Decision Support

Robust portfolio modeling with incomplete cost information and project interdependencies

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Abstract

Robust portfolio modeling (RPM) [Liesiö, J., Mild, P., Salo, A., 2007. Preference programming for robust portfolio modeling and project selection. European Journal of Operational Research 181, 1488–1505] supports project portfolio selection in the presence of multiple evaluation criteria and incomplete information. In this paper, we extend RPM to account for project interdependencies, incomplete cost information and variable budget levels. These extensions lead to a multi-objective zero-one linear programming problem with interval-valued objective function coefficients for which all non-dominated solutions are determined by a tailored algorithm. The extended RPM framework permits more comprehensive modeling of portfolio problems and provides support for advanced benefit–cost analyses. It retains the key features of RPM by providing robust project and portfolio recommendations and by identifying projects on which further attention should be focused. The extended framework is illustrated with an example on product release planning.
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1. Introduction

Resource allocation and project portfolio selection are focal strategic decisions in public administration (e.g., Golabi et al., 1981; Ewing et al., 2006; Kleinmuntz, 2007) and industrial firms (e.g., Stummer and Heidenberger, 2003). Often these decisions – which can pertain to any discrete proposals from which only a subset will be implemented – are made with regard to multiple evaluation criteria on the basis of incomplete information. There may also exist project interdependencies (e.g., mutually exclusive projects) that put constraints on feasible portfolios, as well as project synergies that offer extra benefits and/or cost savings when particular project combinations are started. Moreover, the budget constraints may be 'soft', in the sense that it is meaningful to adjust the budget depending on what benefits can be secured at different levels of resource expenditure (Lindstedt et al., in press, among others). In the presence of these complex considerations,

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the deployment of systematic methods and tools for portfolio selection increases the rigor and transparency of decision making. Ideally, such models should capture all relevant aspects of portfolio decision making so that they can be deployed in different contexts; yet they should be simple and flexible enough so that they can be readily accepted by practitioners (e.g., Archer and Ghasemzadeh, 1999; Cooper et al., 1999).

Robust portfolio modeling (RPM) (Liesiö et al., 2007) is a framework for project portfolio selection with independent projects and fixed budget. In RPM, an additive multi-criteria value model (see, e.g., Keeney and Raiffa, 1976; Belton and Stewart, 2001) is employed to capture the values of both individual projects and project portfolios (Golabi et al., 1981; Ewing et al., 2006). Building on the concepts of *Preference Programming* (e.g., Salo and Hämäläinen, 1992; Salo and Hämäläinen, 2001; Mustajoki et al., 2005), incomplete information about criterion weights is captured through linear inequalities (Salo and Punkka, 2005, among others), while projects' criterion-specific evaluations are described through lower and upper bounds. Based on the computation of all non-dominated portfolios, RPM features a project-specific *core index*, which helps identify those projects that should be surely selected or rejected in view of incomplete information. The core index also suggests borderline projects as candidates for the elicitation of additional information. These features of RPM have been appreciated by decision makers (DMs) in applications in contexts such as strategic product portfolio selection (Lindstedt et al., in press), screening of innovation ideas (Könnölä et al., 2007), development of a national research agenda (Könnölä et al., in press), and evaluation of newly established spin-off companies (Salo et al., 2006).

In this paper, we extend the RPM framework by: (i) admitting a wide range of project interdependencies (cf., Stummer and Heidenberger, 2003), (ii) modeling incomplete information about project costs, and (iii) considering variable budget levels. The key results on decision recommendations and additional information in Liesiö et al. (2007) still hold with these extended features. The modified model structure necessitates changes in the computation of non-dominated portfolios: instead of the knapsack-type problem in the original RPM model, we have a general multi-objective zero-one linear programming (MOZOLP) problem with interval-valued objective function coefficients. This problem has not been widely studied, wherefore we develop a new algorithm for computing the non-dominated portfolios.

These conceptual and computational advances permit new modes of support for portfolio decision making. First, the framework helps determine the budget level through an advanced portfolio-level benefit–cost analysis in view of project interdependencies and incomplete information on projects' benefits and costs. The analysis also shows how budget variation impacts the relative status of projects. Second, project synergies can be estimated via an interval, which can model situations where synergies *may* arise but are not certain. The role of the synergies can also be analyzed by core index values to see whether or not they should be pursued. Third, the RPM methodology supports interactive analyses: it offers robust recommendations subject to incomplete information and helps identify projects and model parameters on which further attention should be focused.

The paper is organized as follows. Section 2 introduces the extended RPM framework and its key concepts. Section 3 presents how incomplete cost information and budget variations can be modeled. Section 4 develops an algorithm for the computation of non-dominated portfolios. Section 5 provides an illustrative example, and Section 6 concludes.

2. Robust portfolio modeling

2.1. Portfolio value

In the RPM framework, m project proposals $X = \{x^1, \ldots, x^m\}$ are evaluated with regard to n criteria. The score vector $v^j = [v_1^j, \ldots, v_n^j]$ contains the evaluation scores of project x^j with regard to the criteria $i = 1, \ldots, n$. These vectors form the rows of the score matrix $v \in \mathbb{R}^{m \times n}$ such that $[v]_{ji} = v_i^j$. The overall value of project x^j is captured through an additive value function, i.e., $V(x^j) = \sum_{i=1}^{n} w_i v_i^j$, where the weight w_i measures the relative importance of the ith criterion. The weights $w = (w_1, \ldots, w_n)^{\mathrm{T}}$ are scaled so that

$$w \in S_w^0 := \left\{ w \in \mathbb{R}^n \mid w_i \geqslant 0, \sum_{i=1}^{n} w_i = 1 \right\}. \tag{1}$$

The weight w_i relates a unit increase in the criterion-specific value to an increase in the overall value. A project is preferred to another if it has higher overall value than the other.

A project portfolio $p \subseteq X$ is a subset of available projects; thus, the set of all possible portfolios is the power set $P := 2^X$. The overall value of a portfolio is the sum of the overall values of its projects. For a given score matrix v and criterion weights w, the overall value of portfolio p is

$$V(p, w, v) := \sum_{x^j \in p} \sum_{i=1}^{n} w_i v_i^j = z(p)^T v w, \tag{2}$$

where $z(\cdot)$ is a bijection $z : P \to \{0, 1\}^m$ such that $z_j(p) = 1$ if $x^j \in p$ and $z_j(p) = 0$ if $x^j \notin p$. For instance, with $m = 5$ projects, portfolio $p = \{x^1, x^2, x^4\}$ corresponds to $z(p) = [1, 1, 0, 1, 0]^T$.

2.2. Portfolio feasibility and project interactions

In RPM, the set of *feasible portfolios* is defined by a set of linear inequalities whose coefficients are recorded in matrix $A \in \mathbb{R}^{q \times m}$ ($a_l^j = [A]_{lj}$) and vector $B = [b_1, \ldots, b_q]^T \in \mathbb{R}^q$. The set of feasible portfolios is

$$P_F := \{p \in P | Az(p) \leqslant B\}, \tag{3}$$

where $\leqslant$ holds componentwise. The following constraint types are common in project portfolio selection, and they can be modeled as linear inequalities (see, e.g., Stummer and Heidenberger, 2003):

- *Budget constraints* with fixed project costs c_j and the budget level R. Note that c_js can be negative if they correspond to cost saving projects.
- *Logical constraints* model mutually exclusive projects and other rigid interdependencies, such as follow-up projects.
- *Positioning constraints* help ensure that the composition of the portfolio is aligned with strategic requirements, such as starting a minimum number of projects in different technological or geographical areas.
- *Threshold constraints* help ensure that the performance of the portfolio and its constituent projects fulfil minimum requirements: for example, the aggregate net present value may have to exceed a minimum acceptable level (cf., Kleinmuntz, 2007).

Project synergies (or cannibalization), i.e., when the overall value and/or the cost of a set of projects differs from the sum of the individual projects' overall values and costs, can also be modeled with linear constraints and dummy projects $x^s \in X$. For instance, assume that synergy occurs if at least m' projects are selected from the set $X' \subseteq X$ ($x^s \notin X'$). The scores v_i^s and cost c_s represent the additional value and cost savings which are obtained if the condition is met. Two constraints are needed to ensure that x^s is included in the portfolio if and only if at least m' projects from X' are included in the portfolio (Stummer and Heidenberger, 2003). Situations where the synergy occurs if at most (exactly) m' projects from X' are selected can be modeled equivalently with structures of two (four) constraints.

Even though synergies are exceptions to value additivity, the RPM optimization model remains linear. From the DM's perspective, decisions are taken only about real projects which determine whether or not synergies occur (i.e., are the corresponding dummy projects included in the portfolio or not). For the given weights w, scores v, costs c and budget R, the most preferred feasible portfolio maximizes the overall value (2). This portfolio is obtained as a solution to the integer linear programming (ILP) problem

$$\max_{p \in P_F} V(p, w, v) = \max_{z(p)} \{z(p)^T v w | Az(p) \leqslant B, z(p) \in \{0, 1\}^m\}. \tag{4}$$

2.3. Incomplete information and dominance

Because elicitation of exact weights and scores can be difficult (e.g., Salo and Punkka, 2005), RPM employs incomplete information modeled by set inclusion: instead of point estimates for weights and scores, the analysis is based on sets of feasible parameters that are consistent with the DM's preference statements. These

statements are converted into a set of feasible criterion weights $S_w \subseteq S_w^0$. We assume that S_w has a polyhedral convex hull which also allows for incomplete ordinal statements (Salo and Punkka, 2005). The extreme points of the convex hull and the extreme point matrix are denoted by

$$\{w^1, \ldots, w^t\} := \text{ext}(\text{conv}(S_w)) \tag{5}$$

$$W_{\text{ext}} := [w^1, \ldots, w^t] \in \mathbb{R}^{n \times t}. \tag{6}$$

Incomplete score information is modeled through score intervals $[\underline{v}_i^j, \bar{v}_i^j]$ that contain the 'true' scores v_i^j. Lower and upper bounds of all score intervals are denoted by matrixes $\underline{v}$ and $\bar{v}$, respectively. The set of feasible scores is the set of matrixes $S_v = \{v \in \mathbb{R}^{m \times n} | \underline{v} \leqslant v \leqslant \bar{v}\}$. The *information set* of feasible weight and score parameters is the Cartesian product $S := S_w \times S_v$. Thus, $(w, v) \in S$ is equivalent to $w \in S_w$ and $v \in S_v$.

When different weights and scores are selected from the information set S, the overall values of portfolios vary so that an interval of values can be associated with each. Even if the value intervals of two portfolios overlap, it may be possible to identify one of them as inferior through the dominance concept.

Definition 1. Let $p, p' \in P$. Portfolio p dominates p' with regard to the information set S, denoted by $p \succ_S p'$, if $V(p, w, v) \geqslant V(p', w, v)$ for all $(w, v) \in S$ and $V(p, w, v) > V(p', w, v)$ for some $(w, v) \in S$.

We denote $p \succ p'$ when there is no risk of confusion about the information set S. Because the overall value is linear in weights and scores, dominance can be checked by comparing the portfolios' values at the extreme points of conv (S_w).

Theorem 1. *Let* $p, p' \in P$ *and information set* $S = S_w \times S_v$. *Then*

$$p \succ_S p' \iff \begin{cases} z(p \setminus p')^{\text{T}} \underline{v} W_{\text{ext}} \geqslant z(p' \setminus p)^{\text{T}} \bar{v} W_{\text{ext}} \\ z(p \setminus p')^{\text{T}} \bar{v} W_{\text{ext}} \neq z(p' \setminus p)^{\text{T}} \underline{v} W_{\text{ext}}, \end{cases}$$

where $\geqslant$ *holds componentwise for the row vectors, W_{ext} is given by* (6) *and* $p \setminus p' := \{x^j \in p | x^j \notin p'\}$.

A rational DM who seeks to maximize the overall portfolio value will not choose a dominated portfolio. Dominated portfolios can thus be discarded and the analysis can be focused on the set of non-dominated portfolios

$$P_{\text{N}}(S) := \{p \in P_{\text{F}} | \nexists p' \in P_{\text{F}} \text{ s.t. } p' \succ_S p\}. \tag{7}$$

We denote $P_{\text{N}} \equiv P_{\text{N}}(S)$, when there is no risk of confusion about the information set S. The computation of the set P_{N} is a key step in supporting project portfolio selection under incomplete preference information – it eliminates unacceptable (dominated) portfolios from further consideration while retaining the interesting (non-dominated) ones.

2.4. Decision recommendations

Dominance between feasible portfolios depends on the information set S. Loose preference statements and wide score intervals typically result in a large number of non-dominated portfolios whereas point estimates give a unique optimal portfolio. In RPM, additional information refers to narrower score intervals and/or additional constraints on the feasible weights. This additional information reduces the initial information set S to $\widetilde{S} \subset S$. For technical reasons, RPM assumes that the revised information set $\widetilde{S}$ is not entirely contained on the border of S, i.e., $\text{int}(S) \cap \widetilde{S} \neq \emptyset$, where $\text{int}(S) := \{s \in S | \forall s' \in S \exists \delta > 0 \text{ such that } s + \varepsilon(s - s') \in S \forall \varepsilon \in [0, \delta]\}$. As a consequence, additional information can eliminate some non-dominated portfolios, but cannot add new ones.

Theorem 2. *Let* $\widetilde{S}, S$ *be information sets such that* $\widetilde{S} \subseteq S$ *and* $\text{int}(S) \cap \widetilde{S} \neq \emptyset$. *Then,* $P_{\text{N}}(\widetilde{S}) \subseteq P_{\text{N}}(S)$.

A major computational implication of Theorem 2 is that the set of non-dominated portfolios needs to be computed with the dynamic programming algorithm (Section 4) for the initial information set S only. Later on, the set $P_{\text{N}}(\widetilde{S})$ can be obtained from $P_{\text{N}}(S)$ by pairwise dominance checks (Theorem 1)

$$P_N(\widetilde{S}) = \{p \in P_N(S) | p' \nsucc_{\widetilde{S}} p \ \forall p' \in P_N(S)\}. \tag{8}$$

The RPM framework helps analyze the robustness of individual projects subject to incomplete information. The notion of the project-specific *core index*, defined as the share of non-dominated portfolios that include the project, is one of the key concepts of RPM.

Definition 2. For a given information set S we define

Core index of project x^j : $\mathrm{CI}(x^j, S) = |\{p \in P_N(S)| x^j \in p\}|/|P_N(S)|$,

Core projects : $X_C(S) = \{x^j \in X | \mathrm{CI}(x^j, S) = 1\}$,

Borderline projects : $X_B(S) = \{x^j \in X | 0 < \mathrm{CI}(x^j, S) < 1\}$,

Exterior projects : $X_E(S) = \{x^j \in X | \mathrm{CI}(x^j, S) = 0\}$.

All core projects can be surely recommended and all exterior projects can be safely rejected, because core projects do and exterior projects do not, respectively, belong to all non-dominated portfolios even if additional information was given. Efforts to specify additional score information can be focused on the borderline projects, because narrower score intervals for core or exterior projects do not have an impact on the set of non-dominated portfolios.

Corollary 1. *Let* $\widetilde{S} \subset S$ *such that* $\mathrm{int}(S) \cap \widetilde{S} \neq \emptyset$. *Then,* $X_C(S) \subseteq X_C(\widetilde{S})$, $X_E(S) \subseteq X_E(\widetilde{S})$. *Furthermore, if* $\widetilde{S}_w = S_w$ *and* $\underline{\widetilde{v}}^j = \underline{v}^j$, $\overline{\widetilde{v}}^j = \overline{v}^j \ \forall x^j \in X_B(S)$, *then* $P_N(S) = P_N(\widetilde{S})$.

RPM also features decision rules for determining robust portfolios. The *maximin* rule recommends the portfolio which yields the highest minimum overall value over the information set. The *minimax-regret* rule recommends the portfolio for which the worst case overall value difference compared to other feasible portfolios is the smallest. These rules coincide with the absolute robustness and robust deviation measures, respectively, in robust discrete optimization (Kouvelis and Yu, 1997).

$$\text{maximin :} \qquad P_{\min} := \arg\max_{p \in P_N} \min_{(w,v) \in S} V(p, w, v)$$

$$= \arg\max_{p \in P_N} \min_{i \in \{1, \dots, t\}} V(p, w^i, \underline{v}) \tag{9}$$

$$\text{minimax-regret :} \quad P_{\mathrm{mmr}} := \arg\min_{p \in P_N} \max_{\substack{p' \in P_N, \\ (w,v) \in S}} [V(p', w, v) - V(p, w, v)]$$

$$= \arg\min_{p \in P_N} \max_{\substack{p' \in P_N, \\ i \in \{1, \dots, t\}}} [V(p' \setminus p, w^i, \overline{v}) - V(p \setminus p', w^i, \underline{v})] \tag{10}$$

The DM is advised to start with loose preference statements, which imply large feasible sets of the parameter values and typically result in a large number of non-dominated portfolios. Core index analysis helps identify core and exterior projects; it also helps focus the efforts of eliciting additional information on borderline projects. Due to narrower score intervals and stricter weight statements, the set of non-dominated portfolios becomes smaller and new core and exterior projects may be identified. Finally, decision rules (9) and (10) can be consulted to recommend one of the remaining non-dominated portfolios.

The concepts of incomplete score information and core index apply to synergies as well. Synergy effects are often hard to estimate, wherefore the use of lower and upper bounds may be convenient. The intervals $[\underline{v}_i^s, \overline{v}_i^s]$ of the corresponding dummy project x^s then contain the 'true' magnitudes of the synergy effect. Specifically, if the lower bound is set to zero, the possibility that the synergy does not occur is taken into account. Furthermore, the upper bound can be set based on an optimistic expectation.

The core index of the dummy project $\mathrm{CI}(x^s, S)$ shows the share of non-dominated portfolios in which the synergy is active. It illustrates how significant the synergy is at the portfolio level: for example, core synergies (for which $\mathrm{CI}(x^s, S) = 1$) are major drivers of portfolio value even with the most pessimistic estimate of their additional value. Note that any portfolio where an exterior synergy ($\mathrm{CI}(x^s, S) = 0$) is active or a core cannibalization effect ($\mathrm{CI}(x^s, S) = 1$) is inactive would be dominated.

3. Incomplete cost and budget information

3.1. Efficient portfolios

In many cases, the budget is a 'soft constraint' that can be adjusted to some extent: for instance, industrial firms can obtain additional funding if there are good enough projects to be funded. Furthermore, projects' exact costs are often uncertain, and optimization with point estimates does not inform the DM about how the results would change if the costs vary. Thus, it is beneficial to: (i) conduct a benefit–cost analysis on the overall value of the portfolio as a function of the budget level, and (ii) analyze the robustness of projects and portfolios in the view of incomplete weight, score, cost and budget information.

To analyze incomplete cost and budget information, we remove the budget constraints from set of feasibility constraints defined by A and B in (3). Thus, P_F denotes the set of portfolios that satisfy the other feasibility constraints (i.e., positioning and threshold requirements, for example). The total cost of portfolio p is denoted by

$$C(p, c) := \sum_{x^j \in p} c_j = z(p)^T c,$$

where $c = [c_1, \ldots, c_m]^T$ are costs of projects $x^1, \ldots, x^m$, respectively.

Project costs are estimated via an interval $[\underline{c}^j, \bar{c}^j]$. In keeping with the notation in Section 2, the set of feasible costs is denoted by $S_c = \{c \in \mathbb{R}^m | \underline{c} \leqslant c \leqslant \bar{c}\}$ and the resulting cost of portfolio p is the interval $[C(p, \underline{c}), C(p, \bar{c})]$. The information set $S = S_w \times S_v$ still corresponds to the feasible regions of weights and scores associated with the n evaluation criteria.

The set of non-dominated portfolios in (7) is determined by the range of overall values that the portfolios can assume over the information set S subject to the (rigid) budget constraint. To incorporate incomplete cost and budget information, we expand the notion of 'interesting portfolios' (cf. non-dominated) to *efficient* portfolios: a portfolio is efficient if no other feasible portfolio gives a higher overall value *at a lower cost*.

Definition 3. The set of efficient portfolios with regard to information set $S = S_w \times S_v$ and cost information S_c is

$$P_E(S, S_c) = \left\{ p \in P_F \middle| \nexists p' \in P_F \text{ such that } \begin{cases} V(p', w, v) \geqslant V(p, w, v) \ \forall (w, v) \in S \\ C(p', c) \leqslant C(p, c) \ \forall c \in S_c \end{cases} \right\},$$

with at least one strict inequality for some $(w, v) \in S$ or $c \in S_c$.

The set of efficient portfolios is analogous to the set of non-dominated portfolios in that a rational DM would not choose a portfolio outside the efficient set. If she were to do so, there would exist an efficient portfolio which yields a higher overall value for all feasible weights and scores, and costs less no matter what the project costs are within their intervals.

To compute the set of efficient portfolios, the total cost $C(p, c)$ is modeled as a criterion to be *minimized*. Thus, the $(n + 1)$th score of project x^j is the opposite of its cost, i.e., $-c_j$. The cost is associated with a weight w_{n+1} which varies so that $w_{n+1} \in [0, 1]$.

Theorem 3. *Consider information set $S = S_w \times S_v$ and cost information S_c. Let the extended information set $\widehat{S} = \widehat{S}_w \times \widehat{S}_v$ be defined by:*

$$\widehat{S}_v = \{[v, -c] \in \mathbb{R}^{m \times (n+1)} | v \in S_v, c \in S_c\}, \tag{11}$$

$$\widehat{S}_w = \left\{ w \in \widehat{S}_w^0 \middle| \frac{1}{1 - w_{n+1}} (w_1, \ldots, w_n)^T \in S_w, w_{n+1} < 1 \right\} \cup \{w \in \widehat{S}_w^0 | w_1 = \cdots = w_n = 0, w_{n+1} = 1\}, \tag{12}$$

where $\widehat{S}_w^0 = \{w \in \mathbb{R}^{n+1} | w_i \geqslant 0, \sum_{i=1}^{n+1} w_i = 1\}$. Then, $P_E(S, S_c) = P_N(\widehat{S})$.

Theorem 3 has important implications. First, the dynamic programming algorithm for solving the set of non-dominated portfolios in Section 4 can be used to solve the set of efficient portfolios, since $\text{conv}(\widehat{S}_w)$ is polyhedral and $\widehat{S}_v$ is a set of $m \times (n+1)$-dimensional score matrixes. Second, additional information $\widetilde{S} \subset \widehat{S}$ can only reduce the set of efficient portfolios, because $P_{\text{N}}(\widetilde{S}) \subseteq P_{\text{N}}(\widehat{S})$ by Theorem 2 and $P_{\text{N}}(\widetilde{S})$ is obtained from $P_{\text{N}}(\widehat{S})$ by pairwise checks between portfolios as in (8). Third, the share of efficient portfolios which contain project x^j can be interpreted as the core index of project x^j for the information set $\widehat{S}$, i.e., $\text{CI}(x^j, \widehat{S})$. Corollary 1 implies that narrower cost intervals for core projects $X_{\text{C}}(\widehat{S})$ or exterior projects $X_{\text{E}}(\widehat{S})$ cannot reduce the set of efficient portfolios.

3.2. Benefit–cost analysis

Efficient portfolios help analyze how the portfolio overall value changes as a function of the budget level. Since both overall values and costs are intervals, efficient portfolios do not result in a unique benefit–cost *curve*; rather they form a *band* in the overall value – total cost plane. For the analysis, we define the set of feasible portfolios $P_{\text{F}}(c,R)$ that are *attainable with fixed $c \in S_c$ and $R \in \mathbb{R}$* and the corresponding set of non-dominated portfolios $P_{\text{N}}(S,c,R)$ with regard to information set S:

$$P_{\text{F}}(c,R) = \{p \in P_{\text{F}} | C(p,c) \leqslant R\}, \tag{13}$$

$$P_{\text{N}}(S,c,R) = \{p \in P_{\text{F}}(c,R) | p' \nsucc_S p \ \forall p' \in P_{\text{F}}(c,R)\}. \tag{14}$$

The set $P_{\text{N}}(S,c,R)$ contains all non-dominated portfolios in terms of overall value, if project costs are c and the budget is R. Thus, dominance is determined by incomplete information on weights and scores $(w,v) \in S$, and $P_{\text{N}}(S,c,R)$ is an alternative notation to $P_{\text{N}}(S)$ defined in (7) to highlight the dependence on c and R.

The set of efficient portfolios corresponds to the union of all sets of non-dominated portfolios in the sense that: (i) every efficient portfolio is included in the set of non-dominated portfolios for some fixed costs $c \in S_c$ and budget R, and (ii) non-dominated portfolios for any fixed $c \in S_c$ and R are efficient. However, there is an exception to the latter property: if two non-dominated portfolios have equal overall value for all feasible weights and scores, only the less expensive one is efficient, by definition. Theorem 4 formalizes the relationship between $P_{\text{E}}(S,S_c)$ and $P_{\text{N}}(S,c,R)$.

Theorem 4. *Consider information set $S = S_w \times S_v$ and cost information S_c. Then*

    (i) $p \in P_{\text{E}}(S,S_c) \Rightarrow \exists R \in \mathbb{R}, c \in S_c$ *such that* $p \in P_{\text{N}}(S,c,R)$

    (ii) $p \in P_{\text{N}}(S,c,R) \Rightarrow \exists p' \sim p, p' \in P_{\text{N}}(S,c,R)$ *such that* $p' \in P_{\text{E}}(S,S_c)$,

where $\sim$ is the equivalence relation $p \sim p' \Longleftrightarrow V(p,w,v) = V(p',w,v) \ \forall(w,v) \in S$.

Theorem 4 implies that $P_{\text{N}}(S,c,R)$ can be obtained from $P_{\text{E}}(S,S_c)$ for any given values of c and R: discard portfolios that do not meet the budget constraint $C(p,c) \leqslant R$ and use pairwise dominance checks (14) in the resulting $P_{\text{F}}(c,R)$ to obtain $P_{\text{N}}(S,c,R)$. Although the budget level R can be adjusted continuously, the composition of $P_{\text{N}}(S,c,R)$ for some fixed c can change at levels $R \in \{C(p,c) | p \in P_{\text{E}}(S,S_c)\}$ only. Thus, the number of such budget levels is limited from above by the number of efficient portfolios.

The benefit–cost band describes the ranges of overall values that non-dominated portfolios can assume at different levels of R, subject to incomplete information $(w,v) \in S$ and $c \in S_c$. For each budget level R, this band is defined by the interval

$$\left[\min_{\substack{p \in P_{\text{N}}(S,\underline{c},R) \\ (w,v) \in S}} V(p,w,v), \ \max_{\substack{p \in P_{\text{N}}(S,\underline{c},R) \\ (w,v) \in S}} V(p,w,v) \right]. \tag{15}$$

The bounds are attained by maximizing/minimizing the overall portfolio value while project costs vary within their intervals $c \in S_c$. The optima are achieved when all costs are at their lower bounds ($c = \underline{c}$), where the budget is least restrictive, i.e., $P_{\text{N}}(S,c,R) \subseteq P_{\text{N}}(S,\underline{c},R) \ \forall c \in S_c$. The lower bound does not necessarily increase with R, because a higher budget may suffice to fund a new portfolio which is non-dominated but has a lower worst case overall value. Instead, we examine how the maximum worst case overall value develops over

$P_N(S, \bar{c}, R)$ when R increases. Note that any portfolio in $P_N(S, \bar{c}, R)$ is feasible no matter what the costs are within their intervals ($C(p, c) \leqslant R \ \forall c \in S_c$). The following measures describe the overall portfolio value as a function of the budget level.

$$\text{Maximal overall value : } MV(R) := \max_{p \in P_N(S, \underline{c}, R)} \max_{(w,v) \in S} V(p, w, v)$$

$$\text{Guaranteed overall value : } GV(R) := \max_{p \in P_N(S, \bar{c}, R)} \min_{(w,v) \in S} V(p, w, v)$$

The $GV(R)$ curve shows the increase in the value of the *maximin* portfolio (9) over $P_N(S, \bar{c}, R)$. Both $MV(R)$ and $GV(R)$ are non-decreasing in R.

3.3. Budget-dependent core index

The sets $P_N(S, \bar{c}, R)$ also help analyze the robustness of individual projects as a function of R. Budget-dependent core index of project x^j measures the share of non-dominated portfolios which contain x^j and are certainly attainable ($c = \bar{c}$) at budget level R.

Definition 4. Budget-dependent core index of project x^j at budget level R is

$$CI(x^j, S, R) = \frac{|\{p \in P_N(S, \bar{c}, R) | x^j \in p\}|}{|P_N(S, \bar{c}, R)|}.$$

If $C(x^j, S, R) = 1$, project x^j is included in the set of non-dominated portfolios attainable with $\bar{c}$ and R regardless of the incomplete information $(w, v) \in S$. Interestingly, the core index is not necessarily increasing in R. For instance, a small increase δ in R may be enough for a new portfolio p to enter $P_N(S, \bar{c}, R + \delta)$. This new portfolio may not contain project x^j that *is* contained in all portfolios in $P_N(S, \bar{c}, R)$. However, the new portfolio p must contain projects that are alternatives to x^j: they fit into the budget $R + \delta$ even if $c = \bar{c}$ and yield a higher overall value than x^j for some $(w, v) \in S$. Thus, project x^j can lose its core status when these alternative projects become attainable at the higher budget $R + \delta$.

3.4. Implications for decision support

The guaranteed overall value curve $GV(R)$ and the budget-dependent core indexes $CI(x^j, S, R)$ are helpful in budgeting and robustness analysis. For instance, the budget may be set at level R where the guaranteed overall value $GV(R)$ first exceeds a given threshold. Guaranteed and maximum benefit–cost ratios are obtained from the $GV(R)$ and $MV(R)$ curves, and they can be employed in determining the final budget. The $CI(x^j, S, R)$ plot links the budget decision to the project-specific yes/no decisions. For instance, the DM can identify a budget level R^* at which a reference project first enters a non-dominated portfolio. The $CI(x^j, S, R)$ plot shows which other projects belong to all, some or none of the non-dominated portfolios at R^*. If the DM were to choose the reference project into the portfolio at a lower budget level than R^*, or she does not choose (reject) all core (exterior) projects at R^*, the resulting portfolio would be dominated. The $CI(x^j, S, R)$ plot also illustrates where the core indexes are sensitive to small budget variations.

A narrower budget range and narrower cost intervals work similarly as stricter weight statements and narrower score intervals, respectively: they leave a smaller (sub)set of efficient portfolios, which retains the established core and exterior projects and may yield new ones from earlier borderline projects. Furthermore, narrower score and cost intervals have an impact on efficient portfolios only when applied to borderline projects.

The RPM framework with incomplete cost and budget information opens further possibilities, too. First, portfolio cost could be used as a weighted criterion among others; the form of the extended information set $\widehat{S}$ in Theorem 3 makes it possible to restrict the feasible values of the weight of the cost criterion w_{n+1}. The DM could set a lower bound, for example, for the marginal rate of added value per dollar on criterion i through the weight ratio w_i / w_{n+1}. Second, the model would allow for multiple resources with incomplete information on consumption (cost) and availability (budget). In this case, Definition 3 (efficiency) can be extended to form

another cost inequality for each additional resource, and Theorem 3 can be applied successively to incorporate resources one by one into the extended information set.

4. Computation of non-dominated portfolios

The computation of non-dominated and efficient portfolios is identical in the sense that the set of efficient portfolios is equal to the set of non-dominated portfolios with regard to the extended information set (Theorem 3). Thus, we only consider the computation of the set P_N without taking stance on whether or not the information set includes the cost criterion.

Solving the set P_N is related to the multi-objective zero-one linear programming (MOZOLP) problem (see, e.g., Kiziltan and Yucaoglu, 1983). With point estimate scores $\underline{v} = v = \bar{v}$, the computation of non-dominated portfolios is equivalent to finding all Pareto-optimal solutions to the t-objective MOZOLP problem

$$\text{vmax}_{z(p)}\{z(p)^\mathrm{T}vW_\text{ext}|Az(p) \leqslant B, z(p) \in \{0,1\}^m\}, \tag{16}$$

where matrix $vW_\text{ext} \in \mathbb{R}^{m \times t}$ contains the overall values of the m projects at the t extreme points of $\text{conv}(S_w)$. A feasible solution $z(p)$ to MOZOLP problem (16) is Pareto-optimal if there exists no feasible solution $z(p')$ such that $z(p')^\mathrm{T}vW_\text{ext} \gneq z(p)^\mathrm{T}vW_\text{ext}$ (where $\gneq$ denotes greater than or equal to on each component and strictly greater on at least one). This is equal to the dominance check in Theorem 1 when $\underline{v} = v = \bar{v}$. Furthermore, if there is no weight information ($S_w = S_w^0$), W_ext is an identity matrix and the condition coincides with the general definition of Pareto-optimality of criterion vectors ($z(p')^\mathrm{T}v \gneq z(p)^\mathrm{T}v$). Thus, the computation of non-dominated portfolios in the RPM framework is a multi-objective zero-one linear programming problem *with interval-valued* objective function coefficients.

Building on the work of Villarreal and Karwan (1981) on MOZOLP algorithms, we build our approach on dynamic programming which, in this case, is equivalent to the breadth-first search strategy. Let us define $P^0 = \{\emptyset\}$ and a recursive iteration scheme

$$P^k = P^{k-1} \cup \{(p \cup \{x^k\})|p \in P^{k-1}\} \tag{17}$$

for $1 \leqslant k \leqslant m$. Clearly, $P^m = P$ and each auxiliary set P^k has the property that if $p \in P^k$ then $p \cap \{x^{k+1}, \ldots, x^m\} = \emptyset$. Since $P_N \subseteq P_F \subseteq P$, a complete enumeration approach would start with P^0 and go through the iteration (17) for all $k \in \{1, \ldots, m\}$ to obtain P. Infeasible portfolios could be discarded using the feasibility check (3), and P_N could be computed from P_F using pairwise dominance checks (7). However, since the size of P is 2^m this approach becomes infeasible in terms of memory requirements and computational time when m grows (see Stummer and Heidenberger, 2003).

Therefore, at each kth stage of the iteration, we identify and discard portfolios $p \in P^k$ that cannot become non-dominated even if projects from the set $\{x^{k+1}, \ldots, x^m\}$ are added to them. Computational benefits of discarding such portfolios are amplified by the iteration scheme, since if $p \in P^k$ is discarded, any portfolios $p' = p \cup p'', p'' \subseteq \{x^{k+1}, \ldots, x^m\}$ are not included in any of the auxiliary sets $P^{k+1}, \ldots, P^m$.

Infeasibility of portfolio $p \in P^k$ is not a sufficient condition for discarding it. Lemma 1 presents a sufficient condition for discarding p on the basis that it cannot become feasible, and thus non-dominated, by including projects from the set $\{x^{k+1}, \ldots, x^m\}$.

Lemma 1. *Let $p \in P^k$. If $\sum_{x^j \in p} a_l^j + \sum_{j=k+1}^m \min\{0, a_l^j\} > b_l$ for some $l \in \{1, \ldots, q\}$, then $(p \cup p'') \notin P_N$ for any $p'' \subseteq \{x^{k+1}, \ldots, x^m\}$.*

Lemma 2 compares the overall values and constraint values of two portfolios $p, p' \in P^k$ to determine if p cannot become non-dominated.

Lemma 2. *Let $p, p' \in P^k$. If $p' \succ p$ and $Az(p') \leqslant Az(p)$ (where $\leqslant$ holds componentwise), then $(p \cup p'') \notin P_N$ for any $p'' \subseteq \{x^{k+1}, \ldots, x^m\}$.*

Lemma 3 states that a feasible portfolio $p \in P^k \cap P_F$ cannot become non-dominated, if it is *full*, i.e., no projects can be added to it without losing feasibility, and there exists a feasible portfolio $p' \in P_F$ that dominates p.

Lemma 3. *Let $p \in P^k \cap P_F$. If $\sum_{x^j \in p} a_l^j + \min_{j \in \{k+1,\ldots,m\}} a_l^j > b_l$ for some $l \in \{1,\ldots,q\}$, then p is full (i.e., $(p \cup p'') \notin P_F$ for any nonempty $p'' \subseteq \{x^{k+1},\ldots,x^m\}$). Furthermore, if there exists $p' \in P_F$ such that $p' \succ p$, then $(p \cup p'') \notin P_N$ for any $p'' \subseteq \{x^{k+1},\ldots,x^m\}$.*

Lemma 4 can be used to discard portfolio $p \in P^k$ based on an upper bound of how much its overall value can still be increased. Thus, we are interested in the maximal overall value of portfolio $p'' \subseteq \{x^{k+1},\ldots,x^m\}$, measured at the extreme point w^i of $\mathrm{conv}(S_w)$, achievable with the slack in constraints $\widetilde{B} = B - Az(p)$. This corresponds to an ILP problem

$$\max_{\tilde{z} \in \{0,1\}^{m-k}} \{\tilde{z}^{\mathrm{T}} \tilde{v} w^i | \widetilde{A}\tilde{z} \leqslant \widetilde{B}\}, \tag{18}$$

where $\tilde{v} = [\bar{v}^{k+1\mathrm{T}},\ldots,\bar{v}^{m\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{(m-k) \times n}$ and $\widetilde{A} = [a^{k+1},\ldots,a^m] \in \mathbb{R}^{q \times (m-k)}$. An exact solution at each extreme point may be too time-consuming. Instead, we solve the Lagrangian dual of (18) with subgradient optimization, which gives an upper bound $U_i^{k+1}(\widetilde{B})$ for the exact solution with less computational effort (see, e.g., Bertsimas and Tsitsiklis, 1997).

In Lemma 4, the overall value of portfolio $p \in P^k$ plus the upper bound of the increase, denoted by vector $U^{k+1}(\widetilde{B}) = [U_1^{k+1}(\widetilde{B}),\ldots,U_t^{k+1}(\widetilde{B})]^{\mathrm{T}}$, is compared to the overall value of a reference portfolio $p' \in P_F$ to determine if p cannot become non-dominated.

Lemma 4. *Let $p' \in P_F$ and $p \in P^k$. If $z(p' \setminus p)^{\mathrm{T}} \underline{v} W_{\mathrm{ext}} \geqq z(p \setminus p')^{\mathrm{T}} \bar{v} W_{\mathrm{ext}} + U^{k+1}(B - Az(p))$ (denoted by $p' \succ^U p$), then $(p \cup p'') \notin P_N$ for any $p'' \subseteq \{x^{k+1},\ldots,x^m\}$.*

For Lemma 4 to discard portfolios effectively, the reference portfolio p' should have a high overall value over the whole information set S. The first step is to generate a set of reference portfolios $P_D \subset P_F$ by solving the ILP-problem (4) with random feasible weights and scores. Algorithm for the computation of non-dominated portfolios (CND) is formulated as follows.

$P_N = \mathbf{CND}(W_{\mathrm{ext}}, \bar{v}, \underline{v}, A, B)\{$
    1. Generate P_D
    2. $P^0 \leftarrow \{\emptyset\}$
    3. **For** $k = 1,\ldots,m$ **do**
        (a) $P^k \leftarrow \{(p \cup \{x^k\}) | p \in P^{k-1}\} \cup P^{k-1}$
        (b) $P^k \leftarrow \{p \in P^k | \sum_{x^j \in p} a_l^j + \sum_{j=k+1}^m \min\{0, a_l^j\} \leqslant b_l \ \forall l \in \{1,\ldots,q\}\}$
        (c) $P^k \leftarrow \{p \in P^k | p' \not\succ^U p \ \forall p' \in P_D\}$
        (d) $P^k \leftarrow \{p \in P^k | p' \not\succ^U p \ \forall p' \in \{p' \in P^k \cap P_F | p'' \not\succ p' \ \forall p'' \in P_D\}\}$
        (e) $P^k \leftarrow \{p \in P^k | \nexists p' \in P^k \text{ such that } p' \succ p, Az(p') \leqslant Az(p)\}$
    4. $P_N \leftarrow \{p \in P^m | p' \not\succ p \ \forall p' \in P^m\}$
$\}$

In Step 3, the algorithm runs through the iteration scheme (17) for all $k = 1,\ldots,m$; in Step 3(a), P^k is structured by taking each portfolio in P^{k-1} with and without project x^k. Step 3(b) discards portfolios that cannot become feasible by Lemma 1. Steps 3(c) and 3(d) discard portfolios by Lemma 4, first using portfolios in P_D as reference portfolios (c) and then using feasible portfolios in P^k that are not dominated by any portfolio in P_D as reference portfolios (d). By Lemma 3, zero upper bounds are used for full portfolios p when determining if $p' \succ^U p$ in Steps 3(c) and 3(d). Step 3(e), discards portfolios within P^k by Lemma 2. Finally in Step 4, $P^m \subseteq P_F$ since all infeasible portfolios have been discarded in Step 3(b). Thus, P_N is obtained from P^m by discarding dominated portfolios.

5. Illustrative example

In product release planning, it is typically necessary to consider several customers with different preferences for the product features, logical requirements and value and cost synergies (Ruhe and Saliu, 2005; Salo and Käkölä, 2005). Here, we present an illustrative example where a technology company has to select a feature portfolio from 40 feature candidates (cf. projects), denoted by $x^1,\ldots,x^{40}$. Each customer (cf. criteria), indexed

by $i = 1, 2, 3$, has given its own evaluations on how valuable each feature would be to it, by using the most preferred feature as a reference. These scores, denoted by v_i^j, are scaled so that the sum of each customer's scores is 1000. For the company, each additional feature increases the total development cost. The uncertain feature-specific costs c_j are evaluated as intervals. The maximum budget is 800,000 euros, but the final expenditure is based on benefit–cost analysis. The features' scores and costs (in thousands of euros) are in Table 1.

The company uses an additive value model for the overall value of each feature. The relative importance of each customer is evaluated based on its size and market power, expected order volume of the new product and the history of business partnership. The company evaluates the importance through the ranking: Customer 1 is the most important, Customer 2 is the second and Customer 3 is the least important. This corresponds to the feasible weight set $S_w = \{w = (w_1, w_2, w_3)^{\mathrm{T}} \in S_w^0 | w_1 \geqslant w_2 \geqslant w_3\}$, i.e., a unit increase in the first customer's

Table 1
Benefit scores and costs of the features

Name	Customer 1 (v_1^j)	Customer 2 (v_2^j)	Customer 3 (v_3^j)	Cost (c_j)
Feature A1	50	0	22	[82,90]
Feature A2	21	10	0	[38,40]
Feature A3	23	9	28	[46,50]
Feature A4	16	6	22	[39,40]
Feature A5	0	0	0	[58,60]
Feature A6	57	0	0	[149,150]
Feature A7	25	22	45	[9,10]
Feature A8	10	50	20	[29,30]
Feature A9	33	23	0	[59,60]
Synergy 1	0	0	0	[−50,−40]
Feature B1	20	0	9	[27,30]
Feature B2	18	30	31	[179,190]
Feature B3	52	3	34	[99,110]
Feature B4	8	60	13	[67,70]
Feature B5	55	6	33	[75,80]
Feature B6	19	52	53	[64,70]
Feature B7	70	0	47	[68,70]
Feature B8	55	0	23	[38,40]
Feature B9	65	27	48	[67,70]
Feature B10	4	0	24	[108,120]
Feature B11	76	26	0	[28,30]
Feature B12	0	16	27	[446,490]
Feature B13	68	61	38	[49,50]
Feature B14	13	60	1	[113,120]
Synergy 2	0	[30,50]	0	[0,0]
Feature C1	0	5	13	[86,90]
Feature C2	0	47	32	[127,130]
Feature C3	33	14	28	[116,120]
Feature C4	0	63	2	[164,180]
Feature C5	11	55	20	[113,120]
Feature C6	24	40	32	[49,50]
Feature C7	0	0	51	[255,260]
Feature C8	0	29	4	[30,30]
Feature C9	0	40	9	[67,70]
Feature C10	0	0	47	[59,60]
Feature C11	40	49	30	[28,30]
Feature C12	0	2	26	[163,170]
Feature C13	4	3	0	[88,90]
Feature C14	17	9	15	[346,380]
Feature C15	89	38	43	[63,70]
Feature C16	6	31	8	[99,100]
Feature C17	0	4	24	[9,10]
Synergy 3	0	0	[0,30]	[0,0]

benefit is preferred to a unit increase in the second customer's benefit, which is preferred to a unit increase in the third customer's benefit. The set S_w also contains the equal weight vector $w = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^{\mathrm{T}}$, which corresponds to the average of the customers' scores.

The overall value of the feature portfolio (2) is to be maximized subject to varying budget levels and incomplete information on the features' benefits and costs. In addition, the following properties have to be accounted for:

- *Product positioning.* To promote novelty and scope of the product, at least three features from each technological area (A,B,C) must be included in the product.
- *Follow-up features.* Technical properties require that Feature A2 cannot be included unless A1 is included. Similarly, B3 is possible only if both B1 and B2 are included, and Feature C4 calls for C1, C2 and C3.
- *Cost synergies.* Features A8 and A9 can be implemented jointly at a lower cost than separately. This is modeled with a dummy feature 'Synergy 1' which has a negative cost interval $[-50, -40]$ (Table 1). That is, savings between 40,000 and 50,000 euros are gained in comparison with the sum of the 'stand-alone' costs of these features.
- *Value synergies.* Customer 2 has stated that Features B13 and B14 together yield an extra benefit to it. This is modeled by dummy feature the 'Synergy 2' with a zero-cost and positive score interval for Customer 2. If Features C14–C17 are all included in the product, Customer 3 *may* gain an extra benefit. This is modeled by the zero-cost dummy feature 'Synergy 3' with a positive score interval ranging from zero for Customer 3. The dummy features are also listed in Table 1.

Cost is included in the model as a criterion to be minimized. Eqs. (11) and (12) in Theorem 3 are used to formulate the extended information set $\widehat{S} = \widehat{S}_w \times \widehat{S}_v$. The extreme points of $\mathrm{conv}(\widehat{S}_w)$ are $(1,0,0,0)^{\mathrm{T}}$, $\left(\frac{1}{2}, \frac{1}{2}, 0, 0\right)^{\mathrm{T}}$, $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0\right)^{\mathrm{T}}$ and $(0,0,0,1)^{\mathrm{T}}$. Computation of the set $P_{\mathrm{E}}(S, S_c)$ took some 7 min on a personal computer (Intel Centrino Duo, 1.83 GHz, 1 GB) and resulted in 737 efficient feature portfolios. A total of 6 core features $X_{\mathrm{C}}(\widehat{S})$ were included in all efficient feature portfolios, while 17 exterior features $X_{\mathrm{E}}(\widehat{S})$ were not included in any of them. By Theorem 2, the company can focus further analysis on the 20 borderline features $X_{\mathrm{B}}(\widehat{S})$. The release planning was started with wide cost intervals for all 43 features/synergies; now only 20 of them can impact the results if they became narrower. With point estimate costs for the borderline features (chosen randomly from the intervals in this example), the number of efficient feature portfolios is reduced to 449.

The overall values and compositions of the 449 efficient feature portfolios are characterized in Figs. 1 and 2. Fig. 1 illustrates how the maximal and guaranteed overall values develop as a function of the budget level R. First, the GV curve shows that $R = 227$ is the lowest budget level which *guarantees* that a feasible portfolio can be constructed (three features from A, B and C). From MV we see that the cost of this first portfolio *may* be as low as $R = 210$. The range $[\mathrm{GV}(R), \mathrm{MV}(R)]$ expands with increasing budget R, since the more expensive portfolios include more features that add to the overall value and cost intervals. Benefit–cost ratios $(\mathrm{MV}(R)/R$ and $\mathrm{GV}(R)/R)$ reach their maximum at $R = 287$, whereafter both ratios decrease in R. This is because the efficient portfolios at lower budget levels include mostly features with the highest individual benefit–cost ratios. As the budget level increases, features with less attractive benefit–cost ratios are added to maximize the overall value of the portfolios.

Fig. 2 characterizes efficient portfolios at different budget levels by showing the budget-dependent core indexes of the 40 features and 3 synergies. Let us highlight some results:

- *Robustness of the features.* Features A7, A8, A9, B11 and C11 (and Synergy 1 from A8 and A9) are core features for all feasible budget levels $R \in [227, 800]$. These features are relatively inexpensive with high benefit–cost ratio intervals. Features C15 and B9, on the other hand, are exterior at low budget levels, but become core from $R = 276$ and $R = 472$ forward, respectively. These two features yield high benefits to the customers; however, they are more expensive and thus become core features only when the budget is high enough. Features B1 and C8, in turn, offer relatively little value at low cost. Their core indexes vary across the budget range, depending on whether there is slack in the budget for these 'small' features; when
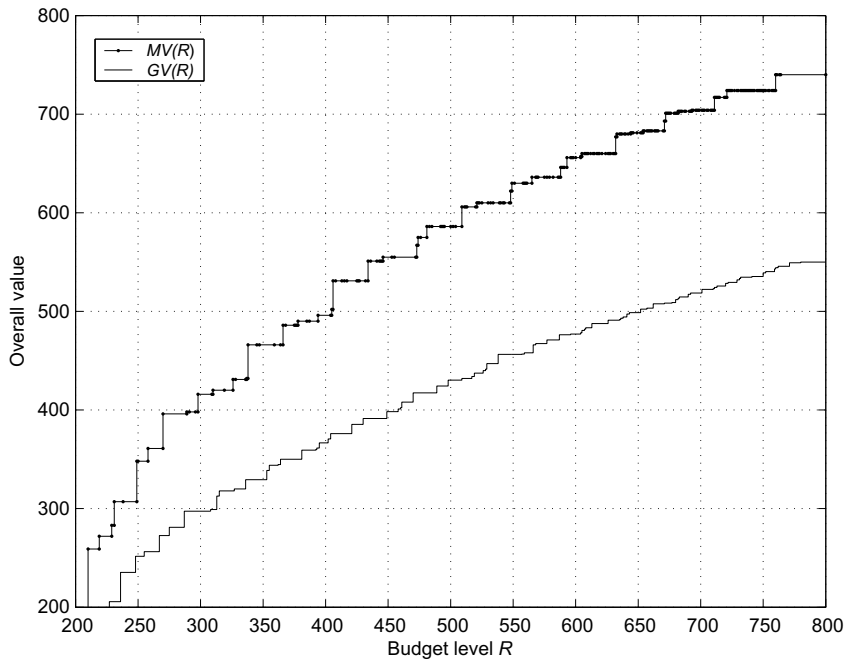
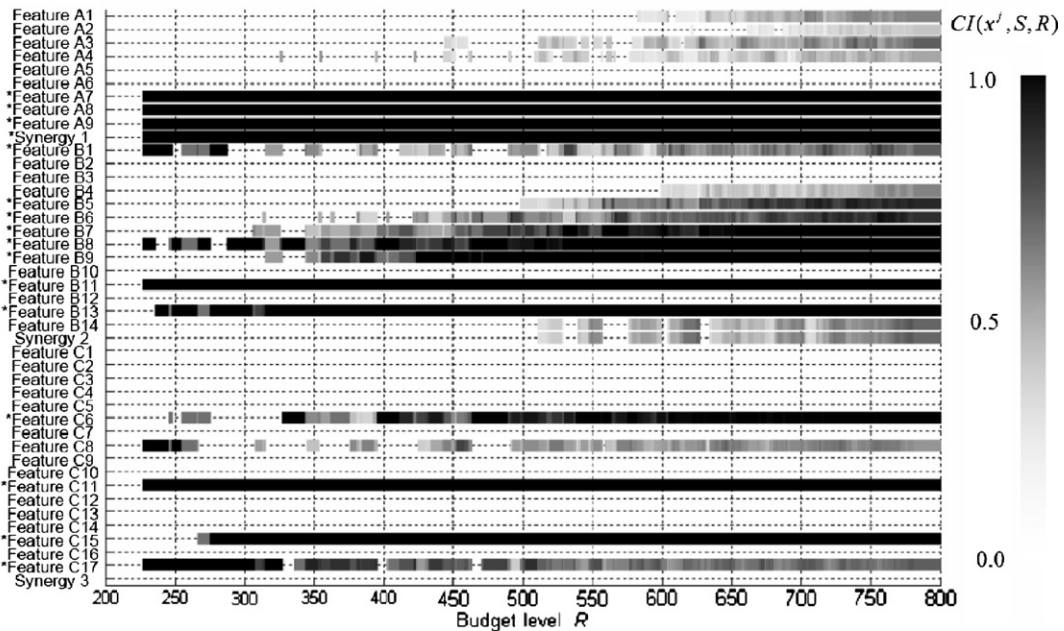Fig. 1. Guaranteed and maximal overall values as a function of the budget level.

Fig. 2. Budget-dependent core indexes $CI(x^j, S, R)$ of the features.

the budget increases further to accommodate another higher value feature, their core indexes decrease until there is additional slack so that they can be implemented.

- *Role of the positioning constraints.* The first efficient portfolio at $R = 227$ consists of the three least expensive features from technological areas A, B and C (taking into account the synergy between A8 and A9). At the budget level $R = 236$ feature B8 is replaced by the more valuable $B13$, whereafter features B1, B8, B13, C6
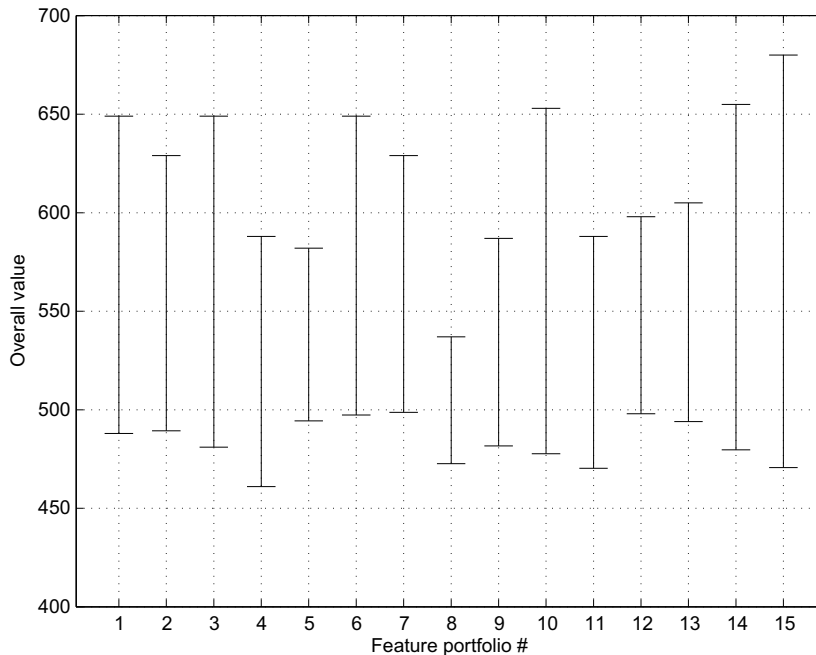
Fig. 3. Overall value intervals of the non-dominated feature portfolios $P_N(S, \bar{c}, R)$ at budget level $R = 650$.

and C8 come and go in short intervals as the budget increases. All positioning constraints remain binding until $R = 315$. As the budget level increases, the positioning constraints become redundant.

• *Role of the synergies.* Synergy 1 provides a substantial cost saving relative to the sum of Features' A8 and A9 stand-alone costs. Feature A9 alone would probably not be a core feature, but combined with A8 and the resulting Synergy 1, these features belong to all efficient portfolios. Synergy 2, which is activated by including Feature B14 on top of B13, holds potential for a relatively high added value for Customer 2. However, B14 and B13 together are expensive and the overall value interval of Synergy 2 is wide. Thus, Synergy 2 (and Feature B14) are not included in any efficient portfolio until the budget level reaches $R = 510$, and even thereafter their core indexes vary. Synergy 3, which requires that all Features C14–C17 are implemented, is exterior throughout the budget range. This is because the group of features C14–C17 is very expensive, and the added value of Synergy 3 is relatively small, uncertain and yields benefit to Customer 3 only. Thus, all synergies are not pursued in the efficient portfolios.

The company sets the final budget at $R = 650$. At this level the sum of the features' scores is above 330 (one third of the maximum) for all customers. Also, the guaranteed overall value exceeds 500 (Fig. 1), which is half of the overall value achieved if all features were included. Third, the core indexes are rather stable around $R = 650$, meaning that small budget variations would not change the sets of core, borderline and exterior features.

The set $P_N(S, \bar{c}, 650)$ contains 15 portfolios, whose overall value intervals are shown in Fig. 3. The core indexes $CI(x^j, S, 650)$, (see Fig. 2 at $R = 650$) show that there are 11 borderline features (and one synergy) at $R = 650$. The maximin decision rule (9) recommends portfolio #7 as the final choice. However, Fig. 3 shows that portfolio #6 offers a higher maximum value while the minimum value is almost the same. Furthermore, the upper bound of the cost of portfolio #6 is 641,000 euros, while it is 643,000 for #7, and the portfolios differ by one feature only (B1 in #6 and C8 in #7). Thus, the company chooses to release the product with feature portfolio #6, marked with asterisks in Fig. 2.

6. Conclusions

In this paper, we have extended the RPM framework (Liesiö et al., 2007) to: (i) incorporate project inter-dependencies, such as synergies and portfolio positioning requirements, (ii) admit interval-valued estimates

about project costs and (iii) consider variations in the budget level and benefit–cost analysis. Based on the computation of all *efficient portfolios*, the budget-dependent core index helps identify which projects are robust choices at any given budget level. It also suggests how to focus further efforts to elicit interval-valued score and cost estimates. Furthermore, portfolio-level measures show how the maximal and guaranteed overall values develop as a function of the budget level. Overall, the extended RPM framework helps examine the effects of incomplete information (weights, scores, costs and budget). It helps focus attention to those projects and model parameters where additional information is likely to matter most.

Computationally, the extended RPM leads to a multi-objective zero-one linear programming problem with interval-valued objective function coefficients. Our experiments with the dedicated dynamic programming algorithm suggest that problems with some 60 projects, 5 criteria and 10 constraints can be solved with a personal computer in a reasonable time, although the solution time is highly dependent on the types of constraints and correlations between project scores and constraint coefficients; here, further simulation studies are needed to characterize how the solution time depends on problem properties. It would also be of interest to examine how depth-first type MOZOLP algorithms (e.g., Kiziltan and Yucaoglu, 1983) can be extended to account for interval-valued objective function coefficients, and how they would perform compared to our breadth-first algorithm. Furthermore, approximative or heuristic algorithms are needed to determine a representative subset of efficient portfolios for large problems (e.g., 200 projects). Also, the first set of efficient portfolios may be computed prior to a decision workshop, for example. The ensuing phase of interactive analysis is computationally straightforward, because all subsets of efficient portfolios are obtained by pairwise comparisons between portfolios. Thus, the decision support process is not too sensitive to the solution time of the dynamic programming algorithm.

This work paves way for future research on large decomposed project portfolio selection problems under incomplete information. For instance, a company may have several business units that are confronted with both local and shared budget constraints. In such cases, the sets of efficient portfolios for each subproblem could be employed to support the allocation of the total budget among the business units and for determining the the optimal project portfolio within each business unit.

Acknowledgement

Appendix

The proofs of Theorems 1,2 and Corollary 1 are presented in Liesiö et al. (2007) using a slightly different notation.

Proof of Theorem 3. Assume $p \succ_{\hat{S}} p$, which by definition is equal to

$$V(p, w, [v, -c]) \gtrapprox V(p, w, [v, -c]) \ \forall (w, [v, -c]) \in \widehat{S}$$
$$\Longleftrightarrow V(p, w, [v, -c]) \gtrapprox V(p, w, [v, -c]) \ \forall w \in \text{ext}(\text{conv}(\widehat{S}_w)), [v, -c] \in \widehat{S}_v,$$

where $\gtrapprox$ means that the inequality is strict for some values. Since the extreme points of $\widehat{S}_w$ are those of S_w together with unit vector e^{n+1} we have

$$\Longleftrightarrow \begin{cases} V(p, w, v) \geqslant V(p, w, v) \ \forall w \in \text{ext}(\text{conv}(S_w)), v \in S_v \\ -z(p)^{\mathrm{T}} c \geqslant -z(p')^{\mathrm{T}} c \ \forall c \in S_c \end{cases}$$
$$\Longleftrightarrow \begin{cases} V(p, w, v) \geqslant V(p, w, v) \ \forall (w, v) \in S \\ z(p)^{\mathrm{T}} c \leqslant z(p')^{\mathrm{T}} c \ \forall c \in S_c \end{cases}$$

with at lest one inequality strict. Thus, $p \in P_{\mathrm{N}}(\widehat{S})$ if and only if p is efficient. $\square$

Proof of Theorem 4

(i) Let $p \in P_E(S, S_c)$. Choose $\hat{c} \in S_c$ such that $\hat{c}^j = \underline{c}, x^j \in p, \hat{c}^j = \bar{c}, x^j \notin p$ and $\widehat{R} = C(p, \hat{c})$. Contrary to the claim assume that $p \notin P_N(S, c, R)$. Then there exists $p' \in P_N(S, c, R)$ such that $p' \succ_S p$ and $C(p', \hat{c}) \leqslant \widehat{R} = C(p, \hat{c})$. This implies that for any $c \in S_c$ we have $C(p', c) - C(p, c) = C(p' \setminus p, c) + C(p' \cap p, c) - C(p \cap p', c) - C(p \setminus p', c) \leqslant C(p' \setminus p, \bar{c}) + C(p' \cap p, \underline{c}) - C(p \cap p', \underline{c}) - C(p \setminus p', \underline{c}) = C(p', \hat{c}) - C(p, \hat{c}) \leqslant 0$. Thus $p' \succ_S p$ and $C(p', c) \leqslant C(p, c) \ \forall c \in S_c$ implying $p \notin P_E(S, S_c)$, which is a contradiction. Thus $p \in P_N(S, c, R)$.

(ii) Let $p \in P_N(S, c, R)$ and denote $[p] = \{p' \in P_N(S, c, R) | V(p', w, v) = V(p, w, v) \ \forall (w, v) \in S\}$ which is non-empty since $p \in [p]$. Contrary to the claim assume that $[p] \cap P_E(S, S_c) = \emptyset$. Then for every $p' \in [p]$ there exists $p'' \in P_E(S, S_c)$ such that $V(p'', w, v) \geqslant V(p', w, v) \ \forall (w, v) \in S$, $C(p'', c) \leqslant C(p', c)$. Also $p'' \in P_F(R, c)$, since $C(p'', c) \leqslant C(p', c) \leqslant R$. If the first inequality is strict for some $(w, v) \in S$ then $p'' \succ_S p'$ which implies $p'' \succ_S p \notin P_N(S, c, R)$ which is a contradiction. On the other hand if $V(p'', w, v) = V(p', w, v) \ \forall (w, v) \in S$, then $p'' \in [p]$ which contradicts $[p] \cap P_E(S, S_c) = \emptyset$. $\square$

Proof of Lemma 1. Let $p \in P^k$. Contrary to the claim assume that for some $l \sum_{x^j \in p} a_l^j + \sum_{j=k+1}^m \min\{0, a_l^j\} > b_l$ and there exists $p'' \subseteq \{x^{k+1}, \ldots, x^m\}$ such that $(p \cup p'') \in P_N$. Since $P_N \subseteq P_F$, $(p \cup p'') \in P_F$ and thus $Az(p \cup p'') = Az(p) + Az(p'') \leqslant B$ (because $p \cap p'' = \emptyset$). Since all feasibility constraints are satisfied, $b_l \geqslant \sum_{x^j \in p} a_l^j + \sum_{x^j \in p''} a_l^j \geqslant \sum_{x^j \in p} a_l^j + \sum_{j=k+1}^m \min\{0, a_l^j\} > b_l$, which is a contradiction. $\square$

Proof of Lemma 2. Let $p, p' \in P^k$. Contrary to the claim assume that $p' \succ p$, $Az(p') \leqslant Az(p)$ and there exists $p'' \subseteq \{x^{k+1}, \ldots, x^m\}$ such that $(p \cup p'') \in P_N$. Since $P_N \subseteq P_F$, $(p \cup p'') \in P_F$ which corresponds to $Az(p \cup p'') = Az(p) + Az(p'') \leqslant B$, since $p \cap p'' = \emptyset$. However by assumption, $Az(p' \cup p'') = Az(p') + Az(p'') \leqslant Az(p) + Az(p'') \leqslant B$ and thus $(p' \cup p'') \in P_F$. The overall value difference $V(p', w, v) - V(p, w, v)$ is equal to $z(p')^T vw - z(p)^T vw = z(p')^T vw - z(p)^T vw + (z(p'')^T vw - z(p'')^T vw) = V(p' \cup p'', w, v) - V(p' \cup p'', w, v)$, since $p' \cap p'' = p \cap p'' = \emptyset$. Thus, $p' \succ p$ implies $(p' \cup p'') \succ (p \cup p'')$ and therefore $(p \cup p'') \notin P_N$ which is a contradiction. $\square$

Proof of Lemma 3

(i) Let $p \in P^k \cap P_F$. Contrary to the claim assume that there exists l such that $\sum_{x^j \in p} a_l^j + \min_{j \in \{k+1, \ldots, m\}} a_l^j > b_l$ and there exists $p'' \subseteq \{x^{k+1}, \ldots, x^m\} \setminus \emptyset$ such that $(p \cup p'') \in P_F$. If $a_l^j < 0$ for some $x^j \in p''$, then, since p is feasible, $b_l < \sum_{x^j \in p} a_l^j + \min_{j \in \{k+1, \ldots, m\}} a_l^j \leqslant \sum_{x^j \in p} a_l^j \leqslant b_l$, which is a contradiction. On the other hand, if $a_l^j \geqslant 0$ for all $x^j \in p''$ then, since p'' is not empty, $b_l \geqslant \sum_{x^j \in p} a_l^j + \sum_{x^j \in p''} a_l^j \geqslant \sum_{x^j \in p} a_l^j + |p''| \min_{x^j \in p''} a_l^j \geqslant \sum_{x^j \in p} a_l^j + \min_{x^j \in p''} a_l^j \geqslant \sum_{x^j \in p} a_l^j + \min_{j \in \{k+1, \ldots, m\}} a_j^l > b_l$ which is a contradiction.

(ii) If p is full, then $(p \cup p'') \notin P_F$ for all $p'' \subseteq \{x^{k+1}, \ldots, x^m\} \setminus \emptyset$, and thus $(p \cup p'') \notin P_N$. If $p'' = \emptyset$ then $P_F \ni p' \succ p = (p \cup p'') \notin P_N$. $\square$

Proof of Lemma 4. Let $p' \in P_F$, $p \in P^k$, $p'' \subseteq \{x^{k+1}, \ldots, x^m\}$ and

$$z(p' \setminus p)^T \underline{v} W_{ext} \gtreqqless z(p \setminus p')^T \bar{v} W_{ext} + U^{k+1}(B - Az(p)). \tag{19}$$

If $(p \cup p'') \notin P_F$ then $(p \cup p'') \notin P_N$ and the lemma holds. Contrary, if $(p \cup p'') \in P_F$ then $Az(p \cup p'') \leqslant B$ which implies $Az(p'') \leqslant B - Az(p)$, since p and p'' are disjoint. Thus, $U^{k+1}(B - Az(p)) \geqslant z(p'') \bar{v} W_{ext}$ which can be used to approximate (19) by noting that p and p'' are disjoint: $z(p' \setminus p)^T \underline{v} W_{ext} \gtreqqless z(p \setminus p')^T \bar{v} W_{ext} + U^{k+1}(B - Az(p)) \gtreqqless z(p \setminus p')^T \bar{v} W_{ext} + z(p'')^T \bar{v} W_{ext} = z((p \cup p'') \setminus p')^T \bar{v} W_{ext} + z(p'' \cap p')^T \bar{v} W_{ext} \gtreqqless z((p \cup p'') \setminus p')^T \bar{v} W_{ext} + z(p'' \cap p')^T \underline{v} W_{ext}$. Rearranging the terms result in the inequality $z(p' \setminus (p \cup p''))^T \underline{v} W_{ext} \gtreqqless z((p \cup p'') \setminus p')^T \bar{v} W_{ext}$. Furthermore, since $z(p' \setminus (p \cup p''))^T \bar{v} W_{ext} \geqslant z(p' \setminus (p \cup p''))^T \underline{v} W_{ext} \neq z((p \cup p'') \setminus p')^T \bar{v} W_{ext} \geqslant z((p \cup p'') \setminus p')^T \underline{v} W_{ext}$ we have $P_F \ni p' \succ (p \cup p'') \notin P_N$. $\square$

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