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# On the origin of divergences in the coincidence probabilities in cavity photodetection experiments

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**Abstract**
The theory of photon correlation is an established part of quantum electronics. However, recently reported divergences in the theory of time correlated detection of photons show that important details of cavity photon statistics are still incompletely understood. The quantum jump superoperators of the SD photon counting model given by Srinivas and Davies (1981 *J. Mod. Opt.* **28** 981–96 ) do not fulfil the assumption of the bounded interaction rate. This has raised doubts about the consistency of the SD photon counting model and especially about the existence of coincidence probability density (CPD) functions (Dodonov *et al* 2005 *J. Opt. B: Quantum Semiclass. Opt.* **7** 99–108). In this work, we start from the first principles of the quantum trajectory theory and show how the different coincidence probability densities and coincidence probabilities (CPs) have to be calculated. CPDs derived by us are well defined, and CPs are finite and correctly normalized for all fields with finite photon number expectation value. Furthermore, we show that the SD model reproduces photon bunching and antibunching phenomena when consistent derivation for the second-order coherence degree is used.

## 1. Introduction

Although photon detection theory is considered as a well-known theory, some details such as the coincidence photon counting probabilities of cavity photon statistics are still incompletely understood. These problems were recently pointed out by Dodonov *et al* [1] who reported divergences in the coincidence probability densities (CPDs) of cavity photon counting.

In their seminal work on cavity photon counting Srinivas and Davies [2] introduced one-count and no-count quantum operators by making a number of intuitive postulates which guaranteed the consistency of their photon counting model (SD model) with the general principles of quantum optics and quantum measurement theory. Later Ueda *et al* [3, 4] elaborated the SD model by calculating the time evolution of photon statistics for a selected single-mode cavity fields and also provided a microscopic theory of the SD model by discussing the interaction of the cavity field with an atomic beam. Their approach, based on the perturbation theory, gave the same expressions for the one-count and no-count

operators as in the original work of Srinivas and Davies [2]. Ueda *et al* [3] also discussed some of the surprising (from the classical physics point of view) features of photon statistics within the framework of quantum optics and quantum theory of measurement. They pointed out, in particular, that the two-fold increase of the expectation value of the number of photons predicted by the SD model after detecting one photon from a thermal field state is both intuitively understandable and quantum-mechanically correct since the measurement will project out the vacuum state from the pertinent mixture of Fock states.

The SD model gives a photon counting rate that is proportional to the expectation value of the number of photons. Recent experiments agree with the predictions of the SD model [5, 6]. Parigi *et al* [5] measured the statistics of a thermal field light pulse after photon addition and photon subtraction. They added or subtracted a photon to/from the light pulse and measured the photon statistics of the pulse with a homodyne detector (see [5] for details). They were able to show that after subtracting a photon from a thermal field the expectation value of the number of photons doubled as predicted by Ueda

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*et al* [3] using the SD model. Furthermore, Parigi *et al* [5] verified experimentally the non-commutativity of the bosonic annihilation and creation operators (i.e., $[\hat{a}, \hat{a}^{\dagger}] \neq 0$) and showed that the simple view of the classical particle addition and subtraction is incorrect in this case. The results of Parigi *et al* show that the photon-added and the photon-subtracted states are given by $\hat{a}^{\dagger}\hat{\rho}\hat{a}$ and $\hat{a}\hat{\rho}\hat{a}^{\dagger}$ (unnormalized), respectively, as predicted by the SD model.

Although the recent experiment [5] agrees with the predictions of the SD model, the SD model has been previously reported to be inconsistent [1, 7][1]. As a consequence, the E model was introduced [1, 7]. The E model is obtained from the SD model by replacing the bosonic annihilation ($\hat{a}$) and creation ($\hat{a}^{\dagger}$) operators by operators [1] $(\hat{a}^{\dagger}\hat{a} + 1)^{-1/2}\hat{a}$ and $\hat{a}^{\dagger}(\hat{a}^{\dagger}\hat{a} + 1)^{-1/2}$, respectively.

The claimed inconsistency is said to be related to the fifth postulate used by Srinivas and Davies [2]. This postulate assumes a bounded interaction rate [2]: there exists a real number $K < \infty$ such that

$$\sum_{m=1}^{\infty} \text{Trace}\{\hat{N}_t(m)\hat{\rho}_f\} < Kt, \tag{1}$$

for all $t > 0$ and all normalized density operators $\hat{\rho}_f$. In equation (1), operators $\hat{N}_t(m)$ have the following meaning: if $\hat{\rho}_f$ is the state of the system at $t = 0$ and $m$ counts are recorded during $[0, t)$, the state of the system at time $t$ is $\hat{N}_t(m)\hat{\rho}_f/\text{Trace}\{\hat{N}_t(m)\hat{\rho}_f\}$. It follows that the probability of $m$ count is $P([0, t), m) = \text{Trace}\{\hat{N}_t(m)\hat{\rho}_f\}$. As already noted by Srinivas and Davies [2], there are fields which do not satisfy the postulate in equation (1) since the trace of the one-count operator, $\gamma_{\text{sd}} \text{Tr}\{\hat{a}\hat{\rho}_f\hat{a}^{\dagger}\}$, is unbounded. They pointed out later in [2] that the fifth postulate is rather a mathematical curiosity needed to show the existence of the one-count and no-count operators.

We have recently proved that the SD model can be derived consistently from the Lindblad equation [8] without further approximations for nonsaturating detectors. In this work, we show that the CPDs given by the SD model exist and are well defined for all initial fields. We explain how the SD model can be used consistently to analyse experiments made by using resolving detectors corresponding to counting exactly one photon and to analyse experiments made by using nonresolving detectors corresponding to counting at least one photon. We also show that the coincidence probabilities (CPs) in the SD model are directly proportional to the factorial moments of the initial field.

[1] In [1] is written: 'Recently [Ref. [27] in Ref. [1]], we have shown that it is possible to get rid of inconsistencies, and still keep the structure of the SD theory, by means of replacing the operators $\hat{a}$ and $\hat{a}^{\dagger}$ in equation (Eq. (1) in Ref. [1]) by the special case of (Eq. (2) in Ref. [1])— the so-called exponential phase operators [Ref. [28–31] in Ref. [1]] $\hat{E}_- = (\hat{n}+1)^{-1/2}\hat{a}$, $\hat{E}_+ = \hat{a}^{\dagger}(\hat{n}+1)^{-1/2}$. Such a change was motivated by the study of the role of the annihilation operator $\hat{a}$ in quantum optics [Ref. [32] in Ref. [1]]. We have drawn attention to the fact (noticed also in [Ref. [10, 13] in Ref. [1]]) that state $\hat{a}|\psi\rangle$ is not always one whose mean number of photons is necessarily less than in $|\psi\rangle$; this occurs, partly, because the presence of the weight $\sqrt{n}$ in $\hat{a}|\psi\rangle = \sqrt{n}|n - 1\rangle$.' In [7] is written: 'Note that using the SD definition, $\hat{J}\hat{\rho} = \hat{a}\hat{\rho}\hat{a}^{\dagger}$, one obtains the weird result $\bar{n} = 2\bar{n}$. So one perceives that using $\hat{a}$ and $\hat{a}^{\dagger}$ for constructing a continuous photocount measurement leads to some inconsistent results.'

Since some of the conclusions on photon correlation predicted by the SD and E models in [1] were not correct we also compare the second-order coherence degrees and waiting times given by the SD and E models. These comparisons show that the SD model reproduces the well-known photon bunching, antibunching and non-bunching phenomena depending on the initial field.

## 2. Coincidence probabilities and photon statistics predicted by the SD model

### 2.1. Quantum trajectories and coincidence probabilities

In the quantum trajectory approach, time is divided into so short intervals $\delta t$ that only the no-count event and the one-count event are possible. For a differential increment $\delta t$, the time development of the density operator is given by

$$\hat{\rho}_f(t + \delta t) = \hat{J}\hat{\rho}_f(t)\delta t + \hat{S}_{\delta t}\hat{\rho}_f(t), \tag{2}$$

where [2, 3, 8] $\hat{J}\hat{\rho}_f(t) = \gamma_{\text{sd}}\hat{a}\hat{\rho}_f(t)\hat{a}^{\dagger}$ is the SD one-count operator and $\hat{S}_{\tau}\hat{\rho}_f(t) = \mathrm{e}^{(-\mathrm{i}\omega\hat{a}^{\dagger}\hat{a} - \frac{\gamma_{\text{sd}}}{2}\hat{a}^{\dagger}\hat{a})\tau}\hat{\rho}_f(t) \, \mathrm{e}^{(+\mathrm{i}\omega\hat{a}^{\dagger}\hat{a} - \frac{\gamma_{\text{sd}}}{2}\hat{a}^{\dagger}\hat{a})\tau}$ is the SD no-count operator with $\gamma_{\text{sd}}$ being the SD model parameter describing the coupling between the field and detector system, and $\omega$ being the mode frequency. Note that in the no-count operator $\delta t$ can also be finite time, while equation (2) is valid only for a differential time $\delta t$ so short that *at most one* photon can be absorbed during this time. Equation (2) describes the evolution of the field as a sum of these two quantum trajectories during time period $[t, t + \delta t]$. The one-count trajectory is defined by $\hat{J}\hat{\rho}_f(t)\delta t$ and the no-count trajectory is defined by $\hat{S}_{\delta t}\hat{\rho}_f(t)$. From this definition it follows that the probability of the one-count event (i.e., detection of a photon) per unit time is given by the count rate $r(t) = \text{Tr}\{\hat{J}\hat{\rho}(t)\} = \gamma_{\text{sd}}\bar{n}(t)$. The product $r(t)\delta t$ gives the average number of counts during $[t, t + \delta t]$. This implies that $\delta t$ is so short that $r(t)$ can be considered constant during $[t, t + \delta t]$ and $r(t)\delta t \leqslant 1$. In order to the probability of absorbing two or more photons during $\delta t$ to be infinitesimal the interval must fulfil $\delta t \ll r^{-1}$ (see also [9]).

We next calculate probabilities for photon counting sequences where one photon is counted at each of the specific non-overlapping intervals $[t_1, t_1 + \mathrm{d}t_1], \ldots, [t_k, t_k + \mathrm{d}t_k]$. Between these intervals, the system is assumed to evolve according to the average evolution operator, i.e., any number of photons can be absorbed from the cavity but the detector is not recording. The average evolution operator is [2]

$$\hat{T}_t = \sum_{m=0}^{\infty} \hat{N}_t(m), \tag{3}$$

where

$$\hat{N}_t(m)\hat{\rho}_f(0) = \int_{t_m=0}^{t} \cdots \int_{t_1=0}^{t_2} \hat{S}_{t-t_m}\hat{J}\hat{S}_{t_m-t_{m-1}} \\ \cdots \hat{J}\hat{S}_{t_1}\hat{\rho}_f(0) \, \mathrm{d}t_1 \cdots \mathrm{d}t_m. \tag{4}$$

Furthermore, $\text{Trace}\{\hat{N}_t(m)\hat{\rho}_f(0)\}$ is the probability of counting $m$ photons during $[0, t)$.

The probability that the system undergoes the average evolution during $[0, t_1)$ and the one count occurs during

$[t_1, t_1 + dt_1)$ is $\text{Tr}\{\hat{J}\hat{T}_{t_1}\hat{\rho}_f(0)\}dt_1$. After the one-count event the system is projected into the state $\hat{\rho}_f(t_1 + dt_1) = \hat{J}\hat{T}_{t_1}\hat{\rho}_f(0)/\text{Tr}\{\hat{J}\hat{T}_{t_1}\hat{\rho}_f(0)\}$. The probability of the second one-count event is $\text{Tr}\{\hat{J}\hat{T}_{t_2-t_1}\hat{\rho}_f(t_1 + dt_1)\}dt_2 = \text{Tr}\{\hat{J}\hat{T}_{t_2-t_1}\hat{J}\hat{T}_{t_1}\hat{\rho}_f(0)\}dt_2/\text{Tr}\{\hat{J}\hat{T}_{t_1}\hat{\rho}_f(0)\}$, which is a preconditional probability that the trajectory corresponding to the operator $\hat{J}\hat{T}_{t_1}$ has occurred previously. The state now becomes $\hat{\rho}_f(t_2 + dt_2) = \hat{J}\hat{T}_{t_2-t_1}\hat{\rho}_f(t_1 + dt_1)/\text{Tr}\{\hat{J}\hat{T}_{t_2-t_1}\hat{\rho}_f(t_1 + dt_1)\} = \hat{J}\hat{T}_{t_2-t_1}\hat{J}\hat{T}_{t_1}\hat{\rho}_f(0)/\text{Tr}\{\hat{J}\hat{T}_{t_2-t_1}\hat{J}\hat{T}_{t_1}\hat{\rho}_f(0)\}$. By using this result recursively we conclude that the probability of the $k$th event and the density operator after this event are, respectively, given by

$$p(t_k|t_{k-1}, \ldots, t_1) = \frac{\text{Tr}\{\hat{J}\hat{T}_{t_k-t_{k-1}}\cdots\hat{T}_{t_2-t_1}\hat{J}\hat{T}_{t_1}\hat{\rho}_f(0)\}\,dt_k}{\text{Tr}\{\hat{J}\hat{T}_{t_{k-1}-t_{k-2}}\cdots\hat{T}_{t_2-t_1}\hat{J}\hat{T}_{t_1}\hat{\rho}_f(0)\}}, \quad (5)$$

$$\hat{\rho}_f(t_k + dt_k) = \frac{\hat{J}\hat{T}_{t_k-t_{k-1}}\cdots\hat{T}_{t_2-t_1}\hat{J}\hat{T}_{t_1}\hat{\rho}_f(0)}{\text{Tr}\{\hat{J}\hat{T}_{t_k-t_{k-1}}\cdots\hat{T}_{t_2-t_1}\hat{J}\hat{T}_{t_1}\hat{\rho}_f(0)\}}. \quad (6)$$

Equation (5) gives the conditional probability of $k$th count with the conditions that $k-1$ one-count events have occurred at $[t_1 + dt_1], \ldots, [t_{k-1} + dt_{k-1}]$, and any number of photons may have been absorbed between these events.

The probability of the $k$-count quantum trajectory is the product of the conditional probabilities giving

$$p(t_1, \ldots, t_k) = \text{Tr}\{\hat{J}\hat{T}_{t_k-t_{k-1}}\cdots\hat{T}_{t_2-t_1}\hat{J}\hat{T}_{t_1}\hat{\rho}_f(0)\}\,dt_1\cdots dt_k. \quad (7)$$

Equation (7) then gives the CP of the $k$ one-count events occurring at $[t_i, t_i + dt_i]$, $i = 1, \ldots, k$ and any number of photon absorptions between these events, i.e., the system is under average evolution between the one-count events. We point out that in defining the probability in equation (7) we have used the one-count operator in such a way that $\hat{J}\hat{\rho}_f(t_i)\delta t_i \ll 1$, $i = 1, \ldots, k$, i.e., each of the one-count probabilities *must be small*.

## 2.2. Coincidence probability densities of the SD model

We next derive a general CPD for the SD model. The CPDs are also derived in [1, 2] but we will derive them using the conditional probabilities and the time dependence of the factorial moments, which allows us to show that the CPs and CPDs are well defined. The CP gives the probability to detect $k$ photons, one at each of the non-overlapping intervals $[t_1, t_1 + dt_1], \ldots, [t_k, t_k + dt_k]$. Between these intervals any number of photons can be absorbed from the cavity. This definition corresponds to an experimental setup where the detector is recording photons during each interval $[t_i, t_i + dt_i)$ and switched off between these measurement intervals.

The probability of absorbing a photon at $[t_{k+1}, t_{k+1} + dt_{k+1}]$ with the condition that $k$ photons have been absorbed at non-overlapping intervals $dt_i$ at specific times $t_1 < t_2 < \cdots < t_k$ (and between the times $t_{i-1} + dt_{i-1}$ and $t_i$ any number of photons may have been absorbed) is given by $\gamma_{sd}\bar{n}(t_{k+1})dt_{k+1}$. Here the expectation value of the number of photons $\bar{n}(t_{k+1})$ (for this particular quantum trajectory) can be written using equation (B.9) (see appendix B.2) as $\bar{n}(t_{k+1}) = \bar{n}(t_k^+)e^{-\gamma_{sd}(t_{k+1}-t_k)}$, where $t_k^+ = t_k + dt_k$. Using equation (B.9)

which describes the average evolution of the factorial moments and equation (B.10) which describes the change of the factorial moments during a one-count event, we obtain

$$p(t_{k+1}|t_k, \ldots, t_1) = \gamma_{sd}\bar{n}(t_k^+)\,e^{-\gamma_{sd}(t_{k+1}-t_k)}\,dt_{k+1}$$
$$= \gamma_{sd}\frac{\overline{n(n-1)}(t_k)}{\bar{n}(t_k)}\,e^{-\gamma_{sd}(t_{k+1}-t_k)}\,dt_{k+1}$$
$$= \gamma_{sd}\frac{\overline{n(n-1)}(t_{k-1}^+)}{\bar{n}(t_{k-1}^+)}\,e^{-\gamma_{sd}(t_{k+1}-t_{k-1})}\,dt_{k+1}. \quad (8)$$

By repeatedly applying equations (B.9) and (B.10) to equation (8) gives

$$p(t_{k+1}|t_k, \ldots, t_1)$$
$$= \gamma_{sd}\frac{\overline{n(n-1)(n-2)}(t_{k-2}^+)}{\overline{n(n-1)}(t_{k-2}^+)}\,e^{-\gamma_{sd}(t_{k+1}-t_{k-2})}\,dt_{k+1}$$
$$= \gamma_{sd}\frac{\overline{n(n-1)(n-2)(n-3)}(t_{k-3}^+)}{\overline{n(n-1)(n-2)}(t_{k-3}^+)}\,e^{-\gamma_{sd}(t_{k+1}-t_{k-3})}\,dt_{k+1}$$
$$\cdots$$
$$= \gamma_{sd}\frac{\overline{n(n-1)(n-2)\cdots(n-k)}(0)}{\overline{n(n-1)(n-2)\cdots(n-(k-1))}(0)}\,e^{-\gamma_{sd}t_{k+1}}\,dt_{k+1}. \quad (9)$$

Equation (9) divided by $dt_{k+1}$ gives the absorption rate at $t_{k+1}$ with the condition that a photon has been absorbed at each of the intervals $[t_1, t_1 + dt_1], \ldots, [t_k, t_k + dt_k]$.

The probability of detecting photons at $[t_1, t_1 + dt_1], \ldots, [t_k, t_k + dt_k]$ is obtained from equation (9) as a product $p(t_k|t_{k-1}, \ldots, t_1) \cdot p(t_{k-1}|t_{k-2}, \ldots, t_1) \cdots p(t_1)$. This gives

$$p(t_1, t_2, \ldots, t_k)$$
$$= \gamma_{sd}^k\overline{n(n-1)\cdots(n-(k-1))}(0)\,e^{-\gamma_{sd}(t_1+\cdots+t_k)}\,dt_1\cdots dt_k. \quad (10)$$

From equation (10) we see that the $k$ photon CP in the SD model is directly proportional to the $k$th factorial moment. The CPDs for the Fock state, the thermal field and the coherent field are obtained from equation (10) by dividing with the product of $dt_1\cdots dt_k$ and substituting the expressions of the factorial moments:

$$f_{\text{Fock}}^{sd}(t_1, \ldots, t_k) = \gamma_{sd}^k\frac{N!}{(N-k)!}\,e^{-\gamma_{sd}(t_1+\cdots+t_k)}, \quad (11)$$

$$f_{\text{ther}}^{sd}(t_1, \ldots, t_k) = \gamma_{sd}^k k!\bar{n}^k(0)\,e^{-\gamma_{sd}(t_1+\cdots+t_k)}, \quad (12)$$

$$f_{\text{coh}}^{sd}(t_1, \ldots, t_k) = \gamma_{sd}^k\bar{n}^k(0)\,e^{-\gamma_{sd}(t_1+\cdots+t_k)}. \quad (13)$$

In equation (11) $N$ is the number of photons in the initial Fock state.

## 2.3. Coincidence probability densities of the E model

As a reference we also give the $k$-photon CPDs for the E model [1] (note that these CPDs depend only on $t_k$!):

$$f_{\text{Fock}}^e(t_k) = \frac{\gamma_e^k}{(N-k)!}\int_{\gamma_e t_k}^\infty x^{N-k}\,e^{-x}\,dx, \quad (14)$$

$$f_{\text{ther}}^e(t_k) = \gamma_e^k\left(\frac{\bar{n}(0)}{\bar{n}(0)+1}\right)^k e^{-\gamma_e t_k/(\bar{n}(0)+1)}, \quad (15)$$

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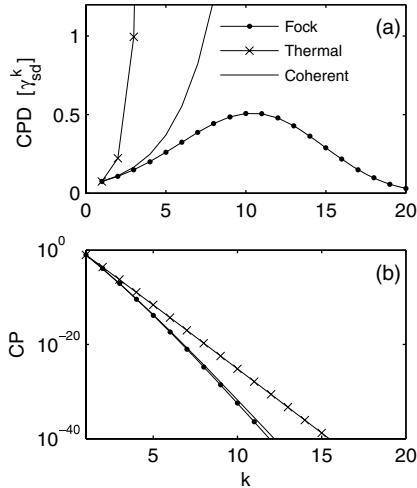
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**Figure 1.** (a) The coincidence probability densities (CPD) and (b) the coincidence probabilities (CP) given by the SD model for the Fock state, the thermal field and the coherent field. Following [1], the measurement intervals $[t_i, t_i + dt_i)$ are chosen so that $t_i = it_1, i = 1, \ldots, k$ with the condition $\gamma_{sd} t_k = 6$. The measurement intervals are $dt_i = \exp(t_1)/(10k\bar{n}(0))$ (so that the condition in equation (17) is fulfilled). The initial expectation value of the number of photons is $\bar{n}(0) = 30$. Note that in figure 2 of [1] the values of the CPDs for the SD model are six orders of magnitude too high.

$$f_{coh}^e(t_k) = \gamma_e^k \left( 1 - e^{-\gamma_e t_k} \sum_{i=0}^{\infty} \frac{(\gamma_e t_k)^i}{i!(i+k-1)!} \right.$$
$$\left. \times \int_{\bar{n}(0)}^{\infty} x^{i+k-1} e^{-x} dx \right). \tag{16}$$

In equation (14), $N$ is the number of photons in the initial Fock state. The probabilities are obtained by multiplying the probability densities with $dt_1 \cdots dt_k$.

### 2.4. Well definiteness of CPs and CPDs

In [1], Dodonov *et al* calculated the CPDs for the SD and E models for the three field types above. They pointed out that the CPDs given by the E model are always less than or equal to unity, while the CPDs given by the SD model may be larger than unity. However, neither *physics* nor the *probability theory* requires the CPDs to be *less than unity* since CPDs are not measurable quantities. It is only the CPs which can be measured and have to be less or equal to one. Therefore, the results reported in [1] neither prove that the SD model is incorrect nor that the E model is correct.

Dodonov *et al* [1] demonstrated the rapid increase of the $k$-photon CPDs given by the SD model by setting $\bar{n}(0) = 30$ and $\gamma_{sd} t_k = 6$. In this case, the absorption instant of the $k$th photon is fixed and the other photons are absorbed at times $t_i = 6i/(\gamma_{sd}k), i = 1, \ldots, k$. With this choice of absorption times $t_i$ the measurement intervals $[t_i, t_i + dt_i)$ are located more and more densely when $k$ increases. The results are given in figure 1(a). Note that Dodonov *et al* (figure 2 in [1]) miscalculated the results and gave CPD values six orders of magnitude too high.
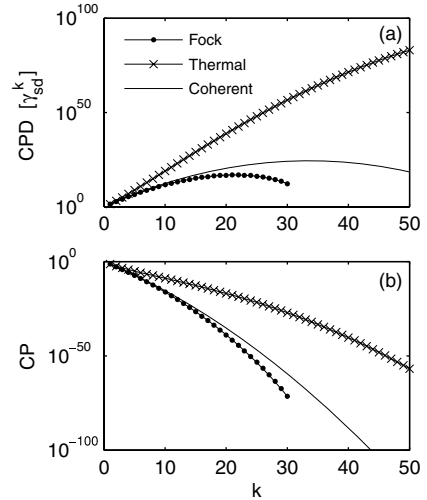


**Figure 2.** (a) The coincidence probability densities and (b) the coincidence probabilities (CP) given by the SD model for the Fock state, the thermal field and the coherent field. The measurement intervals are equally spaced so that $[t_i, t_i + d\tau), t_i = i\tau + (i-1)d\tau,$ $i = 1, \ldots, k$ with $\tau = 1/(10\gamma_{sd})$ and $d\tau = \tau/(2\bar{n}(0))$. The initial expectation value of the number of photons is $\bar{n}(0) = 30$.

From equations (10) to (13), it is seen that the $k$-photon CPD is proportional to $k$th factorial moment. For example, for the thermal field the $k$th factorial moment is $k!\bar{n}^k(0)$, which grows without a limit when $k$ grows. However, one must bear in mind that in the derivation of these CPDs it was assumed that the time intervals must be so short that only zero or one photon is counted at each time intervals, i.e., the probabilities of the one-count events are much less than unity. The condition $p(t_{k+1}|t_k, \ldots, t_1) = \gamma_{sd}(k+1)\bar{n}(0) e^{-\gamma_{sd}t_{k+1}} \ll 1$ for probability in equation (9) gives for the thermal field:

$$\gamma_{sd} dt_{k+1} \ll \frac{e^{+\gamma_{sd}t_{k+1}}}{(k+1)\bar{n}(0)}. \tag{17}$$

Condition (17) does not limit the model but guarantees that the CPs given by equation (10) for the three example fields are well defined and less than unity even though the CPD values may diverge for $k \to \infty$ (see figures 1(b) and 2(b)).

As a second example, we consider CPDs and CPs of observing $k$ photons during intervals $[t_i, t_i + d\tau)$ (where $t_i = i\tau + (i-1) d\tau, i = 1, \ldots, k$) which are equally spaced and have equal lengths. For comparison, we have taken the measurement intervals to be $d\tau = \tau/(2\bar{n}(0))$, which fulfils the condition in equation (17) for all the measurement intervals. Therefore, as seen in figure 2, the $k$-photon CPs are well defined for the thermal field even though the corresponding CPD grows rapidly and obtains very high values at large $k$. The condition given in equation (17) ensures also well-defined CPs for the coherent field and the Fock states (see figure 2), but also less stringent conditions can be obtained using a similar procedure.

### 2.5. Coincidence probabilities of counting at least one photon and counting exactly one photon

If the measurement intervals are not differentially small, the probability of detecting more than one photon during a single

measurement interval is not vanishingly small. In a realizable measurement the intervals are not necessarily differential. Therefore, we calculate the CPs of detecting at least one photon at each of the non-differential measurement intervals.

The operator describing counting of $m$ photons during a time $[0, t)$ is the operator $\hat{N}_t(m)$ in equation (4). It can be shown that [2]

$$\hat{N}_t(m)\hat{\rho}_f(0) = \sum_{n=m}^{\infty} \frac{n!}{m!(n-m)!}(1 - e^{-\gamma_{sd}t})^m$$

$$\times (e^{-\gamma_{sd}t})^{n-m} p_n(0)|n-m\rangle\langle n-m|, \quad (18)$$

where $p_n$ is the probability of state $|n\rangle$ in the mixture. The operator describing the average evolution is $\hat{T}_t$ given in equation (3) and the operator corresponding to counting at least one photon is given by $\hat{C}_t = \sum_{m=1}^{\infty} \hat{N}_t(m)$. In general, we define the counting of $m_0$ or more photons as $\sum_{m=m_0}^{\infty} \hat{N}_t(m)$ which gives for diagonal elements of the density operator:

$$\sum_{m=m_0}^{\infty} \hat{N}_t(m)\hat{\rho}_f(0) = \sum_{m=m_0}^{\infty} \sum_{n=0}^{\infty} \frac{(n+m)!}{m!n!}(1 - e^{-\gamma_{sd}t})^m$$

$$\times (e^{-\gamma_{sd}t})^n p_{n+m}(0)|n\rangle\langle n|. \quad (19)$$

Therefore, after operating with the operator $\sum_{m=m_0}^{\infty} \hat{N}_t(m)$, the probability of the $n$ photon Fock state is given by

$$p_n(t) = \sum_{m=m_0}^{\infty} \frac{(n+m)!}{m!n!}(1 - e^{-\gamma_{sd}t})^m (e^{-\gamma_{sd}t})^n p_{n+m}(0), \quad (20)$$

which must be normalized with $\sum_{n=0}^{\infty} p_n(t)$.

We define the measurement intervals as before by $[t_i, t_i + \Delta t_i), i = 1, \ldots, k$. The probability of detecting at least one photon at the first measurement interval is $\text{Tr}\{\hat{C}_{\Delta t_1}\hat{T}_{t_1}\hat{\rho}_f(0)\}$, and the density operator after the measurement is given by $\hat{\rho}_f(t_1 + \Delta t_1) = \hat{C}_{\Delta t_1}\hat{T}_{t_1}\hat{\rho}_f(0)/\text{Tr}\{\hat{C}_{\Delta t_1}\hat{T}_{t_1}\hat{\rho}_f(0)\}$. By replacing $\hat{J}$ with $\hat{C}_{\Delta t_i}$ we can now use equations (5)–(7) for the calculation of the CPs of counting one or more photons at each of the $k$ intervals $[t_i, t_i + \Delta t_i)$. Note for future reference that the operator corresponding to measurement of exactly one photon during a non-differential time interval $\Delta t$ is $\hat{N}_{\Delta t}(1)$.

In figure 3, we show a comparison of (a) CPs given by equation (7), (b) CPs obtained using the operator $\hat{N}_{\Delta t}(1)$, i.e., counting exactly one photon and (c) CPs of counting at least one photon. The $k$ measurement intervals are chosen so that $[t_i, t_i + \Delta\tau), t_i = i\tau + (i - 1)\Delta\tau, i = 1, \ldots, k$ with $\tau = 1/(5\gamma_{sd})$ and $\Delta\tau = \tau$. Note that now the conditions in equation (17) *is not fulfilled* so the CPs given by equation (7) are *not well defined* since the measurement intervals are not differential (see figure 3(a)). In contrast, the CPs obtained using operators $\hat{N}_{\Delta t}(1)$ and $\hat{C}_{\Delta t}$ are well defined (see figures 3(b) and (c)). These probabilities correspond to detecting exactly one and at least one photon, respectively, at each of the non-differential measurement intervals.

We have also tested using numerical calculations that for differential $\Delta t$ all the three counting operators ($\hat{J}$, $\hat{N}_{\Delta t}(1)$ and $\hat{C}_{\Delta t}$) give equal results. This is understandable since at a differential measurement interval only the one-count and the no-count event are possible, and the probability of the one-count event is small.
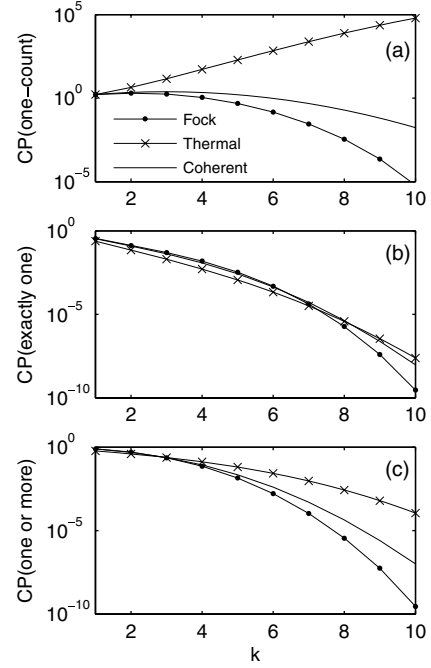


**Figure 3.** (a) The CPs of counting $k$ photons one at each measurement interval using the operator $\hat{J}$ (equation (7)), (b) the CPs of counting exactly one photon at each measurement interval using the operator $\hat{N}_t(1)$ and (c) the CPs of counting at least one photon at each measurement interval using the operator $\hat{C}_t$ given by the SD model for the Fock state, the thermal field and the coherent field. The measurement intervals are chosen so that $[t_i, t_i + \Delta\tau)$, $t_i = i\tau + (i - 1)\Delta\tau, i = 1, \ldots, k$ with $\tau = 1/(5\gamma_{sd})$ and $\Delta\tau = \tau$. The initial expectation value of the number of photons is $\bar{n}(0) = 10$.

## 3. Conclusions

We have derived the coincidence photon counting probabilities using the quantum trajectory theory. The quantum trajectory theory gives well-defined conditional counting probabilities and, therefore, the CP obtained as product of the conditional probabilities is also well defined and correctly normalized. In particular, we have shown that even if the CPDs grow without a limit the CPs are well defined and normalized if (1) the measurement durations are chosen to be so short that only the no-count and one-count trajectories are possible during a single counting interval, or (2) operators that include also the other trajectories are used.

We have also shown how to define the CPs corresponding to counting exactly one photon (resolving detector) and at least one photon (nonresolving detector) during a non-differential measurement intervals. Again the CPs are well-defined and normalized probabilities.

The comparison of the waiting times based on the photon counting theory given by the SD model and the E model for the cavity fields initially in the Fock state, the thermal field and the coherent field is included for completeness in appendix A.2. We point out that the waiting times given by the SD model reproduce the photon bunching, non-bunching and antibunching phenomena. In appendix A.1 we have also given a consistent derivation of second-order coherence degrees. The results given by the SD model also reproduce the photon

**Table A1.** The second factorial moments $\overline{(n(n-1)}(0))$ of the fields and the second-order coherence degrees, $(g^{(2)}_{sd}(t_1, t_2))$, given by the SD model and $(g^{(2)}_e(0,0))$ given by the E model. The Fock state has initially $N \geqslant 2$ photons. Note that $g^{(2)}_{sd}$ is independent of time (see equation (A.1)).

| Initial state | $\overline{n(n-1)}$ | $g^{(2)}_{sd}$ | $g^{(2)}_e(0,0)$ |
|---|---|---|---|
| Fock | $N(N-1)$ | $1 - 1/N$ | $1$ |
| Thermal | $2\bar{n}^2(0)$ | $2$ | $1$ |
| Coherent | $\bar{n}^2(0)$ | $1$ | $\dfrac{e^{\bar{n}(0)} - (\bar{n}(0)+1)}{e^{\bar{n}(0)} + e^{-\bar{n}(0)} - 2}$ |

bunching, non-bunching and antibunching phenomena while those of the E model do not.

The main result of the paper is the operators defined in section 2.5. These operators give the necessary tools to analyse measurements done using resolving and nonresolving detectors also in non-differential measurement intervals.

## Appendix A. Correlation of photons

### A.1. Second-order coherence degrees

The relation of the second-order coherence degree $g^{(2)}(t, t+\tau)$ to photon bunching and antibunching phenomena is the following: if $0 \leqslant g^{(2)}(t, t) < 1$ the light is antibunched, if $g^{(2)}(t, t) = 1$ the light is non-bunched or random, and if $g^{(2)}(t, t) > 1$ the light is bunched [10]. Since a false formula was used in [1] we give the derivation of the second-order coherence degree formula in appendix B.3. By using equations (10) and (B.13) we obtain for the SD model

$$g^{(2)}_{sd}(t_1, t_2) = \frac{\overline{n(n-1)}(0)}{\bar{n}^2(0)}. \quad (A.1)$$

Equation (A.1) states that for the single-mode field the second-order coherence degree predicted by the SD model is governed by the initial field photon statistics and it is independent of time. The general expressions of $g^{(2)}_e(t_1, t_2)$ are time dependent and complicated; the reader can elaborate them from equations (14) to (16) and (B.13). To facilitate the comparison of the SD and E models we, therefore, compare $g^{(2)}_{sd}$ and $g^{(2)}_e$ by taking $t_1 = t_2 = 0$. From table A1 we can conclude that according to the SD model photons in the single-mode Fock state are antibunched, in the thermal field photons are bunched, and in the coherent field photons are non-bunched.

Note that the definition for the second-order coherence degree given in [1] $g^{(2)}_{Dod}(t_1, t_2) = \text{Trace}\{\hat{J}\hat{T}_{t_2-t_1}\hat{J}\hat{T}_{t_1}\hat{\rho}_f\}/\text{Trace}\{\hat{J}\hat{T}_{t_1}\hat{\rho}_f\}^2$ is incorrect as shown in appendix B.3. This definition gives for the SD model $p(t_1, t_2)/p^2(t_1) = \overline{n(n-1)}(0)/\bar{n}^2(0)\,e^{-\gamma_{sd}(t_2-t_1)}$. Thus it is incorrectly concluded in [1] that the SD model always gives the photon bunching phenomenon. The correct definition is (see appendix B.3) $g^{(2)}(t_1, t_2) = \text{Trace}\{\hat{J}\hat{T}_{t_2-t_1}\hat{J}\hat{T}_{t_1}\hat{\rho}_f\}/(\text{Trace}\{\hat{J}\hat{T}_{t_1}\hat{\rho}_f\}\,\text{Trace}\{\hat{J}\hat{T}_{t_2}\hat{\rho}_f\})$.

### A.2. Waiting times

The photon correlation can also be considered by comparing the waiting time of next one-count event and the time interval

between the one-count events. The waiting time $\langle t \rangle_W$ is the time from an arbitrary starting point to the next one-count event while the time interval $\langle t \rangle_I$ is the time span between two consecutive one-count events. The waiting times were previously calculated for the SD model by Lee [11]. Lee [11] obtained the following results for the SD model: (1) if the field is initially in the coherent state, $\langle t \rangle_W = \langle t \rangle_I$ and the photons are non-bunched or random. (2) If the field is initially in the thermal state, $\langle t \rangle_W > \langle t \rangle_I$ and the photons are bunched. (3) If the field is initially in the Fock state, $\langle t \rangle_W < \langle t \rangle_I$ and the photons are antibunched. These results agree with our calculations of photon correlations in appendix A.1.

Dodonov *et al* [1] showed that the E model gives, in contrast to the SD model, $\langle t \rangle_W = \langle t \rangle_I$ for all initial fields. Therefore, the E model cannot reproduce the cavity photon bunching and antibunching phenomena in the one-count event waiting times. We expect this to be a consequence of the inherent saturation of the experimental detector setup the E model is based on (see [8]).

## Appendix B. Derivations

### B.1. One-count and no-count operators

The one-count operator is $\hat{J}_A\hat{\rho}_f(t) = \gamma_A\hat{A}\hat{\rho}_f(t)\hat{A}^\dagger$ and the no-count operator is $\hat{S}_\tau\hat{\rho}_f = e^{\hat{Y}_A\tau/\hbar}\hat{\rho}_f(t)e^{\hat{Y}^\dagger_A\tau/\hbar}$, where $\hat{Y}_A = -i\hbar\omega\hat{a}^\dagger\hat{a} - \frac{1}{2}\hbar\gamma_A\hat{A}^\dagger\hat{A}$. Furthermore, in the SD model $\hat{A} \equiv \hat{a}$ and in the E model $\hat{A} \equiv (\hat{a}^\dagger\hat{a}+1)^{-1/2}\hat{a}$. The no-count and one-count operations for the SD and E models give [1–3]

$$\hat{S}^{sd}_\tau\hat{\rho}(t) = \sum_{n,n'=0}^{\infty} e^{-i\omega_0(n-n')\tau - \gamma_{sd}\frac{n+n'}{2}\tau} p_{n,n'}|n\rangle\langle n'|, \quad (B.1)$$

$$\hat{J}^{sd}\hat{\rho}(t) = \gamma_{sd}\hat{a}^\dagger\hat{\rho}(t)\hat{a} = \gamma_{sd}\sum_{n,n'=0}^{\infty} \sqrt{nn'}\, p_{n,n'}|n-1\rangle\langle n'-1|, \quad (B.2)$$

$$\hat{S}^e_\tau\hat{\rho} = p_{0,0}|0\rangle\langle 0| + \sum_{n=1}^{\infty}\left[ p_{0,n}|0\rangle\langle n|\, e^{+i\omega_0 n\tau - \frac{1}{2}\gamma\tau}\right.$$
$$\left. + p_{n,0}|n\rangle\langle 0|\, e^{-i\omega_0 n\tau - \frac{1}{2}\gamma\tau}\right]$$
$$+ \sum_{n,n'=1}^{\infty} p_{n,n'}\, e^{-i\omega_0(n-n')\tau - \gamma\tau}|n\rangle\langle n'|, \quad (B.3)$$

$$\hat{J}^e\hat{\rho}(t) = \gamma_e\hat{E}^\dagger\hat{\rho}(t)\hat{E} = \gamma_e\sum_{n,n'=1}^{\infty} p_{n,n'}|n-1\rangle\langle n'-1|. \quad (B.4)$$

### B.2. Evolution of factorial moments in the SD model

The density matrix evolves according to [1–3]

$$\frac{d\hat{\rho}}{dt} = -i\omega(\hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a})$$
$$+ \left(\gamma_A\hat{A}\hat{\rho}\hat{A}^\dagger - \frac{\gamma_A}{2}(\hat{A}^\dagger\hat{A}\hat{\rho} + \hat{\rho}\hat{A}^\dagger\hat{A})\right). \quad (B.5)$$

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The probabilities of *n*-photon states are given by the diagonal elements $\langle n | \cdot | n \rangle$. Thus we obtain for the SD model

$$\frac{\mathrm{d}p_n(t)}{\mathrm{d}t} = \gamma_{\mathrm{sd}}((n+1)p_{n+1}(t) - p_n(t)n) \tag{B.6}$$

and for the E model

$$\frac{\mathrm{d}p_{n\geqslant 1}(t)}{\mathrm{d}t} = \gamma_e(p_{n+1}(t) - p_n(t)), \tag{B.7}$$

$$\frac{\mathrm{d}p_0(t)}{\mathrm{d}t} = \gamma_e p_1(t). \tag{B.8}$$

The *k*th factorial moment is defined as $\overline{n(n-1)\cdots(n-(k-1))} = \sum_{n=0}^{\infty} n(n-1)\cdots(n-(k-1))p_n$. Thus, using the master equation (B.6) for photon number, we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{n(n-1)\cdots(n-m)}(t) = \sum_{n=0}^{\infty} n(n-1)\cdots(n-m)\frac{\mathrm{d}p_n(t)}{\mathrm{d}t}$$

$$= \gamma_{\mathrm{sd}}\sum_{n=0}^{\infty} n(n-1)\cdots(n-m)\left[(n+1)p_{n+1}(t) - np_n(t)\right]$$

$$= \gamma_{\mathrm{sd}}\left(\sum_{n=0}^{\infty} n(n-1)\cdots(n-m)(n+1)p_{n+1}(t) - \sum_{n=0}^{\infty} n(n-1)\cdots(n-m)np_n(t)\right)$$

$$= \gamma_{\mathrm{sd}}\left(\sum_{n=1}^{\infty} (n-1)(n-2)\cdots(n-m-1)np_n(t) - \sum_{n=1}^{\infty} n(n-1)\cdots(n-m)np_n(t)\right)$$

$$= \gamma_{\mathrm{sd}}\sum_{n=1}^{\infty} (n(n-1)\cdots(n-m))(n-m-1-n)p_n(t)$$

$$= -\gamma_{\mathrm{sd}}(m+1)\sum_{n=1}^{\infty} n(n-1)\cdots(n-m)p_n(t)$$

$$= -\gamma_{\mathrm{sd}}(m+1)\overline{n(n-1)\cdots(n-m)}(t),$$

which gives for the *k*th factorial moment,

$$\overline{n(n-1)\cdots(n-(k-1))}(t)$$
$$= \mathrm{e}^{-k\gamma_{\mathrm{sd}}t}\overline{n(n-1)\cdots(n-(k-1))}(0). \tag{B.9}$$

The probability of *n*-photon state after the one-count event is (see appendix B.1) $p_n(t^+) = (n+1)p_{n+1}(t)/\bar{n}(t)$. Thus we can find the following relation between the factorial moments before and after the one-count event:

$$= \overline{n(n-1)\cdots(n-m)}(t^+)$$

$$= \sum_{n=0}^{\infty} n(n-1)\cdots(n-m)p_n(t^+)$$

$$= \frac{1}{\bar{n}(t)}\sum_{n=0}^{\infty} n(n-1)\cdots(n-m)(n+1)p_{n+1}(t)$$

$$= \frac{1}{\bar{n}(t)}\sum_{n=1}^{\infty} (n-1)(n-2)\cdots(n-m-1)np_n(t)$$

$$= \frac{1}{\bar{n}(t)}\sum_{n=1}^{\infty} n(n-1)(n-2)\cdots(n-m)(n-m-1)p_n(t)$$

$$= \frac{\overline{n(n-1)(n-2)\cdots(n-m)(n-m-1)}(t)}{\bar{n}(t)}. \tag{B.10}$$

Equation (B.9) describes the average evolution of factorial moments whereas equation (B.10) describes how the factorial moments immediately after the one-count event are related to those before the event.

### B.3. Second-order coherence degree

The second-order coherence degree is [10, 12]

$$g^{(2)}(\mathbf{r}_1, t_1, \mathbf{r}_2, t_2, \mathbf{r}_2, t_2, \mathbf{r}_1, t_1)$$
$$= \frac{G^{(2)}(\mathbf{r}_1, t_1, \mathbf{r}_2, t_2, \mathbf{r}_2, t_2, \mathbf{r}_1, t_1)}{G^{(1)}(\mathbf{r}_1, t_1, \mathbf{r}_1, t_1)G^{(1)}(\mathbf{r}_2, t_2, \mathbf{r}_2, t_2)}, \tag{B.11}$$

where

$$G^{(2)}(\mathbf{r}_1, t_1, \mathbf{r}_2, t_2, \mathbf{r}_2, t_2, \mathbf{r}_1, t_1)$$
$$= \mathrm{Tr}\{\hat{\rho}_f \hat{\mathbf{E}}^{(-)}(\mathbf{r}_1, t_1)\hat{\mathbf{E}}^{(-)}(\mathbf{r}_2, t_2)\hat{\mathbf{E}}^{(+)}(\mathbf{r}_2, t_2)\hat{\mathbf{E}}^{(+)}(\mathbf{r}_1, t_1)\},$$

with $\hat{\mathbf{E}}^{(-)}(\mathbf{r}, t)$ and $\hat{\mathbf{E}}^{(+)}(\mathbf{r}, t)$ being the negative and positive frequency parts of the electric field operator. The two-fold delayed coincidence rate, i.e., the counting rate per unit time squared is given by [12]

$$f(\mathbf{r}_1, t_1, \mathbf{r}_2, t_2, \mathbf{r}_2, t_2, \mathbf{r}_1, t_1)$$
$$= s^2 G^{(2)}(\mathbf{r}_1, t_1, \mathbf{r}_2, t_2, \mathbf{r}_2, t_2, \mathbf{r}_1, t_1), \tag{B.12}$$

where *s* is the sensitivity of the detector. We consider only the temporal correlation so we assume that all of the position vectors are equal and drop the spatial coordinate. We can now use the well-known formula of conditional probability: the probability that an event *B* occurs with the condition that *A* has happened is $p(B|A) = p(B \cap A)/p(A)$ (see, for example, [13]). Thus $p(B \cap A) = p(B|A)p(A)$ giving $f(t_1, t_2)(\mathrm{d}t)^2 = f(t_2|t_1)\,\mathrm{d}t f(t_1)\,\mathrm{d}t$. Furthermore, we can write the second-order coherence degree using the count rates ($p(t) = f(t)\,\mathrm{d}t$ and $p(t_1, t_2) = f(t_1, t_2)\,\mathrm{d}t_2\,\mathrm{d}t_2$):

$$g^{(2)}(t_1, t_2) = \frac{p(t_1, t_2)}{p(t_1)p(t_2)} = \frac{f(t_1, t_2)}{f(t_1)f(t_2)}$$
$$= \frac{\mathrm{Tr}\{\hat{J}\hat{T}_{t_2-t_1}\hat{J}\hat{T}_{t_1}\hat{\rho}_f\}}{\mathrm{Tr}\{\hat{J}\hat{T}_{t_1}\hat{\rho}_f\}\mathrm{Tr}\{\hat{J}\hat{T}_{t_2}\hat{\rho}_f\}}. \tag{B.13}$$

Note that using the conditional probabilities we can also write $g^{(2)}(t_1, t_2) = p(t_2|t_1)/p(t_2)$. Therefore, we can use the CP formula in equation (9) or the CPDs in equations (11)–(13) and (14)–(16) to calculate the second-order coherence degree.

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