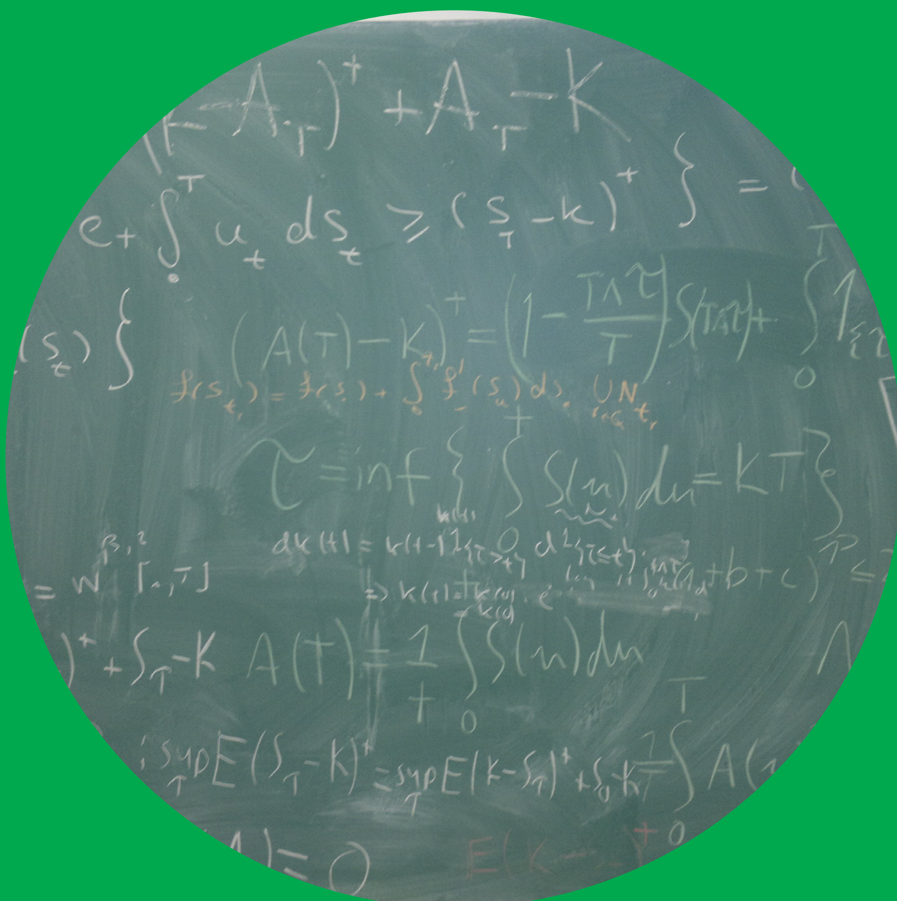
Department of Mathematics and Systems Analysis

Fractional processes, pathwise stochastic analysis and finance

Heikki Tikanmäki



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Fractional processes, pathwise stochastic analysis and finance

Heikki Tikanmäki

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Abstract

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Abstract

This thesis is about fractional processes, their pathwise stochastic analysis and financial applications. Firstly, we introduce a new definition for fractional Lévy processes using the integral representation of fractional Brownian motion on a compact interval. The properties of such processes, as well as connections to the earlier definitions of fractional Lévy processes, are studied. Secondly, the thesis contains integral representations for functionals depending on various averages of (geometric) fractional Brownian motion. These integral representations can be used for obtaining hedges for Asian options in fractional pricing models. Finally, we introduce the core financial contribution of the thesis that is to study the connections between pathwise functional calculus and robust hedging in so-called mixed models. In such models, the price of an asset is modeled as an exponential of the sum of Brownian motion and a fractional process.

Keywords fractional Brownian motion, fractional Lévy process, functional Ito calculus, vertical derivative, Asian options, arithmetic average, robust hedging, non-semimartingale models

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Tiivistelmä

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Heikki Tikanmäki

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Fraktionaaliset prosessit, poluttainen stokastinen analyysi ja rahoitusteoria

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Tiivistelmä
Tämä väitöskirja käsittelee fraktionaalisia prosesseja, niiden poluttaista stokastista analyysia sekä rahoitussovelluksia. Aluksi työssä esitellään uusi määritelmä fraktionaalisille Lévy-prosesseille käyttäen fraktionaalisen Brownin liikkeen integraaliesitystä kompaktilla välillä. Näiden prosessien ominaisuuksia käsitellään sekä tutkitaan yhteyksiä aikaisempiin fraktionaalisten Lévy-prosessien määritelmiin. Seuraavaksi väitöskirjassa käsitellään (geometrisen) fraktionaalisen Brownin liikkeen keskiarvoista riippuvien funktionaalien integraaliesityksiä ja näiden sovelluksia aasialaisten optioiden suojaukseen fraktionaalisissa hinnoittelumalleissa. Lopuksi esitellään työn varsinainen finanssikontribuutio, joka on poluttaisen funktionaalisen kalkuluksen yhteys robustiin suojaukseen niin sanotuissa sekoitetuissa hinnoittelumalleissa. Tällaisissa malleissa osakkeen hinta on geometrinen Brownin liikkeen ja fraktionaalisen prosessin summa.

Avainsanat fraktionaalinen Brownin liike, fraktionaalinen Lévy-prosessi, funktionaalinen Ito-kalkyyli, vertikaalinen derivaatta, aasialaiset optiot, aritmeettinen keskiarvo, robusti suojaus, ei-semimartingaalimallit

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Preface

This thesis is my fractional contribution to the theory of stochastic processes and mathematical finance. I started the thesis project in the group of stochastics of Helsinki University of Technology, Institute of Mathematics in 2007 and finished it five years later in Aalto University, School of Science, Department of Mathematics and Systems Analysis. The group is still the same despite of various changes in the university administration.

I am very grateful to my supervisor and advisor professor Esko Valkeila for all the great ideas, comments and guidance during my PhD project. Because of him, I have had the chance to work in such a stimulating environment as the group of stochastics is.

I would also like to thank the pre-examiners of the thesis, professors Francesco Russo and Tommi Sottinen for their reports and all the valuable comments.

I am grateful to all my colleagues who have helped me during my studies. In particular, I want to thank my co-author Yuliya Mishura for valuable discussions, Ehsan Azmoodeh and Lasse Leskelä for teaching me a lot on stochastics and mathematics in general, Lauri Viitasaari for sharing the office in a fair way, my sister Johanna Tikanmäki for helping me in the beginning of my studies and Milla Kibble for correcting my English language.

Many thanks also to the colleagues who I forgot to mention by name. The atmosphere in the institute of mathematics has always been great because of you all. Especially I would like to thank the members of our coffee room gang.

My PhD project has been partly funded by Finnish Academy of Science and Letters, Vilho, Yrjö and Kalle Väisälä Foundation, Finnish Doctoral Programme in Stochastics and Statistics (FDPSS) and Academy of Finland grant 21245.

Finally, I want to express my greatest thanks to my friends and family for supporting me during my studies.

Espoo, April 10, 2012

Heikki Tikanmäki

Contents

List of Publications

This thesis consists of an overview and of the following publications which are referred to in the text by their Roman numerals.

I H. Tikanmäki and Yu. Mishura. Fractional Lévy processes as a result of compact interval integral transformation. *Stoch. Anal. Appl.*, 29:1081-1101, 2011.

II H. Tikanmäki. Integral representations of some functionals of fractional Brownian motion. *Commun. Stoch. Anal.*, accepted, 19 pages, 2012.

III H. Tikanmäki. Robust hedging and pathwise calculus. *http://arxiv.org/abs/1110.5202*, 19 pages, 09 Dec 2011.

Author's Contribution

Publication I: "Fractional Lévy processes as a result of compact interval integral transformation"

The article is partly based on the author's licentiate's thesis. The definition of fLpMG and most of the results are individual contributions of the author. The exceptions are: Theorem 3.2. is a joint work with Yuliya Mishura. The lemmas in section 7 are by Mishura.

All the illustrations are by the author. The article is mainly written by the author.

Publication II: "Integral representations of some functionals of fractional Brownian motion"

This is an individual work of the author.

Publication III: "Robust hedging and pathwise calculus"

This is an individual work of the author.

1. Introduction

1.1 Fractional processes

This thesis considers certain stochastic processes, namely fractional processes, their pathwise stochastic analysis and possible financial applications. In this thesis, fractional processes cover processes sharing the covariance structure with fractional Brownian motion. As well as fractional Brownian motion, we consider two different fractional Lévy process concepts. The first concept is fractional Lévy processes by Mandelbrot-Van Ness representation ([5, 32]). The other, introduced by the author, is fractional Lévy processes by Molchan-Golosov representation (Publication I).

Fractional processes have several useful properties. Their covariance structure allows for modeling long range dependence, which is a desired property in various application areas. Besides financial applications, fractional processes have been used in statistical modeling, hydrology, telecommunication and population biology, just to mention a few.

1.2 Pathwise stochastic analysis

The stochastic analysis part of the thesis covers several pathwise stochastic integration concepts: the Young integral, generalized Lebesgue-Stieltjes integral and forward type integrals.

As well as stochastic integration, we consider the functional change of variables formula of [13]. For this formula we need two functional derivative concepts for the path: horizontal and vertical derivatives. The functional change of variables formula yields a martingale representation theorem ([14]). In some cases this formula can also be understood in a pathwise sense. The pathwise martingale representation formula is slightly extended in Publication III.

Pathwise stochastic integration, fractional Besov space techniques and the functional change of variables formula are combined in Publication II. As a result we obtain integral representation results for functionals of a path of a fractional Brownian motion.

1.3 Finance

The abovementioned stochastic analysis techniques are also used for solving the hedging problem of path dependent options in models including fractional processes. The models considered in the thesis cover the fractional Black-Scholes and fractional Bachelier models as well as several mixed models. By a mixed model, we mean a Black-Scholes type model driven by the sum of a Brownian motion and a zero quadratic variation process (for example a fractional process). In other words, the price processes are exponentials of certain Dirichlet processes.

One of the financial contributions of this thesis is the hedging of Asian options in fractional Black-Scholes and fractional Bachelier models (Publication II). These two purely fractional models are more or less toy models but the mixed models that we mention next have a serious financial interpretation.

In the mixed models, the Brownian motion part corresponds to the ordinary short time market fluctuations, whereas the fractional part corresponds to the long term trends in the stock market. By the choice of the fractional process, we can include different stylized facts in the model: short or long range dependence and arbitrarily heavy or light tails. In Publication I we propose a mixed Brownian motion and fractional Lévy process market model.

A no-arbitrage theorem as well as a robust replication result for mixed models can be found in [9]. In Publication III we combine the ideas in [9] with the functional calculus of [13, 14] and obtain new results on robust hedging in mixed models.

1.4 Notation

We now introduce the notation that is used in the introductory part of this thesis and, with slight changes, in Publications II and III. The notation of Publication I is somewhat different, the main difference being that the convention for point values and paths of stochastic processes is different.

We will work in a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$ if not mentioned otherwise. The filtration is usually the filtration generated by the process under consideration, if not specified otherwise.

We denote by $D([a,b], A)$ the cadlag paths defined on the interval $[a,b]$ and taking values in A. Abbreviation $D([a,b])$ denotes $D([a,b], \mathbb{R})$. We denote the spaces of continuous functions by $C([a,b], A)$ and $C([a,b])$, respectively.

Let $T > 0$ be fixed and $x \in D([0,T])$. Then $x(t)$ denotes the value of function x at point t and $x_t = (x(s))_{s \in [0,t]}$ denotes the whole path of x from 0 to t. We denote by x_{t-} the path where the end point is replaced by the left limit. More rigorously,

$$\begin{aligned} x_{t-}(u) =& x(u), \quad u \in [0,t), \\ x_{t-}(t) =& x(t-), \end{aligned}$$

where $x(t-) = \lim_{s \uparrow t} x(s)$ is the left limit. Note that in general x_{t-} is not the same path as $(x_-)(t) = (x(u-))_{u \in (0,t]}$. The right limit is defined as $x(t+) = \lim_{s \downarrow t} x(s)$.

The same conventions apply to the stochastic processes. That is, $X(t)$ denotes the value of process X at time t and X_t denotes the whole path of X up to time t.

2. Stochastic processes

In this chapter we will introduce the stochastic processes considered in the thesis.

2.1 Fractional Brownian motion

Fractional Brownian motion (fBm) is, as the name suggests, a generalization of classical Brownian motion. While Brownian motion has independent increments, the increments of fBm are dependent. Because of its dependence structure, fBm is widely used as a model in various application areas.

Definition 2.1.1 (Fractional Brownian motion)**.** *A zero mean Gaussian process* $(B^H(t))_{t\geqslant 0}$ *is a fractional Brownian motion with Hurst index* $H \in (0,1)$ *if*

$$\mathbb{E}B^H(t)B^H(s) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}), \quad t,s \geqslant 0.$$

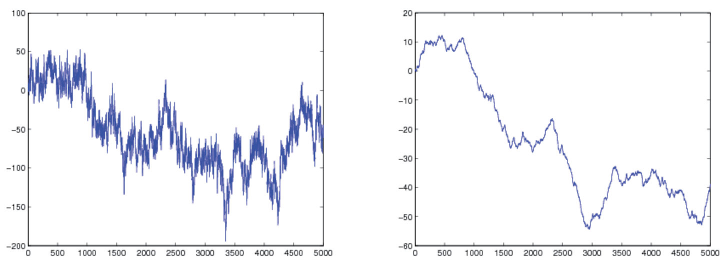

Figure 2.1. Paths of fBm, when $H = 0.25$ (left) and $H = 0.75$ (right).

When $H = \frac{1}{2}$ this reduces to the definition of ordinary Brownian motion. When $H \neq \frac{1}{2}$, fractional Brownian motion is neither a Markov process nor a semimartingale. However, fBm has several useful properties: the increments of fBm are stationary. An fBm with Hurst index H is self-similar with index H. When $H > \frac{1}{2}$, fBm has the long range dependence property. For $H < \frac{1}{2}$, the increments are negatively correlated. The dependence structure is completely characterized by parameter H.

The paths of fBm with Hurst index H are Hölder continuous of any order $\delta \in (0, H)$. In other words, the paths of fBm get more regular when the Hurst index grows, see figure 2.1.

2.1.1 Integral representations of fBm

Fractional Brownian motion can be represented as an integral of a deterministic kernel with respect to an ordinary Brownian motion in several ways. In this thesis, we consider a compact interval integral representation by Molchan and Golosov and an infinite interval integral representation by Mandelbrot and van Ness.

Molchan-Golosov representation

The following compact interval integral representation of fBm is by Molchan and Golosov [37]. First we will define the Molchan-Golosov kernel.

Definition 2.1.2 (Molchan-Golosov kernel)**.** *Let $H > \frac{1}{2}$, then we define*

$$z_H(t,s) = \left(H - \frac{1}{2}\right) c_H s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{1}{2}} (u-s)^{H-\frac{3}{2}} du, \quad 0 < s < t < \infty$$

and $z_H(t,s) = 0$ otherwise.

Let $H \leqslant \frac{1}{2}$, then we define

$$\begin{aligned} z_H(t,s) = & c_H \left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} \\ & - \left(H - \frac{1}{2}\right) c_H s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du, \quad 0 < s < t < \infty \end{aligned}$$

and $z_H(t,s) = 0$ otherwise. The constant is given as

$$c_H = \frac{1}{\Gamma(H+\frac{1}{2})} \left(\frac{2H\Gamma(H+\frac{1}{2})\Gamma(\frac{3}{2}-H)}{\Gamma(2-2H)} \right)^{\frac{1}{2}},$$

where Γ denotes the Gamma function.

There are also other ways to represent the kernel z_H, see for example [27]. Using the kernel z_H we have the following representation of fBm.

Theorem 2.1.3. *Let $H \in (0,1)$. Then it holds that*

$$(B^H(t))_{t\in[0,\infty)} \stackrel{d}{=} \left(\int_0^t z_H(t,s) dW(s) \right)_{t\in[0,\infty)},$$

where $(W(t))_{t\in[0,\infty)}$ is a Brownian motion.

The proof can be found in [27, 38, 37]. Here $\stackrel{d}{=}$ denotes equality in finite dimensional distributions.

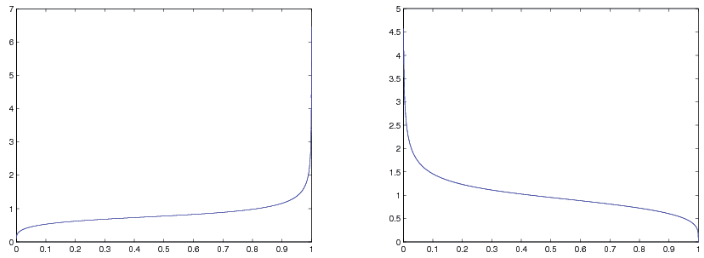

Figure 2.2. MG kernel $z_H(1,s)$ for $H = 0.25$ (left) and for $H = 0.75$ (right).

Mandelbrot-Van Ness representation

Another integral representation for fBm is by Mandelbrot and van Ness [31]. For this representation we need two-sided processes. Let W^1, W^2 be two independent Brownian motions. A two-sided Brownian motion $(W(t))_{t\in\mathbb{R}}$ is defined as

$$\begin{aligned} W(t) =& W^1(t), \quad t \geqslant 0 \quad \text{and} \\ W(t) =& W^2(-t), \quad t < 0. \end{aligned}$$

A two-sided fBm $(B^H(t))_{t\in\mathbb{R}}$ is defined for $H \in (0,1)$ as a centered Gaussian process with covariance

$$\mathbb{E}B^H(t)B^H(s) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right), \quad t, s \in \mathbb{R}.$$

The Mandelbrot-Van Ness kernel is defined as follows:

Definition 2.1.4 (Mandelbrot-Van Ness kernel)**.** *For* $H \in (0,1)$

$$f_H(t,s) = c_H\left((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\right), \quad t, s \in \mathbb{R}.$$

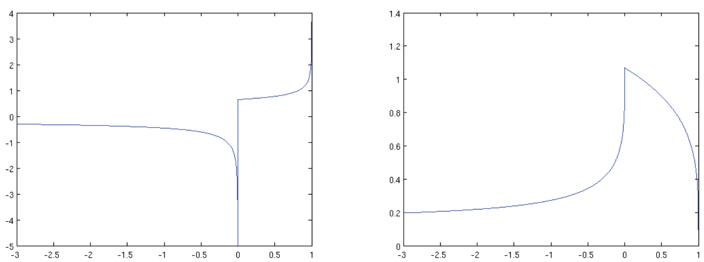

Figure 2.3. MvN kernel $f_H(1,s)$ with $H = 0.25$ (left) and $H = 0.75$ (right).

It is proved in Publication I that the constant c_H is the same as in the definition of the Molchan-Golosov kernel. Now we are ready to write the Mandelbrot-Van Ness representation theorem for fBm.

Theorem 2.1.5. *Let $H \in (0,1)$, then*

$$(B^H(t))_{t\in\mathbb{R}} \overset{d}{=} \left(\int_{\mathbb{R}} f_H(t,s)dW(s)\right)_{t\in\mathbb{R}},$$

where $(W(t))_{t\in\mathbb{R}}$ is a two-sided Brownian motion.

2.2 Lévy processes

Lévy processes are stochastic processes with independent and stationary increments. The best known examples of Lévy processes are Brownian motion and Poisson processes. Note that fractional Brownian motion is not a Lévy process.

Definition 2.2.1 (Lévy process)**.** *A real valued stochastic process $(X(t))_{t\geqslant 0}$ is a Lévy process if the following conditions hold*

1. *$X(t) - X(s) \overset{d}{=} X(t-s)$, for $t \geqslant s \geqslant 0$,*
2. *$X(t) - X(s)$ is independent of X_s, for $t \geqslant s \geqslant 0$,*
3. *$\mathbb{P}(X(0) = 0) = 1$ and*
4. *the paths of X are right-continuous with left limits almost surely.*

Here $\overset{d}{=}$ denotes equality in distribution. Lévy processes are completely characterized by the Lévy-Khinchin theorem. Before stating the theorem we define the concepts of characteristic triplet and characteristic exponent.

Definition 2.2.2 (Characteristic triplet)**.** *Let $\gamma \in \mathbb{R}$, $\sigma \geqslant 0$ and ν be a measure on $\mathbb{R}$ satisfying $\nu(\{0\}) = 0$ and*

$$\int_{\mathbb{R}} (1 \wedge x^2)\nu(dx) < \infty.$$

Then ν is a Lévy measure and (γ, σ, ν) is a characteristic triplet or Lévy triplet.

Definition 2.2.3 (Characteristic exponent)**.** *Let (γ, σ, ν) be a Lévy triplet. Then*

$$\Psi(u) = i\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} \left(e^{ixu} - 1 - ixu1_{(-1,1)}(x)\right)\nu(dx) \tag{2.1}$$

is called a characteristic exponent.

Now we are ready to state the Lévy-Khinchin theorem.

Theorem 2.2.4 (Lévy-Khinchin). *Let X be a Lévy process. Then there exists a unique characteristic triplet (γ, σ, ν) such that the characteristic function of $X(t)$ is given by*

$$\mathbb{E}e^{iuX(t)} = e^{t\Psi(u)}, \tag{2.2}$$

where Ψ is defined as in equation (2.1) and $t \geqslant 0$.

The converse also holds: for any characteristic triplet (γ, σ, ν) there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where one can define a Lévy process X such that equation (2.2) holds.

The proof of the above theorem can be found for example in [30] or [47]. The following Lévy-Itô decomposition gives some intuition on the characteristic triplet. First we need the definition of Poisson random measures.

Definition 2.2.5 (Poisson Random measure). *Let $(S, \mathcal{S})$ be a measurable space. A mapping $\eta : \Omega \times S \mapsto 0, \ldots, \infty$ is called a Poisson random measure with intensity measure μ if*

1. *measure μ is σ-finite,*

2. *$\eta(\omega, \cdot)$ is measure $\mathbb{P}$-a.s.,*

3. *$\eta(\cdot, B)$ is a random variable for $B \in \mathcal{S}$,*

4. *for disjoint $B_1, \ldots, B_n \in \mathcal{S}$, random variables $\eta(\cdot, B_1), \ldots, \eta(\cdot, B_n)$ are independent and*

5. *$\eta(\cdot, B) \sim$ Poisson$(\mu(B))$, when $B \in \mathcal{S}$ s.t. $\mu(B) < \infty$.*

Now we are ready for the Lévy-Itô theorem. The main message of the theorem is that any Lévy process can be decomposed into a sum of independent Brownian motion, compound Poisson process and a pure jump martingale.

Theorem 2.2.6 (Lévy-Itô decomposition). *Let (γ, σ, ν) be a characteristic triplet. Then there exists a probability space in which there exist three independent Lévy processes X^1, X^2, X^3 satisfying the following conditions:*

$$X^1(t) = \sigma W(t) + \gamma t$$

is a Brownian motion.

$$X^2(t) = \sum_{j=1}^{N(t)} Y_j,$$

is a compound Poisson process. Here N is a Poisson process with intensity $\nu((-\infty, -1] \cup [1, \infty))$ and $(Y_j)_{j=1}^{\infty}$ is an i.i.d. sequence with distribution

$$1_{(-\infty,-1]\cup[1,\infty)}(x)\frac{\nu(dx)}{\nu((-\infty, -1] \cup [1, \infty))}$$

independent of N. If $\nu((-\infty, -1] \cup [1, \infty)) = 0$, then X^2 is defined to be identically zero.

$$X^3(t) = \lim_{\epsilon\downarrow 0} \int_0^t \int_{\epsilon\leqslant|x|<1} x(\eta(\cdot, ds \times dx) - ds\nu(dx))$$

is a square integrable martingale with at most countably many jumps in each compact interval almost surely. Here η is a Poisson random measure with intensity measure $Leb \times \nu$.

The characteristic triplet of the Lévy process

$$X(t) = X^1(t) + X^2(t) + X^3(t)$$

is (γ, σ, ν).

The proof of the theorem can be found for example in [30]. Note that the Lévy processes X^1, X^2, X^3 are unique in distribution.

2.3 Fractional Lévy processes

In this thesis the term fractional Lévy processes refers to integral transformations of Lévy processes defined using either the Mandelbrot-Van Ness or Molchan-Golosov kernel. Such fractional Lévy processes have been studied in the literature by [5], [32, 33] and [49], among others. Note that fractional Lévy processes are not Lévy processes.

There is plenty of literature on various fractional processes that go beyond the scope of the present thesis. Such processes are for example fractional stable motions ([46]) or fractional Poisson processes ([4]).

2.3.1 FLp by Mandelbrot-Van Ness representation

Fractional Lévy processes by Mandelbrot-Van Ness representation were first introduced by [5] and further studied by [32]. For the definition, we need the concept of two-sided Lévy processes.

Definition 2.3.1 (Two-sided Lévy processes). *Let* $(X^1(t))_{t\geqslant 0}$ *and* $(X^2(t))_{t\geqslant 0}$ *be two independent Lévy processes with characteristic triplet* (γ, σ, ν). *Define*

$$L(t) = X^1(t) \quad t \geqslant 0,$$
$$L(t) = -X^2(-t-) \quad t < 0.$$

Then $(L(t))_{t\in\mathbb{R}}$ *is called a Lévy process on* $\mathbb{R}$ *or a two-sided Lévy process with characteristic triplet* (γ, σ, ν).

A fractional Lévy process by Mandelbrot-Van Ness representation (fLp-MvN) can now be defined.

Definition 2.3.2 (FLp by MvN representation). *Let* L *be a zero-mean square integrable two-sided Lévy process without Brownian component and* $H \in (0,1)$ *a Hurst index. Then*

$$X(t) = \int_{\mathbb{R}} f_H(t,s)dL(s)$$

is called a fractional Lévy process by Mandelbrot-Van Ness representation.

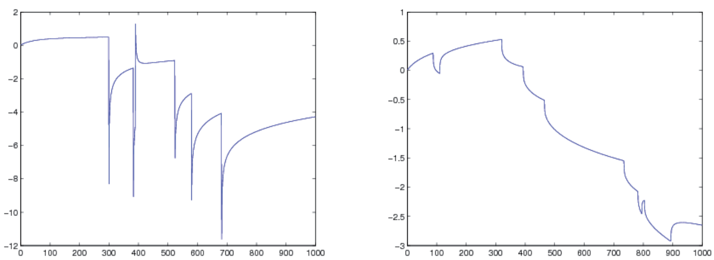

Figure 2.4. Paths of fLpMvN with Lévy measure $\lambda(\delta_1 + \delta_{-1})$ in the case $H = 0.25$ (left) and $H = 0.75$ (right).

The integral is understood as a limit in probability of elementary integrands or in the L^2 sense. For more details, see [32] or [41]. For the case when $H > \frac{1}{2}$, the integral can be understood pathwise as an improper Riemann integral, see [32].

The covariance structure of fLpMvN is the same as that of fBm. The existence of the higher order moments depends on the driving Lévy process L (and on H).

When $H > \frac{1}{2}$, X has almost surely Hölder continuous paths of any order less than $H - \frac{1}{2}$ even though the driving Lévy process has jumps. The increments of X are stationary but X is never self-similar. The semimartingale property of X depends on the driving Lévy process, see [3] and [7] for the complete characterization. For $H > \frac{1}{2}$, the paths of X have zero quadratic variation along a dyadic sequence of partitions.

When $H < \frac{1}{2}$, one can prove using [44, theorem 4] that the paths of an fLpMvN are unbounded and discontinuous with positive probability.

2.3.2 FLp by Molchan-Golosov representation

Fractional Lévy processes by Molchan-Golosov representation (fLpMG) are first introduced in Publication I. However, in a related setup, the Molchan-Golosov kernel has been used before for defining fractional subordinators in [8].

Fractional Lévy processes by Molchan-Golosov representation are defined analogously to fLpMvN, but instead of the Mandelbrot-Van Ness kernel we use the Molchan-Golosov kernel. The advantage of this definition is that the infinite history of the driving Lévy process is not needed.

Definition 2.3.3 (Fractional Lévy process by Molchan-Golosov representation)**.** *Let L be a zero mean square integrable Lévy process and $H \in (0, 1)$. A fractional Lévy process by Molchan-Golosov representation is defined as*

$$Y(t) = \int_0^t z_H(t, s) dL(s).$$

The integral can be understood as a limit in probability of elementary integrals, in the L^2 sense and in some cases also pathwise. FLpMG shares

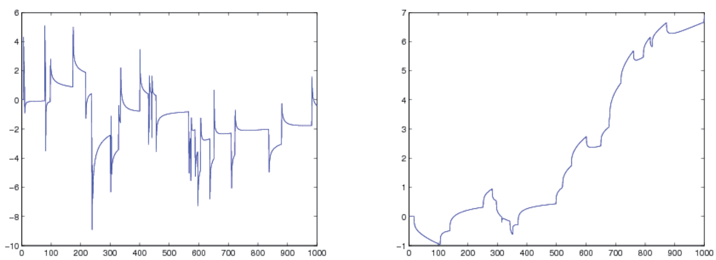

Figure 2.5. Simulated paths of fLpMG when the Lévy measure is $\lambda(\delta_1+\delta_{-1})$ and $H = 0.25$ (left) or $H = 0.75$ (right).

the covariance structure with fBm. The existence of the higher order moments depends on L and on H. For $H > \frac{1}{2}$, the paths of fLpMG are Hölder continuous of any order less than $H - \frac{1}{2}$ and thus they have zero quadratic variation along a dyadic sequence of partitions. When $H < \frac{1}{2}$, the paths are discontinuous and unbounded with positive probability.

An fLpMG cannot be self-similar. The semimartingale property depends on the driving Lévy process, but there is not a complete characterization yet. One drawback of fLpMG compared to fLpMvN is that the increments of fLpMG are not stationary in general.

3. Pathwise techniques

In this chapter we discuss pathwise techniques in stochastic analysis. First we introduce several pathwise integration techniques. After that we introduce pathwise differential operations and a pathwise functional change of variables formula.

3.1 Pathwise integration

In the semimartingale case, the standard integration procedure uses the Itô integral. However, fractional Brownian motion is never a semimartingale and fractional Lévy processes are not always semimartingales. To cover these processes, we need some other concept of integration. One option is to define the stochastic integrals pathwise "ω by ω". In the following, we introduce three different pathwise integration concepts: the Young integral, generalized Lebesgue-Stieltjes integral and forward integral.

3.1.1 Young integration

Young integration is a Riemann-Stieltjes type integral concept for Hölder continuous functions. The following construction of the integral is taken from [45]. In [23], the Young integral is defined slightly more generally for functions of finite p-variation.

Let us denote by C_α the set of Hölder continuous functions of order $\alpha \in (0, 1]$ defined on $[0, T]$. First we define the Young integral in the Lipschitz case.

Definition 3.1.1 (Young integral for Lipschitz functions)**.** *Let $f, g \in C_1$. The Young integral of f with respect to g on $[a, b] \subset [0, T]$ is defined as*

$$\int_a^b f(t)dg(t) = \int_a^b f(t)g'(t)dt.$$

Now we have the following proposition that extends the definition of

Young integrals to Hölder continuous functions that are not Lipschitz continuous.

Proposition 3.1.2 (Young integral)**.** *Let $\alpha, \beta \in (0,1]$ s.t. $\alpha + \beta > 1$. The map*

$$C_1 \times C_1 \mapsto C_\alpha, \quad (f,g) \mapsto \int_0^\cdot f(t)dg(t)$$

extends to a continuous bilinear map $C_\alpha \times C_\beta \mapsto C_\alpha$. The value of this extension at point $(f,g) \in C_\alpha \times C_\beta$ is denoted by

$$\int_0^\cdot f(t)dg(t)$$

and is called the Young integral of f with respect to g.

The proof of the above can be found in [45].

Remark 3.1.3. *The Young integral can be understood also as a limit of Riemann-Stieltjes sums, where the integral is independent of the sequence of partitions, see [23].*

3.1.2 Generalized Lebesgue-Stieltjes integrals and fractional Besov space techniques

This approach to integration was first introduced in [51, 52] and further developed in [40]. Our main reference here is [35]. First we need the concept of fractional derivatives. Left and right Riemann-Liouville fractional derivatives of order $\alpha \in (0,1)$ on the interval $[a,b]$ are defined as

$$(D^\alpha_{a+}f)(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x f(t)(x-t)^{-\alpha}dt$$

and

$$(D^\alpha_{b-}f)(x) = \frac{-1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^b f(t)(t-x)^{-\alpha}dt,$$

where f is such that the integral and derivative can be defined in the usual sense.

Let us now define the fractional Besov spaces.

Definition 3.1.4 (Fractional Besov space of order $1,\beta$)**.** *Let $\beta \in (0,1)$ and $f : [0,T] \mapsto \mathbb{R}$ satisfy*

$$\|f\|_{1,\beta} = \sup_{0 \leqslant s < t \leqslant T}\left(\frac{|f(t)-f(s)|}{(t-s)^\beta} + \int_s^t \frac{|f(u)-f(s)|}{(u-s)^{1+\beta}}du\right) < \infty.$$

Then $f \in W_1^\beta$. The space W_1^β is called the fractional Besov space of order $1,\beta$.

Note that $\|\cdot\|_{1,\beta}$ is not a proper norm but a seminorm. The other Besov space is defined as follows.

Definition 3.1.5 (Fractional Besov space of order $2,\beta$)**.** *Let $\beta \in (0,1)$ and $f:[0,T] \mapsto \mathbb{R}$ be such that*

$$\|f\|_{2,\beta} = \int_0^T \frac{|f(s)|}{s^\beta} ds + \int_0^T \int_0^s \frac{|f(s)-f(u)|}{(s-u)^{\beta+1}} duds < \infty.$$

Then $f \in W_2^\beta$. The space W_2^β is the fractional Besov space of order $2,\beta$.

The space W_2^β is a Banach space. For $\epsilon \in (0, \beta \wedge 1-\beta)$ we have the following inclusions

$$C_{\beta+\epsilon} \subset W_1^\beta \subset C_{\beta-\epsilon}$$

and

$$C_{\beta+\epsilon} \subset W_2^\beta.$$

Now we are ready for the definition of the generalized (fractional) Lebesgue-Stieltjes integral (gLS).

Definition 3.1.6 (Generalized Lebesgue-Stieltjes integral)**.** *Let $[a,b] \subset [0,T]$, $\beta \in (0,1)$ and $f \in W_2^\beta$ and $g \in W_1^{1-\beta}$. Set*

$$\begin{aligned} f_{a+}(x) &= (f(x)-f(a+))1_{(a,b)}(x) \quad \text{and} \\ g_{b-}(x) &= (g(b-)-g(x))1_{(a,b)}(x). \end{aligned}$$

The generalized Lebesgue-Stieltjes integral of f with respect to g is defined as

$$\int_a^b f(x)dg(x) = \int_a^b (D_{a+}^\beta f_{a+})(x)(D_{b-}^{1-\beta} g_{b-})(x)dx + f(a+)(g(b-)-g(a+)).$$

It holds for $u \in (s,t)$ that

$$\int_s^t f dg = \int_s^u f dg + \int_u^t f dg,$$

which is not obvious from the definition. The definition does not depend on the selection of β either. See [35]. We also have the following norm inequality.

Proposition 3.1.7. *Let $f \in W_2^\beta$ and $g \in W_1^{1-\beta}$. Then*

$$\left| \int_0^t f dg \right| \leqslant C \|g\|_{1,1-\beta} \|f\|_{2,\beta}.$$

This norm inequality is very useful, because it tells us that the convergence of the integrands implies convergence of the integrals as well. The following result gives the connection between the Young integral and generalized Lebesgue-Stieltjes integral.

Proposition 3.1.8. *If $\alpha, \beta \in (0,1]$ such that $\alpha + \beta > 1$, $f \in C_\alpha$ and $g \in C_\beta$, then*

$$\text{(gLS)} \int_a^b f dg = \text{(Young)} \int_a^b f dg.$$

3.1.3 Forward type integrals

Next we consider forward integrals to the extent that is needed for the robust hedging approach of [9]. In this thesis we will also consider the so-called Föllmer integral, which is a slight modification of the forward integral. The Föllmer integral will be introduced later in the context of the functional change of variables formula.

The set $\pi = \{0 = t_0 < t_1 < \cdots < t_k = T\}$ is called a partition of the interval $[0, T]$. The size of the partition is defined as

$$|\pi| = \max_{j=1,\ldots,k} |t_j - t_{j-1}|.$$

The concept of partition is needed in the definition of the forward integral.

Definition 3.1.9 (Forward integral). *Let $(\pi_n)_{n=1}^\infty$, where $\pi_n = \{0 = t_0^n < \cdots < t_{k(n)}^n = T\}$, be a sequence of partitions of $[0, T]$ such that $|\pi_n| \to 0$, $t < T$ and $(X(s))_{s\in[0,T]}$ be a continuous stochastic process. The forward integral of process $(Y(s))_{s\in[0,T]}$ along the sequence $(\pi_n)_{n=1}^\infty$ with respect to X is defined as*

$$\int_0^t Y(s) dX(s) = \lim_{n\to\infty} \sum_{t_j^n \in \pi_n \cap (0,t]} Y(t_{j-1}^n)(X(t_j^n) - X(t_{j-1}^n)),$$

where the limit is assumed to exist almost surely.

The forward integral over the whole interval $[0, T]$ is defined as the improper integral

$$\int_0^T Y(s) dX(s) = \lim_{t\uparrow T} \int_0^t Y(s) dX(s).$$

The sequence of partitions is usually fixed and thus is not included in the notation.

Proposition 3.1.10. *Let $X \in C_\alpha$ and $Y \in C_\beta$ almost surely, where $\alpha, \beta \in (0,1]$ and $\alpha + \beta > 1$. Then*

$$\text{(Young)} \int_0^{\cdot} Y(s) dX(s) = \text{(forward)} \int_0^{\cdot} Y(s) dX(s).$$

This is a consequence of remark 3.1.3. Hence, when the Young integral is defined, the generalized Lebesgue-Stieltjes integral, forward integral and Young integral all take the same value.

3.2 Functional change of variables formula

The functional calculus approach was initiated by [19] and further developed by [13, 14, 22]. This section is mainly based on the articles [13] and [14] and the PhD thesis [22], which work in $\mathbb{R}^n$. In this thesis we will only need the theory in one dimension.

The concept of non-anticipative functionals is essential in this approach. Let $U \subset \mathbb{R}$ be open.

Definition 3.2.1 (Non-anticipative functional). *A non-anticipative functional on $D([0,T],U)$ is a family of mappings*

$$F = (F_t)_{t\in[0,T]} \quad \textit{s.t.} \quad F_t : D([0,t],U) \mapsto \mathbb{R}.$$

Note that F can be seen as a functional on the vector bundle

$$\Psi = \bigcup_{t\in[0,T]} D([0,t],U).$$

For the measurability details, see [13]. Sometimes we also consider non-anticipative functionals indexed by interval $[0,T)$. The definition in that case is analogous.

Next we will construct horizontal and vertical derivatives of a path.

Definition 3.2.2 (Horizontal extension). *Let $x \in D([0,T],U)$ and $0 \leqslant t < t+h \leqslant T$. The horizontal extension of x_t to $[0,t+h]$ is defined as*

$$\begin{aligned} x_{t,h}(u) =& x(u), \quad u \in [0,t], \\ x_{t,h}(u) =& x(t), \quad u \in (t,t+h]. \end{aligned}$$

The horizontal derivative is now defined.

Definition 3.2.3 (Horizontal derivative). *The horizontal derivative of a non-anticipative functional F at $x \in D([0,t],U)$ is defined as*

$$\mathcal{D}_t F(x) = \lim_{h\downarrow 0} \frac{F_{t+h}(x_{t,h}) - F_t(x)}{h}$$

if the limit exists. If the limit is defined for all $x \in \Psi$, then

$$\mathcal{D}_t F : D([0,t],U) \mapsto \mathbb{R}, \quad x \mapsto \mathcal{D}_t F(x)$$

defines a non-anticipative functional $\mathcal{D}F = (\mathcal{D}_t F)_{t\in[0,T)}$, the horizontal derivative of functional F.

Another perturbation of the path is needed for the definition of the vertical derivative.

Definition 3.2.4 (Vertical perturbation). *Let* $x \in D([0,T],U)$, $t \in [0,T]$ *and* $h \in \mathbb{R}$ *be small enough. The vertical perturbation of* $x_t \in D([0,t],U)$, *namely* $x_t^h \in D([0,t],U)$ *is defined as*

$$\begin{aligned} x_t^h(u) &= x(u), \quad u \in [0,t), \\ x_t^h(t) &= x(t) + h. \end{aligned}$$

Now we are ready to define the vertical derivative.

Definition 3.2.5 (Vertical derivative). *The vertical derivative of a non-anticipative functional* F *at* $x \in D([0,t],U)$ *is defined as*

$$\nabla_x F_t(x) = \lim_{h \to 0} \frac{F_t(x_t^h) - F_t(x)}{h}$$

if the limit exists. If the limit is defined for all $x \in \Psi$, *then*

$$\nabla_x F : D([0,t],U) \mapsto \mathbb{R}, \quad x \mapsto \nabla_x F_t(x)$$

defines a non-anticipative functional $\nabla_x F = (\nabla_x F_t)_{t \in [0,T]}$, *the vertical derivative of* F.

We will need several different continuity concepts for the non-anticipative functionals. Continuity at fixed times is the simplest of these.

Definition 3.2.6 (Continuity at fixed times). *A non-anticipative functional* F *is continuous at fixed times if for all* $t \in [0,T]$ *the functional* $F_t : D([0,t],U) \mapsto \mathbb{R}$ *is continuous for the supremum norm.*

We will need a new metric for introducing the other continuity concepts. The metric is an extension of the metric induced by the supremum norm. Now the paths do not have to be defined on the same interval.

Definition 3.2.7 (Metric d_∞). *Let* $0 \leqslant t \leqslant t+h \leqslant T$, $x \in D([0,t],U)$ *and* $x' \in D([0,t+h],U)$. *We define*

$$d_\infty(x,x') = \sup_{u \in [0,t+h]} \left| x_{t,h}(u) - x'(u) \right| + h.$$

Now we are ready for the definition of left-continuous functionals.

Definition 3.2.8 (Left-continuous functionals). *A non-anticipative functional* F *is left-continuous if*

$$\begin{aligned} &\forall t \in [0,T], \forall \epsilon > 0, \forall x \in D([0,t],U), \exists \delta > 0, \forall h \in [0,t], \forall x' \in D([0,t-h],U), \\ &d_\infty(x,x') < \delta \implies \left| F_t(x) - F_{t-h}(x') \right| < \epsilon. \end{aligned}$$

The set of left-continuous functionals is denoted by $\mathbb{F}_l^\infty$.

Analogously one can define right-continuous functionals.

Definition 3.2.9 (Right-continuous functionals)**.** *A non-anticipative functional F is right-continuous if*

$$\forall t \in [0,T], \forall \epsilon > 0, \forall x \in D([0,t],U), \exists \delta > 0, \forall h \in [0,T-t], \forall x' \in D([0,t+h],U),$$
$$d_\infty(x,x') < \delta \implies \left|F_t(x) - F_{t+h}(x')\right| < \epsilon.$$

We denote by $\mathbb{F}_r^\infty$ the class of right-continuous functionals.

The class of continuous non-anticipative functionals is defined as $\mathbb{F}^\infty = \mathbb{F}_r^\infty \cap \mathbb{F}_l^\infty$.

The following concepts guarantee that the value of a non-anticipative functional does not explode in a neighborhood of any path.

Definition 3.2.10 (Boundedness-preserving functionals)**.** *Let us define the class of boundedness-preserving functionals $\mathbb{B}$. Let F be a non-anticipative functional. Then $F \in \mathbb{B}$ if for all compact $K \subset U$*

$$\exists C > 0, \forall t \leqslant T, \forall x \in D([0,t],K) \implies |F_t(x)| \leqslant C.$$

Boundedness-preserving functionals satisfy the following weaker local boundedness condition, i.e. they are bounded in the neighborhood of any given path.

Definition 3.2.11 (Locally bounded functionals)**.** *A non-anticipative functional F is locally bounded if*

$$\forall x \in D([0,T],U), \exists C > 0, \delta > 0, \forall t \in [0,T], \forall x' \in D([0,t],U),$$
$$d_\infty(x_t,x') < \delta \implies \forall t \in [0,T], \left|F_t(x')\right| \leqslant C.$$

For the statement of the functional change of variables formula we need the concept of quadratic variation. Quadratic variation will also play a role in the financial part of the thesis.

Definition 3.2.12 (Quadratic variation)**.** *The quadratic variation of a path $x \in C([0,T])$ along the sequence of partitions $(\pi_n)_{n=1}^\infty$ is defined as*

$$[x](t) = \lim_{n\to\infty} \sum_{t_k^n \in (0,t]\cap\pi_n} \left(x(t_k^n) - x(t_{k-1}^n)\right)^2$$

if the limit exists and is continuous in t.

A stochastic process is called a quadratic variation process along sequence $(\pi_n)_{n=1}^\infty$ if the pathwise quadratic variation exists almost surely.

We will also need the following classes of functionals.

Definition 3.2.13 (Smooth non-anticipative functionals $\mathbb{C}^{1,2}$)**.** *The class $\mathbb{C}^{1,2}$ contains the non-anticipative functionals F that admit one horizontal and two vertical derivatives and*

1. *$F, \nabla_x F, \nabla_x^2 F \in \mathbb{F}_l^\infty$ and*

2. *$\mathcal{D}F$ is continuous at fixed times.*

Note that left-continuity implies continuity at fixed times. At some point we will also need the following class

$$\mathbb{C}_b^{1,2} = \{F \in \mathbb{C}^{1,2} | F, \mathcal{D}F, \nabla_x F, \nabla_x^2 F \in \mathbb{B}\}.$$

Now we are ready to state the functional change of variables formula which is the main theorem of the section.

Theorem 3.2.14 (Functional change of variables formula)**.** *Let $x \in C([0,T], U)$ s.t. x has finite quadratic variation along a sequence of partitions $(\pi_n)_{n=1}^\infty$, where $\pi_n = \{0 = t_0^n < \cdots < t_{k(n)}^n = T\}$ s.t. $|\pi_n| \to 0$. Let us define the discretized version of x as*

$$x^n(t) = \sum_{j=0}^{k(n)-1} x\left(t_{j+1}^n\right) 1_{[t_j^n, t_{j+1}^n)}(t) + x(T) 1_{\{T\}}(t).$$

Let $F \in \mathbb{C}^{1,2}$ and $\nabla_x^2 F, \mathcal{D}F$ be locally bounded. Then the limit

$$\int_0^T \nabla_x F(x_u) dx(u) = \lim_{n\to\infty} \sum_{j=0}^{k(n)-1} \nabla_x F_{t_j^n}(x_{t_j^n-}^n)\left(x(t_{j+1}^n) - x(t_j^n)\right)$$

exists and is called the Föllmer integral. Moreover, we have the functional change of variables formula

$$\begin{aligned} F_T(x_T) - F_0(x_0) &= \int_0^T \nabla_x F_u(x_u) dx(u) \\ &\quad + \int_0^T \mathcal{D}_u F(x_u) du + \frac{1}{2}\int_0^T \nabla_x^2 F_u(x_u) d[x](u). \end{aligned}$$

The proof can be found in [13]. The Föllmer integral is a forward type integral that is named after Hans Föllmer who proved the ordinary (i.e. non-functional) Itô formula pathwise using a forward type integral concept for integrands of the form $f'(S(u))$; see [21]. The Föllmer integral in this thesis is defined for integrands that are vertical derivatives of smooth non-anticipative functionals as in [13]. This integral, as the forward integral, depends on the sequence of partitions. However, usually the sequence is fixed and thus it is not included in the notation.

3.2.1 Functional martingale representation theorem

The main reference in this subsection is [14]. Let X be a continuous martingale with respect to its own filtration $(\mathcal{F}_t^X)_{t\in[0,T]}$. We will need the following class of processes.

Definition 3.2.15 (Class $\mathcal{C}_b^{1,2}$)**.** *We define*

$$\mathcal{C}_b^{1,2}(X) = \{Y \textit{ adapted to } \mathcal{F}_t^X | \exists F \in \mathbb{C}_b^{1,2}(X), Y(t) = F_t(X_t) \textit{ a.s.}\}.$$

For $Y \in \mathcal{C}_b^{1,2}(X)$ one can define the vertical derivative of a process Y with respect to X as

$$\nabla_X Y(t) = \nabla_x F_t(X_t).$$

In this case $\nabla_X Y$ is uniquely defined up to an evanescent set, see [14].

We have the following martingale representation theorem.

Theorem 3.2.16 (Functional martingale representation theorem)**.** *Let $H \in \mathcal{F}_T^X$ s.t. $\mathbb{E}|H| < \infty$ and define $Y(t) = \mathbb{E}[H|\mathcal{F}_t^X]$. If $Y \in \mathcal{C}_b^{1,2}(X)$ with $Y(t) = F_t(X_t)$, then*

$$H = Y(T) = \mathbb{E}Y(T) + \int_0^T \nabla_x F_t(X_t) dX(t),$$

where the stochastic integral is an Itô integral.

Proof. The proof is an application of the functional change of variables formula combined with the uniqueness of the semimartingale decomposition. □

Theorem 3.2.16 can also be seen as a pathwise Clark-Ocone type result if the stochastic integral is understood in a pathwise sense. Another approach for pathwise Clark-Ocone formulas can be found in [17, 16]. The integration concept therein is the integration by regularization approach of [45].

In the following, we generalize the vertical derivative with respect to a process for a broader class of processes Y, but only in the weak sense. First we need to define some spaces.

Definition 3.2.17 (Space of integrands)**.** *Let $\mathcal{L}^2(X)$ denote the Hilbert space of progressively measurable processes ϕ satisfying*

$$\|\phi\|_{\mathcal{L}^2(X)}^2 = \mathbb{E}\left(\int_0^T \phi^2(s) d[X](s)\right) < \infty.$$

A space of stochastic integrals is defined as:

Definition 3.2.18 (Space of integrals)**.**

$$\mathcal{I}^2(X) = \left\{ \int_0^{\cdot} \phi(s)dX(s) | \phi \in \mathcal{L}^2(X) \right\}$$

equipped with the norm $\| \cdot \|_2^2 = \mathbb{E}(\cdot(T))^2$.

We restrict our consideration to a subspace of $\mathcal{I}^2(X)$ to obtain a dense set of test processes.

Definition 3.2.19 (Space of test processes)**.**

$$D(X) = \mathcal{I}^2(X) \cap \mathcal{C}_b^{1,2}(X).$$

We know that the vertical derivative with respect to process X is defined in $D(X)$. The following theorem extends the definition to the space $\mathcal{I}^2(X)$.

Theorem 3.2.20. *The vertical derivative* $\nabla_X : D(X) \mapsto \mathcal{L}^2(X)$ *is closable on* $\mathcal{I}^2(X)$. *Its closure defines a bijective isometry*

$$\nabla_X : \mathcal{I}^2(X) \mapsto \mathcal{L}^2(X), \qquad \int_0^{\cdot} \phi dX \mapsto \phi,$$

where the stochastic integral is the Itô integral.

The proof of the theorem can be found in [14]. Note that ∇_X is the adjoint operator of the Itô integral.

4. Finance

This chapter is about the use of fractional processes in various financial models. The main focus is on hedging path dependent options. The hedging problem is one of the main questions in mathematical finance.

4.1 Options

Options are financial contracts whose payoffs depend on the underlying asset. The most common options are European ones. Their payoffs depend on the value of the underlying asset at a given maturity date. Hence, the payoffs are of the form $f(S(T))$, where f is a function depending on the contract, S is the underlying asset and T the time of maturity. For example, the payoff for a European call is

$$(S(T) - K)_+$$

and the payoff for a European put is

$$(K - S(T))_+,$$

where K is the strike price. In the case of the classical Black-Scholes model, the prices and hedges for European calls and puts are very well known, see for example [26] or [50].

Moreover, there exist American options which allow for early exercise. This means that the holder of an American option may choose to get

$$f(S(\tau))$$

at any stopping time $\tau \in [0, T]$. Hence, considering American options yields optimal stopping problems. American options are not considered in this thesis.

There also exist path dependent options. In that case the payoff of the option depends on the whole path of the underlying asset. This means that the payoff is of the form $f(S_T)$.

4.1.1 Path dependent options

Path dependent options, as the name suggests, are options for which the payoff depends on the path of the underlying asset. For example, the payoff may depend on the maximum or minimum of the asset (lookback options) or the average price of the underlying (Asian options).

There are also barrier options whose payoff depends on the end value of the asset and whether the asset price crossed a pre-specified level or not. For a more complete picture on barrier options, see [26].

The key examples throughout this thesis are Asian options. There are both geometric and arithmetic Asian options. The geometric Asian options depend on the geometric average and the arithmetic Asian options on the arithmetic average of the underlying.

Thus, the payoff of an arithmetic Asian option is

$$f\left(\frac{1}{T}\int_0^T S(s)ds\right)$$

and the payoff of a geometric Asian option is given as

$$f\left(\exp\left(\frac{1}{T}\int_0^T \log(S(s))ds\right)\right).$$

Arithmetic Asian options are more often traded in practice. However, the hedges and prices for such options are not known explicitly even in the classical Black-Scholes model. See for example [34] for the difficulties arising. Conversely, geometric Asian options are rarely traded in the market but are easier to consider analytically.

Among these continuously sampled averages, there exist discretely sampled counterparts for both geometric and arithmetic averages.

In this thesis we hedge Asian options in the fractional Black-Scholes model in Publication II. In Publication III, Asian options are our main examples of path dependent options in the robust hedging context.

4.2 Hedging problem

Hedging of options is one of the central problems in mathematical finance and it has been studied extensively in various setups. The idea of hedging is to replicate a claim $f(S_T)$ by trading only the underlying asset. In mathematical terms, we are interested in finding a predictable $\mathcal{H}$ such that

$$f(S_T) = C + \int_0^T \mathcal{H}(s)dS(s), \tag{4.1}$$

where the deterministic constant C is called the hedging cost. In arbitrage free models the hedging cost can be interpreted as the fair price of the option.

In general, one has to make additional assumptions on allowed strategies $\mathcal{H}$ to prove no-arbitrage results. For example in the classical Black-Scholes model one has to assume the so-called no doubling strategies condition. If this condition is relaxed, then any random variable in the filtration of driving Brownian motion (for example constants) can be represented as an Itô integral ([18]). Thus, it is obviously impossible to have a no-arbitrage theorem.

In the case of mixed models, the class of allowed strategies is defined in [9]. One can prove a no-arbitrage result for this class of trading strategies. The class is relatively big including hedges for many European, Asian and loockback options. In the fractional Black-Scholes model it seems that one has to forbid continuous trading to get rid of the arbitrage strategies.

Completeness is another question of mathematical finance. For example the Black-Scholes model is complete because any square integrable claim can be replicated uniquely by the martingale representation theorem, which can be found e.g. in [42, 26]. In some special cases the martingale representation theorem can be understood pathwise as in theorem 3.2.16. This formula even gives the explicit form of the hedging strategy.

As we see from equation (4.1), the hedging problem depends strongly on the stochastic integration concept used. In this thesis we concentrate on the hedging problem with pathwise integration concepts. In Publication II we use the generalized Lebesgue-Stieltjes integral and in Publication III we use forward type integrals.

4.3 Fractional models

By a fractional model we mean a market model including some fractional process. The simplest of these is the fractional Bachelier model, where the asset price is modeled by a fractional Brownian motion. Another simple fractional model is the fractional Black-Scholes model, where the stock price is modeled by a geometric fractional Brownian motion.

In addition to these two purely fractional models we consider mixed models, where the asset price is modeled as

$$S(t) = \exp\{\epsilon W(t) + \sigma Z(t) + \mu t\},$$

where W is a standard Brownian motion and Z is a zero quadratic variation process, for example a fractional process. By choosing Z to be fBm, fLp or some other zero quadratic variation process, we obtain different statistical properties for our model.

See [10] for a survey on fBm in finance.

4.3.1 Fractional Black-Scholes model

In the fractional Black-Scholes model the asset price is modeled by a geometric fractional Brownian motion (gfBm)

$$S(t) = \exp(B^H(t)),$$

where $\frac{1}{2} < H < 1$. Here the integral is the generalized Lebesgue-Stieltjes integral. It is well known that the fractional Black-Scholes model provides arbitrage opportunities ([43, 12, 48]). There exist even strong arbitrage strategies ([36]). However, the arbitrage opportunities disappear when proportional transaction costs are added to the model ([25]). Because of the arbitrage problems, it is questionable to use fractional Black-Scholes as a reasonable model in finance.

In a recent work [36] (conducted after Publication II), it is proved that any claim in the fractional Black-Scholes model is hedgeable in a specific weak sense. They also prove, with some additional assumptions on the claim, that the hedging strategy exist in the sense of generalized Lebesgue-Stieltjes integrals. What is a bit surprising in their results is that the hedging cost can be arbitrary in some cases. In spite of these existence results, finding explicit hedges might be of some interest.

In the setup of the fractional Black-Scholes model, hedges for a broad class of European type options are known, [1]. The hedging strategies are the same as if fBm were a process of bounded variation. However, it is impossible to hedge for example lookback options as gLS or Young integrals using analogous hedging strategies ([2]).

Thus, it is not obvious that we can hedge explicitly Asian options in the fractional Black-Scholes model as we do in Publication II. It turns out that we are able to hedge a broad class of arithmetic Asian options including for example arithmetic Asian calls and puts. The hedges for such options are not known explicitly even in the case of the ordinary Black-Scholes model. Our results show that arithmetic Asian options themselves are not difficult, even though they are difficult to hedge in the ordinary Black-Scholes model.

4.3.2 Mixed models

If one wants to introduce an economically meaningful market model with long-range dependent returns, mixed models are a good option. In such models, one can have both long-range dependence and no-arbitrage in the sense of [9]. In a mixed model the price of an asset is modeled as

$$S(t) = \exp\left(\epsilon W(t) + \sigma Z(t) + \mu t\right),$$

where W is a Brownian motion and Z is a zero quadratic variation process, for example a fractional process. The Brownian part corresponds to the short term market fluctuations and dominates in the short time scale. On the other hand, the zero quadratic variation process, determining the memory properties, dominates in a larger time scale.

It is not known in general whether the sum of a Brownian motion and a fractional process is semimartingale or not. The sum of mixed Brownian and fractional Brownian motion is semimartingale if and only if $H > \frac{3}{4}$ (or in the trivial case $H = \frac{1}{2}$), see [11]. Thus, in general it is not possible to use the Itô integral so we will use the forward integral instead.

It is possible to choose Z quite freely. Possible options are for example fBm, fLp and fractional Ornstein-Uhlenbeck processes ([29]). In fact, it is possible to choose Z in such a way that it has either long or short range dependence and light or heavy tails, see Publication III.

A no-arbitrage result and a robust replication result covering mixed models can be found in [9]. Their main message is that the prices of options as well as the hedges depend mainly on the path properties of the model. Other no-arbitrage results in related setups can be found for example in [6] and [15].

5. Summaries of the articles

I. Fractional Lévy processes as a result of compact interval integral transformation

In this article we introduce a new fractional Lévy process concept, fractional Lévy processes by Molchan-Golosov transformation (fLpMG) for Hurst parameter $H \in (0, 1)$. We study the properties of such processes. We also compare fLpMG to a previous definition of fractional Lévy processes (fLpMvN) by [5] and [32]. The main difference between fLpMG and fLpMvN is that the infinite history is not needed for the definition of fLpMG, because the integral kernel is compactly supported.

Fractional Lévy processes by Molchan-Golosov representation share their covariance structure with fBm. The paths of fLpMG with $H \in (\frac{1}{2}, 1)$ are Hölder continuous of any order less than $H - \frac{1}{2}$ and thus the paths have the zero quadratic variation property. In the case when $H \in (0, \frac{1}{2})$, the paths are unbounded. The increments of fLpMG are not stationary in general and fLpMG is never self-similar for any order. We introduce Wiener integration of deterministic integrands with respect to an fLpMG.

Fractional Lévy processes by the two different integral transformations are different in distribution. With slight additional moment assumptions we can show that fLpMvN and fLpMG are the same only when the driving Lévy process is Brownian motion.

Theorem 1. *Let $H \in \left(\frac{1}{2}, 1\right)$ and L be a Lévy process on $\mathbb{R}$ with non-degenerate Lévy measure and $\mathbb{E}L(1) = 0$. FLpMvN and fLpMG driven by L have different finite dimensional distributions if one of the following two conditions holds:*

1. $\mathbb{E}|L(1)|^3 < \infty$ *and* $\mathbb{E}L(1)^3 \neq 0$.

2. $\mathbb{E}L(1)^4 < \infty$.

Even though the finite dimensional distributions of different fractional Lévy processes differ, we prove that fLpMG and fLpMvN driven by the same Lévy process L are close to eachother in an L^2-like sense. The result is the same as in the fBm case, [28].

II. Integral representations of some functionals of fractional Brownian motion

The main contribution of this paper is to obtain integral representations as generalized Lebesgue-Stieltjes integrals for functionals depending on different averages of (geometric) fractional Brownian motion.

Let B^H be fractional Brownian motion with Hurst index $H > \frac{1}{2}$, $S(t) = \exp(B^H(t))$ and f be a convex function on $\mathbb{R}$. Then we have for functionals depending on the arithmetic average of B^H that

$$\begin{aligned} & f\left(\frac{1}{T}\int_0^T S(s)ds\right) \\ =& f(S(0)) + \int_0^T f'_-\left(\frac{T-s}{T}S(s) + \frac{1}{T}\int_0^s S(u)du\right)\frac{T-s}{T}S(s)dB^H(s), \end{aligned}$$

where f'_- is the left derivative of f. For the geometric average of gfBm we obtain analogously

$$f\left(\exp\left(\frac{1}{T}\int_0^T B^H(s)ds\right)\right) = f(S(0)) + \int_0^T \frac{T-s}{T} f'_-(G(s))G(s)dB^H(s),$$

where

$$G(t) = \exp\left(\frac{1}{T}\int_0^t B^H(s)ds\right) S(t)^{\frac{T-t}{T}}.$$

For the arithmetic average of the fBm we obtain

$$\begin{aligned} & f\left(\frac{1}{T}\int_0^T B^H(s)ds\right) \\ =& f(B^H(0)) + \int_0^T \frac{T-s}{T} f'_-\left(\frac{T-s}{T}B^H(s) + \frac{1}{T}\int_0^s B^H(u)du\right) dB^H(s). \end{aligned}$$

The integral representations can be interpreted as hedging strategies in fractional Black-Scholes and fractional Bachelier models.

The main mathematical tools in this paper are the functional calculus of [13], fractional Besov space techniques ([35]), the Garsia-Rodemich-Rumsey inequality ([24]) and Malliavin calculus ([39]).

An alternative approach would be to use Itô formula of [20] instead of the functional calculus. This approach would need less prerequisities but the proofs themselves would not be essentially simpler. The drawback would be that the result of [20] uses regularization approach for the stochastic integration and the Itô formula should be reproved for the discretization approach used in this thesis.

III. Robust hedging and pathwise calculus

The motivation behind this paper is to study the connections between two different pathwise hedging approaches. These approaches are the functional calculus ([13, 14]) of Cont-Fournié and robust replication ([9]) by Bender-Sottinen-Valkeila.

Let X be a continuous martingale, $H \in \mathcal{F}_T^X$ be integrable and $Y(t) = \mathbb{E}[H|\mathcal{F}_t^X] \in \mathcal{C}_b^{1,2}(X)$. Now Y can be represented as $Y(t) = F_t(X_t)$ and by [14]

$$Y(t) = F_0(X_0) + \int_0^t \nabla_x F_s(X_s) dX(s),$$

where the stochastic integral is an Itô integral. We prove that it holds for all processes Z satisfying a non-degeneracy property and sharing the functional form of quadratic variation with X that

$$F_t(Z_t) = F_0(Z_0) + \int_0^t \nabla_x F_s(Z_s) dZ(s),$$

where the integral is a Föllmer integral. Note that the quadratic variation of mixed Brownian-fBm and Brownian-fLp models have the same functional dependence on the path as the quadratic variation of the classical Black-Scholes model.

The other main result is that whenever there exist both Cont-Fournié type and Bender-Sottinen-Valkeila type pathwise hedging strategies for a given claim, then both of the strategies are the same in the functional sense. This observation allows us to extend the definition of the pathwise vertical derivative with respect to a process. Let Y be adapted to $(\mathcal{F}_t^S)_{t\in[0,T]}$. Assume that $Y(t)$ can be hedged using a Bender-Sottinen-Valkeila type strategy $\psi_s(S_s)$ for all $t \in [0,T]$. Then under some technical assumptions we can define

$$(\nabla_S Y)(s) = \psi_s(S_s), \quad s \in [0,T].$$

We also give several examples.

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