mu(B(x, 2r)) \leq c_D \mu(B(x, r))$$

for all $x \in X$ and $r > 0$. The measure is *$Q$–regular* if there exists a constant $c_A \geq 1$ such that

$$\frac{1}{c_A} r^Q \leq \mu(B(x, r)) \leq c_A r^Q$$

for all $x \in X$ and $0 < r < \operatorname{diam}(X)$. The $n$–dimensional Lebesgue measure on $\mathbb{R}^n$ is $n$–regular. The Hausdorff $s$–content of $E \subset X$ is

$$\mathcal{H}^s_\infty(E) = \inf \sum_{i \in I} r_i^s, \tag{2.1}$$

where the infimum is taken over all countable covers $\{B(x_i, r_i)\}_{i \in I}$ of $E$, with each $B(x_i, r_i) \cap E$ non-empty. In addition, we may assume that $x_i \in E$ for every $i \in I$, because that may increase the Hausdorff content at most by a multiplicative factor $2^s$.

A non-negative Borel measurable function $g_u$ on $X$ is said to be a *$p$–weak upper gradient* of a function $u$ on $X$ if there is a non-negative Borel measurable function $\rho \in L^p(X)$ such that for all rectifiable curves $\gamma$ in $X$, denoting the end points of $\gamma$ by $x$ and $y$, we have either

$$|u(x) - u(y)| \leq \int_\gamma g_u \, ds,$$

or $\int_\gamma \rho \, ds = \infty$. Let $1 \leq p < \infty$. If $u$ is a function that is integrable to power $p$ in $X$, let

$$\|u\|_{N^{1,p}(X)} = \left( \int_X |u|^p \, d\mu + \inf_{g_u} \int_X g_u^p \, d\mu \right)^{\frac{1}{p}},$$

where the infimum is taken over all $p$–weak upper gradients of $u$. The *Newtonian space* on $X$ is the quotient space

$$N^{1,p}(X) = \{u : \|u\|_{N^{1,p}(X)} < \infty\} / \sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$, see [17]. We define $N_0^{1,p}(\Omega)$ to be the set of functions $u \in N^{1,p}(\Omega)$ that can be extended to $N^{1,p}(X)$ so that the extensions are zero on $X \setminus \Omega$ $p$–quasieverywhere.

We say that $X$ supports a *weak $(1, p)$–Poincaré inequality* if there exist constants $c_p > 0$ and $\tau \geq 1$ such that for all balls $B(x, r)$ of $X$, all locally integrable functions $u$ on $X$ and for all $p$–weak upper gradients $g_u$ of $u$, we have

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \leq c_p r \left( \fint_{B(x,\tau r)} g_u^p \, d\mu \right)^{\frac{1}{p}}, \tag{2.2}$$

where

$$u_B = \fint_B u \, d\mu = \frac{1}{\mu(B)} \int_B u \, d\mu.$$

We point out here that if $X$ is the Euclidean space $\mathbb{R}^n$ equipped with the $n$–dimensional Lebesgue measure and the Euclidean metric, then $N^{1,p}(X) = W^{1,p}(\mathbb{R}^n)$, the classical Sobolev space. Moreover, $\mathbb{R}^n$ supports a weak $(1, 1)$–Poincaré inequality.

**Definition 2.3.** Let $\Omega$ be an open set in $X$ and $E$ be a closed subset of $\Omega$. The *$p$–capacity* of $E$ with respect to $\Omega$ is

$$\operatorname{cap}_p(E, \Omega) = \inf \int_X g_u^p \, d\mu,$$

where the infimum is taken over all functions $u$ with $p$–weak upper gradients $g_u$ such that $u_{|E} = 1$ and $u_{|X \setminus \Omega} = 0$. Should there be no such function $u$, then $\operatorname{cap}_p(E, \Omega) = \infty$.

A metric space $X$ is said to be *linearly locally connected* (LLC) if there is a constant $C \geq 1$ so that for each $x \in X$ and $r > 0$, the following two conditions hold:

(1) any pair of points in $B(x, r)$ can be joined in $B(x, Cr)$,
(2) any pair of points in $X \setminus \overline{B}(x, r)$ can be joined in $X \setminus \overline{B}(x, r/C)$.

By *joining* we mean joining by a path. Note that if a complete $Q$–regular space, $Q > 1$, supports a weak $(1, Q)$–Poincaré inequality, then it satisfies the LLC–condition, see for example [6] or [11].

**Definition 2.4.** We say that a set $E \subset X$ is *uniformly perfect* if $E$ is not a singleton set, and there is a constant $c_{UP} \geq 1$ so that for each $x \in E$ and $r > 0$ the set $E \cap B(x, c_{UP} r) \setminus B(x, r)$ is nonempty whenever the set $E \setminus B(x, c_{UP} r)$ is nonempty.

For more information about uniform perfectness, see for example [7] and [19].

A set $E \subset X$ is said to be *uniformly $p$–fat* if there exists a constant $c_0 > 0$ so that for every point $x \in E$ and for all $0 < r < \infty$,

$$\frac{\operatorname{cap}_p(B(x, r) \cap E, B(x, 2r))}{\operatorname{cap}_p(B(x, r), B(x, 2r))} \geq c_0. \tag{2.5}$$

This condition is stronger than the Wiener criterion. Uniform $p$–fatness is a capacitary version of the uniform measure thickness condition, see for example [9].

**Definition 2.6.** Let $1 < p < \infty$. The set $\Omega \subset X$ satisfies $p$–Hardy's inequality if there exists $0 < c_H < \infty$ such that for all $u \in N_0^{1,p}(\Omega)$,

$$\int_\Omega \left( \frac{|u(x)|}{\operatorname{dist}(x, X \setminus \Omega)} \right)^p \, d\mu(x) \le c_H \int_\Omega g_u(x)^p \, d\mu(x). \tag{2.7}$$

Here $g_u$ is a $p$–weak upper gradient of $u$. Here we use $\operatorname{dist}(x, X \setminus \Omega)$ instead of $\operatorname{dist}(x, \partial\Omega)$ since in general the latter quantity can be larger than the former one.

Hardy's inequality has been studied for example in [4], [5], [14], and [20]. Hardy's inequality has been used also to characterize Sobolev functions with zero boundary values, see [10] and [14].

## 3. Main results

In this section, we show that $Q$–Hardy's inequality on $\Omega$ implies uniform $Q$–fatness of the complement. Our method also shows that $Q$–fatness is a self–improving property. To simplify notation, we will assume $X$ to be unbounded throughout this section. However, for our arguments, it is immaterial what happens outside $\overline{\Omega}$, and therefore our arguments work also if $X$ is bounded, provided we adjust the conditions of uniform perfectness and uniform fatness to the bounded setting. Notice also that if $\Omega$ is a domain, then $X \setminus \Omega$ can be replaced by $\partial\Omega$ in our arguments.

**Theorem 3.1.** *Let $(X, d, \mu)$ be a complete $Q$–regular metric measure space supporting a weak $(1, Q)$–Poincaré inequality, and $\Omega \subset X$ be an open subset. If $\Omega$ satisfies $Q$–Hardy's inequality, then $X \setminus \Omega$ is uniformly $(Q - \varepsilon)$–fat for some $\varepsilon > 0$.*

We split the proof into two parts. First in Lemma 3.2, we show that Hardy's inequality implies uniform perfectness of the complement. Then in Theorem 3.6, we show that uniform perfectness implies $(Q-\varepsilon)$–fatness with some $\varepsilon > 0$. Recall that we assume $X$ to be unbounded.

**Lemma 3.2.** *Let $X$ be as in Theorem 3.1. If $\Omega \subset X$ satisfies $Q$–Hardy's inequality, then $X \setminus \Omega$ is uniformly perfect and unbounded.*

*Proof.* Fix $m > 4$ and suppose that $\Omega$ satisfies Hardy's inequality (2.7) and that $X \setminus \Omega$ is not uniformly perfect with respect to the constant $m$ or that $X \setminus \Omega$ is bounded. In both cases, there exists $x_0 \in X \setminus \Omega$ and $r_0 > 0$ such that $B(x_0, mr_0) \setminus B(x_0, r_0) \subset \Omega$. We will deduce an upper bound for such $m$ independent of $x_0$ and $r_0$, and hence conclude that $X \setminus \Omega$ is uniformly perfect for any constant larger than this upper bound and that $X \setminus \Omega$ cannot be bounded.

Define $u : X \to [0, \infty)$ so that

$$u(x) = \begin{cases} \left( \frac{d(x_0,x)}{r_0} - 1 \right)_+, & d(x_0, x) \le 2r_0, \\ 1, & 2r_0 < d(x_0, x) < \frac{m\, r_0}{2}, \\ \left( 2 - \frac{2d(x_0,x)}{m\, r_0} \right)_+, & \frac{m\, r_0}{2} \le d(x_0, x). \end{cases}$$

Now the minimal upper gradient of $u$ satisfies

$$\int_\Omega g_u^Q \, d\mu \leq \left(\frac{1}{r_0}\right)^Q \mu(B(x_0, 2r_0)) + \left(\frac{2}{mr_0}\right)^Q \mu(B(x_0, mr_0)) \leq c_A 2^{Q+1}. \tag{3.3}$$

Next, we show that

$$\int_\Omega \frac{u(x)^Q}{\operatorname{dist}(x, X \setminus \Omega)^Q} \, d\mu(x) \geq c \log(m/4), \tag{3.4}$$

where $c > 0$ is a constant that depends only on $c_A$ and $Q$. For $x \in X$ and $0 < r < R$, we denote the annulus $A(x, r, R) = B(x, R) \setminus B(x, r)$. Let $n \in \mathbb{N}$ be the unique number such that $2^n \leq m < 2^{n+1}$. Since $m > 4$, we have $n \geq 2$. Then

$$A(x_0, 2r_0, mr_0/2) \supset \bigcup_{k=1}^{n-1} A(x_0, 2^k r_0, 2^{k+1} r_0).$$

As $X$ is quasiconvex (which follows from the Poincaré inequality, see for example [11]) and hence path-connected, and as $X \setminus B(x_0, 2^{k+1} r_0)$ is non-empty, there is a point $y_k \in A(x_0, 2^k r_0, 2^{k+1} r_0)$ such that $d(x_0, y_k) = \dfrac{3}{2} 2^k$; hence the ball $B(y_k, 2^{k-1} r_0) \subset A(x_0, 2^k r_0, 2^{k+1} r_0)$. Thus

$$\begin{aligned}
\int_\Omega \frac{u(x)^Q}{\operatorname{dist}(x, X \setminus \Omega)^Q} \, d\mu &\geq \int_{A(x_0, 2r_0, mr_0)} \frac{1}{d(x_0, x)^Q} \, d\mu \\
&\geq \sum_{k=1}^{n-1} \int_{A(x_0, 2^k r_0, 2^{k+1} r_0)} \frac{1}{d(x_0, x)^Q} \, d\mu \\
&\geq \sum_{k=1}^{n-1} \int_{B(y_k, 2^{k-1} r_0)} \frac{1}{d(x_0, x)^Q} \, d\mu \\
&\geq \sum_{k=1}^{n-1} \frac{1}{(2^{k+1} r_0)^Q} \mu(B(y_k, 2^{k-1} r_0)) \\
&\geq \frac{n-1}{4^Q c_A}.
\end{aligned}$$

Since $n > \dfrac{\log(m/2)}{\log(2)}$, we see that $n - 1 > \dfrac{\log(m/4)}{\log(2)}$. Thus,

$$\int_\Omega \frac{u(x)^Q}{\operatorname{dist}(x, X \setminus \Omega)^Q} \, d\mu > \frac{\log(m/4)}{4^Q c_A \log(2)} = c \log(m/4).$$

By combining (3.3) and (3.4), and the fact that $u$ satisfies Hardy's inequality (2.7), it follows that

$$c \log(m/4) < 2^{Q+1} c_H \, c_A.$$

Hence $m < 4 \exp(2^{Q+1} c_H \, c_A / c)$, and therefore $X \setminus \Omega$ is uniformly perfect with constant $c_{UP} = 4 \exp(2^{Q+1} c_H c_A / c)$ and $X \setminus \Omega$ is unbounded. $\square$

The following example shows that $p$–Hardy's inequality with $p \neq Q$ does not imply uniform perfectness.

**Example 3.5.** If $X = \mathbb{R}^n$, $1 < p < n$, and $\Omega = B(0,1) \setminus \{0\}$, then $\Omega$ supports $p$–Hardy's inequality even though $X \setminus \Omega$ is neither uniformly perfect nor uniformly $p$–fat, see [14, p. 179]. When $p > n$, even single points have positive $p$–capacity and hence $X \setminus \Omega$ is uniformly $p$–fat and supports $p$–Hardy's inequality but $X \setminus \Omega$ is not uniformly perfect.

The uniform perfectness of the boundary implies uniform $q$–fatness of the complement for all $q > Q - \varepsilon$. We get a quantitative estimate for $\varepsilon > 0$ that depends only on $c_{UP}$.

**Theorem 3.6.** *Let $(X, d, \mu)$ be a complete $Q$–regular metric measure space. Suppose that $X$ supports a weak $(1, Q)$–Poincaré inequality. Let $\Omega \subset X$ be an open subset. If $X \setminus \Omega$ is uniformly perfect and unbounded, then there exists a constant $\varepsilon > 0$ such that $X \setminus \Omega$ is uniformly $(Q - \varepsilon)$–fat.*

We begin the proof with an elementary inequality.

**Lemma 3.7.** *For every $C > 0$ there exists $0 < \varepsilon_C < 1$ such that for all $0 < \varepsilon < \varepsilon_C$ and $a, b > 0$,*

$$a^\varepsilon + b^\varepsilon \geq (a + b + C \min\{a, b\})^\varepsilon .$$

*Proof.* We may assume that $a \geq b = 1$. Therefore, it is enough to prove that

$$(a + C + 1)^\varepsilon - a^\varepsilon \leq 1$$

when $0 < \varepsilon < 1$ is sufficiently small and $a \geq 1$. As $f(a) = (a + C + 1)^\varepsilon - a^\varepsilon$ is a decreasing function, and hence for every $a \geq 1$

$$(a + C + 1)^\varepsilon - a^\varepsilon \leq (1 + C + 1)^\varepsilon - 1^\varepsilon,$$

it is enough to choose $\varepsilon$ so that

$$\varepsilon \leq \varepsilon_C = \frac{\log 2}{\log(C + 2)}.$$

□

In the proof of Theorem 3.6, we first obtain an estimate for the Hausdorff–content of the boundary. Then the following result is needed to get capacitary estimates. For a proof, see Theorem 5.9 in [6].

**Lemma 3.8.** *Suppose that $(X, d, \mu)$ is a $Q$–regular space. Suppose further that $X$ admits a weak $(1, p)$–Poincaré inequality for some $1 \leq p \leq Q$. Let $E \subset B(x, r)$ be a compact set. If*

$$\mathcal{H}^s_\infty(E) \geq \lambda r^s$$

*for some $s > Q - p$ and $\lambda > 0$, then*

$$\mathrm{cap}_p(E, B(x, 2r)) \geq c \lambda \mathrm{cap}_p(B(x, r), B(x, 2r)).$$

*The constant $c$ depends only on $s$ and on the data associated with $X$.*

*Proof of Theorem 3.6.* Let $\Omega \subset X$ be open, $X \setminus \Omega$ uniformly perfect with constant $c_{UP} > 1$, and $\alpha > 1$. Fix $x_0 \in X \setminus \Omega$ and $r_0 > 0$. Let $A = \overline{B}(x_0, r_0) \setminus \Omega$, and $0 < \varepsilon < \varepsilon_{\alpha c_{UP}}$, where $\varepsilon_{\alpha c_{UP}}$ is as in Lemma 3.7. First we estimate the Hausdorff $\varepsilon$–content of $A$. Let $\mathcal{F}$ be a family of balls covering $A$. Because $A$ is compact, we may assume that $\mathcal{F}$ consists of a finite number of balls. We may also assume that all the balls in $\mathcal{F}$ are centered at $A$.

If there exists balls $B(x_i, r_i)$ and $B(x_j, r_j)$ in $\mathcal{F}$ such that

$$r_i \le \alpha r_j \tag{3.9}$$

and

$$B(x_i, c_{UP} r_i) \cap B(x_j, r_j) \neq \emptyset, \tag{3.10}$$

then (when $r_j \le r_i$)

$$(B(x_i, r_i) \cup B(x_j, r_j)) \subset B\left(x_i, r_i + r_j + \alpha c_{UP} \min\{r_i, r_j\}\right)$$

or (when $r_i \le r_j$)

$$(B(x_i, r_i) \cup B(x_j, r_j)) \subset B\left(x_j, r_i + r_j + \alpha c_{UP} \min\{r_i, r_j\}\right),$$

and by Lemma 3.7,

$$r_i^\varepsilon + r_j^\varepsilon \ge (r_i + r_j + \alpha c_{UP} \min\{r_i, r_j\})^\varepsilon.$$

Thus, we may replace balls $B(x_i, r_i)$ and $B(x_j, r_j)$ with

$$B(x_i, r_i + r_j + \alpha c_{UP} \min\{r_i, r_j\}) \text{ or } B(x_j, r_i + r_j + \alpha c_{UP} \min\{r_i, r_j\})$$

in the covering $\mathcal{F}$ so that the sum

$$\sum_{B(x,r) \in \mathcal{F}} r^\varepsilon$$

does not increase. We continue this process until there is no pair of balls satisfying (3.10) and (3.9). Because the number of balls in $\mathcal{F}$ decreases in each step and $\mathcal{F}$ consists of a finite number of balls, the process ends after a finite number of replacements.

Let $B(x_1, r_1) \in \mathcal{F}$ be the ball containing $x_0$. Because $X \setminus \Omega$ is uniformly perfect and unbounded, the set

$$(B(x_1, c_{UP}\, r_1) \setminus B(x_1, r_1)) \cap (X \setminus \Omega)$$

is nonempty. Now there are two possibilities: either

$$(B(x_1, c_{UP}\, r_1) \setminus B(x_1, r_1)) \cap \left(X \setminus \overline{B}(x_0, r_0)\right) \neq \emptyset$$

or

$$(B(x_1, c_{UP}\, r_1) \setminus B(x_1, r_1)) \cap A \neq \emptyset,$$

because $(X \setminus \Omega) \subset A \cup (X \setminus \overline{B}(x_0, r_0))$. In the latter case, there exists $B(x_2, r_2) \in \mathcal{F}$ such that $B(x_1, r_1) \neq B(x_2, r_2)$ and

$$B(x_2, r_2) \cap B(x_1, c_{UP}\, r_1) \neq \emptyset,$$

because $\mathcal{F}$ covers $A$. Now the balls $B(x_1, r_1)$ and $B(x_2, r_2)$ satisfy condition (3.10). Hence (3.9) fails, that is, $r_2 < r_1/\alpha$.

We continue inductively in the same way: For a ball $B(x_i, r_i) \in \mathcal{F}$, either

$$B(x_i, c_{UP} r_i) \cap (X \setminus \overline{B}(x_0, r_0)) \neq \emptyset$$

or there exists a ball $B(x_{i+1}, r_{i+1}) \in \mathcal{F}$ such that $r_{i+1} \le r_i/\alpha$ and $B(x_i, r_i)$ and $B(x_{i+1}, r_{i+1})$ satisfy the condition (3.10).

Thus we obtain a chain of distinct balls $\{B(x_i, r_i)\}_{i=1}^n \subset \mathcal{F}$ such that $r_i \le \alpha^{1-i} r_1$, (since $r_i \le r_{i-1}/\alpha$, we have $B(x_i, r_i) \ne B(x_j, r_j)$ if $i \ne j$).

$$B(x_i, c_{UP}\, r_i) \cap B(x_{i+1}, r_{i+1}) \ne \emptyset$$

for every $i = 1, \dots, n-1$, and

$$B(x_n, c_{UP}\, r_n) \cap \big(X \setminus \overline{B}(x_0, r_0)\big) \ne \emptyset.$$

It follows that

$$r_0 \le \sum_{i=1}^n (c_{UP} + 1) r_i \le (c_{UP} + 1) \frac{\alpha}{\alpha - 1}\, r_1$$

and we have a lower bound for $r_1$:

$$r_1 \ge \frac{\alpha - 1}{\alpha(c_{UP} + 1)}\, r_0.$$

We may choose $\alpha = 2$ and thus

$$\sum_{B(x,r) \in \mathcal{F}} r^\varepsilon \ge r_1^\varepsilon \ge \frac{1}{(2c_{UP} + 2)^\varepsilon} r_0^\varepsilon.$$

By [8], there exists $\varepsilon > 0$ such that $X$ satisfies a weak $(1, Q - \varepsilon)$–Poincaré inequality. Fix such an $\varepsilon < \varepsilon_{\alpha c_{UP}}$. Now by Lemma 3.8,

$$\operatorname{cap}_p(B(x_0, r_0) \setminus \Omega, B(x_0, 2r_0)) \ge c \operatorname{cap}_p(B(x_0, r_0), B(x_0, 2r_0))$$

for every $Q - \varepsilon < p \le Q$, where $c$ depends on $\varepsilon$ and $c_{UP}$, but is independent of $x_0$ and $r_0$. $\square$

It is known that uniform $p$–fatness is a self–improving phenomenon, see [2].

**Theorem 3.11.** *Let $X$ be a proper linearly locally convex metric space endowed with a doubling Borel regular measure supporting a weak $(1, q_0)$–Poincaré inequality for some $q_0$ with $1 \le q_0 < \infty$. Let $p > q_0$ and suppose that $E \subset X$ is uniformly $p$–fat. Then there exists $q < p$ so that $E$ is uniformly $q$–fat.*

*Remark* 3.12. The proof of Theorem 3.6 gives a new and easier proof for the self–improvement when $p = Q$.

*Remark* 3.13. To complete the picture, note that uniform $p$–fatness for any $p \le Q$ implies uniform perfectness. To see this, suppose that $X$ supports a $(1, p)$–Poincaré inequality for some $1 \le p < Q$, and that $X \setminus \Omega$ is uniformly $p$–fat. We will show that $X \setminus \Omega$ is uniformly perfect. Fix $x_0 \in \partial\Omega$ and $0 < r < 1$. Suppose that $B(x_0, r) \setminus B(x_0, r/m) \subset \Omega$ for some $m > 1$. We will demonstrate that $m$ has an upper bound that is independent of $x_0$ and $r$. Indeed,

$$\operatorname{cap}_p(B(x_0, r), B(x_0, 2r)) \ge \frac{1}{C}\, r^{Q-p}.$$

Also, as the function

$$g(x) = \frac{1}{\log(r/\rho)} \frac{1}{d(x_0, x)} \chi_{B(x_0,r) \setminus B(x_0,\rho)}$$

is an upper gradient of the function

$$u(x) = \min\left\{1, \max\left\{0, \frac{\log(d(x_0,x)/\rho)}{\log(r/\rho)}\right\}\right\},$$

with $u = 0$ on $B(x_0,\rho)$ and $u = 1$ on $X \setminus B(x_0,r)$; hence

$$\begin{aligned}\mathrm{cap}_p(B(x_0,r)\setminus\Omega, B(x_0,2r)) &\le \mathrm{cap}_p(B(x_0,r/m), B(x_0,2r))\\ &\le \frac{C}{\log(m)^p}\, r^{Q-p}.\end{aligned}$$

The last estimate can be proved in the same way as in the proof of Lemma 3.8. We have by uniform $p$–fatness of $X \setminus \Omega$ that

$$\begin{aligned}\frac{C}{\log(m)^p}\, r^{Q-p} &\ge \mathrm{cap}_p(B(x_0,r)\setminus\Omega, B(x_0,2r))\\ &\ge \frac{1}{c_0}\mathrm{cap}_p(B(x_0,r), B(x_0,2r)) \ge \frac{1}{c_0 C}\, r^{Q-p},\end{aligned}$$

where $c_0$ is the uniform fatness constant, and therefore $m \le e^C$. Thus $X\setminus\Omega$ is uniformly perfect.

*Remark* 3.14. In the proof of Theorem 3.6, we need to assume that the space supports a weak $(1, Q-\varepsilon)$–Poincaré inequality. This follows by [8] for some positive $\varepsilon$ if the space supports a $(1,Q)$–Poincaré inequality. However, if we assume a priori the stronger Poincaré inequality, then our proof gives a quantitative estimate for $\varepsilon$. More precisely, if

$$\max\left\{Q - \frac{\log(2)}{\log(3)}, 1\right\} < p < Q$$

and $X$ supports a $(1,p)$–Poincaré inequality, then there exists $c_p > 1$ such that whenever $X \setminus \Omega$ is uniformly perfect for some uniform perfectness constant $1 \le c_{UP} < c_p$, then $X \setminus \Omega$ is uniformly $p$–fat and hence $\Omega$ supports a $p$–Hardy inequality by Theorem 3.17. The proof of Theorem 3.6 implies the claim if

$$p > Q - \frac{\log(2)}{\log(\alpha c_{UP} + 2)}$$

with some $\alpha > 1$. So it is enough to have $c_{UP} < c_p = 2^{\frac{1}{Q-p}} - 2$. By the assumption on $p$, it is clear that $c_p > 1$.

The following examples illustrate the sharpness of Remark 3.14.

**Example 3.15.** If $1 < p < Q$, there is a Cantor set $E_p \subset \mathbb{R}^n$ such that $\mathrm{cap}_p(E_p) = 0$, see [6, p. 40]. Thus the domain $\mathbb{R}^n \setminus E_p$ has uniformly perfect complement, which is not uniformly $p$–fat.

**Example 3.16.** If $1 \le p < Q-1$, then any rectifiable curve $\gamma$ in $X$ is of zero $p$–capacity. In this case, with $\Omega = X \setminus \gamma$, we have that $X \setminus \Omega$ is uniformly perfect with constant $c_{UP} = 1$, but it is not uniformly $p$–fat.

The following theorem shows that Hardy's inequality follows from uniform fatness for all $1 < p \le Q$, see Corollary 6.1 in [2]. Note that the LLC–condition is not a serious restriction in our case, since it follows from the $(1,Q)$–Poincaré inequality, see for example [11].

**Theorem 3.17.** *Let $X$ be a proper LLC metric space endowed with a doubling Borel regular measure supporting a weak $(1,p)$–Poincaré inequality, and suppose that $\Omega$ is a bounded open set in $X$ with $X \setminus \Omega$ uniformly $p$–fat. Then $\Omega$ satisfies $p$–Hardy's inequality.*

The converse of Theorem 3.17 is not true in general, see Example 3.5. As a corollary of Lemma 3.2 and Theorems 3.6 and 3.17, we obtain the following result. Note that uniform $p$–fatness implies uniform $q$–fatness for all $q > p$.

**Theorem 3.18.** *Let $(X, d, \mu)$ be a complete $Q$–regular metric measure space with $Q > 1$. Suppose that $X$ supports a weak $(1, Q)$–Poincaré inequality. Let $\Omega \subset X$ be a bounded open subset. Then the following conditions are quantitatively equivalent.*

(1) *$\Omega$ satisfies $Q$–Hardy's inequality.*
(2) *$X \setminus \Omega$ is uniformly perfect.*
(3) *$X \setminus \Omega$ is uniformly $Q$–fat.*
(4) *$X \setminus \Omega$ is uniformly $(Q - \varepsilon)$–fat for some $\varepsilon > 0$.*

Theorem 3.17 is stated only for bounded sets but the proof works also in the unbounded setting. Hence Theorem 3.18 holds also when $\Omega$ is unbounded if we require additionally that $X \setminus \Omega$ is unbounded in conditions (2) and (3).

## 4. Maz′ya type characterization

In this section, we present one more characterization of open sets that is equivalent with the Hardy's inequality. For more information about this kind of characterizations, see Chapter 2.3 in [15].

**Theorem 4.1.** *Let $X$ be a complete metric space endowed with a doubling measure and supporting a weak $(1,p)$–Poincaré inequality. Let $1 < p \leq Q$. Then $\Omega \subset X$ satisfies $p$–Hardy's inequality if and only if for every $K \subset\subset \Omega$, we have*

$$\int_K \operatorname{dist}(x, X \setminus \Omega)^{-p} \, d\mu(x) \leq c \operatorname{cap}_p(K, \Omega). \tag{4.2}$$

*Proof.* First assume that $\Omega$ satisfies $p$–Hardy's inequality. Let $u \in N_0^{1,p}(\Omega)$ such that $u = 1$ in $K$. Then

$$\int_K \operatorname{dist}(x, X \setminus \Omega)^{-p} \, d\mu(x) \leq \int_\Omega \frac{|u(x)|^p}{\operatorname{dist}(x, X \setminus \Omega)^p} \, d\mu(x) \leq c_H \int_\Omega g_u^p \, d\mu.$$

By taking infimum over all such functions $u$, we obtain (4.2).

Now assume that equation (4.2) is satisfied. We will first prove the claim for Lipschitz–functions that have compact support in $\Omega$. By Theorems 2.12 and 4.8 in [18], such functions form a dense subclass of $N_0^{1,p}(\Omega)$, and thus we get the result for all functions in $N_0^{1,p}(\Omega)$.

Let $u \in N_0^{1,p}(\Omega)$ be compactly supported Lipschitz function, and denote

$$E_k = \{x \in \Omega \,:\, |u(x)| > 2^k\}, \; k = 1, 2, \ldots .$$

Thus by (4.2), we have

$$\begin{aligned}\int_\Omega \frac{|u(x)|^p}{\operatorname{dist}(x, X\setminus\Omega)^p}\,d\mu(x) &\le \sum_{k=-\infty}^{\infty} 2^{(k+1)p}\int_{E_k\setminus E_{k+1}} \frac{1}{\operatorname{dist}(x, X\setminus\Omega)^p}\,d\mu(x)\\ &\le c\sum_{k=-\infty}^{\infty} 2^{(k+1)p}\operatorname{cap}_p(\overline{E}_{k+1},\Omega)\\ &\le c\sum_{k=-\infty}^{\infty} 2^{(k+1)p}\operatorname{cap}_p(\overline{E}_{k+1},E_k).\end{aligned}$$

Let

$$u_k = \begin{cases} 1, & \text{when } |u|\ge 2^{k+1},\\ \frac{|u|}{2^k}-1, & \text{when } 2^k<|u|<2^{k+1},\\ 0, & \text{when } |u|\le 2^k.\end{cases}$$

Then $u_k = 1$ in $\overline{E}_{k+1}$ and $u_k = 0$ in $X\setminus E_k$. Therefore,

$$\operatorname{cap}_p(\overline{E}_{k+1},E_k)\le \int_{E_k\setminus E_{k+1}} g_{u_k}^p\,d\mu \le 2^{-pk}\int_{E_k\setminus E_{k+1}} g_u^p\,d\mu.$$

Consequently,

$$\begin{aligned}c\sum_{k=-\infty}^{\infty} 2^{(k+1)p}\operatorname{cap}_p(\overline{E}_{k+1},E_k) &\le c2^p\sum_{k=-\infty}^{\infty}\int_{E_k\setminus E_{k+1}} g_u^p\,d\mu\\ &= c2^p\int_\Omega g_u^p\,d\mu,\end{aligned}$$

and the claim follows with $c_H = 2^p c$. □

*Acknowledgements.* The authors would like to thank Juha Kinnunen for introducing the subject and for valuable comments.

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