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Rotational frequency shifts in partially coherent optical fields

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We study the frequency shifts taking place when a random, stationary optical field rotates with respect to an observer. The field is expanded in terms of fully coherent Laguerre–Gaussian basis modes, for which the rotational frequency shifts have been studied previously. We demonstrate the formalism by considering the spectrum of a Gaussian Schell-model field, and show that for a spatially highly incoherent field, significant spectral changes can be expected. © 2006 Optical Society of America

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1. INTRODUCTION

A circularly polarized light beam possessing a helical phase front experiences a frequency shift when the beam and the observer are set in relative rotational motion.[1–7] In this phenomenon, which is distinct from the translational Doppler effect, the angular frequency of light is shifted by $(l+\sigma)\Omega$, where Ω is the angular frequency of rotation, l is an integer determining the helicity of the phase front, and $\sigma=\pm 1$ corresponds to the two orthogonal states of circular polarization. The shifts have found practical use, for instance, in determining the relative weights of the helicity components in a light beam,[8] also at the single-photon level,[9] and in creating moving interference patterns for optical micromanipulation.[10]

In this paper, we consider random optical fields and investigate the effect of partial spatial coherence on the spectral changes due to the rotational frequency shifts. We begin in Section 2 by expressing the space–time realizations of an optical wave field in terms of Laguerre–Gaussian basis modes that have helical phase fronts of the form of $\exp(il\theta)$, with θ denoting the azimuthal angle. For treating the relative rotational motion of the field and the observer, we adopt a straightforward coordinate transformation.[11,12] We then derive an expression for the spectral density of a stationary, uniformly circularly polarized optical field, which is stationary also in the rotation. In Section 3, we exemplify the analysis by investigating the rotational frequency shifts in a two-dimensional Gaussian Schell-model (GSM) field. Finally, the results of the work are briefly summarized in Section 4.

2. THEORY

We consider a stationary, uniformly circularly polarized optical beam propagating along the positive z direction. A realization of such a field can be expanded in terms of orthonormal Laguerre–Gaussian modes, which form a complete set of functions.[13] In the space–time domain, the expansion at the plane of the waist $z=0$ reads as

$$\psi(\boldsymbol{\rho},t)\mathbf{u}_{u,v}^{\pm}=\sum_{l=-\infty}^{\infty}\sum_{p=0}^{\infty}a_{l,p}(t)\phi_{l,p}(\boldsymbol{\rho})\exp(-i\omega_0 t)\mathbf{u}_{u,v}^{\pm}, \tag{1}$$

where $\boldsymbol{\rho}=(\rho,\vartheta)$ denotes the spatial variable written in polar coordinates, $a_{l,p}(t)$ is a time-dependent modal weight with zero mean, ω_0 is the mean angular frequency of light, and t denotes time. The left- or right-handed circular polarization states (upper and lower signs, respectively) are described by the vectors $\mathbf{u}_{u,v}^{\pm}=(\mathbf{u}_u\pm i\mathbf{u}_v)/\sqrt{2}$, in which the unit vectors $\mathbf{u}_u$ and $\mathbf{u}_v$ point along the Cartesian coordinate axes. Furthermore, the spatial part of the basis mode is written as

$$\phi_{l,p}(\boldsymbol{\rho})=f_{l,p}(\rho)\exp(il\vartheta), \tag{2}$$

with the radial part given by[1,13]

$$f_{l,p}(\rho)=(-1)^p\left(\frac{2}{\pi w_0^2}\right)^{1/2}\left[\frac{p!}{(|l|+p)!}\right]^{1/2}\left(\frac{\sqrt{2}\rho}{w_0}\right)^{|l|}\times\exp(-\rho^2/w_0^2)L_p^{|l|}\left(\frac{2\rho^2}{w_0^2}\right), \tag{3}$$

where w_0 is the beam-waist parameter, and $L_p^{|l|}$ is the associated Laguerre polynomial. The basis modes satisfy the orthonormality relation[14]

$$\int_0^{\infty}\int_0^{2\pi}\phi_{l,p}^{*}(\boldsymbol{\rho})\phi_{l',p'}(\boldsymbol{\rho})\mathrm{d}^2\boldsymbol{\rho}=\delta_{l,l'}\delta_{p,p'}, \tag{4}$$

where the asterisk denotes complex conjugation, and $\delta_{m,m'}$ $(m=l,p)$ is the Kronecker delta.

We then assume that the field and the observer rotate with respect to each other about the z axis with angular frequency Ω. In what follows, the coordinates (u,v), or equivalently (ρ,ϑ), refer to the frame of the field, whereas (x,y), or (r,θ), refer to the frame of the observer (see Fig. 1). In the rotation, the Cartesian unit vectors transform according to

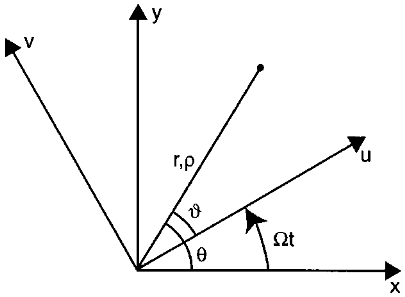

Fig. 1. Coordinate frame (u,v) rotates with angular frequency Ω with respect to the frame (x,y). The cylindrical coordinates of the frames are (ρ,ϑ) and (r,θ), respectively.

$$\begin{bmatrix}\mathbf{u}_u(t)\\ \mathbf{u}_v(t)\end{bmatrix}=\begin{bmatrix}\cos(\Omega t) & \sin(\Omega t)\\ -\sin(\Omega t) & \cos(\Omega t)\end{bmatrix}\begin{bmatrix}\mathbf{u}_x\\ \mathbf{u}_y\end{bmatrix}, \quad (5)$$

and the polar coordinates in the two frames are related by

$$(\rho,\vartheta)=(r,\theta-\Omega t). \quad (6)$$

The realization of the field given in Eq. (1) can now be written in the frame of the observer as

$$\psi(\mathbf{r},t;\Omega)\mathbf{u}^{\pm}_{x,y}=\sum_{l=-\infty}^{\infty}\sum_{p=0}^{\infty}a_{l,p}(t)\phi_{l,p}(\mathbf{r})\times\exp\{-i[\omega_0+(l+\sigma)\Omega]t\}\mathbf{u}^{\pm}_{x,y}, \quad (7)$$

where $\mathbf{r}=(r,\theta)$; $\sigma=+1$ and $\sigma=-1$ for left- and right-handed circular polarization, respectively; and the polarization vectors are given by $\mathbf{u}^{\pm}_{x,y}=(\mathbf{u}_x\pm i\mathbf{u}_y)/\sqrt{2}$.

The coherence properties of the random, uniformly polarized optical field given in Eq. (7) can be described by the (scalar) mutual coherence function characterizing the field correlations at two space–time points $(\mathbf{r},t)$ and $(\mathbf{r}',t')$. The coherence function is explicitly written as

$$\Gamma(\mathbf{r},\mathbf{r}',t,t';\Omega)=\langle\psi^*(\mathbf{r},t;\Omega)\psi(\mathbf{r}',t';\Omega)\rangle, \quad (8)$$

where the angle brackets denote the ensemble average taken over all possible field realizations. By use of Eq. (7) in Eq. (8), we obtain the following formula for the mutual coherence function:

$$\Gamma(\mathbf{r},\mathbf{r}',t,t';\Omega)=\sum_{l=-\infty}^{\infty}\sum_{p=0}^{\infty}\sum_{l'=-\infty}^{\infty}\sum_{p'=0}^{\infty}\langle a^*_{l,p}(t)a_{l',p'}(t')\rangle\phi^*_{l,p}(\mathbf{r})\phi_{l',p'}(\mathbf{r}')\times\exp[-i\omega_0(t'-t)]\exp[-i\Omega(l't'-lt)]\times\exp[-i\sigma\Omega(t'-t)], \quad (9)$$

with the angle brackets having an effect only on the expansion coefficients $a_{l,p}(t)$. The above expression indicates that a field that is stationary for $\Omega=0$ (i.e., a field for which the function Γ depends on time only through the difference $t'-t\equiv\tau$) can be nonstationary for $\Omega\neq 0$. This is due to the phase term $\exp[-i\Omega(l't'-lt)]$ in Eq. (9), which expresses the fact that for each mode the rotational changes in the mean oscillation frequency depend on the index l [see Eq. (7)]. We note that the phase term is present when the term

$$\langle a^*_{l,p}(t)a_{l',p'}(t')\rangle\equiv g_{l,l',p,p'}(t,t') \quad (10)$$

does not vanish for $l\neq l'$, i.e., when the modes with indices l and l' are at least partially correlated. However, if the modes are completely uncorrelated, the terms with $l=l'$ are the only ones that contribute to the mutual coherence function. In such a case the field is stationary also for $\Omega\neq 0$ and one can write

$$g_{l,l',p,p'}(t,t')\equiv g_{l,l,p,p'}(\tau)\delta_{l,l'}, \quad (11)$$

where the τ-dependence results from the stationarity of the field for $\Omega=0$. In this work we consider fields of this type.

The spectral properties of a random field are described by the cross-spectral density function, which for stationary fields is obtained by use of the (generalized) Wiener–Khintchine theorem[15]:

$$W(\mathbf{r},\mathbf{r}',\omega;\Omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma(\mathbf{r},\mathbf{r}',\tau;\Omega)\exp(i\omega t)\mathrm{d}\tau, \quad (12)$$

where $\Gamma(\mathbf{r},\mathbf{r}',\tau;\Omega)$ denotes the mutual coherence function of a stationary field. On substituting Eqs. (9)–(11) into Eq. (12), we find that

$$W(\mathbf{r},\mathbf{r}',\omega;\Omega)=\sum_{l=-\infty}^{\infty}\sum_{p=0}^{\infty}\sum_{p'=0}^{\infty}\phi^*_{l,p}(\mathbf{r})\phi_{l,p'}(\mathbf{r}')\times\tilde{g}_{l,l,p,p'}[\omega-\omega_0-(l+\sigma)\Omega], \quad (13)$$

where $\tilde{g}_{l,l,p,p'}(\omega)$ is the Fourier transform of the function $g_{l,l,p,p'}(\tau)$, i.e.,

$$\tilde{g}_{l,l,p,p'}(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}g_{l,l,p,p'}(\tau)\exp(i\omega\tau)\mathrm{d}\tau. \quad (14)$$

Furthermore, the spectral density of the field is readily obtained as

$$S(\mathbf{r},\omega;\Omega)\equiv W(\mathbf{r},\mathbf{r},\omega;\Omega)=\sum_{l=-\infty}^{\infty}\sum_{p=0}^{\infty}\sum_{p'=0}^{\infty}f_{l,p}(r)f_{l,p'}(r)\times\tilde{g}_{l,l,p,p'}[\omega-\omega_0-(l+\sigma)\Omega], \quad (15)$$

and we see that any effect of rotation on the spectral density is manifested via a shift in the frequency argument of the function $\tilde{g}_{l,l,p,p'}(\omega)$.

3. FREQUENCY SHIFTS IN A GAUSSIAN SCHELL-MODEL FIELD

In this section, we give an example of the spectral changes due to rotation by considering a two-dimensional, uniformly circularly polarized GSM field. The cross-spectral density function for a nonrotating $(\Omega=0)$ GSM field at the plane $z=0$ is of the form[15,16]

$$W(\mathbf{r},\mathbf{r}',\omega;0)=[S(\mathbf{r},\omega;0)S(\mathbf{r}',\omega;0)]^{1/2}\exp(-|\mathbf{r}-\mathbf{r}'|^2/L_c^2), \quad (16)$$

where the spectral density is given by

$$S(\mathbf{r},\omega;0)=\left(\frac{2}{\pi w_G^2}\right)\exp(-2r^2/w_G^2)\tilde{g}(\omega-\omega_0). \quad (17)$$

The parameters L_c and w_G are, respectively, the coherence length and the waist size, which are both assumed to be independent of frequency. Furthermore, we choose the spectrum to be Lorentzian in shape, i.e.,

$$\tilde{g}(\omega-\omega_0)=\frac{\Delta\omega/2\pi}{(\omega-\omega_0)^2+(\Delta\omega/2)^2}, \quad (18)$$

where the parameter $\Delta\omega$ denotes the linewidth (full width at half-maximum).

Next we express the cross-spectral density function of a rotating GSM field in the form of Eq. (13). However, to simplify the analysis we assume that the Laguerre–Gaussian basis modes are mutually completely uncorrelated (also in indices p and p'), and that each mode has the same spectrum for $\Omega=0$. These two assumptions are accounted for by writing

$$\tilde{g}_{l,l,p,p'}[\omega-\omega_0-(l+\sigma)\Omega]=h(l,p)\delta_{p,p'}\tilde{g}[\omega-\omega_0-(l+\sigma)\Omega], \quad (19)$$

where the coefficient $h(l,p)$ describes the relative weight of a basis mode. Inserting Eq. (19) into Eq. (13), setting $\Omega=0$, and comparing the resulting expression with Eq. (16), one finds, with the help of the orthogonality relation of Eq. (4) and Eq. 7.421(4) of Ref. 17, that[16]

$$h(l,p)=\left(\frac{2}{\pi w_G^2}\right)\left(\frac{\pi}{1/w_G^2+1/L_c^2+1/w_0^2}\right)\times\left(\frac{1/L_c^2}{1/w_G^2+1/L_c^2+1/w_0^2}\right)^{2p+|l|}, \quad (20)$$

where

$$w_0=\left(\frac{w_G^2}{1/w_G^2+2/L_c^2}\right)^{1/4}. \quad (21)$$

Hence, the cross-spectral density function of the rotating GSM field can be written as

$$W(\mathbf{r},\mathbf{r}',\omega;\Omega)=\sum_{l=-\infty}^{\infty}\sum_{p=0}^{\infty}\lambda_{l,p}(\omega,\Omega)\phi_{l,p}^*(\mathbf{r})\phi_{l,p}(\mathbf{r}'), \quad (22)$$

where

$$\lambda_{l,p}(\omega,\Omega)=h(l,p)\tilde{g}[\omega-\omega_0-(l+\sigma)\Omega] \quad (23)$$

describes the energy in a mode at frequency ω for a certain value of Ω. Equation (22) is written in the form of a coherent-mode representation with $\lambda_{l,p}(\omega,\Omega)$ and $\phi_{l,p}(\mathbf{r})$ denoting the eigenvalues and eigenfunctions of the related Fredholm integral equation.[15] We note here that due to the rotation, the eigenvalues $\lambda_{l,p}(\omega,\Omega)$ will be spectrally redistributed from their values for $\Omega=0$.

By making use of Eqs. (2), (3), (20), (21), and (23), one can perform the summation over the index p in Eq. (22) in an analytic form.[17] This calculation yields for the cross-spectral density

$$W(\mathbf{r},\mathbf{r}',\omega;\Omega)=\left(\frac{2}{\pi w_G^2}\right)\exp[-(\eta^2+\eta'^2)(1+\xi^2)]\times\sum_{l=-\infty}^{\infty}\exp[il(\theta'-\theta)]I_{|l|}(2\xi^2\eta\eta')\times\tilde{g}[\omega-\omega_0-(l+\sigma)\Omega], \quad (24)$$

where a normalized radial coordinate $\eta=r/w_G$ is introduced, and $I_{|l|}$ denotes the modified Bessel function of the first kind of order $|l|$. The parameter $\xi=w_G/L_c$ can be considered as a measure for the degree of global spatial coherence of the field.[15] Accordingly, the field is completely coherent for $\xi=0$ and completely incoherent for $\xi\to\infty$.

The spectral density is obtained from Eq. (24) as

$$S(\mathbf{r},\omega;\Omega)=\left(\frac{2}{\pi w_G^2}\right)\exp[-2\eta^2(1+\xi^2)]\sum_{l=-\infty}^{\infty}I_{|l|}(2\xi^2\eta^2)\times\tilde{g}[\omega-\omega_0-(l+\sigma)\Omega]. \quad (25)$$

In this expression, the factor in front of the sum term comprises a spatially Gaussian envelope function that depends on the coherence parameter ξ, but not on the frequency ω. The sum term itself depends on both of the parameters ξ and ω and it consists of Lorentzian lines whose center frequencies $\omega_0+(l+\sigma)\Omega$ vary with the summation index l. The behavior of the weights $I_{|l|}(\zeta)$ of these lines is illustrated in Fig. 2 with a few different values of the argument $\zeta=2\xi^2\eta^2$. For the value $\zeta=0$, only the Lorentzian line with $l=0$ contributes to the spectral density, the line being centered at the frequency $\omega_0+\sigma\Omega$. This case corresponds either to a fully coherent field ($\xi=0$) for which the spectrum is the same at any distance from the optical axis, or to a field with any state of spatial coherence observed at the optical axis ($\eta=0$). For the values $\zeta\gg 1$ on the other hand, Fig. 2 indicates that several Lorentzian lines with a large value of $|l|$ can significantly contribute to the spectral density. This is the case when the field is very incoherent (ξ is large), or when the field is observed far from the optical axis (η is large).

Figure 3 illustrates the effect of the angular frequency of rotation Ω on the spectral density with two weight functions $I_{|l|}(10)$ and $I_{|l|}(200)$. The spectrum of a nonrotating beam ($\Omega=0$), i.e. the Lorentzian line, is shown as a

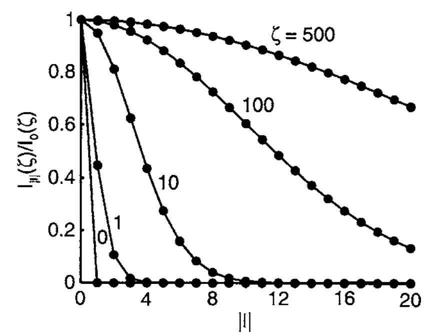

Fig. 2. Behavior of the modified Bessel function of the first kind $I_{|l|}(\zeta)$ as a function of the index $|l|$. The solid curves are plotted to guide the eye.

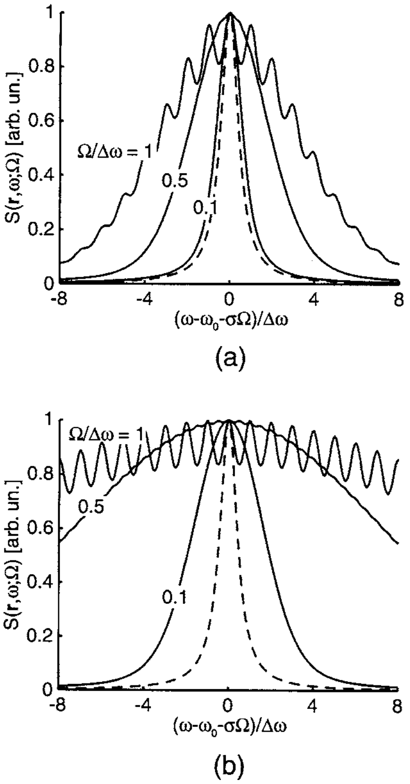

Fig. 3. Spectral density $S(\mathbf{r},\omega;\Omega)$ in the plane of the waist ($z=0$) of the GSM field for different rotational frequencies Ω with weight functions (a) $I_{|l|}(10)$ and (b) $I_{|l|}(200)$. The Lorentzian lineshape function of width $\Delta\omega$ is shown by the dashed curves. All the curves are normalized to have the same maximum value.

dashed curve. For $\Omega \neq 0$, the center frequency ω_0 of the spectrum is shifted to the frequency $\omega_0+\sigma\Omega$, and the overall spectral width is seen to increase as the value of Ω is increased. For example, near the optical axis with the parameter η fixed to a value of $\eta \lesssim 1$, one can consider the differences between Figs. 3(a) and 3(b) to occur due to changes in spatial coherence through the parameter ξ. The spectral changes are hence seen to be more significant in the more incoherent field of Fig. 3(b) for a fixed value of Ω. On the other hand, for a certain state of coherence with the parameter ξ fixed, the spectral changes are more significant farther from the optical axis, i.e., with a large value of the parameter η. The shape of the spectrum can be characterized as a broadened spectral line when the inequality $\Omega \lesssim \Delta\omega$ is satisfied. However, the peaks of the constituent Lorentzian lines become visible for the values of $\Omega \approx \Delta\omega$; and for still higher values of Ω, the Lorentzians would be distinguished separately. In such cases, the center frequency is also at $\omega_0+\sigma\Omega$ and the width of the envelope of the spectrum increases with Ω.

4. SUMMARY

We have studied the effect of partial spatial coherence on the spectral changes occurring when a random, uniformly circularly polarized field and the observer rotate with respect to each other. In our analysis, we expanded the field in terms of Laguerre–Gaussian basis modes and derived an expression for the cross-spectral density for a certain type of fields, which are stationary in the rotation. The formalism was exemplified by considering a two-dimensional Gaussian Schell-model field. We conclude that, the more (spatially) incoherent the field, the more significant the spectral changes. In particular, observable changes in the spectral line shape can occur with rotation rates smaller than the linewidth facilitating an experimental detection of the spectral changes.

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REFERENCES

1. L. Allen, S. M. Barnett, and M. J. Padgett, eds., *Optical Angular Momentum* (IOP Publishing, 2003).
2. M. J. Padgett and L. Allen, "The angular momentum of light: optical spanners and the rotational frequency shift," Opt. Quantum Electron. **31**, 1–12 (1999).
3. B. A. Garetz and S. Arnold, "Variable frequency shifting of circularly polarized laser radiation via a rotating half-wave retardation plate," Opt. Commun. **31**, 1–3 (1979).
4. J. Courtial, K. Dholakia, D. A. Robertson, L. Allen, and M. J. Padgett, "Measurement of the rotational frequency shift imparted to a rotating light beam possessing orbital angular momentum," Phys. Rev. Lett. **80**, 3217–3219 (1998).
5. I. V. Basistiy, V. V. Slyusar, M. S. Soskin, M. V. Vasnetsov, and A. Ya. Bekshaev, "Manifestation of the rotational Doppler effect by use of an off-axis optical vortex beam," Opt. Lett. **28**, 1185–1187 (2003).
6. M. J. Padgett, "The mechanism for energy transfer in the rotational frequency shift of a light beam," J. Opt. A, Pure Appl. Opt. **6**, S263–S265 (2004).
7. O. Steuernagel, "Comment on 'The mechanism for energy transfer in the rotational frequency shift of a light beam,' regarding longitudinal momentum transfer," J. Opt. A, Pure Appl. Opt. **6**, 1058–1059 (2004).
8. M. V. Vasnetsov, J. P. Torres, D. V. Petrov, and L. Torner, "Observation of the orbital angular momentum spectrum of a light beam," Opt. Lett. **28**, 2285–2287 (2003).
9. R. Dasgupta and P. K. Gupta, "Measurement of orbital angular momentum of a single photon: a non-interferrometric approach," in *Proceedings of the 7th International Conference on Optoelectronics, Fiber Optics, and Photonics*, December 9–11, 2004, Cochin, Kerala, India.
10. J. Arlt, M. P. MacDonald, L. Paterson, W. Sibbett, K. Dholakia, and K. Volke-Sepulveda, "Moving interference patterns created using the angular Doppler effect," Opt. Express **10**, 844–852 (2002).
11. B. Mashhoon, R. Neutze, M. Hannam, and G. E. Stedman, "Observable frequency shifts via spin-rotation coupling," Phys. Lett. A **249**, 161–166 (1998).
12. I. Bialynicki-Birula and Z. Bialynicka-Birula, "Rotational frequency shift," Phys. Rev. Lett. **78**, 2539–2542 (1997).

13. L. Torner, J. P. Torres, and S. Carrasco, “Digital spiral imaging,” Opt. Express **13**, 873–881 (2005).
14. G. B. Arfken and H. J. Weber, *Mathematical Methods for Physicists*, 4th ed. (Academic, 1995).
15. L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge U. Press, 1995).
16. B. Zhang and B. Lü, “Transformation of Gaussian Schell-model beams and their coherent-mode representation,” J. Opt. (Paris) **27**, 99–103 (1996).
17. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 5th ed., CD-ROM version 1.0, A. Jeffrey ed. (Academic, 1995).