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Analysis of the Convergence of the Fixed-Point Method Used for Solving Nonlinear Rotational Magnetic Field Problems

Emad Dlala and Antero Arkkio

***Abstract*—The fixed-point method has not been widely used for solving nonlinear electromagnetic field problems, except for the hysteretic problem, for which it is the prevailing method. The method converges stably with a slow rate, exactly opposite to the Newton-Raphson method, which can easily suffer from instability, but which, if it converges, does so remarkably fast. This article is aimed to analyze the convergence of the fixed-point method, look into the barriers behind the slow convergence, and find out how to overcome them. The analysis has proven to be fruitful and provided sound techniques for speeding up the convergence of the fixed-point method. The 2D and 3D problems have been given special attention and a general formula for the fastest convergence has been derived. A time-stepping finite-element formulation has been performed to test the convergence of the fixed-point method in which the Newton-Raphson method has been also used for the sake of comparison. The magnetic field simulations of two rotating electrical machines have been carried out where certain factors including the time-step size have been investigated.**

***Index Terms*—Finite elements, fixed-point iteration, nonlinear material, rotating electrical machines, time-stepping, 2D, 3D magnetic field.**

I. Introduction

DESPITE rapid advances in both computer capabilities and mathematical algorithms, solving nonlinear systems still remains difficult and challenging. Many of the most important modern problems, such as magnetic hysteretic field problems, require the employment of iterative methods whose mere successes to handle nonlinearity are often questionable. Indeed, although the standard Newton-Raphson method rapidly converges near the solution, it cannot always ensure convergence since it is fundamentally based upon the derivative and thus can easily suffer from instability. The classical fixed-point method, on the other hand, can produce stable solutions but very slow ones [1]–[4]. It is stable because the method does not, in principle, need the derivative and the condition of contraction mapping on which the method is based can be easily satisfied; it is slow, however, because the contraction factor resulted from using the classical approach [3] is close to one, allowing significantly slow convergence. Therefore, since the fixed-point method is stable, it is important to explore and improve its convergence rate.

Recently, the authors have proposed a locally convergent fixed-point method for solving time-stepping nonlinear field problems in which the focus was on the 1D problem [5]. The method has proven to be instrumental in accelerating the convergence of the fixed-point iteration. This article extends the study to account for the 2D and 3D problems. The approach made in [5] is only suitable to cope with the 1D nonlinear problem. The analysis of the convergence of the fixed-point method for the general case (1D, 2D, and 3D) requires special treatment. In this article, since the problem is multidimensional, the power of linear algebra is adopted for assessing the convergence of the fixed-point method. A Jacobian matrix that characterizes the convergence of the problem is constructed from the fixed-point formulations of the magnetic problem. It is shown that by utilizing the information of the Jacobian matrix, one can arrive at the conditions of the optimal convergence of the magnetic problem.

The fixed-point method is distinguished by its independence from calculating the derivative, but by utilizing the derivative, the method can produce dramatically faster iteration. It is important to remember that the Newton-Raphson method uses the derivative to *find* the solution and thus the method is highly sensitive to the derivative. On the other hand, the fixed-point method may apply the derivative to *accelerate* the convergence, and therefore, in the worst-case scenario, the derivative can even be approximated if it is difficult to determine.

In this article, the authors analyze the convergence and efficiency of the fixed-point method in the 2D and 3D rotational field problems. At first, the underlying theory of the fixed-point method is discussed and then applied to study nonlinear magnetic systems. The proposed methods have been applied to solve the magnetic problem in the 2D finite-element analysis of electrical motors.

II. The Fixed-Point Theory

The conditions and criteria for the convergence of the fixed-point theory are well documented in mathematics [6], [7]. Here only the most relevant and important ones are summarized.

In general, a nonlinear iterative system can be devised as

$$\boldsymbol{u}^{(k+1)} = \boldsymbol{G}(\boldsymbol{u}^{(k)}) \tag{1}$$

where $\boldsymbol{G} : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a real multidimensional function. The solution of (1), if one exists, is a fixed-point vector $\boldsymbol{u}^* = \boldsymbol{G}(\boldsymbol{u}^*)$ and is a discrete collection of points $\boldsymbol{u}^{(k)} \in \mathbb{R}^m$, which start at a specified initial iterate $\boldsymbol{u}^{(0)}$.

It should be assumed that the multidimensional function $\boldsymbol{G} : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is defined on all of $\mathbb{R}^m$ so that the successive

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iterates $\boldsymbol{u}^{(k)}$ never leave its domain of definition. Thus, the function $\boldsymbol{G}$ must be at least continuous. Even though the procedure of solving (1) is trivial, understanding its behavior is certainly not. The solution may converge to a particular value, which is the key requirement for numerical solution methods.

The fixed-point iteration is based on the contraction mapping principle. This principle guarantees the existence and uniqueness of fixed points and provides a constructive method to find those fixed points. A function $\boldsymbol{G} : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a contraction on a domain $\boldsymbol{G} : \boldsymbol{\Omega} \rightarrow \boldsymbol{\Omega}$ at a point $\boldsymbol{u}^* \in \boldsymbol{\Omega}$ if there exists a constant $0 \leq \alpha < 1$ such that

$$||\boldsymbol{G}(\boldsymbol{u}) - \boldsymbol{G}(\boldsymbol{u}^*)|| \leq \alpha ||\boldsymbol{u} - \boldsymbol{u}^*||. \tag{2}$$

Here, since $\boldsymbol{G}$ is a contraction, $\boldsymbol{u}^*$ is an asymptotically stable fixed point. In general, condition (2) can be perceived for multidimensional functions by using the power of linear algebra. If the spectral radius of the Jacobian matrix $\rho(\boldsymbol{G}^{'}(\boldsymbol{u}^*)) < 1$, then $\boldsymbol{G}$ is a contraction at $\boldsymbol{u}^*$. Hence, the fixed point $\boldsymbol{u}^*$ is asymptotically stable and a basin of attraction exists. The spectral radius is defined as $\rho = \max_{1 \leq i \leq m}(|\lambda_i|)$, where λ_i are the eigenvalues of $\boldsymbol{G}^{'}$. In any case, one can replace the spectral radius by any type of norm $||\boldsymbol{G}^{'}(\boldsymbol{u}^*)||$ without violating the convergence condition since the inequality $\rho(\boldsymbol{G}^{'}) \leq ||\boldsymbol{G}^{'}||$ always holds for any matrix.

The smaller the spectral radius or matrix norm of the Jacobian matrix at the fixed point, the faster the iterates will converge to it. The fastest convergence, which is *quadratic*, will occur when the Jacobian matrix $\boldsymbol{G}^{'}(\boldsymbol{u}^*) = \mathbf{O}$ is the zero matrix, i.e., all first-order partial derivatives of the components of $\boldsymbol{G}$ vanish at the fixed point. (This is analogous to the condition of the quadratic convergence for the 1D function in which $G^{'}(u) = 0$ is enforced.) The quadratic convergence estimate is given by

$$||\boldsymbol{u}^{(k+1)} - \boldsymbol{u}^*|| \leq \tau ||\boldsymbol{u}^{(k)} - \boldsymbol{u}^*||^2 \tag{3}$$

for some constant $\tau > 0$.

In practical numerical systems, the norm or spectral radius of the Jacobian matrix cannot be known because one does not know in advance where the fixed point is. This apparent difficulty can be circumvented by requiring $||\boldsymbol{G}^{'}(\boldsymbol{u})||$ be < 1 for all $\boldsymbol{u}$. This is the beauty of the fixed-point method: it permits to construct different iterated functions $\boldsymbol{G}$ without changing the essence of the problem. Therefore, one can further allow to satisfy $||\boldsymbol{G}^{'}(\boldsymbol{u})|| < 1$ in a local domain containing the fixed point. In such a case, a locally convergent fixed point is guaranteed for all $\boldsymbol{u}^{(0)}$ sufficiently close to $\boldsymbol{u}^*$.

III. Analysis of The Fixed-Point method in Magnetic Systems

Electromagnetic field problems are governed by Maxwell's equations. In the scope of nonlinearity, one only needs to write the following:

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} \tag{4}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{5}$$

where

$\boldsymbol{H} =$ magnetic field strength $(\mathrm{A/m})$,
$\boldsymbol{J} =$ electric current density $(\mathrm{A/m^2})$,
$\boldsymbol{B} =$ magnetic flux density (T).

The nonlinear vectorial relationship between $\boldsymbol{H}$ and $\boldsymbol{B}$ in a magnetic material can be represented by

$$\boldsymbol{H} = \boldsymbol{F}(\boldsymbol{B}) \tag{6}$$

where $\boldsymbol{F}$ may be a hysteretic or single-valued function and should be Lipschitz continuous.

To formulate the magnetic problem by the fixed-point method, the following constitutive equation should be introduced:

$$\boldsymbol{H} = \nu_{\mathrm{FP}} \boldsymbol{B} + \boldsymbol{M} \tag{7}$$

where $\boldsymbol{M}$ is a magnetization-like quantity. The fixed-point coefficient ν_{FP} is a reluctivity-like quantity and must be constant. There is freedom associated with the choice of the coefficient ν_{FP} since only the convergence matters here. Therefore, ν_{FP} must be properly chosen to ensure contraction. The coefficient ν_{FP} is vital and this article is dedicated to investigate its importance.

By substituting (7) in (4), one obtains the following:

$$\nabla \times \nu_{\mathrm{FP}} \boldsymbol{B} = \boldsymbol{J} - \nabla \times \boldsymbol{M}. \tag{8}$$

Equation (7) can also be written as

$$\boldsymbol{M}(\boldsymbol{B}) = \boldsymbol{F}(\boldsymbol{B}) - \nu_{\mathrm{FP}} \boldsymbol{B}, \tag{9}$$

then, one secures from (8) the following iterated function:

$$\nabla \times \boldsymbol{B} = \frac{1}{\nu_{\mathrm{FP}}} (\boldsymbol{J} - \nabla \times \boldsymbol{M}(\boldsymbol{B})). \tag{10}$$

Equation (10) is analogous to (1) and it represents the magnetic problem in the form of the fixed-point method as

$$\boldsymbol{B} \equiv \boldsymbol{G}(\boldsymbol{B}) \equiv \frac{\boldsymbol{H}(\boldsymbol{B})}{\nu_{\mathrm{FP}}} - \boldsymbol{B}. \tag{11}$$

The current density $\boldsymbol{J}$ is omitted from (11) because it has no influence on the shape of the function $\boldsymbol{G}$. In other words, the partial derivatives of $\boldsymbol{G}$ and the norm of the Jacobian matrix $||\boldsymbol{G}^{'}(\boldsymbol{B})||$ will not be altered by removing $\boldsymbol{J}$ from the equation. The variation of $\boldsymbol{J}$ in time will only cause an offset to $\boldsymbol{G}$ and thus create a new fixed-point solution at a time.

For 3D problems ($m = 3$), the three components of (11) in the x-, y-, and z-directions can be written as

$$\begin{aligned} G_x(B_x, B_y, B_z) &= \frac{H_x(B_x, B_y, B_z)}{\nu_{\mathrm{FP}}} - B_x \\ G_y(B_x, B_y, B_z) &= \frac{H_y(B_x, B_y, B_z)}{\nu_{\mathrm{FP}}} - B_y \\ G_z(B_x, B_y, B_z) &= \frac{H_z(B_x, B_y, B_z)}{\nu_{\mathrm{FP}}} - B_z \end{aligned}$$

and the Jacobian matrix of $\boldsymbol{G}$ follows as

$$\boldsymbol{G}^{'}(\boldsymbol{B}) = \begin{pmatrix} \frac{1}{\nu_{\mathrm{FP}}}\frac{\partial H_x}{\partial B_x} - 1 & \frac{1}{\nu_{\mathrm{FP}}}\frac{\partial H_x}{\partial B_y} & \frac{1}{\nu_{\mathrm{FP}}}\frac{\partial H_x}{\partial B_z} \\ \frac{1}{\nu_{\mathrm{FP}}}\frac{\partial H_y}{\partial B_x} & \frac{1}{\nu_{\mathrm{FP}}}\frac{\partial H_y}{\partial B_y} - 1 & \frac{1}{\nu_{\mathrm{FP}}}\frac{\partial H_y}{\partial B_z} \\ \frac{1}{\nu_{\mathrm{FP}}}\frac{\partial H_z}{\partial B_x} & \frac{1}{\nu_{\mathrm{FP}}}\frac{\partial H_z}{\partial B_y} & \frac{1}{\nu_{\mathrm{FP}}}\frac{\partial H_z}{\partial B_z} - 1 \end{pmatrix}. \tag{12}$$

The Jacobian matrix (12) contains the most important information about the convergence of the 3D function $\boldsymbol{G}(\boldsymbol{B})$, whether it is convergent or not. It is interesting and easy to observe that when (12) is reduced to the 1D case, the optimal value $\nu_{\mathrm{FP}} = \frac{\mathrm{d}H}{\mathrm{d}B}$ that allows the fastest convergence can be immediately determined, a result that has been already obtained using a different approach [5]. However, for the 3D function, it is more difficult to obtain the latter result. As stated earlier, the spectral radius (or the norm) of the Jacobian matrix (12) must be less than one so that the 3D function $\boldsymbol{G}(\boldsymbol{B})$ is convergent. Therefore, the freedom for the choice of the coefficient ν_{FP} has to be restricted to a considerable degree. For the sake of simplicity, the off-diagonal entries in (12) will be assigned to zero; the mutual magnetic field effects of the three components, B_x, B_y, and B_z are small enough and the latter assumption can be appreciated. At saturation, this assumption might become physically less acceptable, but, mathematically, it does not significantly influence the methods involved. The diagonal entries of the matrix remain dominant in any case, because the division by the coefficient ν_{FP} plays another important role. Therefore, one has to deal only with the diagonal elements (which are the eigenvalues of the matrix) to assess the convergence of the function $\boldsymbol{G}(\boldsymbol{B})$. In such a case, the zero matrix can be enforced on the Jacobian matrix only in the following special case:

$$\nu_{\mathrm{FP}} = \frac{\partial H_x}{\partial B_x} = \frac{\partial H_y}{\partial B_y} = \frac{\partial H_z}{\partial B_z}.$$

This desirable case ensures quadratic convergence but is achieved only at certain instants if the magnetized body is purely isotropic and magnetized under a purely circular flux. Another special case, but an undesirable one, arises when any of the partial derivatives, $\frac{\partial H_x}{\partial B_x}, \frac{\partial H_y}{\partial B_y}, \frac{\partial H_z}{\partial B_z}$, equals zero, i.e., $\rho(\boldsymbol{G}^{'}) = 1$. This case makes the fixed point iteration stable but not asymptotically stable, meaning that the iterates will remain oscillating around the fixed point not converging or diverging. The borderline case ($\rho(\boldsymbol{G}^{'}) = 1$) will not be further analyzed here because of the assumption that it will not take place, and one is mostly interested in the general case. The problem that arises next is how to generally determine the optimal value of ν_{FP} that can permit the smallest norm of the Jacobian matrix, and hence the fastest convergence possible. This optimization problem can be approached by different techniques including Lagrange multipliers, but the authors prefer to seek a simpler analytical technique.

There is a constraint emerging immediately from the Jacobian matrix (12), affirming that for every diagonal element, the following inequality must hold

$$\Big|\frac{1}{\nu_{\mathrm{FP}}}\frac{\partial H}{\partial B} - 1\Big| < 1$$

and hence

$$\frac{1}{\nu_{\mathrm{FP}}}\frac{\partial H}{\partial B} < 2. \tag{13}$$

This inequality must hold for the three partial derivatives $\frac{\partial H_x}{\partial B_x}, \frac{\partial H_y}{\partial B_y}, \frac{\partial H_z}{\partial B_z}$ and must always be respected. The inequality is useful and will be the basis for finding the optimum of ν_{FP}. The inequality (13) is satisfied only when the three partial derivatives $\frac{\partial H_x}{\partial B_x}, \frac{\partial H_y}{\partial B_y}, \frac{\partial H_z}{\partial B_z}$ have the same sign and only when the following condition is applied:

$$\Big|\frac{1}{\nu_{\mathrm{FP}}}\frac{\partial H}{\partial B}\Big|_{\min} - 1\Big| = \Big|\frac{1}{\nu_{\mathrm{FP}}}\frac{\partial H}{\partial B}\Big|_{\max} - 1\Big|. \tag{14}$$

This condition is implied from the diagonal entries of the Jacobian matrix. The only components that play direct roles are the maximum and minimum of the three partial derivatives, and the one in the middle has no effect and does not contribute. The only case that permits (13) is when the condition (14) is imposed. Therefore, the optimal coefficient ν_{FP} that gives the fastest convergence is found as

$$\nu_{\mathrm{FP}} = \frac{\frac{\partial H}{\partial B}\big|_{\min} + \frac{\partial H}{\partial B}\big|_{\max}}{2}, \tag{15}$$

where

$$\frac{\partial H}{\partial B}\Big|_{\min} = \min\{\frac{\partial H_x}{\partial B_x}, \frac{\partial H_y}{\partial B_y}, \frac{\partial H_z}{\partial B_z}\},$$

$$\frac{\partial H}{\partial B}\Big|_{\max} = \max\{\frac{\partial H_x}{\partial B_x}, \frac{\partial H_y}{\partial B_y}, \frac{\partial H_z}{\partial B_z}\}.$$

In 2D problems, because there are only two partial derivatives for the x- and y-directions, the optimal coefficient ν_{FP} is found as

$$\nu_{\mathrm{FP}} = \frac{\frac{\partial H_x}{\partial B_x} + \frac{\partial H_y}{\partial B_y}}{2}. \tag{16}$$

In 1D problems, the optimal coefficient ν_{FP} is obtained when

$$\frac{1}{\nu_{\mathrm{FP}}}\frac{\mathrm{d}H}{\mathrm{d}B} - 1 = 0$$

and the optimal value of ν_{FP}, which ensures quadratic convergence in this case, is found as

$$\nu_{\mathrm{FP}} = \frac{\mathrm{d}H}{\mathrm{d}B}. \tag{17}$$

The validity of (15) is also checked for the two special cases mentioned above, particularly when $\rho(\boldsymbol{G}^{'}) = 0$ and $\rho(\boldsymbol{G}^{'}) = 1$. Interestingly, the optimal coefficient ν_{FP} of (15) was observed by Hantila [3] in a totally different way. Because, in practice, the partial derivatives cannot be known prior to the fixed-point solution, Hantila suggested precalculating ν_{FP} and keeping it fixed throughout the computations. In this classical way, a global coefficient is used for the whole analysis and, thus, the method is referred to as the global-coefficient method

(GCM). The resultant spectral radius from the GCM is very close to one, $\rho(\boldsymbol{G}^{'}) \approx 0.99$, consistently leading to slow convergence. However, the authors have recently proposed to optimally calculate the coefficient $\nu_{\rm FP}$ for time-domain problems.

IV. Locally convergent fixed-point scheme

In time-stepping analysis, the initial value $\boldsymbol{B}^{(k=0)}$ for time step n is known from the first solution of the current time step and is sufficiently close to the fixed point. Thus, a basin of attraction that contains $\boldsymbol{B}^{(0)}$ and fulfills (2) can be found by rewriting (15), (16), and (17), for the 3D, 2D, and 1D problems, respectively, as

$$\nu_{\rm FP}^{(n)}|_{\rm 3D} = C\frac{\frac{\partial H}{\partial B}^{(k=0)}|_{\rm min} + \frac{\partial H}{\partial B}^{(k=0)}|_{\rm max}}{2}, \quad (18)$$

$$\nu_{\rm FP}^{(n)}|_{\rm 2D} = C\frac{\frac{\partial H_x}{\partial B_x}^{(k=0)} + \frac{\partial H_y}{\partial B_y}^{(k=0)}}{2}, \quad (19)$$

$$\nu_{\rm FP}^{(n)}|_{\rm 1D} = C\frac{{\rm d}H}{{\rm d}B}^{(k=0)}, \quad (20)$$

where C is a constant that must be conveniently chosen to ensure fast convergence so that the function $\boldsymbol{G}(\boldsymbol{B})$ is strictly attractive in some interval containing $\boldsymbol{B}^{(0)}$. The constant C is necessary and must be greater than one because the derivative at the present time step has been approximated from the first iteration. C is dependent on the size of the time step and on the temporal behavior of the system involved. A simple method using linear search was adopted in this work and was found to be effective. For any problem, one may start by giving C some value slightly greater than one and then testing the convergence for a complete simulation. If converged, the value of C is fixed; if not, C is increased by a small value until an appropriate value of C is obtained. For simple problems, it is generally found that C takes values in the range of $1 < C < 2$. On the other hand, in more complicated situations, such as in electrical machines, the factor C is usually greater than 2. In these types of problems, a time-stepping scheme such as Crank-Nicholson's is often used. The initial values resulted by the time-stepping scheme from the previous time step cannot always be close to the fixed-point solution because of the system dynamics. Therefore, the interval $\boldsymbol{\Omega}$ is enlarged by increasing C in order to account for these effects. For a magnetic system, the factor C must be calculated once and then used throughout. Thus, it is best to optimize C for the worst-case scenario. In electrical machines, the start-up (transient state) should be used to optimize C, since the system dynamics in this case are significant. This only requires the simulation of one or two periods of the supply frequency for testing C.

The methods used above in Equations (18), (19), and (20) compute local coefficients $\nu_{\rm FP}$ at each time step and produce locally convergent iterations. Thus, the methods are referred to as local-coefficient methods (LCM). LCMs are well suited to solving nonlinear time-domain discretized partial differential equations. For example, solving Maxwell equations by the time-stepping finite-element method will result in calculating local coefficients $\nu_{\rm FP}$ as a function of time and space $\nu_{\rm FP}(t, x, y, z)$. This means that efficient iteration is achieved because all the finite elements in the mesh are optimized to produce the fastest convergence at each time step.

It should be noted that LCMs make use of the information about the differential reluctivities at each time step. This is similar to the Newton-Raphson method (NRM), which updates the differential reluctivity at each iteration step. However, the NRM is principally based on calculating the derivative while the fixed-point iteration is not. Therefore, LCMs are not sensitive to non-differentiability, even in case of hysteresis.

V. Time-stepping Finite-Element Analysis

The purpose of the finite-element analysis is to evaluate the efficiency of the LCM and GCM as well as test their convergence. The dependence of the constant C on the time-step size is also investigated.

The authors have implemented a 2D time-stepping finite-element program for rotating electrical machines. The main formulation of the field equations by the fixed-point method is presented here, whereas the finite-element procedures and the circuit equations representation are omitted. A more complete description of the techniques used in the finite-element solution of the magnetic field with circuit equations is available in [5], [8], [9].

If (8) is formulated by the magnetic vector potential $\boldsymbol{A}$ as the unknown and considering the eddy-currents generated in a material with conductivity σ, the following nonlinear magnetic system is obtained:

$$\nabla \times \nu_{\rm FP}(\nabla \times \boldsymbol{A}) + \sigma\frac{\partial \boldsymbol{A}}{\partial t} = \boldsymbol{J} - \nabla \times \boldsymbol{M}. \quad (21)$$

Equation (21) is discretized using finite-elements and solved by the Crank-Nicholson time-stepping scheme. Single-valued characteristics were used in order to allow the NRM [8] to be implemented in the 2D time-stepping finite-element program, applying the same stopping criteria. In the 2D problem, the components B_x, B_y, H_x, and H_y must be used in any case. When using the single-valued curve, the magnetic flux density $\boldsymbol{B}$ and the magnetic field strength $\boldsymbol{H}$ are assumed to be collinear (see [10]). Thus, the nonlinear relation is handled as follows:

1) Compute the components of the flux density B_x and B_y from the solution of the finite-element analysis formulated by the magnetic vector potential.
2) Compute the magnitude of the flux density as $B = \sqrt{B_x^2 + B_y^2}$ and the magnitude of the magnetic field strength from the single-valued relation as $H = F(B)$.
3) Determine the components of the field strength, H_x and H_y as $H_x = H\cos(\arctan(B_y/B_x))$ and $H_y = H\sin(\arctan(B_y/B_x))$.

Thus, the use of the single-valued magnetization curve does not contradict with the general purpose of the paper because the magnetic field quantities remain vectors (rotational).

VI. Application example

Rotating electrical machines are chosen as an application to test the convergence because they represent one of the

most difficult nonlinear problems even at steady-state conditions. For example, the magnetic flux density in conventional (squirrel-cage) induction motors rise to values higher than 2 T in some parts of the teeth as well as in the rotor-slot openings if they are made of iron. Furthermore, the flux can be varying nonsinusoidally, taking various patterns, alternating and rotating, in different parts of the machine. In this work, two induction motors, referred to as Motor I and Motor II, with different specifications (see Table I) have been simulated by the numerical techniques developed. The time-stepping analysis has been performed after a time-harmonic method [8] was used to obtain the steady-state quickly.

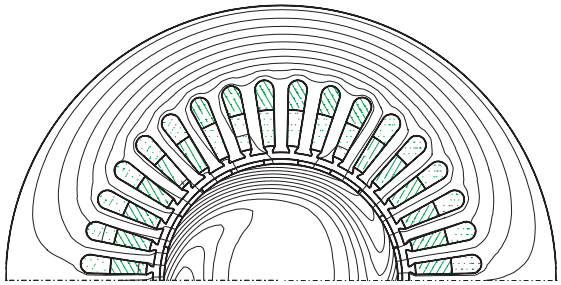

Fig. 1. Flux distribution in the computed geometry of Motor I.

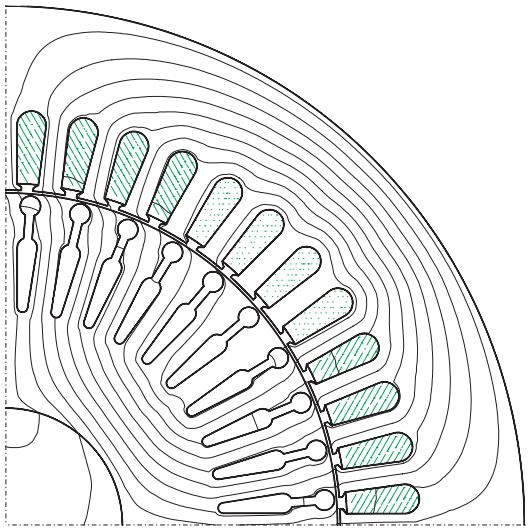

Fig. 2. Flux distribution in the computed geometry of Motor II.

The same Fortran codes were used in the finite-element analysis for the GCM, LCM, and the NRM, differing only in the formulation that each method requires. Furthermore, the simulations were performed under the same criteria and conditions shown in Table II, for the GCM, LCM, and the NRM. The convergence factor C was optimally found using sequential search.

Table III shows comparative results obtained by implementing the methods in the finite-element procedures. The average number of iterates per time-step, the average CPU-time spent on a time-step, and the total CPU-time spent on the entire simulation are tabulated for each motor. Clearly, for the two motors, the LCM and NRM were converging with comparable speed, whereas the GCM was evidently slow. The problem that significantly affects the speed of the GCM is that the maximum differential reluctivity calculated from the maximum flux density in the mesh has to be imposed on the entire mesh even for the elements with low flux densities.

Although the LCM takes more iterates than the NRM to converge, the LCM consumes relatively the same CPU-time as the NRM. The NRM requires that the global matrix of the finite-element method be updated at each iterate k. In contrast, when the LCM is applied, the global matrix remains unchanged during iteration and it needs to be factored once at each time-step; only the right-hand side is updated at each iterate k. One can argue that the GCM should require the global matrix be factored once and for all in the entire simulation because the coefficient ν_{FP} is kept fixed. However, when applying the GCM to a rotating machine in which the mesh in the air gap is modified at each time-step, the global matrix needs to be updated anyway to allow for the rotation.

TABLE I
THE MAIN PARAMETERS OF MOTOR I AND MOTOR II.

Parameter	Motor I	Motor II
Rated voltage (V)	400	400
Rated power (kW)	250	37
Rated frequency (Hz)	500	50
Connection	delta	star
Number of poles	2	4
Number of phases	3	3
Number of stator slots	36	48
Number of rotor slots	0	40

TABLE II
THE SIMULATION INPUT DATA OF MOTOR I AND MOTOR II.

Data	Motor I	Motor II
The convergence constant C	2.2	5.8
Input supply	sinusoidal	sinusoidal
Loading	full	full
Finite-element order	first	first
Number of finite-elements	3194	1516
Number of nodes	1635	785
Number of simulated periods	2	2
Number of time-steps per period	400	400

TABLE III
RESULTS OF THE COMPUTATION TIME.

Test results	Motor type	LCM	GCM	NRM
Iterates per time-step	Motor I	7	33	4
	Motor II	20	123	6
CPU-time per time-step (sec)	Motor I	0.063	0.252	0.061
	Motor II	0.072	0.416	0.070
Total CPU-time (sec)	Motor I	50.4	201.6	48.8
	Motor II	57.6	332.8	56.0

It is observed that the convergence in the case of Motor I is relatively faster for all the methods. Motor I, used for high-speed applications, has a relatively large airgap and its rotor is slotless. (Motor I has no rotor bars where only a thin layer of coper is placed on its solid iron to reduce

eddy currents [11], [12]). The maximum flux density in this motor is rather low especially in the stator core (less than 1 T). On the other hand, the flux density in Motor II reaches over 2 T because of the rotor-slot openings. In this type of motors, the starting torque can be significant only when the iron of the rotor-slot opening is saturated. Therefore, Motor II is considered to be a more difficult nonlinear problem. Moreover, the slotless geometry of the rotor of Motor I is helping the Crank-Nicholson time-stepping scheme to obtain rather undistorted flux density waveforms (see the flux density waveforms in Fig. 3). Thus, the initial values resulted from Motor I are relatively closer to the fixed-points than those resulted from Motor II. This dynamical effect is observed in the values of the constant C in Table II. Motor II required a greater value of C than Motor I to ensure that all initial values are within the basin of attraction.

The time-step size, Δt, has a remarkable influence on the value of C. It is mentioned in [5] that C is dependent on the time-step size but the dependence was not studied. Fig. 4 demonstrates this effect in the case of Motor II by using different time-step sizes and computing accordingly the values of C using sequential search. It is clear that the smaller time-step size, the smaller C one can allow, and thus, the faster the iteration converges. The relation between C and Δt can be approximated by a second-order (quadratic) polynomial.

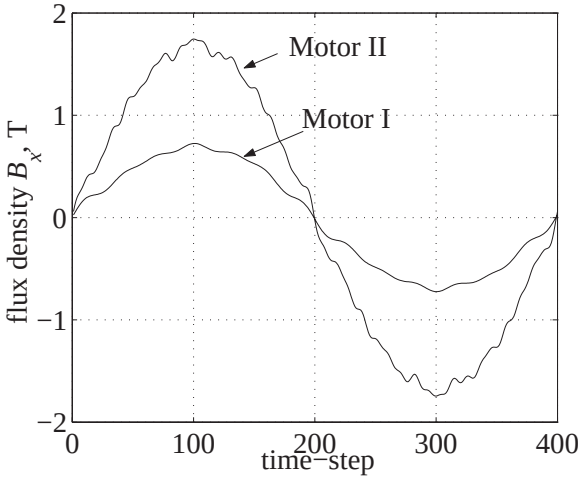

Fig. 3. The flux density waveforms in the teeth of the stator core.

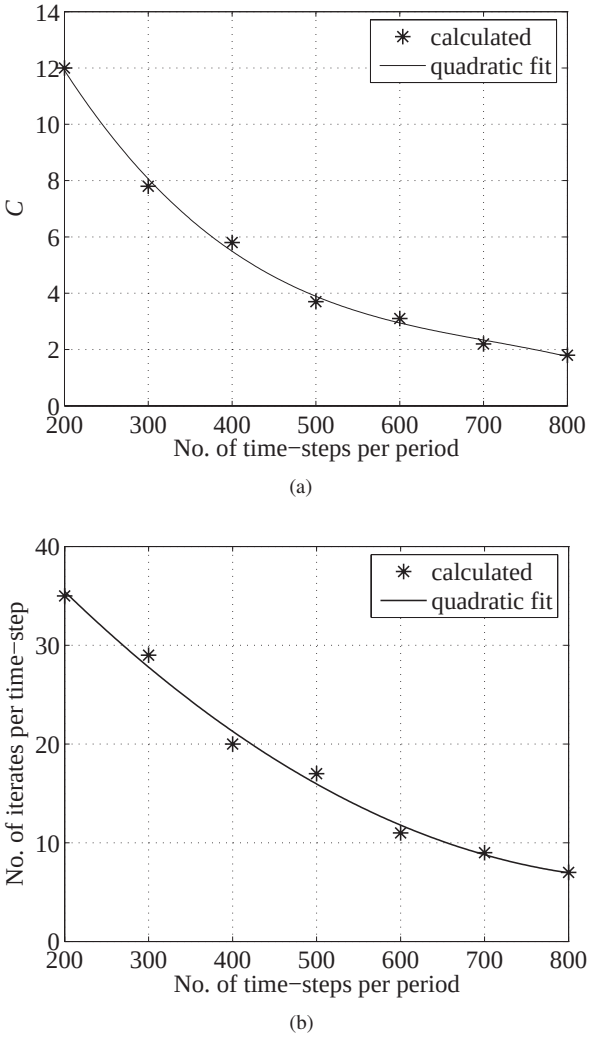

Fig. 4. The effect of the time-step size on the constant C and the speed of iteration.

VII. Conclusion

The fixed-point method of multidimensional functions converges stably provided that the spectral radius of the Jacobian matrix is less than one. In magnetic systems, the latter condition can only be satisfied if proper iterated functions are constructed. The intrinsic freedom of the fixed-point method for creating different iterated functions can be utilized to obtain fast convergence, and in certain instants, the fastest convergence can be secured. In such a case, the fixed-point method converges quadratically with high rate, comparable to the Newton-Raphson method. In practice, one cannot know in advance where the fixed-point solution would occur, and thus, how to construct the optimal iterated functions that allow the fastest convergence possible is difficult. This difficulty is circumvented in time-stepping analysis where the solution of the previous time-step can contribute to the current solution by estimating the optimum of the iterated functions. This allows significantly fast local convergence.

The time-stepping finite-element analysis conducted in this article revealed that the time-step size plays an important role in speeding up the convergence. This is obvious because the smaller time-step size used in the calculation, the better information about the current time-step solution is obtained.

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