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Decision Support

# An adjustment scheme for nonlinear pricing problem with two buyers

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ABSTRACT

We examine a contracting problem with asymmetric information in a monopoly pricing setting. Traditionally, the problem is modeled as a one-period Bayesian game, where the incomplete information about the buyers' preferences is handled with some subjective probability distribution. Here we suggest an iterative online method to solve the problem. We show that, when the buyers behave myopically, the seller can learn the optimal tariff by selling the product repeatedly. In a practical modification of the method, the seller offers linear tariffs and adjusts them until optimality is reached. The adjustment can be seen as gradient adjustment, and it can be done with limited information and so that it benefits both the seller and the buyers. Our method uses special features of the problem and it is easily implementable.


## 1. Introduction

We consider a monopoly pricing problem where the market consists of a seller and buyers with different preferences. The buyers are sorted into two classes, and the demand behavior of each class is specified by a utility function. The seller designs a single price schedule as a function of quantity to maximize his profit, from which the buyers select the quantity they wish to consume. In economics and game theory literature this problem is known as the nonlinear pricing problem. More broadly, such a problem falls into the class of principal-agent games where a principal (here a seller) proposes a contract to an agent (a buyer) whose preferences are the agent's private information. In addition to nonlinear pricing and monopoly pricing [12,14,21,22], other examples of such games are optimal taxation [13], regulation [1], and the design of auctions [15]. In the literature all these games are called adverse selection or mechanism design problems; [8,18] are good textbook presentations on the topic.

An essential feature of all adverse selection problems is incomplete information: the principal does not know the exact values of agents' type parameters, although he knows their probability distributions and the functional forms of the agents' utility functions depending on these parameters. Hence, the problem is solved mathematically as a one-shot Bayesian game.

In nonlinear pricing a practical approach to handle incomplete information in an offline manner[1] was suggested by Spence [21], who noted that the buyers' demand functions can be estimated by offering unit prices to the buyers. Wilson [23,24] took the idea further by formulating the problem so that it could be solved by using the demand data that is estimated from the buyers' responses to linear tariffs; see also Räsänen et al. [17] for one such application in electricity markets. The Wilson's approach may, however, require an extensive data collection that can be rather costly; in the case of Räsänen et al., it took three years to collect reasonable consumer demand data to solve a three quantity, two buyer class pricing problem. In Braden and Oren [6] a Bayesian learning formulation over a finite time horizon was studied in an optimal control fashion to estimate the type for one consumer class. As the authors say, the paper provides more insights than numbers to a rather involved problem containing continuous random variables.

Currently, Internet is taking a vital role as an e-commerce platform. Internet is also used for extensive customer data gathering for pricing services and goods. At the same time, however, customer privacy considerations attached to data collection matter and should be taken into account in the analysis [9]. This fact favors development of efficient online[1] pricing schemes that acquire data incrementally rather than offline pricing methods which usually need large customer data set to be applicable. In papers dealing with dynamic pricing of goods, where in addition to varying demand also inventory considerations may count, various online learning methods have been used to forecast the correct customer behavior and future demand curve [11,16]. Brooks et al. [5] consider adjustment of different pricing schedules, e.g., linear, two-part, nonlinear, etc. tariffs, in nonlinear pricing setting where monopolist offers consumers a new set of articles in each time period. One question they emphasize is that learning customer preferences takes time during which the seller earns less than the optimal profit. In addition to OR literature, the development of

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[1] Using an offline algorithm to solve a problem at hand requires that the whole problem data is available from the beginning. In contrast, an online algorithm can process its data incrementally, without having the entire data available from the start.

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computational algorithms for games that use limited amount of information about the other agents' preferences, e.g., multiagent learning algorithms and combinatiorial auction algorithms, have recently been under active research in AI literature, too [4,19,20].

In this paper we assume that the seller knows the number of different buyers, but does not have knowledge on their utility functions. Instead we assume that the product is sold repeatedly to myopic buyers. By observing the realized sales the seller plans a better pricing policy for the next period. We first present a discrete step adjustment scheme to solve the problem in an online fashion. Actually we come to this scheme intuitively by requiring that the seller increases the amount to be sold a little bit in every period and that in every such period both the seller and the buyers should gain. It turns out that the resulting method is a steepest ascent method. To avoid too many price changes in every iteration step we then consider a practical modification of the method by making use of linear tariffs. This kind of adjustment problem has not been studied in the literature earlier; see however [7,10] where linear tariff adjustment scheme for one buyer type was studied. This paper shows that there are intuitively appealing computational schemes for solving the problem with several buyer types, too.

The contents of the paper are as follows. In Section 2, we formulate the nonlinear pricing problem and study its optimality conditions. We also study an illustrative example in detail. In Section 3, we present a discrete step heuristic method and discuss its properties. It turns out that the invented method can be considered a discrete step gradient adjustment scheme. In Section 4, we define and analyze a modified method based on the use of linear tariffs. In Section 5, we simulate numerically the performance of our method, and finally in Section 6, we offer further considerations to the issue.

## 2. Model

A firm, the seller denoted by $S$, produces a product $x$, $x \geqslant 0$, to a population of buyers. The seller differentiates the buyers by offering them different quantities of the product. We assume that there are just two types of buyers in the population: a low buyer and a high buyer denoted from now on by $L$ and $H$, respectively. The buyers' utilities are quasi-linear,

$$U_i(x,t) = V_i(x) - t, \quad i = L,H, \tag{1}$$

where $t$ is the price of the product and $V_i(x)$ is buyer $i$'s gross surplus of consuming quantity $x$. The utilities are scaled so that $V_i(0) = 0, i = L,H$. The gross surplus $V_i(x)$ is assumed to be twice continuously differentiable, increasing and strictly concave, i.e., $V_i'(x) \geqslant 0$, $V_i''(x) < 0$, when $x \geqslant 0$.

The seller offers the buyers two types of quantity-price bundles, $(x_L, t_L)$ and $(x_H, t_H)$, and gets a total profit

$$\pi(x_L, x_H, t_L, t_H) = p_L(t_L - c(x_L)) + p_H(t_H - c(x_H)), \tag{2}$$

where $p_i$ is the relative number of buyers $i$ in the population, and $c(x)$ is the seller's cost of producing quantity $x$. Without loss of generality, we assume that there is only one $L$ buyer and one $H$ buyer with weights $p_L$ and $p_H$, respectively. Furthermore, we assume that the production cost is of the form $c(x) = cx$, where $c \geqslant 0$ is a constant. We note, however, that the production cost could be convex as well and this would result only in minor changes in the rest of the paper.

In the market the buyers self-select the bundle they wish to consume. In maximizing his profit, the seller therefore faces two kinds of constraints: *individual rationality* (IR) constraints

$$U_i(x_i, t_i) = V_i(x_i) - t_i \geqslant U_i(0,0) = 0, \quad i = L,H, \tag{3}$$

and *incentive compatibility* (IC) constraints

$$U_i(x_i, t_i) = V_i(x_i) - t_i \geqslant V_i(x_j) - t_j = U_i(x_j, t_j), \quad j \neq i. \tag{4}$$

The IR constraints state that a buyer should get positive utility when choosing the bundle intended for him. The IC constraints let the buyers *self-select* the bundle for them; the buyers prefer their own bundle the most. Now, the seller's problem is maximization of $\pi(x_L, x_H, t_L, t_H)$ with respect to the constraint equations (3) and (4).

### 2.1. Necessary and sufficient optimality conditions

We derive first-order conditions to the problem by making a common assumption used in literature, which states that the buyers' utility functions can be sorted.

**Assumption 1.** $V_H'(x) > V_L'(x), \quad \forall x \geqslant 0.$

This assumption is called the *single-crossing property* and it has two major implications. First, the optimal quantities are increasing in buyer type, $x_H^* \geqslant x_L^*$, where from now on $^*$ refers to the optimality. Second, the optimal prices are

$$t_L^* = V_L(x_L^*), \tag{5}$$
$$t_H^* = t_L^* + V_H(x_H^*) - V_H(x_L^*). \tag{6}$$

These results are derived in Spence [22]. Using these results, we can simplify the seller's problem to

$$\begin{aligned} \max_{x_L,x_H,t_L,t_H} \quad & \pi(x_L, x_H, t_L, t_H) = p_L(t_L - cx_L) + p_H(t_H - cx_H) \\ \text{s.t.} \quad & t_L = V_L(x_L), \\ & t_H = t_L + V_H(x_H) - V_H(x_L), \\ & x_H \geqslant x_L \geqslant 0. \end{aligned} \tag{7}$$

**Assumption 2.** There is $x_i^E > 0$ so that $V_i'(x_i^E) = c$, $i = L,H$.

This assumption rules out the possibility that selling nothing to both buyers is optimal for the problem. If buyer $i$ was alone in the market, he would be served with the amount $x_i^E$, which is called the *first-best solution*. In this case, when the cost is linear and $V_i$ is strictly concave, this amount is unique.

Let us define $f_L(x) = p_L(V_L(x) - cx) - p_H(V_H(x) - V_L(x))$ and $f_H(x) = p_H(V_H(x) - cx)$. Then substituting the equality constraints in (7) into the objective function, we get $\pi(x_L, x_H, t_L, t_H) = f_L(x_L) + f_H(x_H)$. Hence, forgetting the constraints $x_H \geqslant x_L \geqslant 0$ for a while, we get the necessary conditions of (7) for a solution $0 < x_L^* \leqslant x_H^*$,

$$f_H'(x_H^*) = p_H(V_H'(x_H^*) - c) = 0, \tag{8}$$
$$f_L'(x_L^*) = p_L(V_L'(x_L^*) - c) - p_H(V_H'(x_L^*) - V_L'(x_L^*)) = 0. \tag{9}$$

Assumptions 1 and 2 imply that $0 < x_L^E < x_H^E < \infty$, and that $f_L'(x) < 0$, for all $x \geqslant x_L^E$. Thus for a solution of (9) we have $0 \leqslant x_L^* < x_L^E$. By (8), $x_H^* = x_H^E$, hence it also holds that $x_L^* < x_H^*$. But (9) may not have solution at all, since $f_L'(x)$ can be strictly negative for all $x \in [0, x_L^E]$. Thus, the problem solution is either to serve both buyers or to exclude the low type and serve only the high type. Which case will happen depends on the buyers' utilities and weights $p_L$ and $p_H$. The latter case will happen if $p_L$ is small, or if the low type values the product considerably less than the high type. If this is the case, the solution is given by $x_H^* = x_H^E$, $t_H^* = V_H(x_H^*)$, and $x_L^* = t_L^* = 0$. In this paper, we shall assume that it is optimal to serve both buyers. Therefore, we make the following assumption.

**Assumption 3.** $f_L'(0) > 0$.

Since $f_L'(x)$ is continuous, there is a solution $x_L^*$, $0 < x_L^* < x_L^E$, to (9). Nevertheless, the solution may not be unique, since $-p_H(V_H'(x_L^*) - V_L'(x_L^*))$ may not be decreasing. To guarantee this we make our final assumption.

**Assumption 4.** $V_L''(x) \leqslant V_H''(x) < 0, \quad \forall x \geqslant 0$.

This assumption means that the curvature for the low type is steeper than for the high type. It also guarantees that $f_L'(x)$ is strictly decreasing in $x$, i.e., $f_L(x)$ is strictly concave, as $f_H'(x)$ and $f_H(x)$ are, respectively. Hence the solution $(x_L^*, x_H^*)$ to maximization problem (7) is unique.

### 2.2. An example

To get an overview of the problem it is instructive to study an example with weights $p_L = p_H = 1/2$ in more detail. Note that Assumption 1 implies that $H$ values the product strictly more since we assumed $V_i(0) = 0$, $i = H, L$. The seller designs amounts $x_H$ and $x_L$ and prices for these amounts $t_H$ and $t_L$ so that the buyers are willing to buy the bundles intended to them, i.e., the bundles should satisfy the IR and IC constraints.

But let us first consider a case, where the seller can perfectly discriminate the buyers by giving them individual offers; thus, we forget the IC constraints from the formulation. Obviously in this case, the seller can extract all the surplus from the buyers. Thus we set $t_i = V_i(x_i)$ in (2), $i = H, L$, and maximize $V_i(x_i) - cx_i$. The optimal amounts are given by the first-order conditions, $V_i'(x_i^E) = c$, $i = H, L$, and together with the optimal prices $t_i^E = V_i(x_i^E)$ these bundles define the *first-best solution*, see bundles $A_L$ and $A_H$ in Fig. 1.

Note that a buyer's indifference curves are of the form $U_i(x, t) = V_i(x) - t = \gamma$, where $\gamma$ is a constant. The indifference curves in Fig. 1 are given by equation $t = V_i(x) - \gamma$, $\gamma \geqslant 0$ a constant. Hence, the greater the $\gamma$ for a buyer, and hence the greater his utility, the lower the corresponding indifference curve is in the figure. Also note, that the slope of a buyer's indifference curve, $-[\partial U_i/\partial x][\partial U_i/\partial t]^{-1}(x, t) = V_i'(x)$, depends on $x$, but not on $t$. In particular, the slopes of $L$'s and $H$'s indifference curves at $A_L$ and $A_H$, respectively, equal the slope $c$ of the seller's cost function.

Now, consider the case, where the buyers may self-select the bundle they wish to consume. If the seller offered the first-best bundles, $A_H$ and $A_L$, $H$ would not choose his own bundle, because he would get strictly positive utility by choosing $L$'s bundle $A_L$. This is because $A_L$ is below the indifference curve going through $A_H$. Thus, the seller must take the IC constraints into account. The seller lowers the price of $A_H$ from $t_H^E$ until $H$ is indifferent between $A_L$ and $A_H$. This happens when $t = t_H^C$, i.e., at the bundle $C_H$. Now, we notice that $H$ has an advantage of the self-selection, because the price has been lowered but his amount of good is the same. We call the price difference $t_H^E - t_H^C$ *informational rent*, and denote it by $\Delta_C$. The feasible incentive compatible bundles $A_L$ and $C_H$ are not yet optimal. They satisfy the requirements (5) and (6) for optimal prices giving zero utility to $L$, and indifferency of $A_L$ and $C_H$ to $H$. At optimal solution the amount $x_H^E$ for $H$ should also be correct. Nevertheless, the amount $x_L^E$ to $L$ is too high as is seen from (9). According to (9) for $p_L = p_H = 1/2$, and Assumptions 1 and 3, $V_L'(x) > c$ at an optimal amount, and not $V_L'(x) = c$ as at $x_L^E$. Hence the optimal bundle for $L$ should be to the left of $A_L$ on $L$'s zero-level indifference curve, such as $B_L$ in Fig. 1. Consequently also the optimal price for $H$ increases when changing $C_H$ to $B_H$; and so the informational rent decreases from $\Delta_C$ to $\Delta_B$. The optimal bundles denoted by $B_H$ and $B_L$, satisfying Eqs. (8) and (9), are called the *second-best solution*. Note that compared to the first-best solution $H$'s utility has been increased. The amount is the same but he gets a price discount $\Delta_B$. On the other hand, $L$'s utility remains at the zero-level although he faces reduction $x_L^E - x_L^*$ in the amount of good as compared to the first-best solution. In general, if $p_L$ is much smaller than 1/2, $L$ may not be served at all, and as $p_L$ increases $x_L^*$ increases.

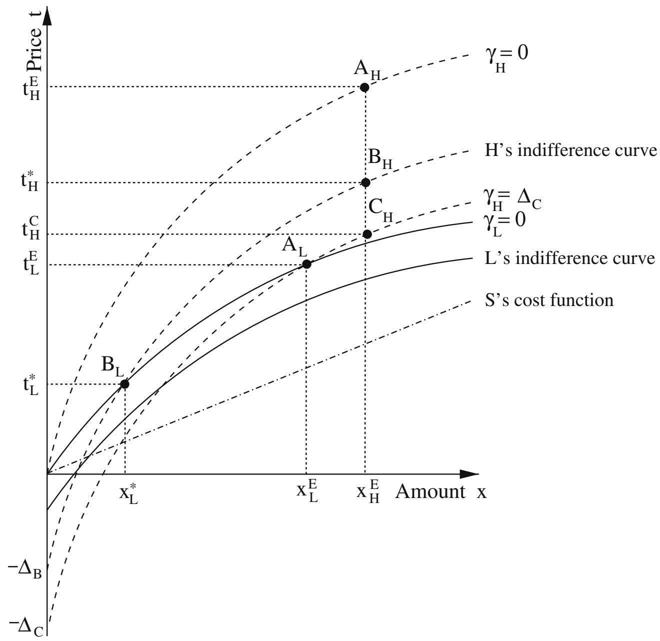


**Fig. 1.** An example with two buyers. The solid, dashed, and dash-dotted lines are buyer $L$'s and buyer $H$'s indifference curves, and the seller's cost function, respectively. $A_L$ and $A_H$ are the first-best solutions, and $B_L$ and $B_H$ are the second-best solutions.

## 3. Discrete step adjustment scheme

Given that the buyers' utility functions are known, it is easy to solve the first-order conditions (8) and (9) numerically. The optimal prices can then be calculated from (5) and (6). Nevertheless, we now assume that the seller does not have *prior* knowledge about the buyers' utility functions $V_L$ and $V_H$. Instead we assume that the seller $S$ is selling his product repeatedly to two myopic buyers $L$ and $H$ by putting different bundles for sale at the same time. Hence, meeting the buyers repeatedly and observing the realized sales, he can plan a better pricing strategy for the next period. Using such online process to adjust prices he can finally produce the optimal bundles provided the process converges. In this section we present a discrete step heuristic adjustment scheme for solving the problem and discuss its good properties. In Section 4, we further elaborate the scheme so that it requires less computational effort.

### 3.1. Heuristic description of the method

Assume first that the weights of the buyers are equal; i.e., $p_H = p_L = 1/2$. The method can be considered to arise through the following process, also illustrated in Fig. 2. An initial bundle $(x^1, t^1)$ is sold to both buyers $L$ and $H$ in period 1. Without loss of generality we assume that $(x^1, t^1)$ is on $L$'s zero-level indifference curve, and $x^1 < x_L^*$. We will return to the question of adjusting $t_1$, for given $x^1$, to $L$'s zero-level curve, without having prior knowledge of the curve, in the end of the section. The approximately optimal bundles created by the method are denoted by $(\bar{x}_L, \bar{t}_L)$ and $(\bar{x}_H, \bar{t}_H)$. The method produces a sequence of bundles $(x^k, t^k)$, $k \geqslant 1$, on $L$'s zero-level indifference curve sold to both $L$ and $H$ in period $k$ until the bundle $(\bar{x}_L, \bar{t}_L)$ is sold. After that, in every iteration, there are two bundles for sale: $(\bar{x}_L, \bar{t}_L)$ for $L$ and a bundle with

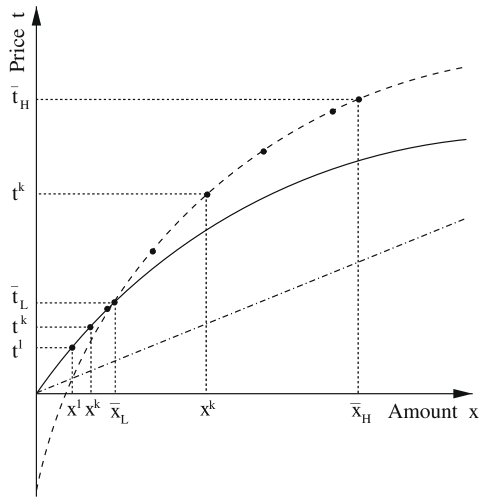


**Fig. 2.** Illustration of the heuristic method.

bigger amount and higher price $(x^k, t^k)$ for $H$. The iteration then continues until $(x^k, t^k)$ reaches $(\bar{x}_H, \bar{t}_H)$.

Let us now examine more carefully how the bundles $(x^k, t^k)$ are created and, in particular, how the optimal bundles are discovered. Suppose $S$ has sold $x^1$ at price $t^1$ to $L$ and $H$ in period 1, and denote $b_1 = (x^1, t^1)$. Now, suppose $S$ wants to increase his profit a little *without making the buyers to be worse off* as when buying $b_1$. Therefore, he increases $x^1$ by a small amount, say lower than or equal to $\Delta x$, and thinks about a correct price. Intuitively, (for rigorous proof see Lemma 1) to get a best profit it suffices to compare one of the feasible bundles $b_S, b_L, b_H$ at $x^1 + \Delta x$ to $b_1$, shown in Fig. 3.

Note that all the bundles *strictly* above the $S$'s indifference line through $b_1$ strictly improve $S$'s profit, but the buyers could prefer to $b_1$ only those below their respective indifference curves through $b_1$. The bundles $b_S$, $b_L$ and $b_H$ in Fig. 3 are preferred (equally or strictly) to $b_1$ either by $H$ or $L$ or both. These bundles are of the form $(x^1 + \Delta x, t^1 + \Delta t)$ and we want $\Delta t$ to be the most profitable for $S$. If $S$ offered $b_S$, both buyers would prefer $b_S$ to $b_1$ but it will not increase his profit. Suppose

$$\Delta y_H < 2\Delta y_L \tag{10}$$

as in the figure. Then along the line from $b_S$ to $b_H$, the best profit increase is obtained at point $b_L$. This is because $L$ and $H$ both prefer $b_L$ to $b_1$ (actually $L$ is indifferent to $b_L$ and $b_1$ but we assume $L$ takes $b_L$) and hence the profit increase to $S$ is $2\Delta y_L$. Buyer $L$ strictly prefers $b_1$ to bundles above $b_L$, on the line from $b_L$ to $b_H$, while $H$ prefers these bundles to $b_1$. Above the $L$'s indifference curve through $b_1$ and $b_L$, $S$ will thus get the best profit increase $\Delta y_H$ by selling $b_H$ to $H$ (and $b_1$ to $L$), but according to inequality (10) this is less than $2\Delta y_L$. Thus, selling $b_L$ to both buyers will benefit $S$ the most, and this way $(x^2, t^2)$ is created. To adjust $b_L$ to $L$'s zero-level curve, i.e., to find a correct price $t^2 = t^1 + \Delta y_L + c\Delta x$ for $b_L$, see the end of this section.

In every period $k$, and as long as inequality (10) holds, $S$ will create the bundles $(x^k, t^k)$, $k \geqslant 2$, in the same way by increasing the amount by a fixed $\Delta x$ and the price by $\Delta t = \Delta y_L + c\Delta x$. Note that although $\Delta x$ remains fixed $\Delta y_L$ and $\Delta y_H$ vary in $k$. Define $(\bar{x}_L, \bar{t}_L)$ to be the first bundle for which $\Delta y_H \geqslant 2\Delta y_L$. Hence, at this bundle,

$$\Delta y_H = 2\Delta y_L, \tag{11}$$

*approximately* holds for $\Delta x$ small.

What is the best way to make more profit at $(\bar{x}_L, \bar{t}_L)$? Consider a bundle $(\bar{x}_L + \Delta x, \bar{t}_L + \Delta t)$, where $\Delta x$ is as earlier. Since, now $\Delta y_H > 2\Delta y_L$, it is easy to conclude that selling $b_H$ to $H$ and $(\bar{x}_L, \bar{t}_L)$ to $L$ benefits $S$ the most. From that on, $S$ makes even more profit by letting the higher bundle $(x^k, t^k)$ move along $H$'s indifference curve through $(\bar{x}_L, \bar{t}_L)$ until the slope of $H$'s indifference curve equals that of his, i.e., is equal to $c$. After that $S$ cannot make more profit. The bundle in question is denoted by $(\bar{x}_H, \bar{t}_H)$.

We now want to argue why the amounts of good $\bar{x}_L$ and $\bar{x}_H$ obtained by the above heuristics are approximately optimal for $\Delta x$ small. First note that from the definition of $\bar{x}_H$, it holds that $V'_H(\bar{x}_H) \cong c$, for $\Delta x$ small. This equation is approximately the optimality condition, Eq. (8). From Fig. 3 we see that $\Delta y_H \cong V'_H(x^1)\Delta x - c\Delta x$, and $\Delta y_L \cong V'_L(x^1)\Delta x - c\Delta x$, for $\Delta x$ small. Thus, since at $\bar{x}_L$ we have $\Delta y_H \cong 2\Delta y_L$, we get $V'_H(\bar{x}_L) - V'_L(\bar{x}_L) \cong V'_L(\bar{x}_L) - c$ for $\Delta x$ small. When $p_L = p_H$, this equation is approximately the optimality condition, Eq. (9). At $x^k$, $k \geqslant 1$, we thus get that $f'_L(x^k)\Delta x \cong 2\Delta y_L - \Delta y_H$ for $\Delta x$ small. Thus, since $f_L(x)$ is strictly decreasing, it follows that inequality (10) holds for $x^k < \bar{x}_L$, provided it holds for $x^1$; Eq. (11) approximately holds at $\bar{x}_L$, and $\Delta y_H > 2\Delta y_L$ holds for $x^k > \bar{x}_L$. Thus, $(\bar{x}_L, \bar{t}_L)$ to $L$ and $(\bar{x}_H, \bar{t}_H)$ to $H$ are approximately optimal bundles for the problem.

With arbitrary weights $p_L$ and $p_H$, Eq. (11) should be replaced by

$$p_H \Delta y_H = (p_L + p_H)\Delta y_L \tag{12}$$

and a similar form for inequality (10). The weighting of $S$'s profit increases should be consistent with the weighting of his total profit.

We now turn to the question of adjusting $t^1$ for given $x^1$ so that $(x^1, t^1)$ is on $L$'s zero-level curve. Suppose this is done with limited information on buyers' preferences by giving price offers and observing the realized sales. We call such process *price testing*, and define it through a process, where the seller raises or lowers the price of $x^1$ using discrete steps (with a fixed step length, or with a variable step length defined, e.g., by the bisection method) and observes whether $L$ takes it or not. Similarly, when testing the price of $b_L$ on $L$'s zero-level curve, see Fig. 3, the seller may raise or lower the price of $x^1 + \Delta x$ and observe whether $L$ takes it or prefers the original bundle $b_1$. If $L$ takes the original bundle, the price for the offered bundle is higher than that of $b_L$. It should be noted that in a practical implementation of the method when testing the price of $b_L$, $S$ should put the two bundles $b_1$ and $b_L$ for sale at the same time. Otherwise, in the case $L$ does not take $b_L$, the losses for $S$ could be high. Price tests to locate $H$'s bundles on his indifference curve through $(\bar{x}_L, \bar{t}_L)$ are made in the same fashion.

We finally make an important remark concerning checking on the status of Eq. (11) in iteration $k$, without explicitly testing the price of $b_H$, see Fig. 3. Namely, after having tested the price of bundle $b_L$, and hence knowing $\Delta y_L$, $S$ may offer $x^k + \Delta x$ at the price $t^{k+1} = t^k + 2\Delta y_L + c\Delta x$ and with a single test observe whether $H$

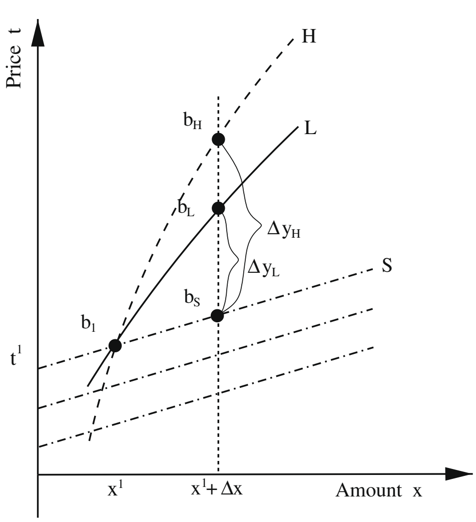


**Fig. 3.** Increasing the amounts by $\Delta x$.

takes this bundle or takes $b_1$, as $L$ does. If $H$ does not take this bundle, then obviously $2\Delta y_L > \Delta y_H$, i.e., (10) holds, and $x^k + \Delta x$ is to the left from $\bar{x}_L$; while if $H$ does take this bundle, then $\Delta y_H \geqslant 2\Delta y_L$ and $x^k + \Delta x$ is to the right from $\bar{x}_L$. To make use of this observation and to avoid extensive price testing, we will make some simple modifications to the method in Section 4.

### 3.2. Interpretation as a steepest ascent method

First observe that when solving Eq. (9) by using an online method such as the one presented in Section 3.1, with initial amounts of good $x_L^1$ and $x_H^1$ not necessarily equal, it is necessary that at some point during the iteration $H$ chooses an amount that is close to the amount chosen by $L$. This is because $x_L^*$ appears in the argument of $V_H'$ in Eq. (9). In our method $L$ and $H$ take the same bundle until $(\bar{x}_L, \bar{t}_L)$ is reached. Moreover, the testing of Eq. (9) is done through the testing of Eq. (11).

Let us now discuss the optimal way to update the bundles. Suppose that the initial amounts $x_L^1$ and $x_H^1$, not necessarily equal, are on the interval $[a, a']$, $a' > a$, and the seller wishes to improve them; yet the new bundles should be such that the amounts stay on the interval $[a, a']$. Then the following lemma gives the optimal new bundles depending on the location of $[a, a']$ with respect to the optimal amounts $x_L^*$ and $x_H^*$.

**Lemma 1.** *Consider the seller's optimization problem with the additional constraint $x_L, x_H \in [a, a']$. Then the optimal bundles $(y_L, t_L)$, $(y_H, t_H)$ are defined as follows:*

(i) *Let $0 \leqslant a < a' \leqslant x_L^*$. Then $y_L = y_H = a'$, $t_L = t_H$, and is defined by $L$'s zero-level curve.*
(ii) *Let $0 \leqslant a < x_L^* < a' \leqslant x_H^*$. Then $y_L = x_L^*$, $t_L = t_L^*$, $y_H = a'$, $t_H$ is defined by $H$'s indifference curve through $(x_L^*, t_L^*)$.*
(iii) *Let $x_L^* \leqslant a \leqslant x_H^* < a'$. Then $y_L = a$, $t_L$ is defined by $L$'s zero-level curve, $y_H = x_H^*$, $t_H$ is defined by $H$'s indifference curve through $(a, t_L)$.*

**Proof.** Let us first observe that by Assumption 1 we have $t_L = V_L(y_L)$ and $t_H = t_L + V_H(y_H) - V_H(y_L)$ as we have for $t_L^*$ and $t_H^*$. In particular, note that $t_L$ is defined by $L$'s zero-level curve. Hence, we can consider the optimization problem (7) with the additional constraint $x_L, x_H \in [a, a']$. Since $\pi(x_L, x_H, t_L, t_H) = f_L(x_L) + f_H(x_H)$, the result follows by observing that $f_i(x_i)$ is strictly increasing (decreasing) on $x_i < x_i^*$ $(x_i > x_i^*)$, $i = L, H$. □

As we can notice, the best way to improve the bundles is to choose the new amounts, say $x_L^2, x_H^2$, as close as possible to $x_i^*$, $i = L, H$, and set the new prices so that $L$ gets zero utility while $H$ is made indifferent between choosing $(x_L^2, t_L^2)$ and $(x_H^2, t_H^2)$. In particular, the heuristic presented in the previous section behaves exactly like this. Hence, it can be seen as a *discrete step steepest ascent method*. Notice that when we begin from $(x_1, t_1)$, $x_1 < x_L^*$, on $L$'s zero-level curve and the step is bounded by $\Delta x$, we have exactly the case (i) of Lemma 1. Consequently, the iteration proceeds as in Fig. 2 until $\bar{x}_L$ is reached. After that the iteration proceeds as can be predicted from case (ii) of Lemma 1. It should be noted that we defined the algorithm without any technical assumptions or complicated mathematics. Our only assumption was that the seller gets the best profit increase, without making the buyers worse off, when moving from $x^k$ to $x^{k+1}$.

## 4. Modified method

In this section we present a method that has two main steps at every iteration: improving step or $\alpha$-step, and test step or $\beta$-step. In the first step the seller offers a linear price-amount tariff. The buyers' optimal choices on the tariff reveal the slopes of their surplus functions $V_i$, $i = L, H$. This idea has been previously presented in [7,10] in the case of one buyer type. The second step checks at every iteration the status of Eq. (12). Recall that when the approximations $\Delta y_H \cong (V_H'(x^k) - c)\Delta x$, and $\Delta y_L \cong (V_L'(x^k) - c)\Delta x$ are used in (12) we get approximately the optimality condition (9).

Instead of offering discrete bundles of the form $(x^k + \Delta x, t^k + \Delta t)$ in period $k$, the seller now offers a linear tariff of the form $t(x) = \alpha^k x + \delta^k$, starting from $(x^k, t^k)$, and letting the buyers select any bundle from it. This is $\alpha$-step. The lowest of these bundles becomes $(x^{k+1}, t^{k+1})$ provided that a test of Eq. (12), the $\beta$-step, shows that we are to the left of $\bar{x}_L$. Further profits can be created, starting from $(x^{k+1}, t^{k+1})$, by decreasing the slope $\alpha^k$ of the linear tariff a little, say an amount $h_\alpha$.

Denote the buyers optimal choices on the tariff $t(x)$ by $\hat{x}_L$ and $\hat{x}_H$. We should have $x^k < \hat{x}_L < \hat{x}_H$. Note that myopic buyers choose amounts that solve

$$\max_x U_i(x, t(x)), \quad i = L, H \tag{13}$$

paying prices $t(\hat{x}_i)$, $i = L, H$. Due to strict concavity of $V_i$'s it holds $V_i'(\hat{x}_i) = \alpha^k$.

We denote

$$\beta^k = \alpha^k + \frac{p_L}{p_H}(\alpha^k - c). \tag{14}$$

Optimality of $L$'s bundle can be tested using the linear tariff starting from $(\hat{x}_L, t(\hat{x}_L))$ with the slope $\beta^k$. Let the buyers now choose amounts $\hat{s}_i$, $i = L, H$, from the tariff. Suppose $\hat{s}_H = \hat{x}_L$. This means that the best bundle on the tariff for both buyers is $(\hat{x}_L, t(\hat{x}_L))$, and hence $\beta^k \geqslant V_H'(\hat{x}_L)$. Using (14) with $\alpha^k = V_L'(\hat{x}_L)$, this inequality implies that $f_L'(\hat{x}_L) < 0$, meaning that $\hat{x}_L \leqslant x_L^*$; c.f., the corresponding discussion in Section 3.1. If, on the other hand, $\hat{s}_H > \hat{x}_L$, then $f_L'(\hat{x}_L) < 0$, which implies $\hat{x}_L > x_L^*$.

We now present the phases of iteration $k$ explicitly to show how the parameters are updated.

***Initial step***. Choose the initial bundle $(x^1, t^1)$; this need not be on $L$'s zero-level curve. Choose a unit price $\alpha^1$, a fixed price $\delta^1$, and a lowering parameter $h_\alpha$.

*$\alpha$-**step***. At iteration $k$, $S$ offers a linear tariff of the form

$$t(x) = \alpha^k x + \delta^k, \quad x \geqslant x^k \tag{15}$$

and observes the amounts $\hat{x}_L$ and $\hat{x}_H$ the buyers take from the tariff. The corresponding prices are $t(\hat{x}_i)$, $i = L, H$.

*$\beta$-**step***. $S$ tests the optimality of $L$'s bundle. He offers a linear tariff

$$t(x) = \begin{cases} \beta^k(x - \hat{x}_L) + \alpha^k \hat{x}_L + \delta^k, & x \geqslant \hat{x}_L, \\ \alpha^k \hat{x}_L + \delta^k, & x \leqslant \hat{x}_L, \end{cases} \tag{16}$$

where $\beta^k$ is as in (14). Let the buyers choose amounts $\hat{s}_i$, $i = L, H$, from the tariff.

If $\hat{s}_H = \hat{x}_L$, then define $x^{k+1} = \hat{x}_L$, $t^{k+1} = t(x^{k+1})$. To increase his profit, $S$ should decrease $\alpha^k$. Let

$$\alpha^{k+1} = \alpha^k - h_\alpha. \tag{17}$$

Also $\delta^k$ is updated so that the new tariff goes through $(x^{k+1}, t^{k+1})$. Thus

$$\delta^{k+1} = \delta^k + x^{k+1} h_\alpha. \tag{18}$$

Then set $k = k + 1$ and go to $\alpha$-step.

If $\hat{s}_H > \hat{x}_L$, then we set $\bar{x}_L = \hat{x}_L$, $\bar{t}_L = t(\bar{x}_L)$ as $L$'s bundle remaining for the future iterations. If $h_\alpha$ is small, $(\bar{x}_L, \bar{t}_L)$ will be close to the optimal one, as will be shown in Section 4.1. We define $x^{k+1} = \hat{s}_H$,

$t^{k+1} = t(x^{k+1})$. Let now $\alpha^{k+1} = \beta^k - h_\alpha$, and $\delta^{k+1} = \delta^k + x^{k+1}h_\alpha$. Set $k = k + 1$ and go to final step.

***Final step***. $S$ offers a bundle $(\bar{x}_L, \bar{t}_L)$ and a linear tariff of the form

$$t(x) = \alpha^k x + \delta^k, \quad x \geqslant x^k \tag{19}$$

and observes the amounts $\hat{x}_L$, $\hat{x}_H$ the buyers take from the tariff. $L$ will always choose $\hat{x}_L = \bar{x}_L$ as long as $h_\alpha$ is moderately small, while $H$ chooses so that $V'_H(\hat{x}_H) = \alpha^k$. Define $x^{k+1} = \hat{x}_H$ and update $\alpha^{k+1}$, $\delta^{k+1}$ as in (17) and (18). Repeat the final step until $\alpha^{k+1} \leqslant c$. Then define $\bar{x}_H = x^{k+1}$, $\bar{t}_H = t(\bar{x}_H)$.

Depending on the size of the parameter $h_\alpha$ the bundles discovered by the method will deviate more and more in the course of the iteration from the path along the indifference curves shown in Fig. 4. Therefore, $S$ makes suitable price tests at the created quantities $\bar{x}_L$ and $\bar{x}_H$. In the method, this is done by raising the price with parameter $h_\delta$ until the current buyer rejects the offer. We finally note that starting the iteration from two initial bundles $(x^1, t^1)$ and $(x^1, s^1)$ that differ only with respect to price, the method produces sequences of bundles $(x^k, t^k)$ and $(x^k, s^k)$, where $s^k - t^k = s^1 - t^1$. This observation reflects the fact that the slopes of the indifference curves do not depend on $t$.

### 4.1. Analysis of the method

The bundles $(\bar{x}_i, \bar{t}_i)$, $i = L, H$, produced by the method can be made arbitrarily close to the optimal ones by choosing $h_\alpha$ and $h_\delta$ sufficiently small. Note also that when $h_\alpha$ is small compared to $h_\delta$, then the error bounds of prices $\bar{t}_L$ and $\bar{t}_H$ are $h_\delta$ and $2h_\delta$, respectively. Hence, the error of $\bar{t}_H$ is presumably larger than the error of $\bar{t}_L$ as could be expected.

**Proposition 1.** *Assume that $x^1 < x_L^*$ and $\bar{x}_L < x_H^*$. The produced bundles have the following bounds:*

$$\begin{aligned}
&x_i^* \leqslant \bar{x}_i < x_i^U,\\
&t_L^* - h_\delta < \bar{t}_L < V_L(x_L^U),\\
&V_L(x_L^U) - h_\delta + V_H(x_H^*) - V_H(x_L^U) - h_\delta < \bar{t}_H < t_L^* + V_H(x_L^U) - V_H(x_L^*),
\end{aligned}$$

*where $x_i^U$ is solved from $V'_i(x_i^U) = \alpha_i^* - h_\alpha$, and $\alpha_i^*$ is defined by $\alpha_i^* = V'_i(x_i^*)$, and $i = L, H$. Assume that there are $M_i$, $M_{ii} > 0$ such that $V'_i(x) \leqslant M_i$ and $|V''_i(x)| \geqslant M_{ii}$ for all $x \geqslant 0$, and $i = L, H$. Then we can approximate these bounds by*

$$\begin{aligned}
&x_i^* \leqslant \bar{x}_i < x_i^* + h_\alpha/M_{ii},\\
&t_L^* - h_\delta < \bar{t}_L < t_L^* + h_\alpha M_L/M_{LL},\\
&t_H^* - 2h_\delta - h_\alpha M_H/M_{LL} < \bar{t}_H < t_H^* + h_\alpha M_H/M_{HH}.
\end{aligned}$$

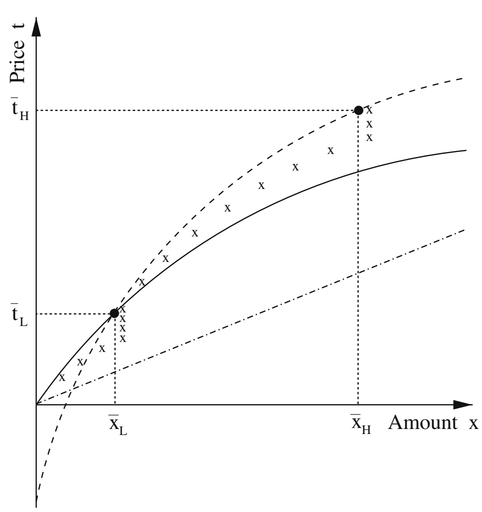


**Fig. 4.** The discrete step modified method.

**Proof.** First note that the method produces quantities that are bigger than the optimal ones. In the worst case the final value of $\alpha_i$ is by $h_\alpha$ lower than the optimal $\alpha_i^* = V'_i(x_i^*)$, $i = L, H$. From this observation, we get the bounds for quantities, i.e., $x_i^U$ defined in the proposition. The bounds for prices depend on the bounds for quantities. At $\bar{x}_L$, it holds that $V_i(\bar{x}_L) - h_\delta < \bar{t}_L \leqslant V_L(\bar{x}_L)$, since the price is raised by $h_\delta$ until $L$ rejects the offer. We get the bounds for $\bar{t}_L$ by noticing that $V_L(x)$ is increasing. Similarly, it holds that $\bar{t}_L + V_H(\bar{x}_H) - V_H(\bar{x}_L) - h_\delta < \bar{t}_H \leqslant \bar{t}_L + V_H(\bar{x}_H) - V_H(\bar{x}_L)$. The bounds for $\bar{t}_H$ follow from the fact that $V_H(x)$ is increasing, and that increasing $\bar{x}_L$ lowers the price for $\bar{x}_H$ since $V'_H(x) > V'_L(x)$, $\forall x \geqslant 0$.

Let us now examine the approximations. By the mean value theorem we have $V'_i(x_i^U) = V'_i(x_i^*) + V''_i(x_i)(x_i^U - x_i^*)$ for some $x_i \in [x_i^*, x_i^U]$. Thus we obtain $x_i^U - x_i^* = -h_\alpha/V''_i(x_i)$, and the result for quantities follows immediately from this relation and the assumption $|V''_i(x)| \geqslant M_{ii}$, $i = L, H$.

We next study the approximations for the prices. We get the upper bound for $\bar{t}_L$ by approximating $\bar{t}_L - t_L^* \leqslant V_L(x_L^U) - V_L(x_L^*)$. Using the mean value theorem, we obtain $V_L(x_L^U) - V_L(x_L^*) = V'_L(x)(x_L^U - x_L^*) \leqslant h_\alpha M_L/M_{LL}$, where $x \in [x_L^*, x_L^U]$. The result then follows.

The bounds for $\bar{t}_H$ are computed in the following way. Recalling that $t_H^* = t_L^* + V_H(x_H^*) - V_H(x_L^*)$, we can simplify the expression for the lower bound

$$\begin{aligned}
&t_H^* - 2h_\delta + V_H(x_L^*) - V_H(x_L^U) + V_L(x_L^U) - V_L(x_L^*)\\
&\quad \leqslant t_H^* - 2h_\delta + V_H(x_L^*) - V_H(x_L^U) < \bar{t}_H.
\end{aligned}$$

Now, we use the mean value theorem, $V_H(x_L^*) - V_H(x_L^U) = V'_H(x)(x_L^* - x_L^U) \geqslant -h_\alpha M_H/M_{LL}$, where $x \in [x_L^*, x_L^U]$. The approximation is not as good as the others since we left out the term $V_L(x_L^U) - V_L(x_L^*)$.

The upper bound of $\bar{t}_H$ is computed in the same way as for $\bar{t}_L$ since the highest values are achieved with $x_L^*$. So fixing $\bar{x}_L = x_L^*$, we get $\bar{t}_H - t_H^* \leqslant V_H(\bar{x}_H) - V_H(x_H^*)$. And we get the result by using once again the mean value theorem. □

## 5. Numerical example

We demonstrate the modified method with two sets of parameters. The first set, denoted by $A$, illustrates a case where the initial bundle lies on the $L$'s zero-level curve and the slope of the tariff is decreased rather slowly. The second set, denoted by $B$, illustrates a start from an initial bundle well below $L$'s zero-level curve and the slope of the tariff is decreased three times faster. Consequently, there will be a considerable rise of prices when the optimal quantities are found.

The data for the example is: $V_L(x) = 2\sqrt[5]{x}$, $V_H(x) = 3\sqrt[4]{x}$, $p_L = 0.7$, $p_H = 0.3$, $c(x) = 1.5x$, $\delta^1 = 0.7$, $\alpha_A^1 = 4$, $h_{\alpha,A} = 0.15$, $h_{\delta,A} = 0$, $\alpha_B^1 = 6$, $h_{\alpha,B} = 0.4$ and $h_{\delta,B} = 0.06$. The utility functions are arbitrary concave functions, and 70% of the population is $L$ type buyers.

The results are presented in Figs. 5 and 6. The optimal bundles $(x_i^*, t_i^*)$, $i = L, H$, are $(0.12, 1.32)$ and $(0.40, 1.92)$, which are depicted as white asterisks in Fig. 5. We can also see that iterations corresponding to set $A$, denoted by crosses, move almost along the "optimal" indifference curves until the quantities $\bar{x}_L, \bar{x}_H$ are found. Since the first bundle is near $L$'s zero-level curve, the prices will be close to the optimal ones and no price test iterations are needed. If the slope update parameter were bigger, it would take less iterations but the price would be lower and price test iterations might be needed. We can see from the latter figure that the profit gets closer to the optimal one with an even pace.

Set $B$ iterations, the dots, start farther away from $(x_L^*, t_L^*)$ than set $A$ iterations. Yet, because of the bigger slope update parameter $\bar{x}_L$ is reached with fewer iterations. The price before the price test

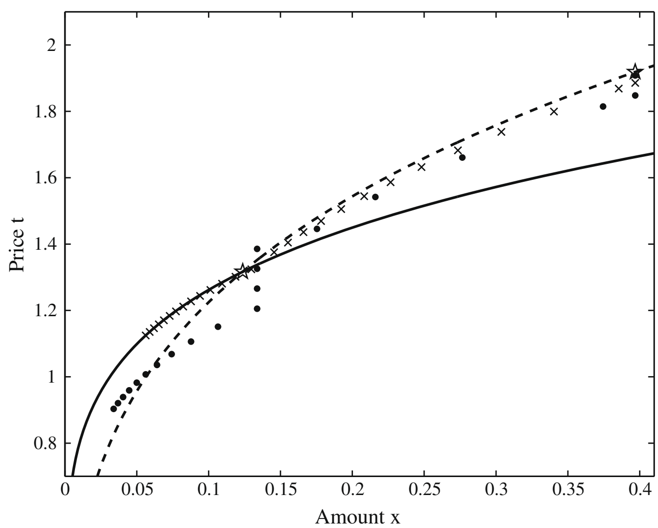


**Fig. 5.** The iterations corresponding to sets *A* and *B* are denoted by crosses and dots, respectively. The optimal bundles are denoted by white asterisks.

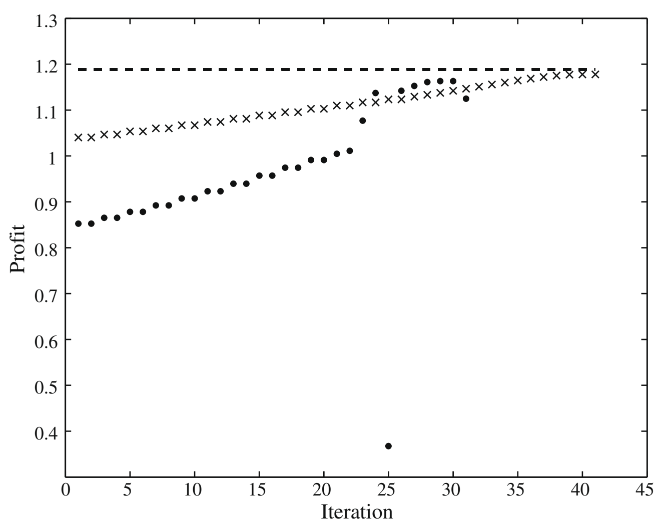


**Fig. 6.** The profits. The dashed line presents the optimal profit.

iterations is well below $\bar{t}_L$. We can see the drawback of price testing in the Fig. 6. In iteration 25 the price is so high that *L* buys nothing at all; the result is the point at (25, 0.36). The latter price testing is not so dramatic in this sense, because *H* will take *L*'s bundle when the price of *H*'s bundle is too high. We can also see from Fig. 6 that the bigger slope update parameter results in faster profit increase.

## 6. Discussion

In this paper, we have considered monopoly pricing problem with two different buyers in its simplest form, namely under the single-crossing property. Basically this property means that the buyers' indifference curves starting from the same bundle do not intersect each other. Hence, the optimal bundles for the problem are increasing in buyer type, and the IR and IC inequality constraints become equality constraints simplifying the solution of the problem considerably. With the single-crossing property in mind we have then shown that the problem can be solved using a simple heuristic online iteration scheme without explicitly referring to the optimality conditions. Our only hint has been that in each period the seller seeks the best profit for himself, without making the buyers to be worse off. These are guaranteed by putting also the old bundle for sale when testing the price for the new one. Recall that the main iteration of the method consists of increasing the amount of good by a fixed $\Delta x$, and then adjusting the price to be on optimal curves, e.g., using the bisection method. Note that similar adjustment could also be done in the *x*-direction when approaching the optimal amounts $\bar{x}_L$ and $\bar{x}_H$.

We have shown (the result follows from Lemma 1) that our method is a steepest ascent, or gradient search method, the iteration path of which proceeds along the optimal indifference curves. The method is robust in the sense that the bundles put for sale (as well as the corresponding profits) remain close to each other in subsequent periods, provided we start the method so that the initial bundles for *L* and *H* are close to each other.

We have then modified the method by making explicit use of linear tariffs and optimality conditions (8) and (9). This method proceeds in discrete steps along "nearly" optimal path, while pure price adjustments are only done in the neighborhood of optimal amounts.

Our online adjustment scheme is the first one presented in the literature for Spence's nonlinear pricing problem [21]. When presenting our method we have also had in mind the solving of more complicated nonlinear pricing problems with schemes that are efficient with respect to data collection during computation. Braden and Oren [6], and more recently Brooks et al. [5], discuss rather lengthly, and address the important question of appropriate customer preference revelation when planning and developing various online learning and adjustment schemes. Using extensive data collection to solve the problem at hand from the start can be quite costly, and also lead to imprecise results, since the customer preferences may vary considerably over time. Brooks et al. [5] also discuss profit losses for a firm as a consequence of usually time-consuming preference learning. They study online adjustment of different price tariffs for a consumer agent society and compare their efficiency and trade-off between complexity and profitability.

The single-crossing property, Assumption 1, with the curvature condition, Assumption 4, is quite a strong assumption. Note that Assumption 4 implies also strict concavity of $V_L$ and $V_H$. One question we will study in the future is, if we can relax these assumptions and still get sensitive results. The answer is, hardly not. In a forthcoming article [2] we will study a model with more than two consumer classes, and it seems that even with the single-crossing property there may arise new phenomena making the numerical characterization of the solution quite a challenging task. One such property is known as *bunching*, which means that different types of consumers get the same bundle in the optimum. Such properties are apt to make the practical implementation of any solution method rather challenging as well. The complexity of the problem solution increases further if we allow several goods to be allocated in addition to several types of consumer classes. In Berg and Ehtamo [3] we derive some preliminary results for this problem using graph theory.

Nevertheless, in its general form, i.e., with more than one quantity or quality, with more than two consumer classes, and without the single-crossing property, the nonlinear pricing problem is a complex optimization problem. Indeed, it is a multi-dimensional bilevel optimization problem (i.e., where the constraints include optimization problems) with large number of local optima. It seems that similar adjustment schemes as we have studied here, gradient search methods and even tabu search for certain combinatorial parts of the problem, might perform well also for these more general problems, at least near local optima. They could be used to produce moderate profit increase for a firm during its sales promotion. This paper gives valuable hints and serves a basis for further research in the area.

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