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SCENARIO-BASED PORTFOLIO SELECTION OF INVESTMENT PROJECTS WITH INCOMPLETE PROBABILITY AND UTILITY INFORMATION

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Abstract

In the selection of investment projects, it is important to account for exogenous uncertainties (such as macroeconomic developments) which may impact the performance of projects. These uncertainties can be addressed by examining how the projects perform across a set of scenarios; but it may be difficult to assign well-founded probabilities to such scenarios, or to characterize the decision makers' risk preferences through a uniquely defined utility function. Motivated by these considerations, we develop a portfolio selection framework which (i) uses set inclusion to admit incomplete information about scenario probabilities and utility functions, (ii) identifies all the non-dominated project portfolios in view of available information, and (iii) offers consequent decision support for the selection and rejection of projects. The proposed framework enables interactive decision support processes where the implications of introducing additional probability and utility information or further risk constraints are shown in terms corresponding decision recommendations.

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1 Introduction

Industrial and public organizations take 'go/no go' decisions about investment projects with uncertain future consequences. Typically these decisions are complicated by the presence of multiple attributes, several resources constraints, project interdependencies, and balance requirements across technologies and business areas. These complications, along with the importance of the project portfolio selection problem, have fostered the development of decision analytic methods which have found high-impact applications in domains such as R&D project portfolio selection (Golabi *et al.*, 1981; Beaujon *et al.*, 2008), capital budgeting in healthcare (Kleinmuntz, 2008), and military resource allocation (Ewing *et al.*, 2006).

In the selection of investment projects, exogenous uncertainties (which are not influenced by the projects, e.g., rate of industry growth) are crucial, because they may impact many projects, even to the point where unfavorable developments decrease the value of every project in the portfolio. Such uncertainties can be captured through a set of scenarios (see, e.g., Bunn and Salo, 1993; Poland, 1999) and by combining scenario-based project analyses with models for risk preferences and risk constraints (see, e.g., Gustafsson and Salo, 2005). However, well-founded information about the scenario probabilities or the decision makers' (DM) risk preferences may be difficult to elicit: for example, the DMs may have conflicting views about these probabilities, and they may also exhibit different risk attitudes. It is therefore important to explore how decisions can be supported on the basis of incomplete information, also in view of positive experiences from the use of such information in multi-attribute project selection problems (Stummer and Heidenberger, 2003; Liesiö *et al.*, 2007, 2008; Lindstedt *et al.*, 2008).

In this paper, we develop a decision analytic framework for scenario-based selection of portfolios of investment projects based on incomplete probability and utility information. We model incomplete information through set inclusion and solve multiple objective zero-one linear programming problems to determine all the corresponding non-dominated portfolios in recognition of relevant logical, resource, and risk constraints as well as project interdependencies. Then, decision recommendations about individual projects are derived by examining which projects are contained in all, some, or none of non-dominated portfolios; this approach applies the concepts of core, borderline and exterior projects from Robust Portfolio Modeling (Liesiö *et al.*, 2007) to scenario-based decision analyses. Furthermore, the DMs are allowed to provide information about scenario probabilities and their risk preferences interactively, and to examine the implications of this for decision recommendations.

The rest of this paper is structured as follows. Section 2 discusses earlier approaches for the project portfolio selection. Section 3 presents our analytical framework. Section 4 extends this framework to settings with incomplete information, presents corresponding dominance structures and describes approaches to decision support. Section 5 develops computational methods for the identification of non-dominated portfolios. Section 6 presents an illustrative example in R&D portfolio selection.

2 Earlier Approaches to Project Portfolio Selection

The selection of investment projects involves usually estimates about the DM's preferences and the projects' future performance. Decision support for this problem needs to be aligned with the possibilities of eliciting such estimates; in particular, the usability of highly sophisticated optimization models that assume complete information and offer unique 'optimal' solutions may be limited if the requisite inputs cannot be elicited with a high level of confidence (Kleinmuntz, 2008; Cooper *et al.*, 1999). For example, Stummer and Heidenberger (2003) note that DMs may find it difficult to provide exact information about their preferences. Motivated by this recognition, they develop an approach for R&D project portfolio selection based on the computation of all Pareto-optimal portfolios in view of multiple attributes (i.e. portfolios that cannot be improved with regard to all attributes). They also describe a dedicated decision support software which allows the DMs to set aspiration levels for the attributes and to seek interactively portfolios that are consistent with their preferences.

Even if consensus estimates about the investments' expected values can be obtained, uncertainties about these estimates contribute to portfolio risk. For example, Beaujon *et al.* (2008) evaluated risks arising from uncertain project values in R&D portfolio selection. Specifically, they first obtained the optimal portfolio with mixed integer programming and then used Monte Carlo simulation to explore how the value of the optimal portfolio varies subject to random errors in the projects' value estimates. In their sensitivity analysis, the portfolio value was close to the optimum even with rather large errors. This is not surprising, because the independently distributed random errors (an assumption which can be questioned in many cases) are likely to cancel out each other, when the portfolio value is computed as the sum of projects' values.

In their case study on strategic product portfolio development, Lindstedt *et al.* (2008) used Robust Portfolio Modeling (RPM; Liesiö *et al.*, 2007, 2008) to capture different views on the products' future values in order to identify most attractive combinations of technologies and market segments. In RPM, the possibility to admit incomplete information through feasible sets of parameters value was harnessed to account for different views within the management team. Based on the computation of all non-dominated portfolios (i.e. portfolios for which a better portfolio for all allowed parameter values cannot be found), decision recommendations for the product strategy were developed in an interactive decision workshop. Because many products were included in all non-dominated portfolios and some in none, the discussion could be focused on those 'borderline' products included in some but not all non-dominated portfolios. Thus, instead of requiring a lengthy debate on what the 'correct' values for model parameters would be, RPM helped demonstrate which product decisions were supported by the available and partly incomplete information.

Poland (1999) reports positive experiences from the use of decision trees in the development of a business portfolio strategy. In his model, the exogenous (or global) uncertainties (e.g., gross domestic product, interest rates) were captured by scenarios and associated probabilities, which were then harnessed in the analysis of 20 businesses. Based on these results, multiple relevant portfolio strategies were structured, which specified strategy within each business. Yet, instead of computing a single portfolio strategy, several nearly optimal portfolios were presented to the senior managers who then selected the final portfolio strategy. In the final selection, they also debated those goals and constraints that were not explicitly included in the preceding analyses.

In dynamic decision problems with dozens of projects, it is impossible to examine all portfolio strategies by inspection, and hence optimization approaches are needed. Contingent Portfolio Programming (CPP; Gustafsson and Salo, 2005), for example, captures exogenous uncertainties through a multi-period scenario tree with known probabilities and investments' scenario-specific cash flows, and determines the optimal portfolio strategy in sense of expected cash position at the terminal period from a mixed integer linear programming problem. In the CPP framework, it is also possible to introduce risk constraints based on various risk measures, for example by imposing bounds on the Conditional Value-at-Risk levels at different time periods or confidence levels (Kettunen and Salo, 2008).

3 Portfolio Selection under Risk

3.1 Portfolio Value and Feasibility

The m investment projects $X^0 = \{x^1, \ldots, x^m\}$ represent discrete 'go/no-go' decision alternatives with outcomes in n disjoint scenarios $\Omega = \{s_1, \ldots, s_n\}$. The value of project x^j in scenario s_i, denoted by $x^j(s_i)$, can represent the net present cash flow of the project in scenario s_i, or the cardinal multi-attribute value of the project, as obtained from conventional MAVT analysis (see, e.g., Keeney and Raiffa, 1976), for instance.

An investment portfolio X is a subset of available investment projects X^0 and hence the set of all possible portfolios is $\mathcal{X} = \{X \mid X \subseteq X^0\}$. Each portfolio X implies a real-valued random variable $X : \Omega \rightarrow \mathbb{R}$ which represents the portfolio's value:

$$X(s_i) = \sum_{x^j \in X} x^j(s_i). \tag{1}$$

The probability of scenario s_i is p_i. The scenario probabilities $p = (p_1, \ldots, p_n)^T$ belong to the set

$$P^0 = \{p \in \mathbb{R}^n \mid p_i \geq 0, \sum_{i=1}^{n} p_i = 1\}. \tag{2}$$

For any $p \in P^0$, the probability of the event $\omega \subseteq \Omega$ is $\mathbb{P}(\omega) = \sum_{s_i \in \omega} p_i$. For brevity, we write $\mathbb{P}(X \leq t) = \mathbb{P}(\{s_i \in \Omega \mid X(s_i) \leq t\})$. The expected value of portfolio X is $\mathbb{E}_p[X] = \sum_{i=1}^{n} p_i X(s_i)$, and $\mathbb{E}_p[X \mid \omega] = \sum_{s_i \in \omega} (p_i / \mathbb{P}(\omega)) X(s_i)$ is the expected portfolio value conditioned on the event $\omega \subseteq \Omega$ (with notational convention $\mathbb{E}_p[X \mid \emptyset] = 0$).

The set of feasible portfolios $\mathcal{X}_F \subseteq \mathcal{X}$ can be restricted by various constraints (e.g., availability of resources, project interdependencies, requirements of balance; see Stummer and Heidenberger, 2003; Liesiö *et al.*, 2008). These constraints are modeled through linear inequalities so that

$$\mathcal{X}_F = \{X \in \mathcal{X} \mid Az(X) \leq B\}, \tag{3}$$

where the coefficients for the q constraints are contained in the matrix $A \in \mathbb{R}^{q \times m}$ and the vector $B \in \mathbb{R}^q$. In (3), the binary vector $z(X) \in \{0,1\}^m$ is such that $z_j(X) = 1$ if and only if $x^j \in X$.

A risk neutral DM with complete information about scenario probabilities (in the sense in a single point estimate) seeks to maximize the expected value of the portfolio. This maximum can be solved from the linear zero-one programming (ZOLP) problem

$$\max_{X \in \mathcal{X}_F} \mathbb{E}_p[X] = \max_{z \in \{0,1\}^m} \{\sum_{j=1}^{m} z_j \sum_{i=1}^{n} p_i x^j(s_i)) \mid Az \leq B\}. \tag{4}$$

3.2 Portfolio Risk

Especially for large non-recurring investment projects the assumption of risk neutrality is not tenable. In the Expected Utility Theory (von Neumann and Morgenstern, 1947), the DM's risk preferences are captured by a strictly increasing utility function u that maps the portfolio values to utilities. Thus, instead of expected value, the DM seeks to maximize the expected utility of the portfolio so that (4) becomes a non-linear zero-one programming problem

$$\max_{X \in \mathcal{X}_F} \mathbb{E}_p[u(X)] = \max_{z \in \{0,1\}^m} \{\sum_{i=1}^{n} p_i u(\sum_{j=1}^{m} z_j x^j(s_i)) \mid Az \leq B\}. \tag{5}$$

In some situations it may be more convenient to use risk-measures and associated risk constraints rather than utility functions to model risk aversion. A risk-measure ρ maps each portfolio to a real-valued measure for risk. Since our model builds on maximization of value rather than minimization of losses, following Dentcheva and Ruszczynski (2006) we define that portfolio X is less or equally risky than X' if $\rho[X] \geq \rho[X']$. We would like to emphasize that this is matter of definition and does not change any practical aspects of measuring risk. Such a risk measure is *coherent* if for any $X, X' \in \mathcal{X}$ it satisfies (Artzner *et al.*, 1999)

$$\begin{aligned} \text{Translation invariance:} \quad & \rho[X+\lambda] = \rho[X] + \lambda \;\forall\; \lambda \in \mathbb{R} \\ \text{Positive homogeneity:} \quad & \lambda\rho[X] = \rho[\lambda X] \;\forall\; \lambda \geq 0 \\ \text{Superadditivity:} \quad & \rho[X+X'] \geq \rho[X] + \rho[X'] \\ \text{Monotonicity:} \quad & X(s_i) \geq X'(s_i) \;\forall\; s_i \in \Omega \Rightarrow \rho[X] \geq \rho[X']. \end{aligned}$$

Translation invariance and positive homogeneity guarantee that the ordering of portfolios based on their riskiness will not change if their values are subjected to positive affine transformations. Superadditivity implies that diversification does not increase risk: for example, if there are two equally risky portfolios X and X' such that $\rho[X] = \rho[X']$, then doubling either one of these portfolios results in a portfolio that is either more or equally risky than a diversified portfolio formed from X and X', because $\rho[2X'] = 2\rho[X'] = \rho[X'] + \rho[X] \leq \rho[X' + X]$. Monotonicity, in turn, ensures that if a portfolio X yields at least as much value as X' in all scenarios, it cannot be the more risky.

In our framework, we use the Conditional Value-at-Risk (CVaR) measure, which is coherent. For a fixed confidence level α, CVaR is the expected portfolio value, conditional to that the value realizes from the worst α-quantile, i.e., $\text{CVaR}[X] = \mathbb{E}[X|X \leq t]$, where t is such that $\mathbb{P}(X \leq t) = \alpha$. However, $\mathbb{P}(X \leq t) = \alpha$ may have no solution, because the cumulative probability distribution $\mathbb{P}(X \leq t)$ is discontinuous in t. Therefore an alternative definition is used, which coincides to the interpretation above if $\mathbb{P}(X \leq t) = \alpha$ has a solution.

Definition 1 *Let portfolio $X \in \mathcal{X}$, probabilities $p \in P^0$ and risk-level $\alpha \in (0, 1]$. Let scenarios be indexed so that $X(s_{i-1}) \leq X(s_i)$ for all $i \in \{2, ..., n\}$ and denote $\omega^0 = \emptyset$,*

$\omega^i = \{s_1, \ldots, s_i\}$ *for all* $i \in \{1, ..., n\}$. *The Conditional Value-at-Risk is defined as*

$$\mathrm{CVaR}_p^\alpha[X] = \lambda \mathbb{E}_p[X|\omega^{k-1}] + (1-\lambda)\mathbb{E}_p[X|\{s_k\}],$$

where $k = \max\{k \in \{1, \ldots, n\} \mid \mathbb{P}(\omega^{k-1}) \leq \alpha\}$ *and* $\lambda = \mathbb{P}(\omega^{k-1})/\alpha \in (0, 1]$.

The CVaR measure can be computed as the minimum expected value over a set of scenario probabilities, which is a linear programming (LP) problem. This follows from Artzner *et al.*(1999) which states that a risk measure is coherent if and only if it can be presented as the minimum expected value over some set of probability measures.

Lemma 1 *Let portfolio* $X \in \mathcal{X}$, *probabilities* $p \in P^0$ *and risk-level* $\alpha \in (0, 1]$. *Then*

$$\mathrm{CVaR}_p^\alpha[X] = \min_{q \in Q_p^\alpha} \mathbb{E}_q[X], \;\; Q_p^\alpha = \{q \in P^0 \mid q \leq \frac{p}{\alpha}\}, \tag{6}$$

or equivalently $\mathrm{CVaR}_p^\alpha[X] = \max_{t\in\mathbb{R}}(t - \frac{1}{\alpha}\mathbb{E}_p[\max\{0, t - X\}])$, *which is the dual of LP problem (6).*

The dual is often used as a definition of CVaR (Rockafellar and Uryasev, 2000; Dentcheva and Ruszczynski, 2006; Kettunen and Salo, 2008). Yet, we use the representation (6) which extends readily to the consideration of incomplete information about scenario probabilities and makes it possible to limit portfolio risk in Problem (4) without need for continuous variables. Because the minimum of (6) is always obtained at an extreme point of Q_p^α, denoted by $ext(Q_p^\alpha) = \{q^1, \ldots, q^t\}$, t additional constraints in Problem (4) ensure that the optimal portfolio's CVaR will exceed a given threshold γ:

$$\begin{aligned} \max_{\substack{X \in \mathcal{X}_F \\ \mathrm{CVaR}_p^\alpha[X] \geq \gamma}} \mathbb{E}_p[X] \;\; = \;\; & \max_{z \in \{0,1\}^m} \sum_{j=1}^m z_j \sum_{i=1}^n p_i x^j(s_i) \\ & Az \leq B \\ & \sum_{j=1}^m z_j \sum_{i=1}^n q_i x^j(s_i) \geq \gamma \;\forall\; q \in ext(Q_p^\alpha). \end{aligned} \tag{7}$$

4 Project Portfolio Selection under Incomplete Information

4.1 Modeling Incomplete Information

It may be difficult to obtain precise probabilities, because the elicitation of these probabilities may involve considerable costs or time delays; moreover, the experts may hold diverging beliefs about which scenarios are more probable than others. Thus, instead of deriving decision recommendations from a single probability estimate, it is instructive to admit incomplete probability information and to examine what implications are suggested by it (cf. e.g., White *et al.*, 1981; Hazen, 1986; Moskowitz *et al.*, 1993; Walley, 1991).

In our framework, the set of feasible probabilities is

$$P := \{p \in P^0 \mid A_p p \leq B_p\}, \tag{8}$$

where the matrix $A_p \in \mathbb{R}^{q_p \times n}$ and the vector $B_p \in \mathbb{R}^{q_p \times 1}$ are derived from statements about scenarios probabilities. For instance, if scenario s_1 is more likely than scenario s_2, we have the constraint $p_1 \geq p_2$. Events with multiple scenarios can also be compared: for example, if the event $\{s_1, s_2\}$ is less likely than the event $\{s_3, s_4, s_5\}$, the constraint $p_3 + p_4 + p_5 \geq p_1 + p_2$ holds. If scenario probabilities are derived from statistical analysis, confidence intervals can be characterized through lower and upper bounds ($\underline{p}_i \leq p_i \leq \overline{p}_i$) and if scenario probabilities are estimated by several experts, the set of feasible probabilities can be defined as the convex hull of their independent estimates (so that the extreme points of P, denoted by $ext(P)$, correspond to the experts' estimates).

Likewise, the set of feasible utility functions is $U \subset U^0$, where

$$U^0 = \{u : \mathbb{R} \rightarrow [0,1] \mid u(t) \geq u(t') \ \forall\ t \geq t'\}. \tag{9}$$

We assume that the set U is convex with at least one strictly increasing utility function (i.e., for any $t > t'$ these exists $u \in U$ such that $u(t) > u(t')$). If the DM is risk averse, the set of feasible utility functions is limited to concave functions

$$U^A := \{u \in U^0 \mid \lambda u(t) + (1-\lambda)u(t') \leq u(\lambda t + (1-\lambda)t') \ \forall\ t, t' \in \mathbb{R} \ \ \lambda \in [0,1]\}. \tag{10}$$

The set of feasible utility functions U can be restricted by standard techniques for the elicitation of risk preferences in which the DM compares alternatives with certain and uncertain outcomes (see, e.g., Clemen, 1996). Arguably, the elicitation of incomplete information can be easier, because responses need not to be adjusted until indifference is reached. Instead any preference for one option over another implies constraints on the model: for example, if a lottery which yields \$1 million with probability of 40% and \$50 thousand with probability of 40% is preferred to a certain outcome which yields \$100 thousand for sure, all feasible utility functions have to satisfy the constraint $u(100) \leq 0.4u(1000) + 0.6u(50)$

The information set (which contains information about probabilities and utility functions) is denoted by $S = P \times U$. The largest such set (which reflects no information on scenario probabilities or utility functions) is denoted by $S^0 = P^0 \times U^0$.

4.2 Dominance structures

When scenario probabilities and utility functions vary over their respective feasible sets, different expected utilities are associated with the portfolio X. In order to make conclusions about which portfolios outperform others, we define *dominance* as follows (cf. White *et al.*, 1981; Hazen, 1986; Moskowitz *et al.*, 1993).

Definition 2 *Portfolio X dominates X' with regard to information set $S = P \times U$, denoted $X \succ_S X'$ if*

$$\begin{aligned} \mathbb{E}_p[u(X)] &\geq \mathbb{E}_p[u(X')] \text{ for all } (p,u) \in S \text{ and} \\ \mathbb{E}_p[u(X)] &> \mathbb{E}_p[u(X')] \text{ for some } (p,u) \in S. \end{aligned}$$

Thus, a portfolio dominates another if and only if (i) its expected utility is at least as great for all feasible scenario probabilities and utility functions and (ii) there exist some scenario probabilities and utility functions for which its expected utility is strictly greater.

Dominance can be checked by comparing the expected utilities at the extreme points of the set of feasible scenario probabilities P. Specifically, if there are no constraints on scenario probabilities (i.e., $P = P^0$), every extreme point of P^0 is associated with a single scenario that occurs with probability one. In this case, portfolio X dominates X' if and only if the value of X is greater than or equal to X' in all scenarios and strictly greater at least in one scenario.

Theorem 1 *Let $P \subseteq P^0, U \subseteq U^0$ and choose portfolios $X, X' \in \mathcal{X}$. Then*

$$\begin{aligned} (i)\ X \succ_{P\times U} X' &\Leftrightarrow X \succ_{ext(P)\times U} X' \\ (ii)\ X \succ_{P^0\times U} X' &\Leftrightarrow X(s_i) \geq X'(s_i)\ \forall\ i \in \{1,\ldots,n\}, \end{aligned}$$

where at least one of the inequalities in (ii) is strict for some $i \in \{1,\ldots,n\}$.

Definition 2 generalizes the notion of stochastic dominance (see, e.g., Levy, 1992) to incompletely defined scenario probabilities. For increasing utility functions $U = U^0$, the definition means that first degree stochastic dominance (FSD) holds for all feasible scenario probabilities. In the modeling of risk aversion with concave utility functions ($U = U^A$), dominance means that second degree stochastic dominance (SSD) must hold for all feasible scenario probabilities.

Lemma 2 *Let $P \subseteq P^0$ and portfolios $X, X' \in \mathcal{X}$. Then*

$$\begin{aligned} \text{FSD} : X \succ_{P \times U^0} X' &\Leftrightarrow \mathbb{P}(X \leq t) \leq \mathbb{P}(X' \leq t) \; \forall \; p \in P, t \in \mathbb{R} \\ \text{SSD} : X \succ_{P \times U^A} X' &\Leftrightarrow \int_{-\infty}^{t} \mathbb{P}(X \leq y) dy \leq \int_{-\infty}^{t} \mathbb{P}(X' \leq y) dy \; \forall \; p \in P, t \in \mathbb{R}, \end{aligned}$$

where for both equivalence relations there exist some $p \in P, t \in \mathbb{R}$ such that the right side inequalities are strict.

Second degree stochastic dominance is closely related to the CVaR measure. For fixed scenario probabilities, portfolio X dominates portfolio X' with regard to second order stochastic dominance if and only if $\text{CVaR}^\alpha[X]$ is greater than $\text{CVaR}^\alpha[X']$ at all risk levels $\alpha \in (0, 1]$ (Dentcheva and Ruszczynski, 2006). This result can be extended to account for incomplete information about scenario probabilities.

Lemma 3 *Let $P \subseteq P^0$ and portfolios $X, X' \in \mathcal{X}$. Then*

$$X \succ_{P \times U^A} X' \Leftrightarrow \text{CVaR}_p^\alpha[X] \geq \text{CVaR}_p^\alpha[X'] \; \forall p \in P, \alpha \in (0, 1],$$

where the inequality is strict for some $p \in P, \alpha \in (0, 1]$.

Hence, the set of feasible utility functions U makes it possible to discard feasible portfolios that seem too risky in view of incomplete probability information (even in the CVaR sense). Further analysis can be focused on those feasible portfolios that are not dominated by any other feasible portfolio: for if a dominated portfolio were selected, it would be possible to identify another portfolio with greater expected utility for *all* feasible probabilities and utility functions.

Definition 3 *The set of non-dominated portfolios with information set $S = P \times U$ is*

$$\mathcal{X}_N(S) = \{X \in \mathcal{X}_F \mid \nexists X' \in \mathcal{X}_F \text{ such that } X' \succ_S X\}$$

As a rule, the introduction of additional constraints on feasible scenario probabilities or utility functions reduces the set of non-dominated portfolios, but cannot generate new non-dominated portfolios, i.e., if $\tilde{S}$ is a subset of S then $\mathcal{X}_N(\tilde{S})$ is also a subset of $\mathcal{X}_N(S)$. However, if $\tilde{S}$ is a subset of the 'border' of S, then there can be two portfolios in $\mathcal{X}_N(\tilde{S})$ that have a equal expected utility in the border while one is strictly inferior if evaluated anywhere else in S and thus does not belong to $\mathcal{X}_N(S)$. To rule out such situations we assume that $\tilde{S}$ includes at least some points from the (relative) interior of S.

Theorem 2 *Let $\tilde{S} \subseteq S$ and $int(S) \cap \tilde{S} \neq \emptyset$, where*

$$\begin{aligned} int(U) &= \{u \in U \mid \forall u^* \in U \exists \epsilon > 0 \text{ such that } u + \epsilon(u - u^*) \in U\} \\ int(P) &= \{p \in P \mid \forall p^* \in P \exists \epsilon > 0 \text{ such that } p + \epsilon(p - p^*) \in P\} \end{aligned}$$

and $int(S) = int(P) \times int(U)$. Then $\mathcal{X}_N(\tilde{S}) \subseteq \mathcal{X}_N(S)$.

Figure 1 summarizes key relationships among non-dominated portfolios for different feasible scenario probabilities and utility functions. If there are no constraints on scenario probabilities and all increasing utility functions are considered, the set of non-dominated portfolios $\mathcal{X}_N(P^0 \times U^0)$ corresponds to feasible portfolios such that any other feasible portfolio has a lower value in at least one scenario. This is implied by Theorem 1, which also states that if $P = P^0$, restrictions on the set of utility functions will not change the set of non-dominated portfolios, i.e, $\mathcal{X}_N(P^0 \times U^0) = \mathcal{X}_N(P^0 \times U)$ for any U.

However, if scenario probabilities are restricted to $P \subset P^0$, the set of feasible utility functions may impact the composition of the set of non-dominated portfolios. For instance, if the set of utility functions is not restricted, then for any feasible portfolio $X' \notin \mathcal{X}_N(P \times U^0)$, there exists a portfolio $X \in \mathcal{X}_N(P \times U^0)$ which dominates X' with regard to first order stochastic dominance that holds for all feasible scenario probabilities (Lemma 2).

The set of non-dominated portfolios $\mathcal{X}_N(P \times U^A)$ includes all portfolios that a risk-averse DM would consider, regardless of whether risk aversion is defined in terms of (i) preferences for certain outcomes over uncertain outcomes with equal expected value or (ii) preferences for an increase in the portfolio's CVaR at any risk level α (Lemma 3). Furthermore, $\mathcal{X}_N(P \times U^A)$ includes all portfolios that maximize the expected portfolio value subject to constraints on the portfolio CVaR, no matter what scenario probabilities in $p \in P$, risk levels $\alpha \in (0, 1]$ and threshold levels $\gamma \in \mathbb{R}$ are chosen.

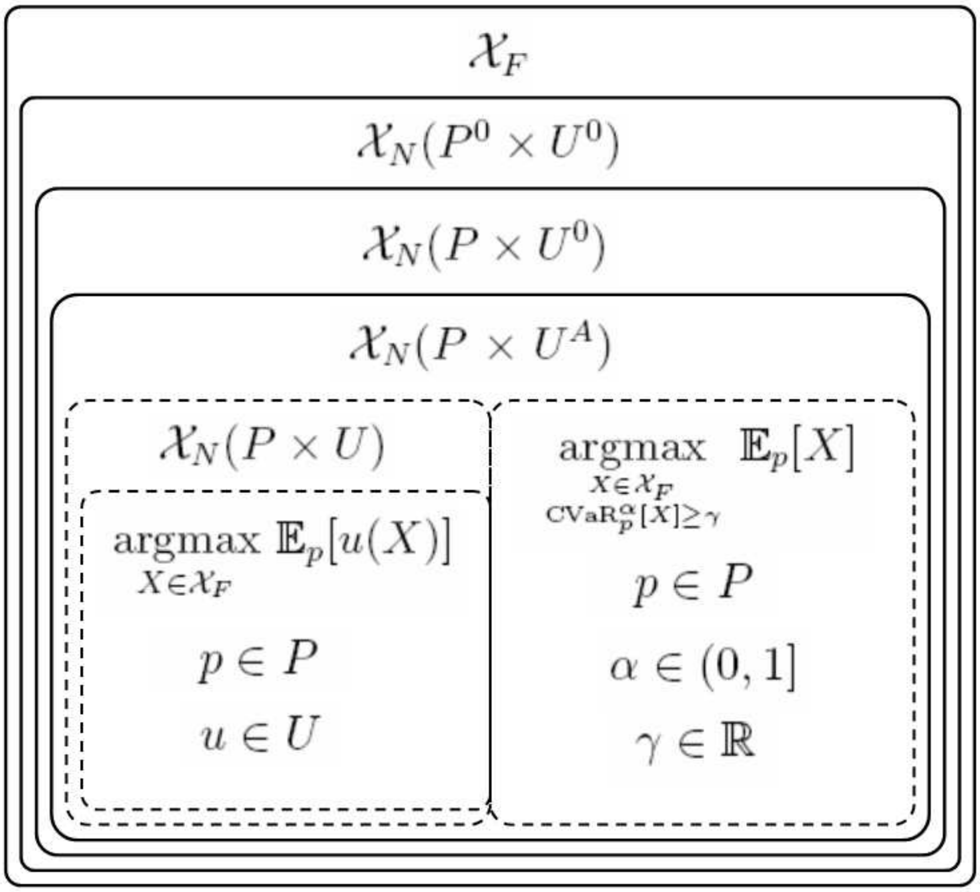

Figure 1: Relationship among sets of non-dominated portfolios when $P \subset P^0$ and $U \subset U^A \subset U^0$. Sets marked with dashed-lines may intersect depending on the problem instance.

4.3 Risk Measures under Incomplete Information

With incomplete information about scenario probabilities, the portfolio risk consists of two components: the portfolio can have a low value in the scenario that obtains (scenario risk), or the 'true' scenario probabilities are such that the expected value of the portfolio will be low (parametric risk). This latter type of risk can be measured by the minimum expected portfolio value when scenario probabilities vary with in the set P. The resulting risk measure is also coherent.

Lemma 4 *Minimum expected value* $\mathrm{ME}_P[X] = \min_{p \in P} \mathbb{E}_p[X]$ *is a coherent risk-measure for any* $P \subseteq P^0$.

With incompletely defined scenario probabilities, the scenario risk can be captured by the minimum CVaR value over all feasible probabilities, i.e, $\mathrm{WCVaR}_P^\alpha[X] = \min_{p \in P} \mathrm{CVaR}_p^\alpha[X]$. Because WCVaR corresponds to a minimum expected value over a specific set of probabilities, it is a coherent measure of risk based on Lemma 4.

Lemma 5 *For a risk level* $\alpha \in (0, 1]$ *and set of feasible probabilities* $P \subseteq P^0$ *the Worst-case Conditional Value-at -Risk for a portfolio* $X \in \mathcal{X}$ *is a solution to the LP-problem*

$$\mathrm{WCVaR}_P^\alpha[X] = \min_{q \in Q_P^\alpha} \mathbb{E}_q[X], \;\; Q_P^\alpha = \{q \in P^0 \mid \exists p \in P \text{ such that } \alpha q \leq p\},$$

which is a coherent risk measure and $P \subseteq Q_P^\alpha$.

Given a probability set P the WCVaR measures the worst case expected portfolio value, not over P, but over a larger set of feasible probabilities $Q_P \supseteq P$. In case of exact probabilities $P = \{p\}$ WCVaR is equal to CVaR.

WCVaR coincides with the *absolute robustness*-measure (i.e., the worst scenario specific value) of robust discrete optimization (Kouvelis and Yu, 1997) when (i) no restrictions are placed on scenario probabilities or (ii) α tends to zero. In these cases, the least risky portfolio (measured through WCVaR) is the maximin portfolio. At the other extreme, portfolio's WCVaR with $\alpha = 1$ is equal to the portfolio's minimum expected value over the feasible scenario probabilities P.

Lemma 6 *Let* $X \in X^0$ *and* $P \subseteq P^0$. *Then*

$$\begin{array}{ll} \text{(i)} & \mathrm{WCVaR}_{P^0}^\alpha[X] = \min_{s_i \in \Omega} X(s_i) \; \forall \; \alpha \in (0, 1] \\ \text{(ii)} & \lim_{\alpha \to 0^+} \mathrm{WCVaR}_P^\alpha[X] = \min_{s_i \in \Omega} X(s_i), \; \text{ if } \; P \cap \mathrm{int}(P^0) \neq \emptyset \\ \text{(iii)} & \mathrm{WCVaR}_P^1[X] = \mathrm{ME}_P[X]. \end{array}$$

4.4 Implications for Decision Support

A risk averse DM can be advised to focus on the set of non-dominated portfolios $\mathcal{X}_N(P \times U^A)$, because (i) this set contains all rational choices for a risk averse DM (see Figure 1) and (ii) all other feasible portfolios have a lower ME and WCVaR_P^α for any $\alpha \in (0,1]$ (Lemma 3).

Next, the DM can be presented with the risk levels of non-dominated portfolios, after which she can interactively screen portfolios by varying WCVaR thresholds for different risk-levels α. At any stage, the projects that belong to non-dominated portfolios can be shown to the DM to highlight which projects are included in all non-dominated portfolios and should therefore be selected; or conversely, which projects are not contained in any non-dominated portfolios, and which should therefore be rejected. Also, by varying the level of risk constraints in these analysis helps illustrate how the recommendations depend on the level of acceptable portfolio risk.

In group settings where the DMs may hold different views about the scenario probabilities, the set of feasible scenario probabilities P can be taken to be the convex combination of the DM's estimates; in essence, this is a conservative approach as it assumes that all DMs may be 'correct' in their estimates. The same approach can be used also in the elicitation of feasible utility functions. In either case, rather than arguing about which scenario probabilities (or risk preferences) should be used, the DMs can negotiate about the performance of non-dominated portfolios $\mathcal{X}_N(P \times U)$. This set includes optimal portfolios in view of each individual probability estimate as well as rational compromises, because the expected utility of any dominated feasible portfolio could be improved with regard to *all* individual scenario probability estimates. Furthermore the composition of the non-portfolios may be similar, in which case the different views on scenario probabilities possibly have an impact on decisions about few investment projects only.

5 Computation of Non-Dominated Portfolios

The non-dominated portfolios $\mathcal{X}_N(P \times U)$ can be determined by first computing the set $\mathcal{X}_N(P^0 \times U^0)$ and by then discarding the portfolios that are dominated with regard to the information set $P \times U$, because $\mathcal{X}_N(P \times U) \subseteq \mathcal{X}_N(P^0 \times U^0)$ by Theorem 2. By Property (ii) of Theorem 1, the set $\mathcal{X}_N(P^0 \times U^0)$ can be determined by computing all the Pareto-optimal solutions to the multiple objective zero-one linear programming (MOZOLP) problem

$$\text{v}-\max_z\{Cz \mid Az \leq B,\ z \in \{0,1\}^m\}, \tag{11}$$

where the coefficient matrix $C \in \mathbb{R}^{n\times m}$ with $[C]_{ij} = x^j(s_i)$ contains the projects' scenario-specific values. Any Pareto-optimal solution z to (11) is a non-dominated portfolio X such that $x^j \in X$ if and only if $z_j = 1$. Several MOZOLP algorithms are available (Villareal and Karwan, 1981; Kiziltan and Yucaoglu, 1983; Liesiö *et al.*, 2007, 2008).

For risk neutral utility functions $U = U^L = \{u(t) = at + b \mid a, b \in \mathbb{R}, a > 0\}$, portfolio X dominates X' if and only if the expected value of portfolio X is greater in every extreme point of P and strictly greater in at least one extreme point (cf. Theorem 1). In this case, the set of non-dominated portfolios $\mathcal{X}_N(P \times U^L)$ can be computed from the MOZOLP problem (11) by replacing the matrix C with $C' \in \mathbb{R}^{t\times m}$ that contains the expected values of the investment projects x^j at the extreme points of set of feasible scenario probabilities P (i.e, $[C']_{ji} = \mathbb{E}_{p^i}(x^j)$, $\{p^1, \ldots, p^t\} = ext(P)$).

In the general case $P \times U \subseteq P^0 \times U^0$, dominated portfolios can be discarded from $\mathcal{X}_N(P^0 \times U^0)$ with a linear programming model that determines the maximum and minimum expected utility difference for pairs of portfolios X and X' at each extreme point $p \in ext(P)$. Specifically, let $\hat{v} \in \mathbb{R}^h$ contain the scenario-specific values of portfolios X and X' in an increasing order so that $\hat{v} = \mathrm{SORT}(\{X(s_i) \mid s_i \in \Omega\} \cup \{X'(s_i) \mid s_i \in \Omega\})$. By construction, $\hat{v}_j < \hat{v}_{j+1}$ for all $j = 1, \ldots, h-1$ and $h \leq 2n$. The expected utility difference of portfolios X and X' is

$$\begin{aligned}\mathbb{E}_p[u(X)] - \mathbb{E}_p[u(X')] &= \sum_{i=1}^{n} p_i u(X(s_i)) - \sum_{i=1}^{n} p_i u(X'(s_i)) \\ &= \sum_{j=1}^{h} u(\hat{v}_j)[\sum_{X(s_i)=\hat{v}_j} p_i - \sum_{X'(s_i)=\hat{v}_j} p_i] = \sum_{j=1}^{h} \Delta_j \hat{u}_j,\end{aligned}$$

where $u(\hat{v}_j) = \hat{u}_j$ and $\Delta_j = \sum_{X(s_i)=\hat{v}_j} p_i - \sum_{X'(s_i)=\hat{v}_j} p_i$. For any given scenario probabilities $p \in ext(P)$, the minimum and maximum expected utility differences over $u \in U^0$ can be obtained from LP-problems

$$\min_{\hat{u}} / \max_{\hat{u}} \{\sum_{j=1}^{h} \Delta_j \hat{u}_j \mid \hat{u} \in [0,1]^h,\ \hat{u}_j \leq \hat{u}_{j+1}\ \forall\ j = 1, \ldots, h-1\} \tag{12}$$

For example, consider a two-scenario problem with feasible scenario probabilities $P = \{p = (p_1, p_2) \in P^0 \mid p_1 \in [0.4, 0.5]\}$ and corresponding extreme points $p^1 = (0.4, 0.6)$ and $p^2 = (0.5, 0.5)$). If the scenario-specific values of portfolios X, X' are $X(s_1) = 5, X(s_2) = 3$ and $X'(s_1) = 2, X(s_2) = 5$, respectively, we have $\hat{v} = (2, 3, 5)^T$. At the extreme point p^1, the difference in the expected utilities of these portfolios is $\mathbb{E}_{p^1}[u(X)] - \mathbb{E}_{p^1}[u(X')] = \sum_{j=1}^{3} \Delta_j \hat{u}_j = -0.4u(2) + 0.6u(3) - 0.2u(5)$, where $\Delta = (-0.4, 0.6, -0.2)^T$. For increasing utility functions, $U = U^0$, this expression attains its minimum -0.2 at $\hat{u} = (0, 0, 1)^T$.

Thus, because the expected utility of portfolio X at p^1 can be strictly lower than that of X', portfolio X does not dominate X'.

The set of feasible utility functions can be restricted with further constraints on (12). For instance, linear constraints

$$\frac{\hat{u}_j - \hat{u}_{j-1}}{\hat{v}_j - \hat{v}_{j-1}} \geq \frac{\hat{u}_{j+1} - \hat{u}_j}{\hat{v}_{j+1} - \hat{v}_j} \ \forall\ j = 2, \ldots, h-1,$$

can be introduced to model risk averse preferences with concave utility functions $U = U^A$. Even in this case, sufficient and necessary condition for dominance can be established by solving the minimum and maximum of LP problem (12) at every extreme point of P.

Theorem 3 *Let $X, X' \in \mathcal{X}$ and $S = P \times U$. Then $X \succ_S X'$ if and only if the minimum of LP-problem (12) is non-negative for all $p \in ext(P)$ and the maximum is strictly positive for some $p \in ext(P)$.*

6 An Illustrative Example

Because the commercial success of R&D projects is often contingent on enabling technologies, we present an illustrative example where a high-technology Company chooses an R&D portfolio from 30 project proposals. The projected cash flows of the projects depend on the two enabling technologies, labeled A and B. Every project is related to one (but not both) of these technologies.

For both technologies, the Company builds three scenarios (pessimistic, neutral, and optimistic) to describe how successful the technology will be over a five-year planning horizon; these scenarios are denoted by $\omega^{A-}, \omega^{A0}, \omega^{A+}$ and $\omega^{B-}, \omega^{B0}, \omega^{B+}$, respectively. Thus, for every proposal, three cash flow estimates are elicited to describe how the project will perform depending on the success of its enabling technology (see Table 1). Taken together, the three scenarios for technologies A and B define nine joint scenarios $s_1, \ldots, s_9$ (see Table 2).

The selection of projects is constrained by the R&D budget (1.2 million euros) and the availability of human resources (50 man-years; see Table 1). In addition there are other project interactions. First, both of projects A7 and B5 can be implemented in two variants, which requires two linear constraints to ensure only one of the variants can be included in the portfolio. Second, Project A4.1 is a follow-up to Project A4.0, and thus Project A4.1 cannot be selected unless Project 4.0 is also selected. Third, the joint

Table 1: Project candidates

Name	$x^j(s), s \in \omega^{A-}$	$x^j(s), s \in \omega^{A0}$	$x^j(s), s \in \omega^{A+}$	Cost	HR
Project A1	0	0	380	44	5
Project A2	0	0	420	31	5
Project A3	0	10	540	30	6
Investment A1-3	0	0	0	80	0
Project A4.0	80	100	520	79	6
Project A4.1	20	130	690	85	6
Project A5	70	110	360	142	2
Project A6	130	230	230	121	2
Project A7a	0	10	460	132	6
Project A7b	30	60	420	111	8
Project A8	40	40	440	87	6
Project A9	60	70	420	132	9
Project A10	60	70	450	117	8
Project A11	150	180	180	96	6
Project A12	40	190	190	145	8
Project A13	100	230	230	101	2
	$x^j(s), s \in \omega^{B-}$	$x^j(s), s \in \omega^{B0}$	$x^j(s), s \in \omega^{B+}$		
Project B1	80	110	200	98	2
Project B2	60	150	190	182	4
Project B3	70	150	240	183	6
Project B4	140	160	270	224	2
Project B5a	40	220	230	105	9
Project B5b	50	170	290	157	10
Project B6	200	230	260	177	3
Project B7	40	240	310	139	4
Project B8	120	260	320	184	9
Project B9	230	270	320	157	7
Project B10	200	260	330	254	2
Project B11	60	250	360	224	4
Project B12	130	240	380	224	5
Project B13	180	370	480	331	8

execution of projects A1, A2, and A3 calls for an additional investment into new research equipment. This is modeled with the help of a dummy project which has a positive cost but no projected cash flows (cf. Investment A1-3) and a linear constraint which ensures that the portfolio can contain any of the projects A1, A2 and A3 only if the dummy project is also included. Taken together, the model has two resource constraints and four

Table 2: Technology success and model scenarios; For instance, $\omega^{A-} = \{s_1, s_4, s_7\}$ $\omega^{B0} = \{s_4, s_5, s_6\}$.

	ω^{A-}	ω^{A0}	ω^{A+}
ω^{B-}	s_1	s_2	s_3
ω^{B0}	s_4	s_5	s_6
ω^{B+}	s_7	s_8	s_9

Table 3: Scenario probability estimates (%)

	p_1^k	p_2^k	p_3^k	p_4^k	p_5^k	p_6^k	p_7^k	p_8^k	p_9^k
p^1	2.5	5	15	5	22.5	30	5	10	5
p^2	2	5	16	5	20	20	10	20	2
p^3	4	2	8	4	35	20	10	16	1
p^4	1	4	12	3	30	22.5	5	20	2.5
p^5	3	7	15	5	25	15	10	15	5

other feasibility constraints.

Without any information about scenario probabilities or risk preferences, the set $\mathcal{X}_N(P^0 \times U^0)$ includes 329 non-dominated portfolios. These were computed in less than a minute on a laptop computer (1.83GHz, 1GB memory) with the MOZOLP algorithm of Liesiö *et al.* (2008). The corresponding aggregate portfolio cash flows vary from \$0.44 to \$4.63 million across the nine scenarios and all non-dominated portfolios.

Information about scenario probabilities $p_1, \ldots, p_9$ is elicited by consulting five experts. These experts believe it is unlikely that the pessimistic scenarios will obtain for *both* technologies, because there is a market for the products enabled by these technologies; but because the technologies compete with each other, the joint occurrence of the optimistic scenarios is also unlikely, too. These considerations are reflected in the low probabilities of scenarios s_1 and s_9 in Table 3.

It is assumed that the probability estimate of each expert may be the correct one. The set of feasible scenario probabilities is therefore defined as the convex combination of these estimates $p^1, \ldots, p^5$ so that $P = \{p = \sum_{i=1}^{5} \lambda_i p^i \mid \sum_{i=1}^{n} \lambda_i = 1, \lambda \geq 0\}$. When the scenario probabilities are restricted to this set P, the number non-dominated optimal portfolio declines to 317 from the initial 329.

When risk-seeking preferences are excluded by considering only linear or concave utility functions $\mathcal{X}_N(P \times U^A)$, the number of non-dominated portfolios drops to sixty. The

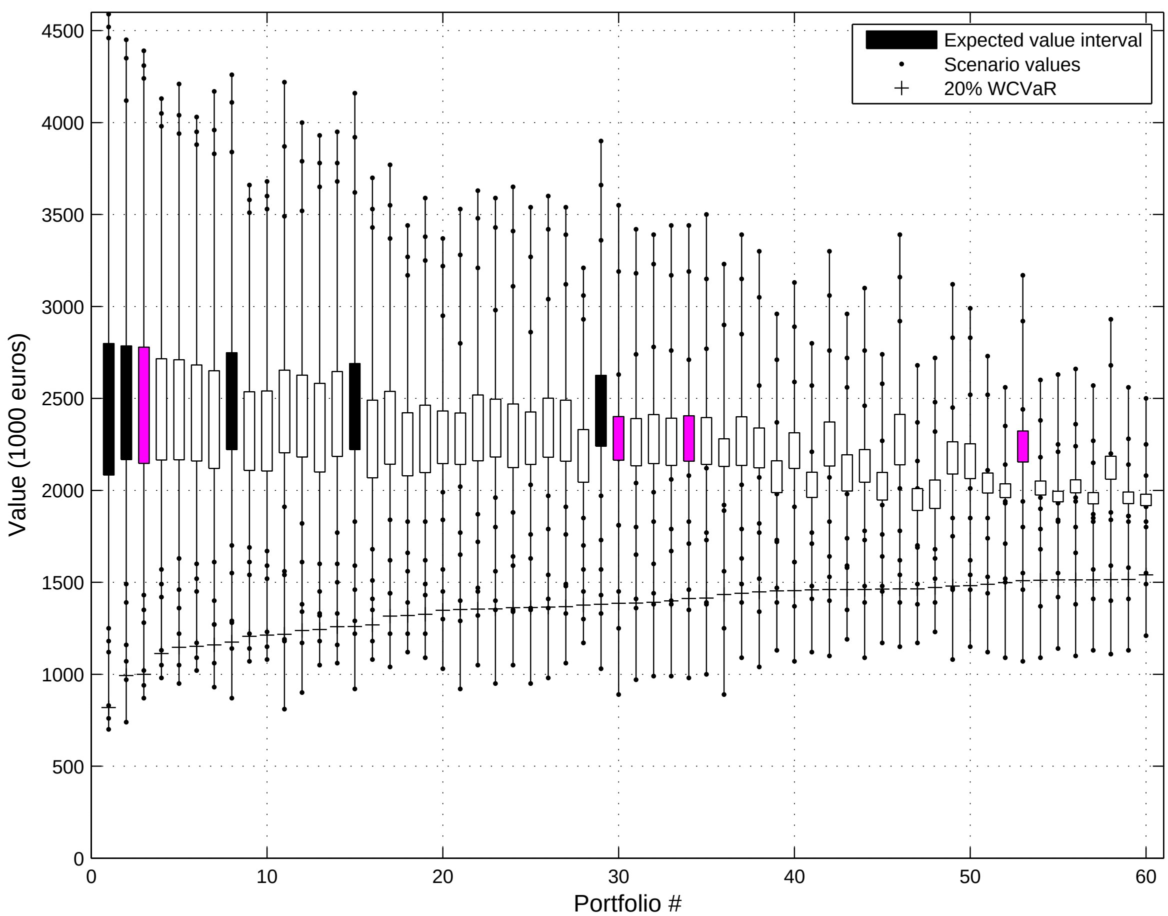

Figure 2: Portfolios in $\mathcal{X}_N(P \times U^A)$. White ones become dominated with U^1, gray ones with the linear utility function.

intervals of corresponding expected portfolio cash flows, scenario-specific portfolio cash flows and $\text{WCVaR}^{0.2}$-values are shown in Figure 2. Here, the portfolios are indexed in an increasing order of $\text{WCVaR}^{0.2}$, which reflects the worst-case expected cash flow of the portfolio, conditioned on the occurrence of the worst 20% of outcomes.

Interestingly, the level of portfolio risk – as measured by $\text{WCVaR}^{0.2}$ – can be be reduced by diversifying among projects that relate to technologies A and B. This can be seen from Figure 3 which shows the proportion of funds that are allocated to Technology A and Technology B related projects: for instance, in portfolio #1 – which has the greatest possible expected cash flow ($2.8 million) — about 80% of the budget is allocated to projects in Technology A; but in the least risky portfolio (#60) only some 33% of budget is allocated to projects in Technology A.

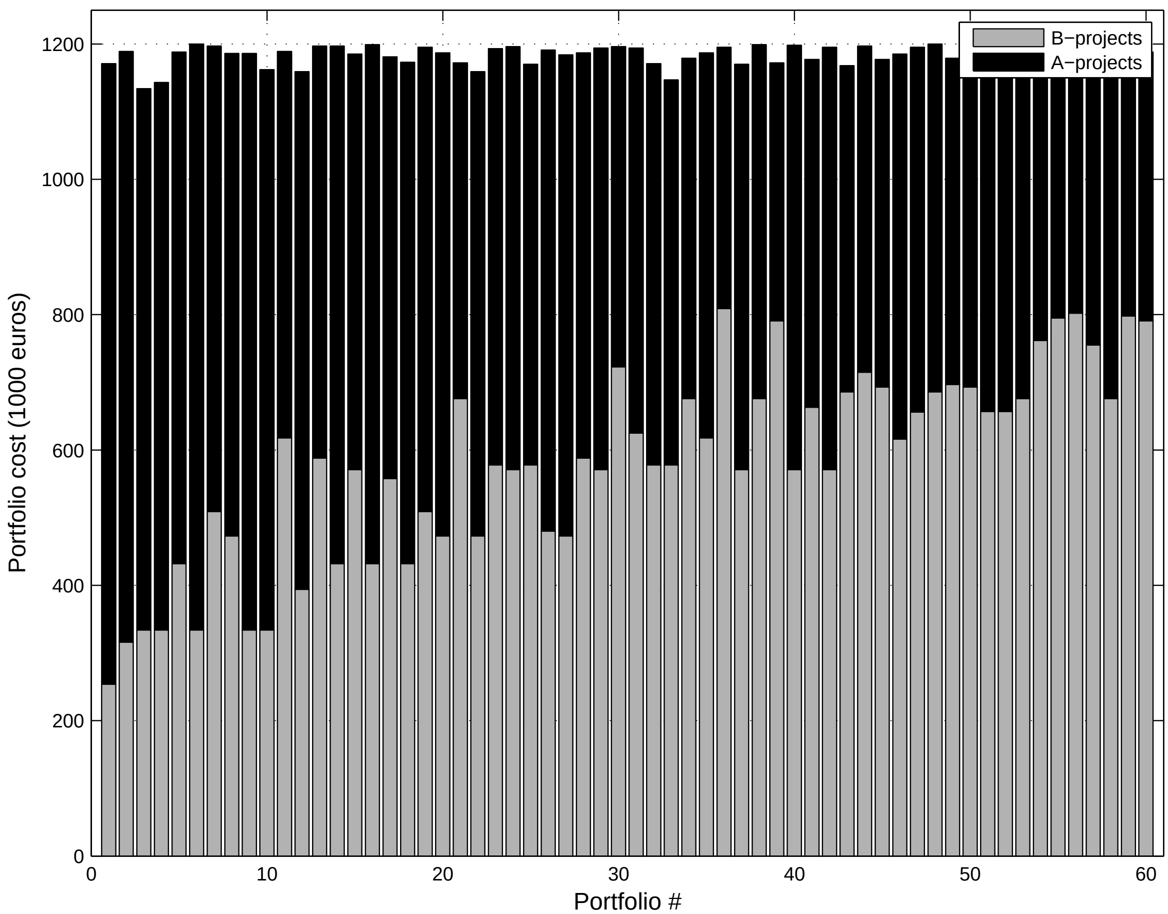

Figure 3: Budget allocation between projects in Technologies A and B in the non-dominated portfolios $\mathcal{X}_N(P \times U^A)$.

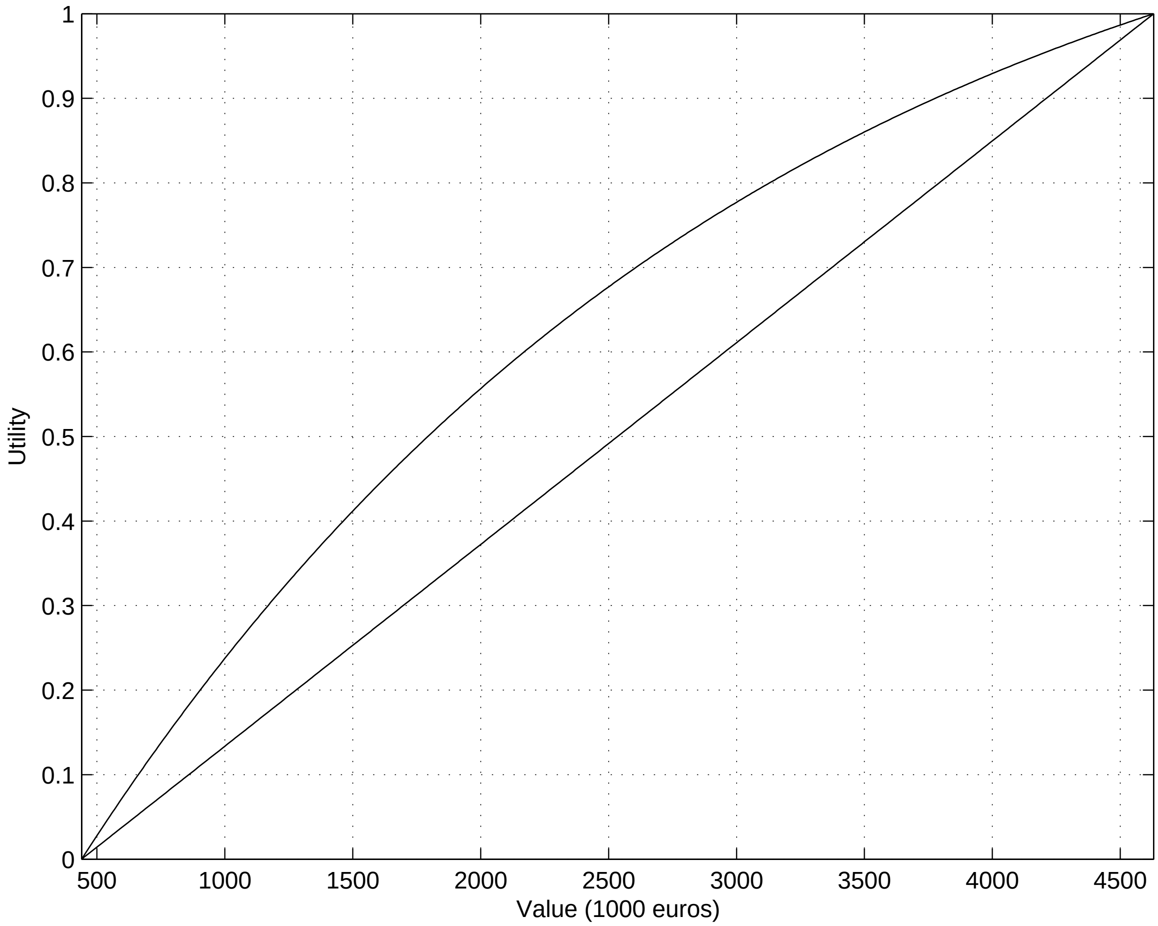

Figure 4: Set of feasible utility functions U.

Next, the Company management places an upper bound on the level of risk aversion by specifying that the certainty equivalent of a fifty-fifty gamble between the worst (\$0.44 million) and the best (\$4.63 million) portfolio cash flows is at least \$1.8 million. In the set of exponential utility functions $u^e(t) = [e^{-440a} - e^{-at}]/[e^{-440a} - e^{-4630a}]$ that normalize the worst and best cash flows onto the range $[0, 1]$, this corresponds to the coefficient $a = 0.00037$ as $u^e(1800) = 0.5$. The set of feasible utility functions thus becomes

$$U^1 = \{u \in U^A \mid 0 \leq u(t) \leq u^e(t) \ \forall \ t \in [440, 4630]\}, \tag{13}$$

as illustrated in Figure 4.

For the utility functions in U^1, nine out of the previously computed portfolios are non-dominated; and five of these are non-dominated when the portfolios are evaluated with a risk-neutral linear utility function (see Figure 5).

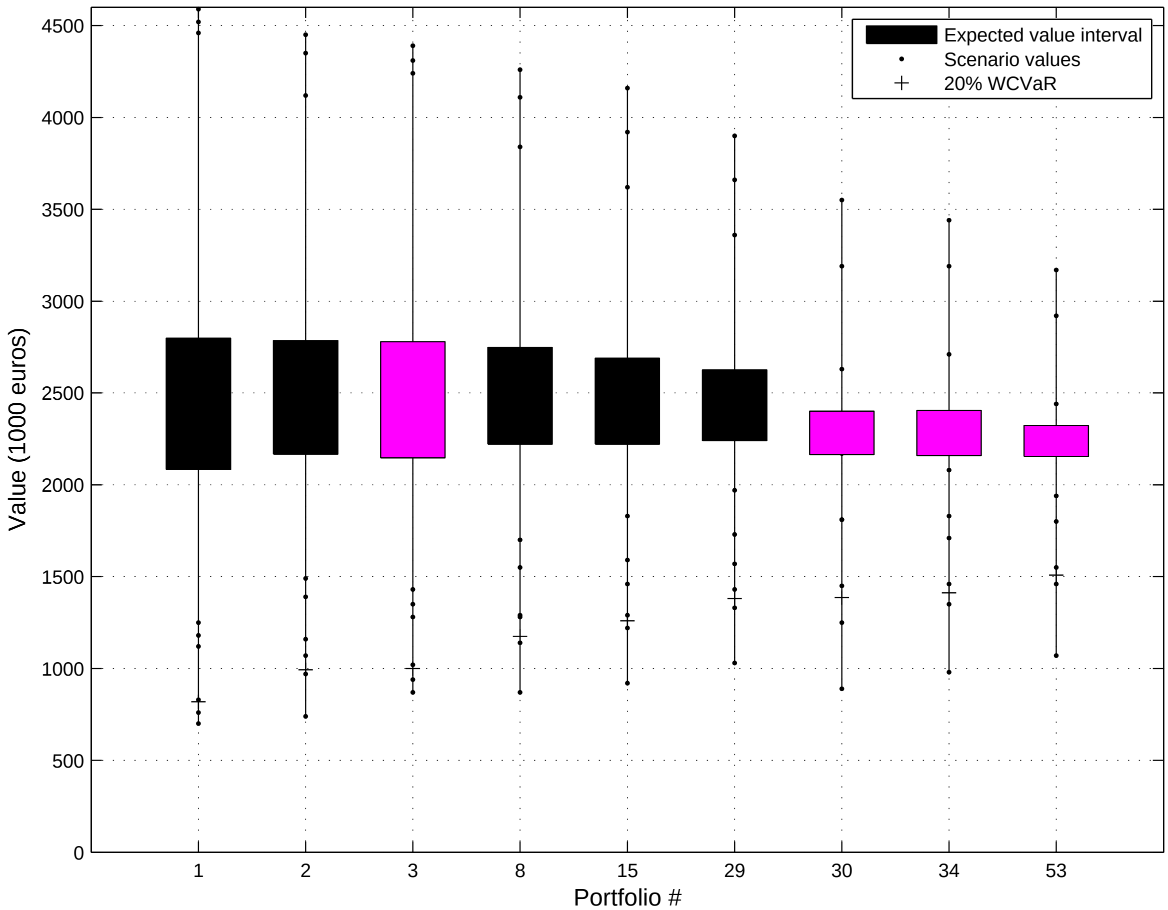

Figure 5: Portfolios in $\mathcal{X}_N(P \times U^1)$. Black ones are non-dominated also with the linear utility function.

The WCVaR$^{0.2}$-values for nine portfolios in $\mathcal{X}_N(P \times U^1)$ range between \$0.8 and \$1.5 million. The Company agrees that an acceptable WCVaR$^{0.2}$ level is over \$1 million, which makes it possible to discard portfolios #1 #2 and #3. Furthermore, portfolios #30, #34 and #53, although acceptable in terms of their risk level, offer much less upside potential with smaller minimum expected cash flows than those of other portfolios. Thus, the final selection is restricted to portfolios #8, # 15 and #29 which differ in terms of few projects only (Figure 6). Specifically, because they all include the investment into new research equipment (Investment A1-3), portfolio #8 is recommended for selection because it makes the most use of this investment by including projects A1, A2 and A3.

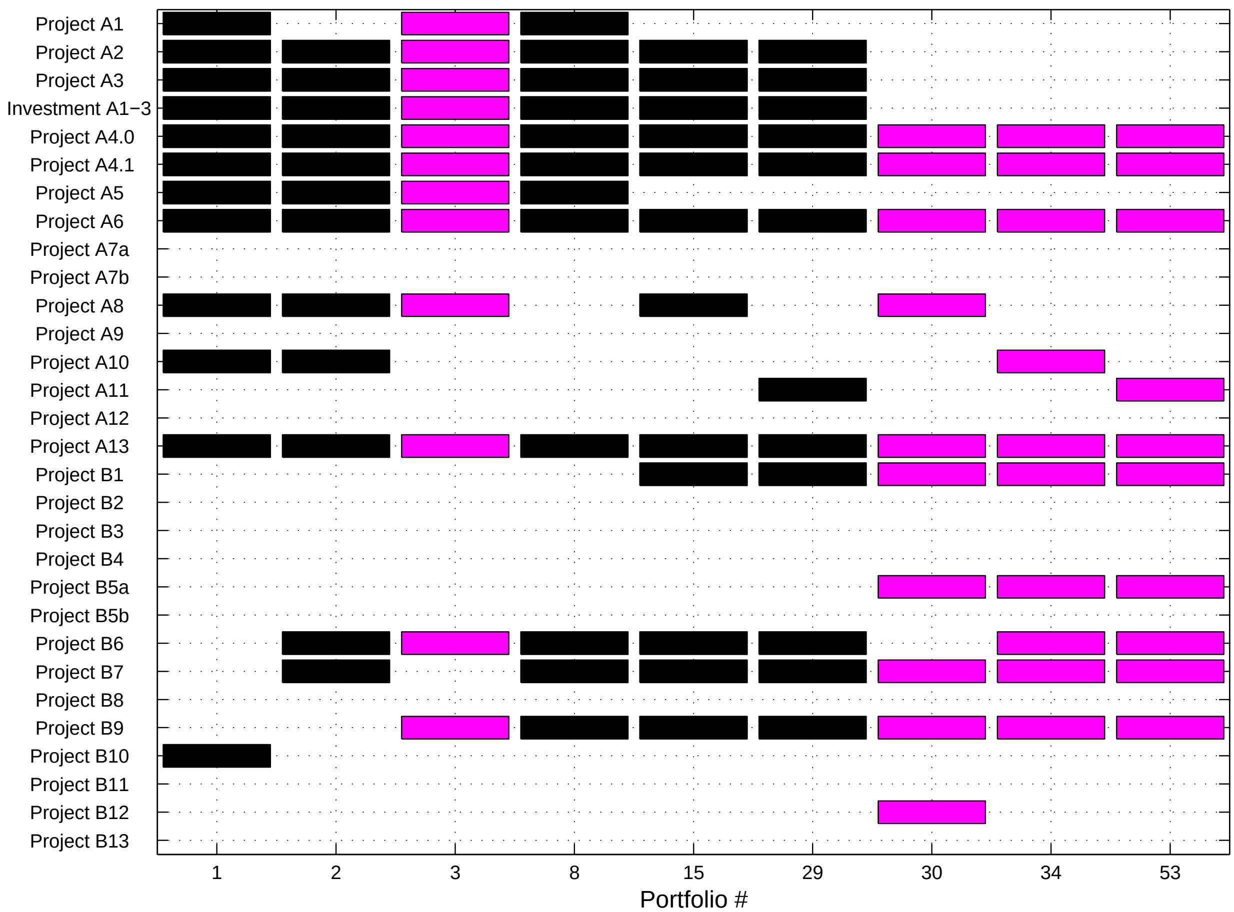

Figure 6: Composition of portfolios in $\mathcal{X}_N(P \times U^1)$. Black ones are non-dominated also with the linear utility function.

7 Conclusions

The framework developed in this paper extends scenario-based project appraisal to the selection of project portfolios in the presence of incomplete information about scenario probabilities and risk preferences. This framework ensures that scenarios (see, e.g., Bunn and Salo, 1993) are explicitly considered in the appraisal of investment projects; it also synthesizes results from such an appraisal into well-founded decision recommendations. In general, the framework thus extends scenario analyses to problems where (i) several investment projects are selected at the same time, (ii) complete information about scenario probabilities or risk preferences is difficult or impossible to obtain, and (iii) the projects may have complex interdependencies.

The proposed framework captures risk preferences through utility functions, which can be elicited with well-established techniques based on the comparison of lotteries with certain and uncertain outcomes. In the comparison of such lotteries, the DM may provide ordinal preference statements or specify upper and lower bounds on certainty equivalents. The framework also permits the specification of risk constraints, most notably through bounds on the CVaR risk measure at different confidence levels.

The framework also supports the interactive exploration of the possibilities offered by the proposed investment projects. Such support can be offered by computing all the project portfolios that are non-dominated in view of available information about feasible scenario probabilities and risk preferences. When additional information is elicited during the decision support process, or when additional risk constraints are introduced, the set of non-dominated portfolios becomes smaller and more conclusive decision recommendations can be given about which projects should be selected or rejected. Even though the determination of non-dominated portfolios may necessitate intensive computations, these computations can be usually carried out in advance with efficient algorithms for general multi-objective integer linear programming problems (see, e.g., Zitzler and Thiele, 1999). In the screening phase, the final dominance checks among the portfolios can be carried out efficiently, which makes it possible to offer interactive decision support in a workshop setting.

This work suggests several avenues for future research. First, the framework could be extended to multi-period portfolio models (cf. Contingent Portfolio Programming; Gustafsson and Salo, 2005) to account for incomplete information about scenario probabilities and risk preferences. Second, the modeling of project-specific uncertainties can be important, too. However, the need to limit the total number of scenarios (resulting from both exogenous and endogenous uncertainties) suggests that project-specific uncertainties may be best characterized not through conventional decision trees but, rather, with the help of intervals for which the corresponding non-dominated portfolios can then be computed (Liesiö *et al.*, 2008). Third, the potential of the proposed framework needs to be explored in the context of case studies that shed light on its benefits in practice.

Acknowledgements

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Appendix

Proof of Lemma 1. Let $k = \max\{k \in \{1,\ldots,n\} \mid \mathbb{P}(\omega^{k-1}) \leq \alpha\}$: The optimum $\min_{q\in Q_p^\alpha} \mathbb{E}_q[X]$ is obtained by setting $q_i^* = p_i/\alpha$ for all $i = 1,...,k-1$, $q_k^* = 1 - \sum_{i=1}^{k-1} q_i^*$ and $q_i^* = 0$ for all $i = k+1,...,n$, which is a feasible solution since $q_i^* < \sum_{i=1}^{k} q_i^* = \sum_{i=1}^{k-1} p_i/\alpha + (1-\sum_{i=1}^{k-1} p_i/\alpha) = 1$. Then $\mathbb{E}_{q^*}[X] = \sum_{i=1}^{k} \frac{p_i}{\alpha} X(s_i) + (1-\sum_{i=1}^{k-1} p_i/\alpha)X(s_k) = \frac{\mathbb{P}(\omega^{k-1})}{\alpha}\frac{\sum_{i=1}^{k} p_i X(s_i)}{\mathbb{P}(\omega^{k-1})} + (1-\frac{\mathbb{P}(\omega^{k-1})}{\alpha})X(s_k) = \frac{\mathbb{P}(\omega^{k-1})}{\alpha}\mathbb{E}_p[X|\omega^{k-1}] + (1-\frac{\mathbb{P}(\omega^{k-1})}{\alpha})\mathbb{E}_p[X|\{s_k\}]$. Thus $\text{CVaR}_p^\alpha[X] = \min_{q\in Q_p^\alpha} \mathbb{E}_q[X]$

$$= \min_{q\in\mathbb{R}^n}\{\sum_{i=1}^{n} q_i X(s_i) \mid 0 \leq q_i \leq \frac{p_i}{\alpha}\ \forall\ i \in \{1,...,n\},\ \sum_{i=1}^{n} q_i = 1\},$$

which is a bounded and feasible LP-problem, and thus its dual

$$\begin{aligned} \max_{\substack{z\in\mathbb{R}^n \\ t\in\mathbb{R}}} \quad & t + \sum_{i=1}^{n} \frac{p_i}{\alpha} z_i \\ & z_i + t \leq X(s_i)\ \forall\ i \in \{1,\ldots,n\} \\ & z_i \leq 0\ \forall\ i \in \{1,\ldots,n\} \end{aligned}$$

yields same optimum. To maximize the dual for a fixed t, variables z_i are set to their upper bound, i.e, $z_i(t) = \min\{0, X(s_i) - t\} = -\max\{0, t - X(s_i)\}$, which gives

$$\text{CVaR}_p^\alpha[X] = \max_{t\in\mathbb{R}}(t - \frac{1}{\alpha}\sum_{i=1}^{n} p_i \max\{0, t - X(s_i)\}) = \max_{t\in\mathbb{R}}(t - \frac{1}{\alpha}\mathbb{E}_p[\max\{0, t - X\}]).$$

□

Proof of Theorem 1. i) '$\Leftarrow$': Assume $X \succ_{\text{ext}(P)\times U} X'$ which implies

$$\mathbb{E}_p[u(X)] \geq \mathbb{E}_p[u(X')] \;\forall\; p \in \{p^1, \ldots, p^t\},\; u \in U$$

where $\{p^1, \ldots, p^t\}$ =ext(P). Any $p \in P$ is a linear combination of these extreme points, i.e., $p = \sum_{k=1}^t \alpha_k p^k$, where $\alpha_k \geq 0$. For any $(p, u) \in P \times U$:

$$\begin{aligned}
\mathbb{E}_p[u(X)] - \mathbb{E}_p[u(X')] &= \sum_{i=1}^n p_i[u(X(s_i)) - u(X'(s_i))] \\
&= \sum_{i=1}^n \sum_{k=1}^t \alpha_k p_i^k [u(X(s_i)) - u(X'(s_i))] \\
&= \sum_{k=1}^t \alpha_k \sum_{i=1}^n p_i^k [u(X(s_i)) - u(X'(s_i))] \\
&= \sum_{k=1}^t \alpha_k (\mathbb{E}_{p^k}[u(X)] - \mathbb{E}_{p^k}[u(X')]) \geq 0
\end{aligned}$$

since all terms of the sum are non-negative. Thus $\mathbb{E}_p[u(X)] \geq \mathbb{E}_p[u(X')]$ for all $(p, u) \in (P \times U)$ and the inequality is strict for some $p \in \text{ext}(P) \subset P$ and $u \in U$, which implies $X \succ_{P\times U} X'$.

'$\Rightarrow$': Assume $X \succ_{P\times U} X'$, which implies $\mathbb{E}_p[u(X)] \geq \mathbb{E}_p[u(X')]$ for all $(p, u) \in \text{ext}(P)\times U$, since ext$(P) \subset P$. Furthermore, exists $p \in P$, $p = \sum_{k=1}^t \alpha_k p^k$, such that $0 < \mathbb{E}_p[u(X)] - \mathbb{E}_p[u(X')] = \sum_{k=1}^t \alpha_k(\mathbb{E}_{p^k}[u(X)] - \mathbb{E}_{p^k}[u(X')])$. Thus $\mathbb{E}_{p^k}[u(X)] > \mathbb{E}_{p^k}[u(X')]$ for some $p^k \in$ext(P) which implies $X \succ_{\text{ext}(P)\times U} X'$.

ii) Since the extreme points $\{p^1, \ldots, p^n\}$ of P^0 are of the form $p_i^i = 1, p_j^i = 0 \;\forall j \neq i$, i) implies that dominance $X \succ_{(P^0\times U)} X'$ holds if and only if $u(X(s_i)) \geq u(X'(s_i)) \;\forall\; i \in \{1, \ldots, n\}, u \in U$ (with the inequality strict for some i and u). Since $U \subseteq U^0$ contains only increasing utility functions (and at least one strictly increasing) the condition is equal to $X(s_i) \geq X'(s_i) \;\forall\; i \in \{1, \ldots, n\}$ with a strict inequality for at least one i. □

Proof of Lemma 2. It is well-known (see, e.g., Hanoch and Levy, 1969) that for a fixed $p \in P^0$

$$\begin{aligned}
\mathbb{E}_p[u(X)] \geq \mathbb{E}_p[u(X)] \;\forall\; u \in U^0 \;&\Leftrightarrow\; \mathbb{P}(X \leq t) \leq \mathbb{P}(X' \leq t) \;\forall\; t \in \mathbb{R} \\
\mathbb{E}_p[u(X)] \geq \mathbb{E}_p[u(X)] \;\forall\; u \in U^A \;&\Leftrightarrow\; \int_{-\infty}^t \mathbb{P}(X \leq y)dy \leq \int_{-\infty}^t \mathbb{P}(X' \leq y)dy \;\forall\; t \in \mathbb{R}.
\end{aligned}$$

If the expected utility inequalities are strict for some u, the right-hand-side inequalities are strict for some t. The lemma results directly from these results. □

Proof of Lemma 3. It is easy to verify that $\int_{-\infty}^{t}\mathbb{P}(X \leq y)dy = \sum_{X(s_i)\leq t} p_i(t - X(s_i)) = \max_{0\leq r\leq p}\sum_{i=1}^{n} r_i(t - X(s_i)) = -\min_{0\leq r\leq p}[\sum_{i=1}^{n} r_i X(s_i) - t\sum_{i=1}^{n} r_i]$. Denoting $G(X,p,t) = \min_{0\leq r\leq p}[\sum_{i=1}^{n} r_i X(s_i) - t\sum_{i=1}^{n} r_i]$ and using property ii) of Lemma 2 gives

$$X \succ_{P\times U^A} X' \Leftrightarrow G(X,p,t) \geq G(X',p,t) \forall p \in P, t\in\mathbb{R},$$

with the inequality strict for some $p \in P, t \in \mathbb{R}$. Furthermore, denote

$$F(X,p,\alpha) = \alpha\mathrm{CVaR}_p^{\alpha}[X] = \min_{\substack{0\leq \alpha q\leq p\\ \sum_{i=1}^{n} q_i=1}} \sum_{i=1}^{n} \alpha q_i X(s_i) = \min_{\substack{0\leq r\leq p\\ \sum_{i=1}^{n} r_i=\alpha}} \sum_{i=1}^{n} r_i X(s_i).$$

For any $p \in P, \alpha \in (0,1], X \in \mathcal{X}$ it holds that

$$\begin{aligned} F(X,p,\alpha) &\geq G(X,p,t) + \alpha t\ \forall t \in \mathbb{R}\\ \exists t^* \in \mathbb{R} \text{ s.t. } F(X,p,\alpha) &= G(X,p,t^*) + \alpha t^*, \end{aligned}$$

since the dual of LP problem $F(X,p,\alpha)$ is $\max_{t\in\mathbb{R}}[G(X,p,t) + \alpha t]$. To prove the lemma it is thus sufficient to show that

$$G(X,p,t) \geq G(X',p,t)\ \forall p \in P, t \in \mathbb{R} \tag{14}$$

$$\Leftrightarrow F(X,p,\alpha) \geq F(X',p,\alpha)\ \forall p \in P, \alpha \in (0,1], \tag{15}$$

and that inequality (14) is strict for some $p \in P, t \in \mathbb{R}$ if and only if the inequality (15) is strict for some $p \in P, \alpha \in (0,1]$.

'(14) $\Rightarrow$ (15)': For any $p \in P, \alpha \in (0,1]$ exists $t^* \in \mathbb{R}$ such that $F(X',p,\alpha) = G(X',p,t^*) + \alpha t^* \leq G(X,p,t^*) + \alpha t^* \leq F(X,\alpha,p)$.

'(15) $\Rightarrow$ (14)': Take $p \in P, t \in \mathbb{R}$. Let $\alpha^* = \sum_{i=1}^{n} r_i^*$, where r^* solves the optimization problem $G(X,p,t)$. Then $G(X,p,t) + \alpha^* t = \sum_{i=1}^{n} r_i^* X(s_i) \geq F(X,p,\alpha^*) \geq F(X',p,\alpha^*) \geq L(X',p,t) + \alpha^* t$.

Finally, correspondence between strict inequalities holds since $\exists p \in P, t \in \mathbb{R}$ s.t. $G(X,p,t) > G(X',\alpha,p,t) \Leftrightarrow \neg(G(X',p,t) \geq G(X,p,t)\ \forall p \in P, t \in \mathbb{R}) \Leftrightarrow \neg(F(X',p,\alpha) \geq F(X,p,\alpha)\ \forall p \in P, \alpha \in (0,1]) \Leftrightarrow \exists p \in P, \alpha \in (0,1]$ s.t. $F(X,p,\alpha) > F(X',p,\alpha)$. $\square$

Proof of Theorem 2. Assume $X' \in \mathcal{X}_F \setminus \mathcal{X}_N(S)$. Then exists $X \in \mathcal{X}_N(S)$ such that $X \succ_S X'$. Since $\tilde{S} \subseteq S$, $\mathbb{E}_p[u(X)] \geq \mathbb{E}_p[u(X)]\ \forall\ (p,u) \in \tilde{S}$ and exists $(u^*,p^*) \in S$ such that $\mathbb{E}_{p^*}[u^*(X)] > \mathbb{E}_{p^*}[u^*(X')]$. For any $(p,u) \in$int$(S) \cap \tilde{S}$, exists $\epsilon > 0$ such that

$$\begin{aligned} p' &= p + \epsilon(p - p^*) \in P\\ u' &= u + \epsilon(u - u^*) \in U. \end{aligned}$$

Since $S = P \times U$, $(p', u^*) \in S$ and $(p^*, u') \in S$. Denoting $\beta = \epsilon/(1+\epsilon)$ yields $u = (1-\beta)u' + \beta u^*$ and $p = (1-\beta)p' + \beta p^*$, wherefore

$$\begin{aligned}
& \mathbb{E}_p[u(X)] \\
= \; & \beta^2 \mathbb{E}_p'[u'(X)] + \beta(1-\beta)\mathbb{E}_p'[u^*(X)] + \beta(1-\beta)\mathbb{E}_{p^*}[u'(X)] + (1-\beta)^2\mathbb{E}_{p^*}[u^*(X)] \\
\geq \; & \beta^2 \mathbb{E}_p'[u'(X')] + \beta(1-\beta)\mathbb{E}_p'[u^*(X')] + \beta(1-\beta)\mathbb{E}_{p^*}[u'(X')] + (1-\beta)^2\mathbb{E}_{p^*}[u^*(X)] \\
> \; & \beta^2 \mathbb{E}_p'[u'(X')] + \beta(1-\beta)\mathbb{E}_p'[u^*(X')] + \beta(1-\beta)\mathbb{E}_{p^*}[u'(X')] + (1-\beta)^2\mathbb{E}_{p^*}[u^*(X')] \\
= \; & \mathbb{E}_p[u(X')],
\end{aligned}$$

since $\beta > 0$ and $1-\beta > 0$. Thus $X \succ_{\tilde{S}} X'$ wherefore $X' \notin \mathcal{X}_N(\tilde{S})$. $\square$

Proof of Lemma 4. Since ME is a LP problem, translation invariance and positive homogeneity are trivial. Superadditivity:

$$\begin{aligned}
\mathrm{ME}_P[X+X'] &= \min_{p\in P} \mathbb{E}_p[X+X'] = \min_{p\in P}(\mathbb{E}_p[X] + \mathbb{E}_p[X']) \\
&= \min_{\substack{p^1,p^2\in P \\ p^1=p^2}} (\mathbb{E}_{p^1}[X] + \mathbb{E}_{p^2}[X']) \geq \min_{p^1,p^2\in P} (\mathbb{E}_{p^1}[X] + \mathbb{E}_{p^2}[X']) \\
&= \min_{p^1\in P} \mathbb{E}_p^1[X] + \min_{p^2\in P} \mathbb{E}_p^2[X'] = \mathrm{ME}_P[X] + \mathrm{ME}_P[X'],
\end{aligned}$$

since $\{p^1, p^2 \in P \mid p^1 = p^2\} \subseteq \{p^1, p^2 \in P\}$. Monotonicity: $X(s_i) \geq X'(s_i)\ \forall\ s_i \in \Omega \Rightarrow \mathbb{E}_p[X] \geq \mathbb{E}_p[X']\ \forall\ p \in P^0 \Rightarrow \min_{p\in P} \mathbb{E}_p[X] = \mathbb{E}_{p^*}[X] \geq \mathbb{E}_{p^*}[X'] \geq \min_{p\in P} \mathbb{E}_p[X'] \Rightarrow \mathrm{ME}_P[X] \geq \mathrm{ME}_P[X']$. $\square$

Proof of Lemma 5 By definition

$$\begin{aligned}
\mathrm{WCVaR}_P^\alpha[X] &= \min_{p\in P} \mathrm{CVaR}_p^\alpha[X] = \min_{p\in P} \min_{q\in Q_p^\alpha} \mathbb{E}_q[X] \\
&= \min_{p\in P, q\in Q_p^\alpha} \mathbb{E}_q[X] = \min_{q\in Q_P^\alpha} \mathbb{E}_q[X],
\end{aligned}$$

which is a LP problem since $Q_P^\alpha = \{q \in P^0 \mid \alpha q \leq p,\ p \in P\}$ is a polyhedral set. Since $\mathrm{WCVaR}_P^\alpha[X] = \mathrm{ME}_{Q_P^\alpha}[X]$, Lemma 4 implies that WCVaR is a coherent risk measure. Furthermore, for any $p \in P, \alpha p \leq p$ wherefore $p \in Q_P^\alpha$ and thus $P \subseteq Q_P^\alpha$. $\square$

Proof of Lemma 6. By Lemma 5

$$\mathrm{WCVaR}_P^\alpha(X) = \min_{q\in Q_P^\alpha} \mathbb{E}_q[X],\ Q_P^\alpha = \{q \in P^0 \mid \alpha q \leq p, p \in P\}$$

and $\min_{q\in P^0} \mathbb{E}_q[X] = \min_{s_i\in\Omega} X(s_i)$ (see proof of by Theorem 1). i) If $P = P^0$ then by Lemma 5 $P^0 \subseteq Q_{P^0}^\alpha$ and by construction $Q_{P^0}^\alpha \subseteq P^0$, which together imply $Q_{P^0}^\alpha = P^0$. ii) Take any $p \in P \cap \mathrm{int}(P^0)$, which implies $p_i > 0 \forall i = 1,..,n$. For any $q \in P^0$, $q_i \leq 1 \leq p_i/\alpha$

for all $i = 1, ...n$, $\alpha \in (0, \min_i p_i]$. Thus, $P^0 \subseteq Q_P^{\alpha \to 0} \subseteq P^0$, i.e., $Q_P^{\alpha \to 0} = P^0$. iii) If $\alpha = 1$, the constraint $\alpha q = q \leq p$ holds only for $p = q$, wherefore $Q_P^1 = P$. $\square$

Proof of Theorem 3. By Theorem 1 $X \succ_{P \times U} X'$ if and only if $\mathbb{E}_p[u(X)] \geq \mathbb{E}_p[u(X')]$ for all $p \in \{p^1, \ldots, p^t\}$, $u \in U$ with a strict inequality for some $p \in \{p^1, \ldots, p^t\}$, $u \in U$. This condition can be checked by minimizing and maximizing the expected utility difference $\mathbb{E}_p[u(X)] - \mathbb{E}_p[u(X')]$ for each $p \in \{p^1, \ldots, p^t\}$ subject to $u \in U$, which is equal to the LP-problem (12). $\square$

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