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Unified quantum jump superoperator for optical fields from the weak- to the strong-coupling limit

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We derive a generalized quantum jump superoperator that can be used in the quantum trajectory description of single photon detectors, light-emitting diodes (LEDs), and lasers. Our model describes an optical single-mode cavity field coupled to a reservoir through a two-state quantum system and includes three physical parameters: the coupling of the field to the two-state system, the coupling of the two-state system to the reservoir, and the cavity loss rate. In this setup, the two-state system can act as a photodetector or as an energy-adding mechanism. In the first case, we assume that the reservoir acts as a damping mechanism for an ideal cavity and derive reduced field operators describing the photon detection events. Our model coincides with the commonly known quantum trajectory based photon counting models at the weak- and strong-coupling limits and is, furthermore, also applicable between their validity regimes. In the second case, we assume that the reservoir injects energy into a lossy cavity through the two-state system. Again we derive the reduced field operators describing photon creation events into the lossy cavity. We show that this setup can act as an LED or as a laser depending on the strength of the injection. We also investigate how the setup operates at the close proximity of the lasing threshold.

I. INTRODUCTION

Recent progress in experimental quantum optics has enabled more detailed quantum optical measurements and experiments that have given new insight into photon counting and photon detection. Several recent experiments have focused on photon creation and destruction [1,2], on the statistics of the collapsed field [3], and on the photon correlation properties [4–6] of the generated photon fields. All these experiments make use of very high quality photodetector schemes [7–9]. Detailed theoretical analysis of these experiments usually requires numerically solving the equation of motion of the full density operator of the optical field interacting with the detector. The introduction of reduced quantum jump superoperators (QJSs) for the optical field in the limiting cases of weak [10,11] and strong [11,12] coupling by tracing over the detector states has given more detailed insight into the properties of the optical field.

In this work we derive the generalized field evolution operators for all coupling parameter regimes by considering the commonly used detector setup [11], where a two-state quantum system is coupled to a cavity field and an absorbing reservoir.

We also derive the operators for a setup where a two-state quantum system is coupled to a cavity field and an energy source. We apply our model to describe a light-emitting diode (LED) and a laser activity of a cavity and show that the reduced operators can reproduce a thermal field or a coherent field depending on the strength of the coupling parameters and losses.

II. DETECTOR MODEL

We derive the generalized field evolution operators for a dissipative system by considering the detector setup where an

optical field (f) is coupled to a two-state quantum system (s) with a ground state $|g\rangle$ and an excited state $|e\rangle$. The reservoir (R) acts as a dissipative energy drain. The detection event in this setup is the transfer of an energy quanta from s to the reservoir, which can be implemented, e.g., by injecting a beam of atoms initially in the ground state through a cavity or with a setup where an atom in a cavity is coupled to the cavity mode and to a reservoir mode.

Evolution of the combined f-s density operator interacting with the reservoir is governed by the Lindblad equation with coupling parameter γ describing the coupling of the field to the two-state system and coupling parameter λ describing the coupling of the two-state system to the reservoir. The Lindblad master equation for this setup is [12]

$$\frac{d\hat{\rho}_s(t)}{dt} = -\frac{i}{\hbar}[\hat{H}\hat{\rho}_s(t) - \hat{\rho}_s(t)\hat{H}^\dagger] + 2\lambda\sigma_-\hat{\rho}_s(t)\sigma_+, \quad (1)$$

where the density operator $\hat{\rho}_s(t)$ describes both the field and the two-state quantum system. In Eq. (1) the system Hamiltonian $\hat{H}$ is the standard Jaynes-Cummings Hamiltonian of a two-state system with eigenstates $\pm\hbar\omega_0/2$ coupled to the photon mode with $E = \hbar\omega$ with an additional dissipative term $-i\hbar\lambda\sigma_+\sigma_-$, given by

$$\hat{H} = \tfrac{1}{2}\hbar\omega_0\sigma_0 + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar\gamma(\hat{a}\sigma_+ + \hat{a}^\dagger\sigma_-) - i\hbar\lambda\sigma_+\sigma_-, \quad (2)$$

where $\sigma_+ = |e\rangle\langle g|$, $\sigma_- = |g\rangle\langle e|$, $\sigma_0 = |e\rangle\langle e| - |g\rangle\langle g|$, and furthermore, we assume exact resonance $\omega = \omega_0$ in the following. Equation (1) is the Lindblad master equation, which is known as the most general form of a memoryless coupling between a system and an (infinite) reservoir [13]. Master equation (1) implies a linear coupling between the system and the reservoir but does not otherwise limit the strength of the interaction as long as the state of the reservoir remains unchanged. In this formalism the events recorded by the detector are the emissions of energy quanta by the two-state

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system into the reservoir which corresponds to the collapse of $\hat{\rho}_s(t)$ to the ground state.

Assuming that the full setup at time t is described by $\hat{\rho}_s(t)$, its evolution in an infinitesimal time dt may be described by a decomposition of the form

$$\hat{\rho}_s(t + dt) = \hat{J}_s \hat{\rho}_s(t) dt + \hat{S}_s(dt) \hat{\rho}_s(t), \tag{3}$$

where the two terms describe the quantum trajectories of the f-s system. These trajectories describe the events where the emission of an energy quantum is either detected $\hat{J}_s \hat{\rho}_s(t) dt$ or not detected $\hat{S}_s(dt) \hat{\rho}_s(t)$ during $[t, t + dt]$. The one-count and no-count operators in this formalism are

$$\hat{J}_s \hat{\rho}_s(t) = 2\lambda \hat{\sigma}_- \hat{\rho}_s(t) \hat{\sigma}_+, \tag{4}$$

$$\hat{S}_s(dt) \hat{\rho}_s(t) = \hat{U}(dt) \hat{\rho}_s(t) \hat{U}^\dagger(dt), \tag{5}$$

where the time evolution operator $\hat{U}(t)$ is defined as $\hat{U}(t) = \exp(-i\hat{H}t/\hbar)$ using the dissipative Hamiltonian in Eq. (2). If we detect the one-count event (i.e., the emission of an energy quantum into the reservoir) the system is projected into state $\hat{\rho}_s(t + dt) = \hat{J}_s \hat{\rho}_s(t)/\text{Tr}\{\hat{J}_s \hat{\rho}_s(t)\}$. Similarly, after an observed no-count event, the system collapses into state $\hat{\rho}_s(t + dt) = \hat{S}_s(dt) \hat{\rho}_s(t)/\text{Tr}\{\hat{S}_s(dt) \hat{\rho}_s(t)\}$.

We have previously shown [11,14] that in the (weak coupling) limit $\lambda \gg \gamma \sqrt{\bar{n}}$, the photon count rate is proportional to the mean number of photons $\bar{n}$, and in the (strong coupling) limit $\lambda \ll \gamma \sqrt{\bar{n}}$, the count rate is proportional to the probability that photons exist, i.e.,

$$r_{\lambda \gg \gamma \sqrt{\bar{n}}} = (2\gamma^2/\lambda)\bar{n}, \tag{6}$$

$$r_{\lambda \ll \gamma \sqrt{\bar{n}}} = \lambda(1 - p_0), \tag{7}$$

where p_0 is the probability of the vacuum state. In the first case, the absorption rate of the detector is not limited by the photon statistics (as long as the condition above is met), and the model in Ref. [10] [the Srinivas-Davies (SD) model] is reproduced. In the second case, the detector is saturated and absorbs photons with maximum rate of λ, and the model in Ref. [15] [the model based on $\hat{E}$ operators, (E) model] is reproduced. Between these regimes, no previously known reduced photon counting models that correctly describe the average time evolution of the field have been introduced. In the next section we will derive a general reduced operator that describes the average evolution of the field.

A. General reduced operator

Next we will identify a new quantum jump superoperator that reproduces the average time-dependent photon count rate and the photon statistics for all coupling strengths. We calculate $\hat{U}(t)$ following the approach given in Ref. [16]. The probability of the first one-count to occur during $[t, t + dt]$ for the system initially in the ground state is dt times the probability density given by the trace of the operator [11]

$$\hat{J}\hat{S}(t)\hat{\rho} = 8\lambda e^{-\lambda t} \sum_{n=0}^{\infty} (n + 1) p_{n+1} |n\rangle\langle n|$$

$$\times \frac{\left| \sin\left[\frac{\gamma t}{2}\sqrt{4(n + 1) - (\lambda/\gamma)^2}\right]\right|^2}{|4(n + 1) - (\lambda/\gamma)^2|}. \tag{8}$$

The probability of the one-count, $\text{Tr}\{\hat{J}\hat{S}(t)\hat{\rho}\}dt$, decays exponentially due to the longer no-count event preceding the one-count, and also it contains an oscillating term due to the interaction of the field with the two-state system.

The reduced quantum jump superoperator is formally defined as the time average $\langle \text{Tr}_s\{\hat{J}_s\hat{S}(t)\hat{\rho}_s\}/\text{Tr}_{s,f}\{\hat{S}_s(t)\hat{\rho}_s\}\rangle$ [11]. This average has an accessible analytic value only in the limit of weak and strong coupling. A good approximation of the reduced quantum jump superoperator can, however, be obtained by considering the following properties of Eq. (8).

Equation (8) states that starting from the initial state $|g,n\rangle$ $(n > 0)$ the probability density for the time of the first one-count event taking place is

$$f_n^D(t) = 8\lambda e^{-\lambda t} n \frac{\left| \sin\left[\frac{\gamma t}{2}\sqrt{4n - (\lambda/\gamma)^2}\right]\right|^2}{|4n - (\lambda/\gamma)^2|}. \tag{9}$$

The probability density is 0 if the initial state is $|g,0\rangle$ and $\int_0^\infty f_n^D(t)dt = 1$ for $n > 0$. The mean value for the time when the first one-count takes place is then

$$\bar{t}_n^D = \int_{t=0}^{\infty} t f_n^D(t)dt = \frac{2\left(\frac{\gamma}{\lambda}\right)^2 n + 1}{2\frac{\gamma^2}{\lambda}n}. \tag{10}$$

The average count rate at which the one-count events occur is the inverse of time $\bar{t}_n^D$. Thus, the average one-count rate from state $|g,n\rangle$ is

$$\bar{r}_n^D = \frac{2\frac{\gamma^2}{\lambda}n}{1 + 2\left(\frac{\gamma}{\lambda}\right)^2 n}. \tag{11}$$

The total average count rate is obtained as a sum of the rates $\bar{r}_n^D$ weighted with the probabilities p_n of the initial states $|n\rangle$

$$\bar{r}_{\text{tot}}^D = \sum_{n=0}^{\infty} \frac{2\frac{\gamma^2}{\lambda}n}{1 + 2\left(\frac{\gamma}{\lambda}\right)^2 n} p_n. \tag{12}$$

Comparing the rate obtained from Eq. (4) $[r(t) = 2\lambda p_e(t)]$ with Eq. (12) shows that the average excited-state probability and the average ground-state probability needed to reproduce Eq. (12) are

$$\bar{p}_e = \sum_{n=0}^{\infty} \frac{\left(\frac{\gamma}{\lambda}\right)^2 n}{1 + 2\left(\frac{\gamma}{\lambda}\right)^2 n} p_n, \tag{13}$$

$$\bar{p}_g = 1 - \bar{p}_e = \sum_{n=0}^{\infty} \frac{1 + \left(\frac{\gamma}{\lambda}\right)^2 n}{1 + 2\left(\frac{\gamma}{\lambda}\right)^2 n} p_n. \tag{14}$$

Note that the rates and probabilities in Eqs. (8)–(14) hold for all coupling strengths and therefore allow us to deduce the form of the general quantum jump superoperator.

Let us now define a new QJS for the reduced system as $\hat{J}\hat{\rho} = A\hat{O}\hat{\rho}\hat{O}^\dagger$ with operator $\hat{O}$ operating only on the field states being defined as (see also the Appendix)

$$\hat{O} = (1 + B\hat{a}\hat{a}^\dagger)^{-1/2}\hat{a}. \tag{15}$$

Using it to evaluate the photon count rate from state $\hat{\rho}$ results in

$$\text{Tr}\{\hat{J}\hat{\rho}\} = \sum_{n=1}^{\infty} \frac{An}{1 + Bn} p_n, \tag{16}$$

$$= \begin{cases} A\bar{n}, & \text{if } B\bar{n} \ll 1, \\ \frac{A}{B}(1 - p_0), & \text{if } B\bar{n} \gg 1. \end{cases} \quad (17)$$

By setting $A = 2\gamma^2/\lambda$ and $B = 2\gamma^2/\lambda^2$ we obtain the same photon counting rates as in Eqs. (6), (7), and (12) and see that operator $\hat{J}\hat{\rho} = A\hat{O}\hat{\rho}\hat{O}^\dagger$ predicts the same count rates as previously known SD and E models at the weak and strong coupling and is additionally able to produce the average rate given by Eq. (12).

The Lindblad master equation (1) can now be reduced to describe only the photon field by using operator $\hat{O}$. The reduced master equation is

$$\frac{d\hat{\rho}(t)}{dt} = -\frac{A}{2}(\hat{O}^\dagger\hat{O}\hat{\rho} - 2\hat{O}\hat{\rho}\hat{O}^\dagger + \hat{\rho}\hat{O}^\dagger\hat{O})$$

$$= -\frac{A}{2}\sum_{n,n'=0}^{\infty}\left[\left(\frac{n}{1+Bn} + \frac{n'}{1+Bn'}\right)p_{n,n'}|n\rangle\langle n'|\right.$$

$$\left. - 2\sqrt{\frac{nn'}{(1+Bn)(1+Bn')}}p_{n,n'}|n-1\rangle\langle n'-1|\right],$$

$$(18)$$

and it can be easily used to calculate, e.g., the photon number master equations in the form

$$\frac{dp_n}{dt} = -\frac{An}{1+Bn}p_n + \frac{A(n+1)}{1+B(n+1)}p_{n+1}. \quad (19)$$

Note that our reduced model is applicable at all regimes of coupling parameters λ and γ. At the limit $\lambda \gg \gamma\sqrt{\bar{n}}$ also the adiabatic approximation [17,18] can be used to eliminate the atomic states from the density operator. The adiabatic approximation of Eq. (1) reproduces the SD model and, therefore, agrees also with our model, since at this regime our model coincides with the SD model.

B. Comparison of the full system and reduced models

To quantify the differences between the previous reduced photon field models and the generalized model, we compare the f-s-R setup and the reduced photon field models by numerical calculations of the count rates and expectation values of the number of photons. Figure 1(a) shows the count rates for a field initially in the Fock state $|25\rangle$ with $\lambda/\gamma = 5$ as a function of time for the three reduced operators and the full setup. Figure 1(b) shows the mean photon numbers for the same models. Figure 2 shows the same data as Fig. 1 but for a field initially in the thermal state with $\bar{n} = 1$ and $\lambda/\gamma = 2$. Note that the initial states have been selected so that the conditions in Eqs. (6) and (7) are not met and neither the SD nor the E model is accurate. Figure 1 shows that our model accurately reproduces the average rate and the mean number of photons. In Fig. 2, due to the Rabi-like oscillation, the rate has a strong peak which decays fast as the mean photon number decays. Our reduced model naturally cannot reproduce the Rabi oscillations, but still our model is able to reproduce the average behavior of the rate and the mean number of photons. In Fig. 3, we show a comparison of the ground-state and the excited-state probabilities of the two-state system calculated (i) numerically from the full setup and (ii)
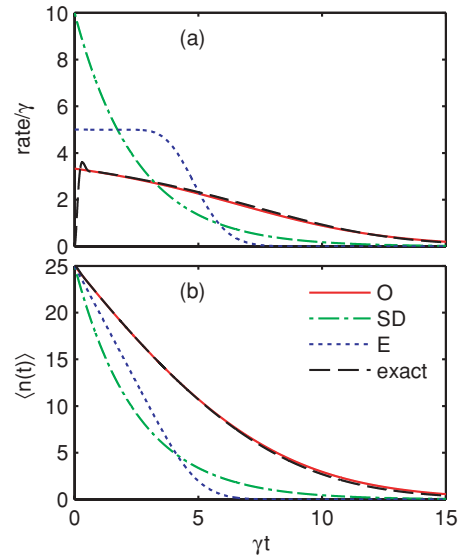

FIG. 1. (Color online) (a) Photon counting rates and (b) expectation values of the number of photons for a setup with $\lambda/\gamma = 5$ and the field initially in the Fock state $|25\rangle$. The coupling strength in the figures are between the strong- and weak-coupling regimes where neither the SD nor the E model are accurate. As a result they are unable to correctly predict the average dissipation rate and the photon number in the cavity. In contrast, our model accurately reproduces these average quantities.

using the averaged formulas (13) and (14). We see that our averaged formulas follow smoothly the average behavior of the oscillating probabilities.
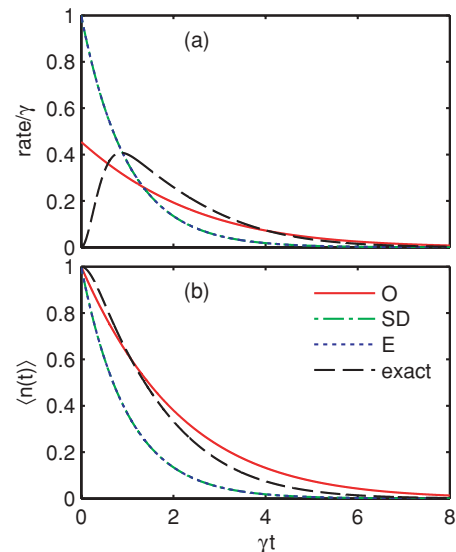

FIG. 2. (Color online) (a) Photon counting rates and (b) expectation values of the number of photons for a setup with $\lambda/\gamma = 2$ and the field initially in the thermal state with $\bar{n}(0) = 1$. Note that in this special case, the SD and the E models coincide. Again, the coupling strength in the figures has been chosen in the region where neither the SD nor the E model applies. However, our model is able to predict qualitatively the field evolution, although its accuracy is slightly reduced due to the relatively strong peak in the rate caused by the Rabi-type oscillation combined with the fast decay rate.
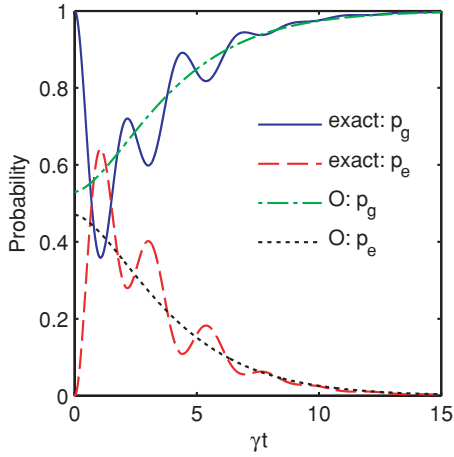
FIG. 3. (Color online) Comparison of the excited-state probabilities and the ground-state probabilities calculated with the full model (exact) and using Eqs. (13) and (14) for a system with $\lambda/\gamma = 0.5$ and a field initially in the Fock state $|2\rangle$.

In summary, Figs. 1–3 show that our reduced detector model is able to reproduce the average of the count rate and the mean number of photons in the cavity.

III. PHOTON CREATION

Let us now modify the setup introduced in the last section so that the two-state system is coupled to an amplifying reservoir. The system Hamiltonian is now of the form

$$\hat{H} = \tfrac{1}{2}\hbar\omega_0\sigma_0 + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar\gamma(\hat{a}\sigma_+ + \hat{a}^\dagger\sigma_-) - i\hbar\lambda\sigma_-\sigma_+, \quad (20)$$

and after the one-count event $[\hat{J}_s\hat{\rho}_s(t) = 2\lambda\sigma_+\hat{\rho}_s(t)\sigma_-]$, which is now the energy injection, the two-state system has collapsed to the excited state. Following the same procedure that previously led to Eq. (8), we obtain the probability density

$$f_n^A(t) = 8\lambda e^{-\lambda t}(n+1)\frac{\left|\sin\left[\frac{\gamma t}{2}\sqrt{4(n+1) - (\lambda/\gamma)^2}\right]\right|^2}{|4(n+1) - (\lambda/\gamma)^2|} \quad (21)$$

for the first quantum jump event, and similarly obtain the average emission rate as

$$\bar{r}_{\text{tot}}^A = \sum_{n=0}^\infty \frac{2\frac{\gamma^2}{\lambda}(n+1)}{1 + 2\left(\frac{\gamma}{\lambda}\right)^2(n+1)}p_n. \quad (22)$$

The amplification rate of the full f-s system is $r(t) = 2\lambda p_g(t)$. Comparing this rate with Eq. (22) shows that the average ground-state probability and the average excited-state probability reproducing Eq. (22) are

$$\bar{p}_g = \sum_{n=0}^\infty \frac{\left(\frac{\gamma}{\lambda}\right)^2(n+1)}{1 + 2\left(\frac{\gamma}{\lambda}\right)^2(n+1)}p_n \quad (23)$$

$$\bar{p}_e = 1 - \bar{p}_g = \sum_{n=0}^\infty \frac{1 + \left(\frac{\gamma}{\lambda}\right)^2(n+1)}{1 + 2\left(\frac{\gamma}{\lambda}\right)^2(n+1)}p_n. \quad (24)$$

Using the operator $\hat{O}$, defined in Eq. (15), and operating only on the field states of the reduced system, the rate can be written as

$$\text{Tr}\{\hat{J}\hat{\rho}\} = \sum_{n=0}^\infty \frac{A(n+1)}{1 + B(n+1)}p_n, \quad (25)$$

$$= \begin{cases} A(\bar{n} + 1), & \text{if } B\bar{n} \ll 1, \\ \frac{A}{B}, & \text{if } B\bar{n} \gg 1. \end{cases} \quad (26)$$

Comparison of Eqs. (22) and (25) shows again that $A = 2\gamma^2/\lambda$ and $B = 2\gamma^2/\lambda^2$, and we find that the same $\hat{O}$ operator applies for both the injection and the dissipation of energy to or from the system.

A. Amplifying setup with dissipation

The $\hat{O}$ operator introduced in the previous sections fills the gap between the previously known SD and E models and allows modeling of optical components also in the regimes where the electronic states interacting with the optical fields are not fully saturated or unsaturated. As an example of an interesting application of the $\hat{O}$ operator, we will next consider a setup where energy is injected from a reservoir to the two-state system interacting with an optical field confined in a nonideal optical cavity. As we will show, this setup can be used to model active optical components such as LEDs and lasers.

In our setup the field interacts with an amplifying medium through the jump term $A\hat{O}^\dagger\hat{\rho}\hat{O}$, and the cavity losses are taken into account through a jump term $C\hat{a}\hat{\rho}\hat{a}^\dagger$ which is linear with respect to the photon number. If only the mirror losses of the cavity are taken into account, then $C = \omega/Q$ [19], where ω is the frequency and Q is the quality factor of the cavity. The (reduced) Lindblad equation governing the evolution of the density operator of the optical field is

$$\frac{d\hat{\rho}(t)}{dt} = -\frac{A}{2}(\hat{O}\hat{O}^\dagger\hat{\rho} - 2\hat{O}^\dagger\hat{\rho}\hat{O} + \hat{\rho}\hat{O}\hat{O}^\dagger)$$
$$- \frac{C}{2}(\hat{a}^\dagger\hat{a}\hat{\rho} - 2\hat{a}\hat{\rho}\hat{a}^\dagger + \hat{\rho}\hat{a}^\dagger\hat{a}). \quad (27)$$

From Eq. (27) we can easily derive the following master equation for the probability p_n of the n photon state:

$$\frac{dp_n(t)}{dt} = -\frac{A(n+1)}{1 + B(n+1)}p_n + \frac{An}{1 + Bn}p_{n-1}$$
$$- Cnp_n + C(n+1)p_{n+1}. \quad (28)$$

Equations similar to Eq. (28) are obtained, for example, in Refs. [19,20] using different approaches.

1. Steady-state solutions

Solving Eq. (28) in the two limiting cases of $B\bar{n} \ll 1$, $A < C$ and $B\bar{n} \gg 1$, $A \gg C$ results in the photon statistics of the thermal and coherent fields. In the following we summarize these familiar photon statistics formulas along with some more general properties of the fields predicted by our model.

The general steady-state solution of Eq. (28) with the detailed balance condition, i.e., requiring that the rate from $|n\rangle$ to $|n+1\rangle$ equals the rate from $|n+1\rangle$ to $|n\rangle$, is given by

$$p_n = p_0 \prod_{k=1}^{n} \frac{A/C}{1 + Bk}. \tag{29}$$

Equation (29) simplifies considerably at the two limits of thermal and coherent fields. First, if $B\bar{n} \ll 1$ and, furthermore, the injection rate is below the rate required to reach the laser threshold (i.e. $A < C$), we obtain

$$p_n = \left(1 - \frac{A}{C}\right)\left(\frac{A}{C}\right)^n, \tag{30}$$

which corresponds to a thermal field with

$$\bar{n} = \frac{1}{C/A - 1} = \frac{1}{\exp\left(\frac{\hbar\omega}{k_B T}\right) - 1}, \tag{31}$$

where k_B is the Boltzmann constant and T is the temperature. In the second limiting case, in which $B\bar{n} \gg 1$ and the injection rate is far above the injection required to reach the laser threshold (i.e., $A \gg C$), we obtain

$$p_n = e^{-A/(BC)} \frac{[A/(BC)]^n}{n!}, \tag{32}$$

which is the Poisson distribution and corresponds to the coherent field with

$$\bar{n} = \frac{A}{BC}. \tag{33}$$

In the general case the steady-state expectation value of the number of photons is (see the Appendix)

$$\bar{n} = \sum_{n=1}^{\infty} n p_n = \frac{1}{B}\left(\frac{A}{C} - 1\right) + \frac{1}{B} p_0 \tag{34}$$

and the second factorial moment is (see the Appendix)

$$\overline{n(n-1)} = \frac{1}{B}\left(\frac{A}{C} - 1\right)\bar{n} + \frac{1}{B}(1 - p_0). \tag{35}$$

Equations (34) and (35) can be used to calculate the second-order coherence degree (see the Appendix) as

$$g^{(2)} = \frac{\left[\frac{1}{B}\left(\frac{A}{C} - 1\right)\right]\left[\frac{1}{B}\left(\frac{A}{C} - 1\right) + \frac{1}{B} p_0\right] + \frac{1}{B}(1 - p_0)}{\left[\frac{1}{B}\left(\frac{A}{C} - 1\right) + \frac{1}{B} p_0\right]^2}. \tag{36}$$

The vacuum-state probability p_0 in Eqs. (34)–(36) is obtained from Eq. (29) as

$$p_0 = \left(\sum_{n=0}^{\infty} \prod_{k=1}^{n} \frac{A/C}{1 + Bk}\right)^{-1}. \tag{37}$$

Equations (34), (36), and (37) above give the steady-state expectation values of the number of photons $\bar{n}$ and the second-order coherence degree $g^{(2)}$ which can be used to qualitatively classify the properties of the optical field. For a thermal field, $g^{(2)} = 2$; and for a coherent field, $g^{(2)} = 1$. Figure 4 shows (a) $\bar{n}$ and (b) $g^{(2)}$ as a function of the parameter B for an amplifying system with $A/C = 0.95$ (below threshold), $A/C = 1.0$ (at threshold), and $A/C = 1.05$ (above threshold). From Fig. 4(a), we can see that for all three A/C values, the number of photons decreases when parameter B increases [cf. Eqs. (29) and (34)].
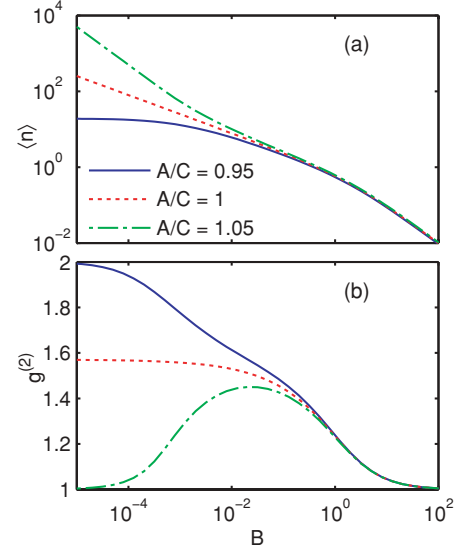

FIG. 4. (Color online) (a) Expectation value of the number of photons and (b) second-order coherence degree as a function of parameter B for the systems at the threshold ($A = C$) and slightly below and above it.

In contrast, Fig. 4(b) shows different features for the three cases. Below the threshold ($A/C = 0.95$), the field is thermal with small B and approaches a coherent field when $B > 1$. At the threshold, the field has super-Poissonian properties with small B, and it approaches a coherent field when B increases. Above the threshold, the field is coherent with $B \gg 1$ and $B \ll 1$ but has super-Poissonian properties in between these limits. Comparing Figs. 4(a) and 4(b), we notice that it could be possible to produce very weak coherent fields near the threshold.

IV. CONCLUSIONS

We have derived a general reduced photon detector model which describes the average time evolution of an optical field interacting with an energy reservoir through a two-state quantum system. Our model is able to fill the gap in the previously introduced reduced photon detector models and describe the average features of the photon field also in between the weak- and strong-coupling regimes.

We have applied our model to studying photon statistics in a conventional detector setup as well as in an amplifying setup describing a laser cavity. The results agree with the expected results, showing that the model can be applied to model LEDs and lasers. We also expect that our approach can be applied to modeling, e.g., the quantum nondemolition measurements based on atom beams passing through a cavity [21] where the initial field is prepared by injecting an exited atom through the cavity. The model should also be applicable to describing the quantum memory experiment in Ref. [22], where the information stored in the optical field can be manipulated by an atom beam, which is used to read or write information to the memory.

In conclusion, our model extends the previously introduced SD model of direct photon counting and the E model of fully saturating detectors to cover also the intermediate range where

neither of the previous models is applicable, and, furthermore, it allows the modeling of various active optical components.

APPENDIX: DERIVATIONS

Operation of $\hat{O}$ and $\hat{O}^\dagger$ into state $|n\rangle$ gives

$$\hat{O}|n\rangle = \frac{1}{\sqrt{1+B\hat{a}\hat{a}^\dagger}}\hat{a}|n\rangle = \sqrt{\frac{n}{1+Bn}}|n-1\rangle,$$

$$\hat{O}^\dagger|n\rangle = \hat{a}^\dagger \frac{1}{\sqrt{1+B\hat{a}\hat{a}^\dagger}}|n\rangle = \sqrt{\frac{n+1}{1+B(n+1)}}|n+1\rangle.$$

Furthermore, $\hat{O}$ and $\hat{O}^\dagger$ obey the following relations

$$\hat{O}^\dagger\hat{O}|n\rangle = \frac{n}{1+Bn}|n\rangle,$$

$$\hat{O}\hat{O}^\dagger|n\rangle = \frac{n+1}{1+B(n+1)}|n\rangle, \qquad (A1)$$

$$[\hat{O},\hat{O}^\dagger]|n\rangle = \frac{1}{(1+Bn)[1+B(n+1)]}|n\rangle.$$

In the general case, the expectation value of number of photons in the steady state is obtained following Ref. [19] and using Eq. (29) as follows

$$\bar{n} = \sum_{n=1}^{\infty} n p_n = p_0 \sum_{n=1}^{\infty} n \left(\frac{A}{BC}\right)^n \prod_{k=1}^{n}\frac{1}{k+1/B}$$

$$= p_0 \sum_{n=1}^{\infty} \left(n+\frac{1}{B}-\frac{1}{B}\right)\left(\frac{A}{BC}\right)^n \prod_{k=1}^{n}\frac{1}{k+1/B}$$

$$= \sum_{n=1}^{\infty} p_0 \left(\frac{A}{BC}\right)^n \left(n+\frac{1}{B}\right)\prod_{k=1}^{n}\frac{1}{k+1/B}$$

$$-\frac{1}{B}\sum_{n=1}^{\infty}\underbrace{p_0\left(\frac{A}{BC}\right)^n\prod_{k=1}^{n}\frac{1}{k+1/B}}_{p_n}$$

$$= \frac{A}{BC}\underbrace{\sum_{n=1}^{\infty}p_0\left(\frac{A}{BC}\right)^{n-1}\prod_{k=1}^{n-1}\frac{1}{k+1/B}}_{\sum_{n=1}^{\infty}p_{n-1}=\sum_{n=0}^{\infty}p_n} - \frac{1}{B}(1-p_0)$$

$$= \frac{A}{BC} - \frac{1}{B}(1-p_0) = \frac{1}{B}\left(\frac{A}{C}-1\right)+\frac{1}{B}p_0. \qquad (A2)$$

Similarly the second factorial moment is

$$\overline{n(n-1)} = p_0\sum_{n=2}^{\infty}n(n-1)\left(\frac{A}{BC}\right)^n\prod_{k=1}^{n}\frac{1}{k+1/B}$$

$$= \sum_{n=2}^{\infty}(n-1)\underbrace{p_0\left(\frac{A}{BC}\right)^n\prod_{k=1}^{n-1}\frac{1}{k+1/B}}_{\frac{A}{BC}p_{n-1}}$$

$$-\frac{1}{B}\underbrace{\sum_{n=2}^{\infty}(n-1)p_0\left(\frac{A}{BC}\right)^n\prod_{k=1}^{n}\frac{1}{k+1/B}}_{\sum_{n=2}^{\infty}(n-1)p_n}$$

$$= \frac{A}{BC}\bar{n} - \frac{1}{B}(\bar{n}-1+p_0)$$

$$= \frac{1}{B}\left(\frac{A}{C}-1\right)\bar{n} + \frac{1}{B}(1-p_0). \qquad (A3)$$

The second-order coherence degree in the steady state is defined as [19,20] $g^{(2)}(t,t) = \overline{n(n-1)}(t)/\bar{n}^2(t)$ and obtained using Eqs. (A2) and (A3).

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