T. Lukkari. Elliptic equations with nonstandard growth involving measures. Hiroshima Mathematical Journal, to appear.

HIROSHIMA MATH. J.
00 (0000), 1–17

Elliptic equations with nonstandard growth involving measures

Teemu LUKKARI

(Received Xxx 00, 0000)

ABSTRACT. We show that given a positive and finite Radon measure μ, there is a $\mathcal{A}_{p(\cdot)}$-superharmonic function u which satisfies

$$-\operatorname{div}\mathcal{A}(x, Du) = \mu$$

in the sense of distributions. Here $\mathcal{A}$ is an elliptic operator with $p(x)$-type nonstandard growth.

1. Introduction

We study the existence of solutions of

$$-\operatorname{div}\mathcal{A}(x, Du) = \mu, \tag{1.1}$$

where $\mathcal{A}$ is an operator with $p(x)$-type nonstandard structural conditions. Our main result is that for positive, finite Radon measures μ, there exists an $\mathcal{A}_{p(\cdot)}$-superharmonic function u which satisfies (1.1) in the sense of distributions. See section 2 for the exact definition of this class of functions. Examples of the operators $\mathcal{A}$ considered here arise from variational integrals like

$$\int |\nabla u|^{p(x)}\,\mathrm{d}x; \tag{1.2}$$

the Euler-Lagrange equation of (1.2) is the $p(x)$-Laplacian equation

$$\operatorname{div}(p(x)|\nabla u|^{p(x)-2}\nabla u) = 0, \tag{1.3}$$

where

$$\mathcal{A}(x, \xi) = p(x)|\xi|^{p(x)-2}\xi.$$

There is an extensive literature on partial differential equations and the calculus of variations with various nonstandard growth conditions, see for example [30, 31, 24, 2, 1, 4] and the references in the survey [26].

We study this problem for two reasons. First, some properties of $\mathcal{A}_{p(\cdot)}$ -superharmonic functions require an additional integrability assumption; see for example [14, Theorem 4.5]. We would like to show the existence of $\mathcal{A}_{p(\cdot)}$-superharmonic functions for which the integrability assumption can be verified. The need for an extra assumption is due to the fact that Harnack estimates for equations with $p(x)$-growth are intrinsic in the sense that they depend on the solution itself, see [2, 3, 14].

Second, we would like to show the existence of solutions with nonremovable isolated singularities. There is a method due to Serrin [29] to construct such solutions. Again because of the intrinsic nature of the Harnack estimates, this method fails. Hence the second purpose of this work is to find an alternative to Serrin's method. This turns out to be quite simple, just choosing the Dirac measure as μ in (1.1) suffices.

Our approach is an adaptation of that of Kilpeläinen and Malý [17]. First, we obtain approximative solutions u_i by approximating μ with more regular measures. Then we prove uniform estimates for u_i and use them to find a limit u

2000 *Mathematics Subject Classification.* Primary 35J70, Secondary 35D05, 31C45.
Key words and phrases. Nonstandard growth, measure data, superharmonic functions.

and to prove the fact that the left hand side of (1.1) makes sense as a distribution. Finally we show that this u is indeed a solution of (1.1). This approach is related to the works of Boccardo and Gallouët [5, 6]; see also [7, 22, 28].

The results we use as tools here do not hold without additional assumptions on the function $p(\cdot)$. Even the variable exponent Lebesgue and Sobolev spaces have very few properties for general, for instance just measurable, exponents. There is a frequently used assumption, called logarithmic Hölder continuity, that seems to be the right one for our purposes. See below for more details.

2. Preliminaries

A measurable function $p\colon \mathbb{R}^n \to (1,\infty)$, $n \geq 2$, is called a variable exponent. We denote

$$p_A^+ = \sup_{x\in A} p(x), \quad p_A^- = \inf_{x\in A} p(x), \quad p^+ = \sup_{x\in\mathbb{R}^n} p(x), \quad p^- = \inf_{x\in\mathbb{R}^n} p(x).$$

We assume, unless otherwise specified, that the exponent $p(\cdot)$ is logarithmically Hölder continuous, i.e. satisfies (2.1) *below and that* $1 < p^- \leq p^+ < \infty$.

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set. The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ consists of all measurable functions u defined on Ω for which the $p(\cdot)$-modular

$$\varrho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}\,\mathrm{d}x$$

is finite. The Luxemburg norm on this space is defined as

$$\|u\|_{p(\cdot)} = \inf\Big\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} \mathrm{d}x \leq 1\Big\}.$$

Equipped with this norm $L^{p(\cdot)}(\Omega)$ is a Banach space. For basic results on variable exponent spaces we refer to [19]. In particular, the dual of $L^{p(\cdot)}(\Omega)$ is the space $L^{p'(\cdot)}(\Omega)$ obtained by conjugating the exponent pointwise, [19, Theorem 2.6]. It follows that $L^{p(\cdot)}(\Omega)$ is reflexive. Furthermore, a version of Hölder's inequality,

$$\int_\Omega fg\,\mathrm{d}x \leq C\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)},$$

holds for functions $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{p'(\cdot)}(\Omega)$.

The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ consists of functions $u \in L^{p(\cdot)}(\Omega)$ whose distributional gradient ∇u exists and belongs to $L^{p(\cdot)}(\Omega)$. The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is a Banach space with the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

Smooth functions are not dense in $W^{1,p(\cdot)}(\Omega)$ without additional assumptions on the exponent $p(\cdot)$. This was observed by Zhikov [30, 31] in the context of the Lavrentiev phenomenon. Zhikov introduced the logarithmic Hölder continuity condition,

$$|p(x) - p(y)| \leq \frac{C}{-\log(|x-y|)} \tag{2.1}$$

for all $x, y \in \Omega$ such that $|x-y| \leq 1/2$, as a criterion for the absence of the Lavrentiev phenomenon. If the exponent satisfies (2.1), smooth functions are dense in variable exponent Sobolev spaces and we can define the Sobolev space with zero boundary values, denoted by $W_0^{1,p(\cdot)}(\Omega)$, as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{1,p(\cdot)}$. We refer to [10] and [16] for the details of this definition.

Since we assume the exponent $p(\cdot)$ to be continuous, the $p(\cdot)$-*Poincaré inequality*

$$\|u\|_{p(\cdot)} \le C\|\nabla u\|_{p(\cdot)}$$

holds for every $u \in W_0^{1,p(\cdot)}(\Omega)$, see [13, Theorem 4.1]. In particular, the $p(\cdot)$-Poincaré inequality implies that the norms $\|u\|_{1,p(\cdot)}$ and $\|\nabla u\|_{p(\cdot)}$ are equivalent on $W_0^{1,p(\cdot)}(\Omega)$.

We use the following compactness properties of $W_0^{1,p(\cdot)}(\Omega)$ in our existence proof. The limit function v belongs to $W_0^{1,p(\cdot)}(\Omega)$ by Mazur's lemma, the first property follows from the reflexivity of $L^{p(\cdot)}(\Omega)$ and the second from the fact that $W_0^{1,p(\cdot)}(\Omega)$ embeds compactly into $L^{p(\cdot)}(\Omega)$, see [19, Theorem 3.10].

THEOREM 2.2. *Assume that the sequence (u_j) is bounded in $W_0^{1,p(\cdot)}(\Omega)$. Then there is a function $v \in W_0^{1,p(\cdot)}(\Omega)$ and a subsequence (u_{j_k}) with the following properties.*

(1) $\nabla u_{j_k} \to \nabla v$ *weakly in* $L^{p(\cdot)}(\Omega)$.
(2) $u_{j_k} \to v$ *pointwise almost everywhere and in* $L^{p(\cdot)}(\Omega)$.

We need the following assumptions to hold for the operator $\mathcal{A} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$.

(1) $x \mapsto \mathcal{A}(x,\xi)$ is measurable for all $\xi \in \mathbb{R}^n$,
(2) $\xi \mapsto \mathcal{A}(x,\xi)$ is continuous for all $x \in \Omega$ and $\mathcal{A}(x,-\xi) = -\mathcal{A}(x,\xi)$ for all $\xi \in \mathbb{R}^n$,
(3) $\mathcal{A}(x,\xi)\cdot\xi \ge \alpha|\xi|^{p(x)}$, where $\alpha > 0$ is a constant, for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$,
(4) $|\mathcal{A}(x,\xi)| \le \beta|\xi|^{p(x)-1}$, where $\beta \ge \alpha > 0$ is a constant, for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$,
(5) $(\mathcal{A}(x,\eta) - \mathcal{A}(x,\xi))\cdot(\eta - \xi) > 0$ for all $x \in \Omega$ and $\eta \ne \xi \in \mathbb{R}^n$.

These are called the structure conditions of $\mathcal{A}$.

We say that a function $u \in W_{loc}^{1,p(\cdot)}(\Omega)$ is a subsolution of the equation

$$-\operatorname{div}\mathcal{A}(x,\nabla u) = 0 \tag{2.3}$$

if

$$\int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla\varphi\,\mathrm{d}x \le 0$$

for all nonnegative test functions $\varphi \in C_0^\infty(\Omega)$. We use the assumption $\mathcal{A}(x,-\xi) = -\mathcal{A}(x,\xi)$ and say that u is a supersolution if $-u$ is a subsolution. Further, u is a solution if it is both a super- and a subsolution. Since smooth functions are dense in $W^{1,p(\cdot)}(\Omega)$, we are allowed to employ test functions $\varphi \in W_0^{1,p(\cdot)}(\Omega)$ with compact support in Ω by the usual approximation argument.

Logarithmic Hölder continuity plays an important role in the calculus of variations and theory of partial differential equations with $p(\cdot)$-growth. Indeed, higher integrability [31], Hölder regularity results [1, 8], and Harnack estimates [2, 12, 14] use condition (2.1). Harnack estimates and boundary regularity theory from [3] are used to prove the properties of supersolutions and $\mathcal{A}_{p(\cdot)}$-superharmonic functions we employ here, i.e. Theorems 2.5 and 2.7 below. Hence the log-Hölder assumption is crucial to us.

The results in [2, 3, 12, 14] are given for $\mathcal{A}(x,\xi) = p(x)|\xi|^{p(x)-2}\xi$ or $\mathcal{A}(x,\xi) = |\xi|^{p(x)-2}\xi$. However, they hold also for the general operators $\mathcal{A}$ considered here. This is due to the fact that the same Caccioppoli type estimates, the comparison principle and convergence results are available. Indeed, using the ellipticity condition (3) and the growth condition (4) the proofs of the Caccioppoli estimates given for operators $\mathcal{A}$ with constant growth exponents given, for example, in [23, Chapter 2] can be adapted to the case we consider here; see Lemma 3.1 below for a simple example of such an adaptation. Further, comparison and convergence results are a consequence of the monotonicity assumption (5) and the continuity assumption (2), replacing the constant exponent Hölder's inequality by a pointwise application of Young's inequality where necessary; see, e.g., the arguments given in [15, Chapter 3].

DEFINITION 2.4. We say that a function $u : \Omega \to (-\infty, \infty]$ is $\mathcal{A}_{p(\cdot)}$-superharmonic in Ω, denoted $u \in \mathcal{S}(\Omega)$, if

(1) u is lower semicontinuous,
(2) u is finite almost everywhere and
(3) The comparison principle holds: Let $D \Subset \Omega$ be an open set. If h is a solution in D, continuous in $\overline{D}$ and $u \geq h$ on ∂D, then $u \geq h$ in D.

Further, we say that u is $\mathcal{A}_{p(\cdot)}$-hyperharmonic if u has the properties (1) and (3) of Definition 2.4.

The properties of supersolutions and $\mathcal{A}_{p(\cdot)}$-superharmonic functions we need below are collected in the next theorem. For the first property, see [12, Theorem 6.1] and [14, Theorem 4.1]. The second property is an easy consequence of the definition, as is the fact that truncations of $\mathcal{A}_{p(\cdot)}$-superharmonic functions are $\mathcal{A}_{p(\cdot)}$-superharmonic. Bounded $\mathcal{A}_{p(\cdot)}$-superharmonic functions are supersolutions by [12, Corollary 6.6], and hence the functions $\min(u, \lambda)$ are also supersolutions. The last two properties follow in the same way as in the constant exponent case, see [15, Corollary 7.23 and Theorem 7.27].

THEOREM 2.5. (1) *If u is a supersolution, then the lower semicontinuous regularization of u, defined as*

$$\widetilde{u}(x) = \operatorname*{ess\,liminf}_{y \to x} u(y),$$

is an $\mathcal{A}_{p(\cdot)}$-superharmonic function and equals $u(x)$ a.e.
(2) *If (u_k) is an increasing sequence of $\mathcal{A}_{p(\cdot)}$-superharmonic functions, then the limit function is $\mathcal{A}_{p(\cdot)}$-hyperharmonic.*
(3) *If u is $\mathcal{A}_{p(\cdot)}$-superharmonic, so is the function $\min(u, \lambda)$ for all $\lambda \in \mathbb{R}$. The truncations $\min(u, \lambda)$ are also supersolutions.*
(4) *If u and v are $\mathcal{A}_{p(\cdot)}$-superharmonic and $u = v$ almost everywhere, then $u = v$ everywhere.*
(5) *Being $\mathcal{A}_{p(\cdot)}$-superharmonic is a local property.*

For an $\mathcal{A}_{p(\cdot)}$-superharmonic function u we define a derivative Du pointwise as

$$Du = \lim_{k \to \infty} \nabla \min(u, k).$$

Note that Du is not necessarily the gradient of u in any sense.

We recall the following integrability lemma. See [17, Lemma 1.11], or [15, Lemma 7.43], for the proof, and [15, Section 1.6, p. 6] for the choice of κ.

LEMMA 2.6. *Let Ω be bounded, $1 < p < \infty$ and let u be a nonnegative function which is finite almost everywhere. Set*

$$\kappa = \begin{cases} \frac{n}{n-p}, & \text{if } p < n, \text{ and} \\ 2, & \text{if } p \geq n. \end{cases}$$

Suppose that for all $k \in \mathbb{N}$

$$\min(u, k) \in W_0^{1,p}(\Omega)$$

and

$$\int_\Omega |\nabla \min(u,k)|^p \,\mathrm{d}x \le Mk$$

for a constant M *independent of* k. *If* $1 \le q < \kappa p/(\kappa(p-1)+1)$, *then*

$$\int_\Omega |\nabla \min(u,k)|^{q(p-1)} \,\mathrm{d}x \le C,$$

where $C = C(n,p,q,M,\operatorname{diam}\Omega)$, *and if* $0 < s < \kappa(p-1)$, *then*

$$\int_\Omega u^s \,\mathrm{d}x \le C,$$

where $C = C(n,p,s,M,\operatorname{diam}\Omega)$.

The previous lemma is used to prove the following result; see [12, Theorem 7.5] and [21, Theorem 4.6]. The extra assumption mentioned in the introduction is the requirement that $u \in L^t_{loc}(\Omega)$.

THEOREM 2.7. *Assume that* u *is* $\mathcal{A}_{p(\cdot)}$*-superharmonic in* Ω. *If* $u \in L^t_{loc}(\Omega)$ *for some* $t > 0$, *there is a number* $q > 1$ *such that* $|u|^{q(p(x)-1)}$ *and* $|Du|^{q(p(x)-1)}$ *are locally integrable.*

Theorem 2.7 seems insufficient to bound the gradients of approximate solutions in the proof of our main theorem. We fix this by using the following lemma. In the lemma, we need the sharp form of the weak Harnack inequality [14, Theorem 3.7]

$$\left(\fint_{B_{2R}} u^\gamma \,\mathrm{d}x\right)^{1/\gamma} \le C(\inf_{x\in B_R} u(x) + R). \tag{2.8}$$

More precisely, we need an exponent $\gamma > p^-_{B_{2R}} - 1$ on the left hand side of (2.8). We can establish this by modifying the iteration argument of [3, Lemma 6.3] to use [14, Lemma 3.4] in a fashion similar to [14, Lemma 3.5]. This way, we see that the weak Harnack inequality (2.8) holds for any exponent $0 < \gamma < \kappa(p^-_{B_{2R}} - 1)$, where $\kappa = k(p^-_{B_{2R}})$ is the Sobolev inequality parameter corresponding to $p^-_{B_{2R}}$, as given in Lemma 2.6.

LEMMA 2.9. *Let* u *be a nonnegative* $\mathcal{A}_{p(\cdot)}$*-superharmonic function such that* $u \in L^t_{loc}(\Omega)$ *for some* $t > 0$. *Then there exist numbers* $q > 1$ *and* $\varepsilon > 0$, *such that*

$$\begin{aligned}\int_{B_R} |Du|^{q(p(x)-1)} \,\mathrm{d}x \le& CR^{n-p^-_{B_{2R}}} \left(\inf_{x\in B_R} u(x) + R\right)^{p^-_{B_{2R}}-1-\varepsilon} \\ &+ CR^n \left(\inf_{x\in B_R} u(x) + R\right)^{(1+\varepsilon)/((p^+_{B_{2R}})'/q-1)} + CR^n\end{aligned}$$

for all sufficiently small balls $B_R = B(x_0,R)$. *The constant depends on* $p(\cdot)$, n, q, ε, *the structural constants* β *and* α, *and the* $L^{q's}(B_{6R})$*-norm of* u *for* $s > p^+_{B_{6R}} - p^-_{B_{6R}}$, *where* q' *is the Hölder conjugate of* q.

PROOF. Let us first pick a number $\lambda > 1$ such that the exponent $\lambda(p^-_{B_{2R}} - 1)$ is admissible in the weak Harnack inequality; for instance, the choice

$$\lambda = \min\left\{\frac{n}{n-1}, \frac{3}{2}\right\}$$

will do. Further, we let $q > 1$ be a number such that

$$q < \min\{(p^+)', \lambda\}$$

and set

$$\varepsilon = \min\left\{\frac{1}{2}(p^- - 1), \frac{\lambda}{1+2\delta} - 1\right\},$$

where $\delta = (\lambda - 1)/4$.

Assume first that u is a supersolution, and pick a cutoff function η compactly supported in B_{2R} such that $0 \le \eta \le 1$, $\eta = 1$ on B_R and $|\nabla \eta| \le C/R$. We use Young's inequality and the Caccioppoli estimate for supersolutions [14, Lemma 4.3], and get that

$$\begin{aligned}\int_{B_R} |\nabla u|^{q(p(x)-1)}\,\mathrm{d}x &= \int_{B_R} |\nabla u|^{q(p(x)-1)} u^{-q(1+\varepsilon)\frac{p(x)-1}{p(x)}} u^{q(1+\varepsilon)\frac{p(x)-1}{p(x)}}\,\mathrm{d}x \\ &\le C\int_{B_{2R}} |\nabla u|^{p(x)} u^{-1-\varepsilon}\eta^{p^+_{B_{2R}}}\,\mathrm{d}x \\ &\quad + C\int_{B_{2R}} u^{(1+\varepsilon)/(p'(x)/q-1)}\,\mathrm{d}x \\ &\le C\int_{B_{2R}} u^{p(x)-1-\varepsilon}|\nabla\eta|^{p(x)}\,\mathrm{d}x \\ &\quad + C\int_{B_{2R}} u^{(1+\varepsilon)/(p'(x)/q-1)}\,\mathrm{d}x.\end{aligned}$$

We estimate the first integral by log-Hölder continuity (see [14, Lemma 3.3]), Hölder's inequality, the weak Harnack inequality and Lemma 3.4 of [14], and obtain

$$\begin{aligned}&\int_{B_{2R}} u^{p(x)-1-\varepsilon}|\nabla\eta|^{p(x)}\,\mathrm{d}x \\ &\le CR^{n-p^-_{B_{2R}}}\left(\fint_{B_{2R}} u^{q'(p(x)-p^-_{B_{2R}})}\,\mathrm{d}x\right)^{1/q'}\left(\fint_{B_{2R}} u^{q(p^-_{B_{2R}}-1-\varepsilon)}\,\mathrm{d}x\right)^{1/q} \\ &\le CR^{n-p^-_{B_{2R}}}\left(\fint_{B_{2R}} 1 + u^{q'(p^+_{B_{2R}}-p^-_{B_{2R}})}\,\mathrm{d}x\right)^{1/q'}\left(\inf_{x\in B_R} u(x) + R\right)^{p^-_{B_{2R}}-1-\varepsilon} \\ &\le CR^{n-p^-_{B_{2R}}}(1+\|u\|^{(p^+_{B_{2R}}-p^-_{B_{2R}})}_{L^{q's}(B_{6R})})\left(\inf_{x\in B_R} u(x) + R\right)^{p^-_{B_{2R}}-1-\varepsilon}.\end{aligned}$$

For the second term, we first estimate

$$\int_{B_{2R}} u^{(1+\varepsilon)/(p'(x)/q-1)}\,\mathrm{d}x \le CR^n + CR^n\fint_{B_{2R}} u^{(1+\varepsilon)/((p^+_{B_{2R}})'/q-1)}\,\mathrm{d}x. \tag{2.10}$$

Next we claim that we can choose $q > 1$ such that the exponent on the right hand side of (2.10) is admissible in the weak Harnack inequality. A sufficient condition for admissibility is

$$\frac{q(p^+_{B_{2R}}-1)(1+\varepsilon)}{p^+_{B_{2R}} - q(p^+_{B_{2R}}-1)} < \lambda(p^-_{B_{2R}} - 1). \tag{2.11}$$

By the continuity of $p(\cdot)$, we can assume that

$$\frac{p^+_{B_{2R}}-1}{p^-_{B_{2R}}-1} \le 1+\delta$$

by considering small enough balls B_{2R}. Whenever B_{2R} is such a ball, (2.11) is satisfied if

$$\frac{q(1+\varepsilon)}{p^+ - q(p^+-1)} < \frac{\lambda}{1+\delta}. \tag{2.12}$$

Here we used the fact that the function $t \mapsto t - q(t-1)$ is decreasing and positive on the interval $[0, p^+]$, since $1 < q < (p^+)'$. The left hand side of (2.12) tends to $1 + \varepsilon$ as q tends to one. Since $1 + \varepsilon \le \lambda/(1 + 2\delta) < \lambda/(1 + \delta)$, it is possible to choose a number $q > 1$ such that (2.12) holds. Now we can estimate the integral average on the right hand side of (2.10) by the weak Harnack inequality. This proves the claim in the case of supersolutions.

For a general $\mathcal{A}_{p(\cdot)}$-superharmonic function u, we apply the estimate for supersolutions to $u_k = \min(u, k)$ and note that we can estimate the norms of u_k appearing in the constants by the norms of u. Letting $k \to \infty$ completes the proof. □

We use the next estimate in proving that the $\mathcal{A}_{p(\cdot)}$-superharmonic solutions of (1.1) we find below are solutions outside the support of the measure μ. A simplified version of the arguments leading to Lemma 3.7 in [21], with the appropriate modifications to take care of the presence of $\mathcal{A}$, establishes the case $q_0 > p^- - 1$. The case $q_0 > 0$ then follows by a standard iteration argument, see for example [15, Lemma 3.38].

LEMMA 2.13. *Let $u \ge 0$ be a solution of* (2.3) *in B_{6R}. Then*

$$\operatorname*{ess\,sup}_{x \in B_R} u(x) \le C \left[\left(\fint_{B_{2R}} u^{q_0} \,\mathrm{d}x \right)^{1/q_0} + R \right],$$

where $q_0 > 0$. The constant depends on n, q_0, $p(\cdot)$ and the $L^{q'r}(B_{6R})$-norm of u, where $1 < q < n/(n-1)$, q' is the Hölder conjugate of q and $r > p^+_{B_{6R}} - p^-_{B_{6R}}$.

3. Compactness of $\mathcal{A}_{p(\cdot)}$-superharmonic functions

In this section we prove a weak compactness property of $\mathcal{A}_{p(\cdot)}$-superharmonic functions, Theorem 3.4. It is our main tool for the next section.

LEMMA 3.1. *Assume that u is a nonnegative subsolution and $\eta \in C_0^\infty(\Omega)$ is such that $0 \le \eta \le 1$. Then*

$$\int_\Omega |\nabla u|^{p(x)} \eta^{p^+} \,\mathrm{d}x \le C \int_\Omega u^{p(x)} |\nabla \eta|^{p(x)} \,\mathrm{d}x.$$

PROOF. We use $u\eta^{p^+}$ as a test function and obtain

$$0 \ge \int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla u \eta^{p^+} \,\mathrm{d}x + \int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla \eta p^+ \eta^{p^+ - 1} u \,\mathrm{d}x.$$

From this we obtain that

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla u \eta^{p^+} \,\mathrm{d}x \le p^+ \int_\Omega |\mathcal{A}(x, \nabla u)| |\nabla \eta| \eta^{p^+ - 1} u \,\mathrm{d}x. \tag{3.2}$$

Next we use structure, (3.2) and Young's inequality and conclude that

$$\begin{aligned}
\int_\Omega \eta^{p^+} |\nabla u|^{p(x)} \,\mathrm{d}x \le & C \int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla u \eta^{p^+} \,\mathrm{d}x \\
\le & C \int_\Omega |\mathcal{A}(x, \nabla u)| |\nabla \eta| \eta^{p^+ - 1} u \,\mathrm{d}x \\
\le & C \int_\Omega |\nabla u|^{p(x)-1} |\nabla \eta| \eta^{p^+ - 1} u \,\mathrm{d}x \\
\le & \frac{1}{2} \int_\Omega |\nabla u|^{p(x)} \eta^{\frac{p^+}{p'(x)} p'(x)} \,\mathrm{d}x \\
& + C \int_\Omega u^{p(x)} |\nabla \eta|^{p(x)} \eta^{(p^+ - 1 - \frac{p^+}{p'(x)}) p(x)} \,\mathrm{d}x,
\end{aligned}$$

from which the claim follows. □

It follows from the inequalities between the Luxemburg norm and the modular [9, Theorem 1.3] that

$$\| |u|^{p(\cdot)-1} \|_{p'(\cdot)} \le \max\{ \|u\|_{p(\cdot)}^{p^+-1}, \|u\|_{p(\cdot)}^{p^--1} \}. \tag{3.3}$$

When using inequality (3.3) in the sequel, we preserve the letter s for the exponent at which the maximum on the right hand side is attained.

THEOREM 3.4. *Let (u_j) be a sequence of positive $\mathcal{A}_{p(\cdot)}$-superharmonic functions. Then there exist a subsequence (u_{j_k}) and an $\mathcal{A}_{p(\cdot)}$-hyperharmonic function u such that $u_{j_k} \to u$ almost everywhere in Ω and $Du_{j_k} \to Du$ almost everywhere in the set $\{u < \infty\}$.*

PROOF. Assume first that $u_j \le M < \infty$, where $M \ge 1$. Then the functions u_j are supersolutions, [12, Corollary 6.6]. Let $U \Subset U' \Subset \Omega' \Subset \Omega$ be open sets and choose cutoff functions $\eta \in C_0^\infty(U')$ and $\varphi \in C_0^\infty(\Omega')$ such that $0 \le \varphi, \eta \le 1$, $\eta = 1$ in U and $\varphi = 1$ in U'. We want to show that the sequence (ηu_j) is bounded in $W_0^{1,p(\cdot)}(U')$. To this end, we estimate

$$\begin{aligned}\int_{U'} |\nabla(\eta u_j)|^{p(x)} \,\mathrm{d}x \le & C\Big(\int_{U'} u_j^{p(x)} |\nabla \eta|^{p(x)} \,\mathrm{d}x + \int_{U'} |\nabla u_j|^{p(x)} \eta^{p(x)} \,\mathrm{d}x\Big) \\ \le & CM^{p^+} \int_{U'} |\nabla \eta|^{p(x)} \,\mathrm{d}x + C \int_{U'} |\nabla u_j|^{p(x)} \eta^{p(x)} \,\mathrm{d}x.\end{aligned}$$

Since $M - u_j$ is a nonnegative subsolution, we obtain for the second term by the Caccioppoli estimate (Lemma 3.1) that

$$\begin{aligned}\int_{U'} |\nabla u_j|^{p(x)} \eta^{p(x)} \,\mathrm{d}x \le & \int_{U'} |\nabla u_j|^{p(x)} \,\mathrm{d}x \\ \le & \int_{\Omega'} |\nabla (M - u_j)|^{p(x)} \varphi^{p^+} \,\mathrm{d}x \\ \le & C \int_{\Omega'} |u_j - M|^{p(x)} |\nabla \varphi|^{p(x)} \,\mathrm{d}x \\ \le & CM^{p^+} \int_{\Omega'} |\nabla \varphi|^{p(x)} \,\mathrm{d}x.\end{aligned}$$

The $p(\cdot)$-Poincaré inequality now implies that the sequence (ηu_j) is bounded in $W_0^{1,p(\cdot)}(U')$. Thus by Theorem 2.2 there is a function $u \in W_0^{1,p(\cdot)}(U')$ and a subsequence, still denoted by (ηu_j), such that $\eta u_j \to u$ in $L^{p(\cdot)}(U')$ and pointwise almost everywhere in U', and finally $\nabla(\eta u_j) \to \nabla u$ weakly in $L^{p(\cdot)}(U')$. Since $\eta = 1$ in U, it follows that $u_j \to u$ in $L^{p(\cdot)}(U)$ and pointwise almost everywhere in U, and $\nabla u_j \to \nabla u$ weakly in $L^{p(\cdot)}(U)$.

Next we claim that u has a representative which is $\mathcal{A}_{p(\cdot)}$-superharmonic in U. To prove this, set $v_i = \inf_{i \le j} u_j$ and for a fixed i, $w_k = \min_{i \le j \le k} u_j$. Then w_k is a supersolution by [12, Theorem 3.3] and the sequence (w_k) is decreasing and bounded below. By [12, Theorem 5.2] this implies that $v_i = \lim_{k\to\infty} w_k$ is a supersolution. Thus the function $\widetilde{v}_i(x) = \operatorname{ess\,liminf}_{y \to x} v_i(y)$ is $\mathcal{A}_{p(\cdot)}$-superharmonic in U. Let $\widetilde{v} = \lim_{i\to\infty} \widetilde{v}_i$. Now $\widetilde{v}$ is the desired representative since it is $\mathcal{A}_{p(\cdot)}$-superharmonic as an increasing limit of $\mathcal{A}_{p(\cdot)}$-superharmonic functions and

$$u(x) = \lim_{j\to\infty} u_j(x) = \lim_{i\to\infty} v_i(x) = \widetilde{v}(x)$$

for almost every $x \in U$.

We have proved that if the sequence (u_j) is bounded and $U \Subset \Omega$, we can find a subsequence that converges pointwise a.e. in U to a function u which is $\mathcal{A}_{p(\cdot)}$-superharmonic in U. To find a limit which is $\mathcal{A}_{p(\cdot)}$-superharmonic in

Ω, choose open sets U_k, $k = 1, 2, \ldots$, such that $U_k \Subset U_{k+1}$ and $\Omega = \cup_k U_k$. Then we can pick a subsequence (u_j^1) and a limit function u^1 which is $\mathcal{A}_{p(\cdot)}$-superharmonic in U_1. We proceed inductively and pick a subsequence (u_j^{k+1}) of (u_j^k) that converges to a function $u^{k+1} \in \mathcal{S}(U_{k+1})$. Then $u^k = u^{k+1}$ almost everywhere in U_k, and by $\mathcal{A}_{p(\cdot)}$-superharmonicity this holds everywhere. Thus we can define the desired limit function as $u = u^k$ in U_k. This function u is $\mathcal{A}_{p(\cdot)}$-superharmonic in Ω since being $\mathcal{A}_{p(\cdot)}$-superharmonic is a local property. Further, $u \le M$ by the boundedness of the original sequence, and in particular u is a supersolution in Ω.

The next step is to prove that we can assume $\nabla u_j \to \nabla u$ almost everywhere in U_k for any $k = 1, 2, \ldots$ by passing to a further subsequence. To this end, fix a number $\varepsilon > 0$ and let

$$\begin{aligned} E_j =& \{x \in U_k : \mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla u_j) \cdot (\nabla u - \nabla u_j) \ge \varepsilon\}, \\ E_j^1 =& \{x \in E_j : |u - u_j| \ge \varepsilon^2\} \end{aligned}$$

and $E_j^2 = E_j \setminus E_j^1$. $|E_j^1| \to 0$ as $j \to \infty$ since $u_j \to u$ in $L^{p(\cdot)}(U_k)$. To estimate $|E_j^2|$, we note that

$$|E_j^2| \le \frac{1}{\varepsilon} \int_{E_j^2} (\mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla u_j)) \cdot (\nabla u - \nabla u_j) \, \mathrm{d}x,$$

pick a cutoff function $\eta \in C_0^\infty(U_{k+1})$ such that $0 \le \eta \le 1$, $\eta = 1$ in U_k and set

$$v_j = \min((u_j - u + \varepsilon^2)_+, 2\varepsilon^2).$$

We use ηv_j as a test function and obtain

$$0 \le \int_{U_{k+1}} \mathcal{A}(x, \nabla u) \cdot v_j \nabla \eta \, \mathrm{d}x + \int_{U_{k+1} \cap \{|u - u_j| < \varepsilon^2\}} \mathcal{A}(x, \nabla u) \cdot \eta (\nabla u_j - \nabla u) \, \mathrm{d}x. \tag{3.5}$$

We pick another cutoff function $\varphi \in C_0^\infty(U_{k+2})$ such that $0 \le \varphi \le 1$, $\varphi = 1$ in U_{k+1}. Using the Caccioppoli estimate (Lemma 3.1) as in the beginning of the proof, we obtain

$$\begin{aligned} \int_{U_{k+1}} |\nabla u_j|^{p(x)} \, \mathrm{d}x \le& \int_{U_{k+2}} |\nabla (M - u_j)|^{p(x)} \varphi^{p^+} \, \mathrm{d}x \\ \le& C \int_{U_{k+2}} |M - u_j|^{p(x)} |\nabla \varphi|^{p(x)} \, \mathrm{d}x \\ \le& C M^{p^+} \int_{U_{k+2}} |\nabla \varphi|^{p(x)} \, \mathrm{d}x. \end{aligned}$$

The same computation can be carried out for u, since we know that u is a supersolution in Ω. Thus we can find a constant C such that

$$\|\nabla u\|_{p(\cdot), U_{k+1}} \le C \text{ and } \|\nabla u_j\|_{p(\cdot), U_{k+1}} \le C. \tag{3.6}$$

We use (3.5), structure of $\mathcal{A}$, the Hölder inequality, (3.3) and (3.6) and get that

$$\begin{aligned} \int_{U_{k+1} \cap \{|u - u_j| < \varepsilon^2\}} \mathcal{A}(x, \nabla u) \cdot \eta (\nabla u - \nabla u_j) \, \mathrm{d}x \le& \int_{U_{k+1}} \mathcal{A}(x, \nabla u) \cdot v_j \nabla \eta \, \mathrm{d}x \\ \le& C \varepsilon^2 \int_{U_{k+1}} |\nabla u|^{p(x) - 1} |\nabla \eta| \, \mathrm{d}x \\ \le& C \varepsilon^2 \|\nabla u\|_{p(\cdot), U_{k+1}}^s \|\nabla \eta\|_{p(\cdot), U_{k+1}} \\ \le& C \varepsilon^2. \end{aligned}$$

Replacing v_j with $\widetilde{v}_j = \min((u - u_j + \varepsilon^2)_+, 2\varepsilon^2)$ allows us to reverse the roles of u_j and u in the above computation. Thus we conclude that

$$|E_j^2| \le \frac{1}{\varepsilon} \int_{E_j^2} (\mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla u_j)) \cdot (\nabla u - \nabla u_j)\, \mathrm{d}x \le C\varepsilon.$$

It follows that

$$|E_j| = |E_j^1| + |E_j^2| \le (C+1)\varepsilon \tag{3.7}$$

for $j \ge j_\varepsilon$.

Estimate (3.7) implies that $\nabla u_j \to \nabla u$ in measure in U_k; this allows us to pick the desired pointwise almost everywhere convergent subsequence. To prove the convergence in measure, we assume the opposite and find positive numbers δ and a such that

$$|\{x \in U_k : |\nabla u_j - \nabla u| \ge \delta\}| \ge a > 0.$$

Pick any sequence (ε_i) such that $\varepsilon_i \to 0$ as $i \to \infty$. We note that

$$\begin{aligned} &|\{x \in U_k : (\mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla u_j)) \cdot (\nabla u - \nabla u_j) \ge \varepsilon_i\}| \\ &\ge |\{x \in U_k : (\mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla u_j)) \cdot (\nabla u - \nabla u_j) \ge \varepsilon_i, |\nabla u_j - \nabla u| \ge \delta\}|. \end{aligned}$$

By measure theory, the structure of $\mathcal{A}$ and the counterassumption, the right hand side tends to a limit $L \ge a$ as $i \to \infty$. Thus there is a number $\varepsilon_0 > 0$ such that

$$|\{x \in U_k : (\mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla u_j)) \cdot (\nabla u - \nabla u_j) \ge \varepsilon\}| \ge a/2 > 0$$

whenever $\varepsilon \le \varepsilon_0$, and this contradicts (3.7).

We can now assume that $\nabla u_j \to \nabla u$ pointwise almost everywhere in Ω. This follows from the pointwise almost everywhere convergence in U_k proved above by an inductive process similar to the one by which we found the superharmonic limit u in Ω.

As the final step we remove the boundedness assumption by another diagonalization argument. By the first part of the theorem, we can find a subsequence (u_j^1) and an $\mathcal{A}_{p(\cdot)}$-superharmonic function u_1 such that

$$\min(u_j^1, 1) \to u_1 \text{ and } \nabla \min(u_j^1, 1) \to \nabla u_1$$

almost everywhere in Ω. We proceed inductively and pick a subsequence (u_j^k) of (u_j^{k-1}) such that

$$\min(u_j^k, k) \to u_k \text{ and } \nabla \min(u_j^k, k) \to \nabla u_k$$

almost everywhere in Ω. We observe that if $l \ge k$ and $u_k(x) < k$, we have $u_l(x) = u_k(x)$. Thus the sequence (u_k) is increasing, and we conclude that the limit $u = \lim_{k\to\infty} u_k$ exists and defines the desired $\mathcal{A}_{p(\cdot)}$-hyperharmonic function in Ω. We note that by construction $\min(u, k) = u_k$, so that for the diagonal sequence (u_k^k) it holds that $\nabla u_k^k \to Du$ almost everywhere in the set $\{u < \infty\}$. □

4. Existence of $\mathcal{A}_{p(\cdot)}$-superharmonic solutions

In this section we prove our main existence result, Theorem 4.7. Throughout, we use T to denote the map defined by

$$(Tu, \varphi) = \int_\Omega \mathcal{A}(x, Du) \cdot \nabla\varphi \,\mathrm{d}x, \tag{4.1}$$

where $\varphi \in C_0^\infty(\Omega)$. By Theorem 2.7 and the structure of $\mathcal{A}$, Tu defines a distribution for $\mathcal{A}_{p(\cdot)}$-superharmonic functions u that belong to $L^t_{loc}(\Omega)$ for some $t > 0$, and $Tu \in (W^{1,p(\cdot)}(\Omega))^*$ if $u \in W^{1,p(\cdot)}(\Omega)$.

THEOREM 4.2. *Let u be an $\mathcal{A}_{p(\cdot)}$-superharmonic function such that $u \in L^t_{loc}(\Omega)$ for some $t > 0$. Then there is a positive Radon measure μ such that*

$$-\operatorname{div}\mathcal{A}(x, Du) = \mu$$

in the sense of distributions.

PROOF. Since $u \in L^t_{loc}(\Omega)$ for some $t > 0$, $|Du|^{p(x)-1} \in L^1_{loc}(\Omega)$ by Theorem 2.7. Pick any $\varphi \in C_0^\infty(\Omega)$ such that $\varphi \geq 0$ and denote $u_k = \min(u, k)$. Then

$$\mathcal{A}(x, \nabla u_k) \cdot \nabla\varphi \to \mathcal{A}(x, Du) \cdot \nabla\varphi$$

pointwise almost everywhere by the continuity of $\xi \mapsto \mathcal{A}(x, \xi)$.

Using the structure of $\mathcal{A}$, we have

$$|\mathcal{A}(x, \nabla u_k) \cdot \nabla\varphi| \leq C|\nabla u_k|^{p(x)-1}|\nabla\varphi| \leq C|Du|^{p(x)-1}|\nabla\varphi|.$$

Using the dominated convergence theorem and the fact that the functions u_k are supersolutions, we conclude that

$$(Tu, \varphi) = \int_\Omega \mathcal{A}(x, Du) \cdot \nabla\varphi \,\mathrm{d}x = \lim_{k\to\infty} \int_\Omega \mathcal{A}(x, \nabla u_k) \cdot \nabla\varphi \,\mathrm{d}x \geq 0.$$

The claim now follows from the Riesz representation theorem, see for example [27, Theorem 2.14]. □

LEMMA 4.3. *Let $u, v \in W_0^{1,p(\cdot)}(\Omega)$ be supersolutions such that*

$$Tu = \mu \leq \nu = Tv.$$

Then $u \leq v$ almost everywhere in Ω.

PROOF. Let $\eta = \min(v - u, 0)$. Since $\mu \leq \nu$, we obtain that

$$\begin{aligned} 0 &\geq \int_\Omega \eta \,\mathrm{d}\nu - \int_\Omega \eta \,\mathrm{d}\mu \\ &= \int_\Omega \mathcal{A}(x, \nabla v) \cdot \nabla\eta \,\mathrm{d}x - \int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla\eta \,\mathrm{d}x \\ &= \int_{\{u>v\}} (\mathcal{A}(x, \nabla v) - \mathcal{A}(x, \nabla u)) \cdot (\nabla v - \nabla u) \,\mathrm{d}x. \end{aligned}$$

By the monotonicity of $\mathcal{A}$, it follows that $\nabla v = \nabla u$ almost everywhere in $\{u > v\}$. Hence $\nabla\eta = 0$ and it follows that $\eta = 0$ almost everywhere, which means that $v \geq u$ almost everywhere. □

To show the existence of solutions in the case $\mu \in (W_0^{1,p(\cdot)}(\Omega))^*$, we use the following theorem. See [20, Théorème 2.1, p. 171] for the proof.

THEOREM 4.4. *Let X be a reflexive, separable Banach space, and assume that $T : X \to X^*$ is*

1. *monotone, i.e. $\langle Tu - Tv, u - v\rangle \geq 0$ for all $u, v \in X$,*
2. *bounded, i.e. if $E \subset X$ is bounded, so is $T(E)$;*
3. *demicountinuous, i.e. $x_j \to x$ implies $(Tx_j, y) \to (Tx, y)$ for all $y \in X$ and*
4. *coercive, i.e. for a sequence $(x_j) \subset X$ such that $\|x_j\|_X \to \infty$ it holds that*

$$\frac{(Tx_j, x_j)}{\|x_j\|_X} \to \infty \text{ as } j \to \infty.$$

Then T is surjective, i.e. the equation $Tx = f$ has a solution $x \in X$ for each $f \in X^*$.

THEOREM 4.5. *Let Ω be a bounded domain and $\mu \in (W_0^{1,p(\cdot)}(\Omega))^*$ be a positive Radon measure. Then there is a unique nonnegative supersolution $u \in W_0^{1,p(\cdot)}(\Omega)$ such that*

$$-\operatorname{div} \mathcal{A}(x, \nabla u) = \mu$$

in the sense of distributions.

PROOF. We prove the existence part by verifying the assumptions of Theorem 4.4 for the map $T : W_0^{1,p(\cdot)}(\Omega) \to (W_0^{1,p(\cdot)}(\Omega))^*$ given by (4.1). First, the monotonicity of T is an immediate consequence of the monotonicity assumption on $\mathcal{A}$.

Using the structure of $\mathcal{A}$, the Hölder inequality and (3.3), we infer that

$$\begin{aligned}|(Tu, v)| \leq& C \int_\Omega |\nabla u|^{p(x)-1} |\nabla v| \,\mathrm{d}x \\ \leq& C \| |\nabla u|^{p(x)-1} \|_{p'(\cdot)} \|\nabla v\|_{p(\cdot)} \\ \leq& C \|u\|_{1,p(\cdot)}^s \|v\|_{1,p(\cdot)}.\end{aligned}$$

This implies that $\|Tu\|_{(W_0^{1,p(\cdot)}(\Omega))^*} \leq C\|u\|_{1,p(\cdot)}^s$, so that T is bounded.

Let $(u_j) \subset W_0^{1,p(\cdot)}(\Omega)$ be such that $u_j \to u$ in $W_0^{1,p(\cdot)}(\Omega)$. We pass to a subsequence and assume that $u_j \to u$ and $\nabla u_j \to \nabla u$ pointwise almost everywhere. By continuity of the map $\xi \mapsto \mathcal{A}(x, \xi)$, it follows that $\mathcal{A}(x, \nabla u_j) \to \mathcal{A}(x, \nabla u)$ almost everywhere. Since

$$\int_\Omega |\mathcal{A}(x, \nabla u_j)|^{p(x)/(p(x)-1)} \,\mathrm{d}x \leq C \int_\Omega |\nabla u_j|^{p(x)} \,\mathrm{d}x \leq M < \infty$$

by the convergence of the sequence (u_j), $(\mathcal{A}(x, \nabla u_j))$ is bounded in $L^{p'(\cdot)}(\Omega)$. Thus we may pass to a further subsequence and assume that $\mathcal{A}(x, \nabla u_j) \to \mathcal{A}(x, \nabla u)$ weakly in $L^{p'(\cdot)}(\Omega)$.

This implies that the whole sequence converges weakly; indeed, assuming the opposite, we find a weak neighbourhood U of $\mathcal{A}(x, \nabla u)$ and a subsequence such that $(\mathcal{A}(x, \nabla u_{j_k})) \subset L^{p'(\cdot)}(\Omega) \setminus U$. We may assume pointwise convergence by passing to a further subsequence, and this sub-subsequence converges weakly in $L^{p'(\cdot)}(\Omega)$ to $\mathcal{A}(x, \nabla u)$ by the earlier argument, which is a contradiction. It follows that

$$(Tu_j, v) = \int_\Omega \mathcal{A}(x, \nabla u_j) \cdot \nabla v \,\mathrm{d}x \to \int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla v \,\mathrm{d}x = (Tu, v).$$

Let (u_j) be a sequence such that $\|u_j\|_{1,p(\cdot)} \to \infty$. Since the norms $\|u\|_{1,p(\cdot)}$ and $\|\nabla u\|_{p(\cdot)}$ are equivalent on $W_0^{1,p(\cdot)}(\Omega)$, we may assume that $\|\nabla u_j\|_{p(\cdot)} \geq 1$. By [9, Theorem 1.3], this implies that $\varrho_{p(\cdot)}(\nabla u_j) \geq \|\nabla u_j\|_{p(\cdot)}^{p^-}$. We use the structure of $\mathcal{A}$, and the $p(\cdot)$-Poincaré inequality and obtain that

$$\frac{(Tu_j, u_j)}{\|u_j\|_{1,p(\cdot)}} \geq C \frac{\int_\Omega |\nabla u_j|^{p(x)} \,\mathrm{d}x}{\|u_j\|_{1,p(\cdot)}} \geq C \frac{\|\nabla u_j\|_{p(\cdot)}^{p^-}}{\|u_j\|_{1,p(\cdot)}} \geq C \|u_j\|_{1,p(\cdot)}^{p^- -1} \to \infty$$

as $j \to \infty$.

Finally, we note that the uniqueness and positivity claims follow from Lemma 4.3. □

We say that a sequence of measures (μ_j) converges weakly to a measure μ if

$$\lim_{j \to \infty} \int_\Omega \varphi \,\mathrm{d}\mu_j = \int_\Omega \varphi \,\mathrm{d}\mu$$

for all $\varphi \in C_0^\infty(\Omega)$. We use the following elementary technique from Mikkonen's thesis [25] to approximate a general, finite positive Radon measure μ by measures $\mu_j \in (W_0^{1,p(\cdot)}(\Omega))^*$.

LEMMA 4.6. *Let Ω be a bounded open set and assume that μ is a finite positive Radon measure on Ω. Then there is a sequence (μ_j) of finite positive Radon measures such that $\mu_j \in (W_0^{1,p(\cdot)}(\Omega))^*$, $\mu_j \to \mu$ weakly and $\mu_j(\Omega) \le \mu(\Omega)$.*

PROOF. Let $Q_{i,j}$, $i = 1, \dots, N_j$, be the dyadic cubes with side length 2^{-j} contained in Ω. For any measurable set $E \subset \Omega$ we define

$$\mu_j(E) = \sum_{i=1}^{N_j} \frac{\mu(Q_{i,j})}{|Q_{i,j}|} |E \cap Q_{i,j}|,$$

and the proof will be completed by showing that the sequence (μ_j) has the desired properties. First we observe that

$$\mu_j(\Omega) = \sum_{i=1}^{N_j} \mu(Q_{i,j}) \le \mu(\Omega),$$

since the cubes $Q_{i,j}$ do not completely cover the set Ω. Given a function $\varphi \in C_0^\infty(\Omega)$ we obtain

$$\begin{aligned} \left|\int_\Omega \varphi \, \mathrm{d}\mu_j\right| =& \left|\sum_{i=1}^{N_j} \frac{\mu(Q_{i,j})}{|Q_{i,j}|} \int_{Q_{i,j}} \varphi \, \mathrm{d}x\right| \\ \le& 2^{nj} \mu(\Omega) \int_\Omega |\varphi| \, \mathrm{d}x \le C\|\varphi\|_{p(\cdot)} \le C\|\varphi\|_{1,p(\cdot)}, \end{aligned}$$

so that $\mu_j \in (W_0^{1,p(\cdot)}(\Omega))^*$. The weak convergence follows in the same way as in [25, Lemma 2.12], and we omit the details. □

THEOREM 4.7. *Let Ω be bounded and μ a finite positive Radon measure. Then there is an $\mathcal{A}_{p(\cdot)}$-superharmonic function u such that $\min(u, k) \in W_0^{1,p(\cdot)}(\Omega)$ for all $k > 0$ and*

$$-\operatorname{div} \mathcal{A}(x, Du) = \mu$$

in the sense of distributions.

PROOF. Let (μ_j) be the sequence of measures belonging to $(W_0^{1,p(\cdot)}(\Omega))^*$ obtained from Lemma 4.6 and denote by (u_j) the sequence of supersolutions satisfying

$$-\operatorname{div} \mathcal{A}(x, \nabla u_j) = \mu_j \tag{4.8}$$

in the sense of distributions; such functions u_j exist by Theorem 4.5.

By Theorem 3.4, there is an $\mathcal{A}_{p(\cdot)}$-hyperharmonic function u such that we can assume $u_j \to u$ and $\nabla \min(u_j, k) \to \nabla \min(u, k)$ almost everywhere by passing to a subsequence. As the first step, we prove that $u \in L^t_{loc}(\Omega)$ for some $t > 0$. To this end, we use structure of $\mathcal{A}$ and (4.8) and infer that

$$\begin{aligned} \int_\Omega |\nabla \min(u_j, k)|^{p(x)} \, \mathrm{d}x \le& C \int_\Omega \mathcal{A}(x, \nabla u_j) \cdot \nabla \min(u_j, k) \, \mathrm{d}x \\ =& C \int_\Omega \min(u_j, k) \, \mathrm{d}\mu_j \\ \le& C\mu_j(\Omega)k \le C\mu(\Omega)k. \end{aligned} \tag{4.9}$$

From (4.9) and the p^--Poincaré inequality, we obtain that

$$\begin{aligned}\int_\Omega |\min(u_j,k)|^{p^-}\,\mathrm{d}x \leq& C\int_\Omega |\nabla \min(u_j,k)|^{p^-}\,\mathrm{d}x\\ \leq& \int_\Omega |1+\nabla\min(u_j,k)|^{p(x)}\,\mathrm{d}x\\ \leq& C|\Omega| + C\mu(\Omega)k \leq C(|\Omega|+\mu(\Omega))k. \end{aligned} \tag{4.10}$$

Since $u_j \to u$ almost everywhere, it follows from Fatou's lemma and (4.10) that

$$\int_\Omega |\min(u,k)|^{p^-}\,\mathrm{d}x \leq Mk,$$

with the constant $M = C(|\Omega|+\mu(\Omega))$ independent of k. This estimate implies that u is finite almost everywhere. Indeed, denoting $E = \{x \in \Omega : u(x) = \infty\}$, we get

$$|E| = \frac{1}{k^{p^-}}\int_E k^{p^-}\,\mathrm{d}x \leq \frac{1}{k^{p^-}}\int_\Omega |\min(u,k)|^{p^-}\,\mathrm{d}x \leq Mk^{1-p^-} \to 0$$

as $k\to\infty$. Estimate (4.9) and the $p(\cdot)$-Poincaré inequality imply that $(\min(u_j,k))$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$. It follows that $\min(u,k) \in W_0^{1,p(\cdot)}(\Omega)$ since weak limits must coincide with pointwise limits. Next, we use pointwise a.e. convergence of the gradients and Fatou's lemma and argue as in (4.10). This leads to the estimate

$$\int_\Omega |\nabla\min(u,k)|^{p^-}\,\mathrm{d}x \leq C(|\Omega|+\mu(\Omega))k.$$

This inequality allows us to use Lemma 2.6 to conclude that $u \in L^t_{loc}(\Omega)$ for some $t>0$, and then we use Theorem 2.7 to conclude that $u, Du \in L^{q(p(x)-1)}_{loc}(\Omega)$ for some $q>1$.

By Theorem 4.2, there is a measure ν such that

$$-\operatorname{div}\mathcal{A}(x,Du) = \nu \tag{4.11}$$

in the sense of distributions. We will complete the proof by showing that $\mu=\nu$ in the sense of distributions. We know that $u\in L^t_{loc}(\Omega)$ for some $t>0$. We consider an arbitrary ball $B = B(x_0,2R)$, chosen sufficiently small that the exponent t is admissible in Lemma 2.9. By the usual partition of unity argument, it suffices to show that

$$\int_B \varphi\,\mathrm{d}\mu = \int_B \varphi\,\mathrm{d}\nu$$

for all $\varphi\in C_0^\infty(B)$.

The constants on the right hand sides of (4.9) and (4.10) are independent of j. Hence the sequence $(\|u_j\|_{L^t(B)})$ is bounded for some $t>0$, by Lemma 2.6. The pointwise convergence of (u_j) implies that $(\inf_{B_R} u_j)$ is bounded. Thus the sequence $(|\nabla u_j|^{p(x)-1})$ is bounded in $L^q(B)$ for some $q>1$ by Lemma 2.9. Next we use the structure of $\mathcal{A}$, and get that

$$\int_B |\mathcal{A}(x,\nabla u_j)|^q\,\mathrm{d}x \leq C\int_B |\nabla u_j|^{q(p(x)-1)}\,\mathrm{d}x \leq C.$$

Thus the sequence $(\mathcal{A}(x,\nabla u_j))$ is also bounded in $L^q(B)$, and it follows from the continuity of $\xi\mapsto\mathcal{A}(x,\xi)$ that $\mathcal{A}(x,\nabla u_j)\to\mathcal{A}(x,Du)$ weakly in $L^q(B)$. We use the weak convergence in L^q and (4.11) to conclude that

$$\lim_{j\to\infty}\int_B \varphi\,\mathrm{d}\mu_j = \lim_{j\to\infty}\int_B \mathcal{A}(x,\nabla u_j)\cdot\nabla\varphi\,\mathrm{d}x = \int_B \mathcal{A}(x,Du)\cdot\nabla\varphi\,\mathrm{d}x = \int_B\varphi\,\mathrm{d}\nu,$$

which completes the proof. □

5. Solutions with isolated singularities

In this section we show the existence of solutions with nonremovable isolated singularities. We assume that the origin belongs to Ω, $1 < p^- \le p^+ < n$ and use δ to denote the unit mass at the origin.

THEOREM 5.1. *If u is a solution of*

$$-\operatorname{div}\mathcal{A}(x, Du) = \delta \tag{5.2}$$

obtained from Theorem 4.7, then u is a solution of

$$-\operatorname{div}\mathcal{A}(x, \nabla u) = 0 \tag{5.3}$$

in $\Omega \setminus \{0\}$.

PROOF. Let (μ_j) be the sequence approximating δ we obtain from Lemma 4.6. From the proof of the lemma we see that the support of μ_j is contained in a ball $B_j = B(0, c\,2^{-j})$, where the constant c is independent of j. Thus the corresponding supersolution u_j is a nonnegative solution of (5.3) in $\Omega \setminus \overline{B}_j$.

Next we pass to the subsequence provided by Theorem 3.4, still denoted by (u_j). Fix a ball $B = B(x_0, R)$ with $x_0 \in \Omega$, $x_0 \neq 0$ and R such that $6B \Subset \Omega\setminus\{0\}$. Then for sufficiently large j it holds that $6B \Subset \Omega \setminus \overline{B}_j$. We discard the values of j that are not sufficiently large and still denote the subsequence we obtain by (u_j).

Since the right hand side of (4.9) is independent of j, Lemma 2.6 implies that the sequence (u_j) is bounded in $L^t(\Omega)$ for some $t > 0$. We can use Lemma 2.13 and the bound in $L^t(\Omega)$ to conclude that the sequence (u_j) is uniformly bounded in B. Indeed, we can pick $r > 0$ such that $t = q'r$ and then assure that r is admissible in Lemma 2.13 by passing to a smaller ball if necessary, since $p(\cdot)$ is continuous. Further, the sequence is also equicontinuous in B, since the bound in $L^t(\Omega)$ also allows us to take a constant independent of j in Harnack's inequality, [14, Theorem 3.9]. The reason for this is the fact that the dependence of the constant of Harnack's inequality on u is the same as in Lemma 2.13. Now by using the Arzela–Ascoli theorem we can assume that (u_j) converges uniformly in B by passing to a further subsequence. This uniform limit must be u by the pointwise convergence, and thus u is a solution in B by [12, Corollary 5.3]. Since this argument can be repeated for any point $x_0 \in \Omega \setminus \{0\}$, it follows that u is a solution in $\Omega \setminus \{0\}$. □

The above proof can be easily modified to show that a solution of

$$-\operatorname{div}\mathcal{A}(x, Du) = \mu$$

constructed by the present method is a solution of (5.3) in $\Omega \setminus \operatorname{spt}(\mu)$. However, solutions of equations involving measures are not necessarily unique without some additional assumptions, even when the exponent is constant; see [18] for an example. Hence our present tools are insufficient to obtain the conclusion of Theorem 5.1 for an arbitrary solution of (5.2).

A solution of (5.2) cannot be a supersolution of (5.3). To see this, note that if the measure μ is such that $\mu \in (W_0^{1,p(\cdot)}(\Omega))^*$, then $\mu(E) = 0$ for all $E \subset \Omega$ such that $\operatorname{cap}_{p(\cdot)}(E, \Omega) = 0$, where $\operatorname{cap}_{p(\cdot)}$ is the variational $p(x)$-capacity, as defined in [11]. This can be proven in the same way as in the constant exponent case, see [25, Lemma 2.4]. Further, recall that the operator T defined by (4.1) maps $W_0^{1,p(\cdot)}(\Omega)$ to $(W_0^{1,p(\cdot)}(\Omega))^*$. This implies that the measure μ associated to a supersolution $u \in W_0^{1,p(\cdot)}(\Omega)$ by Theorem 4.2 must belong to $(W_0^{1,p(\cdot)}(\Omega))^*$. Clearly $\operatorname{cap}_{p(\cdot)}(\{0\}, \Omega) = 0$ if $p^+ < n$, so that $\delta \notin (W_0^{1,p(\cdot)}(\Omega))^*$; hence a solution of (5.2) cannot be a supersolution.

Acknowledgements

The author was supported in part by the Academy of Finland, and would also like to thank professor Fumi-Yuki Maeda for his comments on an earlier version of this work, and the anonymous referee for carefully reading the manuscript.

References

[1] E. Acerbi and G. Mingione, Regularity results for a class of functionals with nonstandard growth, Arch. Rational Mech. Anal., **156** (2001), 121–140.

[2] Y. A. Alkhutov, The Harnack inequality and the Hölder property of solutions of nonlinear elliptic equations with a nonstandard growth condition, Differential Equations, **33** (1997), 1653–1663.

[3] Y. A. Alkhutov and O. V. Krasheninnikova, Continuity at boundary points of solutions of quasilinear elliptic equations with nonstandard growth, Izv. Ross. Akad. Nauk. Ser. Mat., **68** (2004), 3–60, translation in *Izv. Math.* **68** (2004), no. 6, 1063-1117.

[4] S. Antontsev and V. Zhikov, Higher integrability for parabolic equations of $p(x,t)$-Laplacian type, Adv. Differential Equations, **10** (2005), 1053–1080.

[5] L. Boccardo and T. Gallouët, Non-linear elliptic and parabolic equations involving measure data, J. Funct. Anal., **87** (1989), 149–169.

[6] L. Boccardo and T. Galloüet, Non-linear elliptic equations with right hand side measures, Comm. Partial Diff. Eq., **17** (1992), 641–655.

[7] P. Bénilan, L. Boccardo, T. Galloüet, R. Gariepy, M. Pierre, and J. L. Vazquez, An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), **22** (1995), 241–274.

[8] X. Fan and D. Zhao, A class of De Giorgi type and Hölder continuity, Nonlinear Anal., **36** (1999), 295–318.

[9] X. Fan and D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, J. Math. Anal. Appl., **263** (2001), 424–446.

[10] P. Harjulehto, Variable exponent Sobolev spaces with zero boundary values, Math. Bohem., **132** (2007), 125–136.

[11] P. Harjulehto, P. Hästö, and M. Koskenoja, Properties of capacities in variable exponent Sobolev spaces, J. Anal. Appl., **5** (2007), 71–92.

[12] P. Harjulehto, P. Hästö, M. Koskenoja, T. Lukkari, and N. Marola, Obstacle problem and superharmonic functions with nonstandard growth, Nonlinear Anal., to appear. Preprint available at `http://mathstat.helsinki.fi/analysis/varsobgroup/`.

[13] P. Harjulehto, P. Hästö, M. Koskenoja, and S. Varonen, The Dirichlet energy integral and variable exponent Sobolev spaces with zero boundary values, Potential Anal., **25** (2006), 205–222.

[14] P. Harjulehto, J. Kinnunen, and T. Lukkari, Unbounded supersolutions of nonlinear equations with nonstandard growth, Bound. Value Probl., **2007** (2007), Article ID 48348, 20 pages, available at `http://www.hindawi.com/GetArticle.aspx?doi=10.1155/2007/48348`.

[15] J. Heinonen, T. Kilpeläinen, and O. Martio, Nonlinear potential theory of degenerate elliptic equations, Oxford University Press, 1993.

[16] P. Hästö, On the density of continuous functions in variable exponent Sobolev space, Rev. Mat. Iberoamericana, **23** (2007), 215 – 237.

[17] T. Kilpeläinen and J. Malý, Degenerate elliptic equations with measure data and nonlinear potentials, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), **19** (1992), 591–613.

[18] T. Kilpeläinen and X. Xu, On the uniqueness problem for quasilinear elliptic equations involving measures, Rev. Mat. Iberoamericana, **12** (1996), 461–475.

[19] O. Kováčik and J. Rákosník, On spaces $L^{p(x)}$ and $W^{1,p(x)}$, Czechoslovak Math. J., **41** (1991), 592–618.

[20] J.-L. Lions, Quelques méthodes de resolution des problèmes aux limites nonlinéaires, Dunod, 1969.

[21] T. Lukkari, Singular solutions of elliptic equations with nonstandard growth, Math. Nachr., to appear. Preprint available at `http://mathstat.helsinki.fi/analysis/varsobgroup/`.

[22] F.-Y. Maeda, Renormalized solutions of Dirichlet problems for quasilinear elliptic equations with general measure data, preprint.

[23] J. Malý and W. P. Ziemer, Fine Regularity of Solutions of Elliptic Partial Differential Equations, American Mathematical Society, 1997.

[24] P. Marcellini, Regularity and existence of solutions of elliptic equations with p, q-growth conditions, J. Differential Equations, **90** (1991), 1–30.

[25] P. Mikkonen, On the Wolff potential and quasilinear elliptic equations involving measures, Ann. Acad. Sci. Fenn. Math. Diss., **104** (1996), 1–71.

[26] G. Mingione, Regularity of minima: an invitation to the dark side of the calculus of variations, Appl. Math., **51** (2006), 355–425.

[27] W. Rudin, Real and Complex Analysis, McGraw-Hill, 1987.

[28] M. Sanchon and J. M. Urbano, Entropy solutions for the $p(x)$-laplace equation, Trans. Amer. Math. Soc., to appear.

[29] J. Serrin, Local behavior of solutions of quasi-linear equations, Acta Math., **111** (1964), 247–302.

[30] V. V. Zhikov, On Lavrentiev's phenomenon, Russian J. Math. Phys., **3** (1995), 249–269.

[31] V. V. Zhikov, On some variational problems, Russian J. Math. Phys., **5** (1997), 105–116.

Teemu Lukkari
Helsinki University of Technology
Institute of Mathematics
P.O. Box 1100
FI-02015 TKK
Finland
E-mail: Teemu.Lukkari@tkk.fi