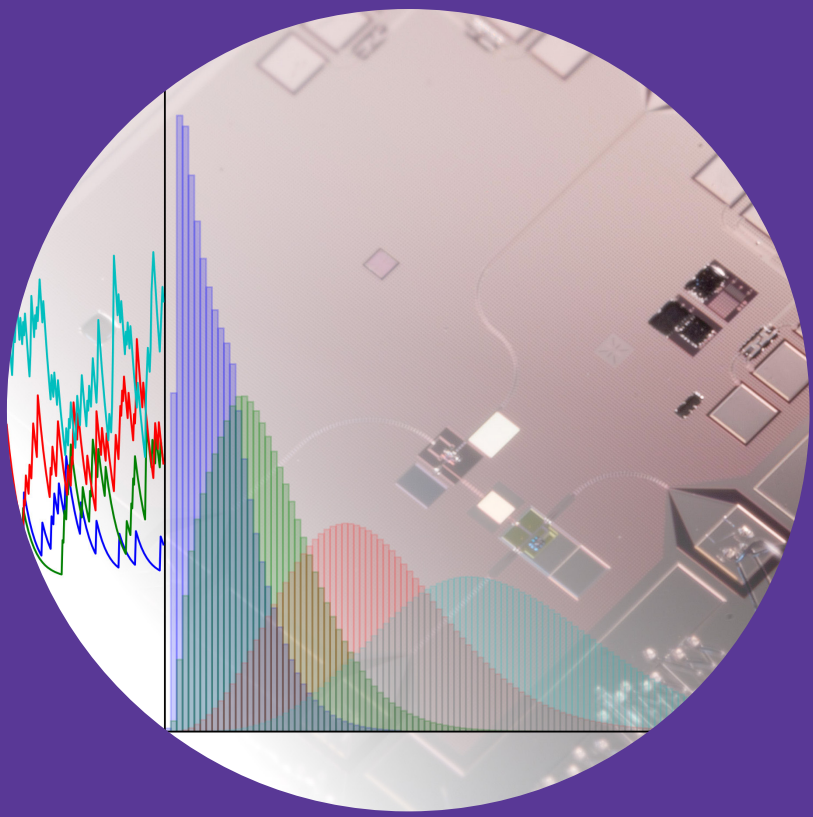
O.V. Lounasmaa Laboratory

Temperature fluctuations and heat transport in nanoelectronics

Matti Laakso



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Temperature fluctuations and heat transport in nanoelectronics

Matti Laakso

Doctoral dissertation for the degree of Doctor of Science in Technology to be presented with due permission of the School of Science for public examination and debate in Auditorium T2 at the Aalto University School of Science (Espoo, Finland) on the 25th of May 2012 at 12 noon.

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Abstract

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Abstract

In metallic structures at very low temperatures heat and energy are carried mostly by electrons. When these structures are small, with dimensions of the order of nanometers, many kinds of quantum phenomena become significant. The combination of heat transport with these effects, with a special focus on the random fluctuations of temperature in these systems, is the topic of this Dissertation.

Using quantum transport theory, we have studied the flow of heat and charge in some prototypical nanoelectronic systems, e.g., single-electron transistor and superconductor–normal metal tunnel structure. We have devised a powerful theoretical method for the calculation of probability distributions of fluctuating quantities, especially temperature, and used it to find cases where the statistics of these fluctuations becomes non-Gaussian. We have also shown how the fluctuations of temperature affect the fluctuations of the electric current. In some cases the temperature-fluctuation induced current noise can be orders of magnitude larger than the intrinsic shot noise of the system. We also address the possibility to detect these fluctuations by monitoring the induced current noise.

Keywords temperature fluctuations, heat transport, noise

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Tiivistelmä

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Tekijä
Matti Laakso

Väitöskirjan nimi
Lämpötilan satunnaisvaihtelut ja lämmönkuljetus nanoelektroniikan rakenteissa

Julkaisija Perustieteiden korkeakoulu

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Tiivistelmä

Hyvin matalaan lämpötilaan jäähdytetyissä metallirakenteissa lämpöä kuljettavat pääasiassa elektronit. Kun nämä rakenteet ovat pieniä, nanometrien kokoluokkaa, monenlaiset kvantti-ilmiöt ovat merkittäviä. Tämän väitöskirjan aiheena on näiden kvantti-ilmiöiden vaikutus lämmön kuljetukseen ja erityisesti niiden vaikutus lämpötilan satunnaisvaihteluihin.

Olemme tutkineet lämmön ja varauksen kuljetusta kvanttikuljetusteoriaa käyttäen muutamissa tyypillisissä nanoelektroniikan rakenteissa, esimerkiksi yksielektronitransistorissa ja suprajohde–normaalimetalli tunneliliitoksessa. Olemme kehittäneet tehokkaan teoreettisen menetelmän lämpötilan ja muiden suureiden todennäköisyysjakaumien laskemiseen ja etsineet sen avulla tapauksia, joissa näiden suureiden satunnaisvaihteluiden statistiikka ei noudata normaalijakaumaa. Olemme myös näyttäneet miten lämpötilan satunnaisvaihtelut vaikuttavat sähkövirran kohinaan. Joissain tapauksissa lämpötilan satunnaisvaihteluiden aikaansaama virtakohina voi olla kertaluokkia suurempi kuin johtimen normaali raekohina. Tutkimme myös miten lämpötilan satunnaisvaihtelut voitaisiin havaita seuraamalla niiden aikaansaamaa sähkövirran kohinaa.

Avainsanat lämpötilan satunnaisvaihtelut, lämmönkuljetus, kohina

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Preface

I started this work in the Low Temperature Laboratory of the Helsinki University of Technology, the first publication dating back to my undergraduate assignment in 2006, and finished it in the O. V. Lounasmaa Laboratory of the Aalto University in 2012. Despite the changes in the names, the laboratory has remained an interesting and rewarding place for me to work in during all these years.

There are several people who have been instrumental to the completion of this work. First and foremost I would like to thank my instructor and head of the nano theory group, Dr. Tero Heikkilä, for his tutelage during the completion of three special assignments, a Master's Thesis, and finally this Dissertation. His enthusiasm and interest in new research topics has provided me an inexhaustible supply of ideas and avenues to pursue. I am grateful to Prof. Mikko Paalanen for giving me the opportunity to work in this traditional, high-profile, research institute. I would also like to extend my thanks to my supervisor Prof. Päivi Törmä for the smooth handling of the bureaucratic processes.

I am especially grateful for the guidance and collaboration of Prof. Yuli Nazarov, who has devoted his time and attention to me like I was his own student. Without his insight this Dissertation would probably not exist. I also thank Prof. Pertti Hakonen and Prof. Jukka Pekola for discussions and ideas regarding the experimental research related to my work.

My colleagues and friends in the various research groups of the laboratory have made this an enjoyable and eventful period of my life. I would especially like to thank Raphaël Khan, Jian Li, Teemu Ojanen, Joonas Peltonen, Antti Puska, Pauli Virtanen, and Juha Voutilainen in this context.

Research work can be lonely and tiresome. Luckily I have friends who have reminded me that there is life outside academia: Thanks to Aleksi,

Juha, Lauri, Pekka, Petri, Teemu, and Timo, I have kept my sanity.

Finally, I would like to thank my dear wife Salla for being my sunshine in rainy days, my small daughter for making me smile all day long, and my parents Raisa and Unto and my little sister Mari for always giving me a home away from home.

Espoo, April 27, 2012,

Matti Laakso

Contents

List of Publications

This dissertation consists of an overview and of the following publications which are referred to in the text by their Roman numerals.

I M. A. Laakso, T. T. Heikkilä, and Yuli V. Nazarov. Fully Overheated Single-Electron Transistor. *Physical Review Letters*, Volume 104, Number 19, 196805, May 2010.

II M. A. Laakso, T. T. Heikkilä, and Yuli V. Nazarov. Giant current fluctuations in an overheated single-electron transistor. *Physical Review B*, Volume 82, Number 20, 205316, November 2010.

III M. A. Laakso, T. T. Heikkilä, and Yuli V. Nazarov. Manifestly non-Gaussian fluctuations in superconductor-normal metal tunnel nanostructures. *Physical Review Letters*, Volume 108, Number 6, 067002, February 2012.

IV M. A. Laakso, T. T. Heikkilä, and Yuli V. Nazarov. Theory of temperature fluctuation statistics in superconductor-normal metal tunnel structures. *Physical Review B*, April 2012.

V J. Voutilainen, M. A. Laakso, and T. T. Heikkilä. Physics of proximity Josephson sensor. *Journal of Applied Physics*, Volume 107, Number 6, 064508, March 2010.

VI M. A. Laakso, P. Virtanen, F. Giazotto, and T. T. Heikkilä. Nonequilibrium characteristics in all-superconducting tunnel structures. *Physical*

Review B, Volume 75, Number 9, 094507, March 2007.

Author's Contribution

Publication I: "Fully Overheated Single-Electron Transistor"

The author obtained most of the analytical results and made all of the numerical computations. The author wrote a large part of the manuscript.

Publication II: "Giant current fluctuations in an overheated single-electron transistor"

The author obtained most of the analytical results and made all of the numerical computations. The author wrote most of the manuscript.

Publication III: "Manifestly non-Gaussian fluctuations in superconductor-normal metal tunnel nanostructures"

The author obtained most of the analytical results and made all of the numerical computations. The author wrote most of the manuscript.

Publication IV: "Theory of temperature fluctuation statistics in superconductor-normal metal tunnel structures"

The author obtained most of the analytical results and made all of the numerical computations. The author wrote most of the manuscript.

Publication V: "Physics of proximity Josephson sensor"

The author derived the model for the electron–phonon interaction. The author gave advice on the other parts of the manuscript.

Publication VI: "Nonequilibrium characteristics in all-superconducting tunnel structures"

The author obtained all the analytical and numerical results, and wrote the entire manuscript.

1. Introduction

The theory of heat conduction dates back to the late 18th century, when the French chemist Antoine Lavoisier was able to disentangle the concept of heat from explanations of combustion. He explained heat with an invisible fluid called *caloric*, flowing spontaneously from hot objects to cold objects, and the quantity of which is constant throughout the universe [1].

In modern thermodynamics heat is understood as a form of energy. Depending on the system, heat can manifest for example as kinetic energy of molecules, vibrational energy of a lattice, or kinetic energy of electrons. Nonetheless, it still flows from hot to cold *on average*, and the principle of conservation of energy is one of the cornerstones of physics. In electronic systems heat is carried mostly by electrons, and in metals at low temperatures the specific heat is dominated by the specific heat of electrons [2]. This is studied in *heattronics*.

Unlike position, momentum, or energy, temperature is not a physical observable. It is a concept from equilibrium statistical mechanics which describes the distribution of energy in a system composed of a large number of particles [3]. In a system driven out of equilibrium temperature is not even well-defined. Still, temperature is routinely measured by various kinds of thermometers, even in non-equilibrium systems. An interesting question is then what actually is measured with a thermometer. The observable of choice is often the electric current, which depends on the distribution of electrons among states with different energies and momenta. The study of non-equilibrium current response can give essential insight to this problem.

The possibility to routinely make nanometer scale electronic structures by the means of electron beam lithography has sparked interest in the topic of *nanoelectronics* in the last couple of decades. Even the commercially manufactured integrated circuits approach the limit of nano-

electronics with their ever decreasing transistor channel lengths [4]. In nanoelectronic devices the effects of quantum mechanics begin to play a significant role, requiring the use of *quantum transport theory* in their description. The quantum phenomena are especially prominent at low temperatures, where they are not masked by thermal fluctuations. Many nanoelectronic device applications are based on these effects, thus relying on cryogenic operating temperatures. In these systems a thorough understanding of heat flows is of utmost importance.

The performance of many nanoelectronic devices is hampered by the presence of fluctuations, i.e., noise. For example, low-temperature thermometers are based on the measurement of electric current. Fluctuations in the current reduce the accuracy of the thermometer. Moreover, fluctuations in one quantity may induce even larger fluctuations in another quantity. On the other hand, the fluctuations themselves can be used to deduce information about the studied system: Noise in the electric current can provide us information about the motion of electrons through the conductor [5].

The research reported in this Dissertation concerns theoretical heattronics in nanoscale structures. We have focused on the description of some structures common in the basic research of nanoelectronics in the framework of quantum transport, studied the flow of heat and charge in these systems, and devised a theoretical method for the calculation of probability distributions of fluctuating quantities, especially temperature. We have also studied how the fluctuations of temperature affect the fluctuations of the electric current. It turns out that in systems with a strongly non-linear temperature dependence of the current this effect can be significant.

Organization of this overview

Chapter 2 introduces the theoretical foundations upon which this Dissertation is built. I discuss the basics of functional field integrals in quantum mechanics, their extension to non-equilibrium situations, and their application to a quantum transport setup — the single-electron transistor, which is also the focus of Publications I and II. I have sought to start from the basics and include intermediate steps that leave nothing implicit. At some points I refer to a good textbook or review for parts that are particularly well-explained in the literature. Chapter 3 focuses on the formulation of the temperature fluctuation statistics in the language

of functional field integrals. I also discuss the assumptions behind the concept of temperature fluctuations and the limits of its applicability to physical systems. I briefly review some of the main results of the publications included in this Dissertation. I also discuss the prospects to measure these fluctuations in a realistic setup. Finally, in Chapter 4, I summarize our findings and conclude with the possible future directions for this research.

2. Functional field integrals

Theoretical methods for the study of quantum field theory out of equilibrium were first developed in the 1960's [6, 7]. These were based on Green functions and their perturbation expansions, similar to the equilibrium Green function methods. Later the Keldysh theory was formulated in the language of Feynman diagrams [8, 9, 10], and it is still widely used in theoretical condensed matter physics. The modern non-equilibrium quantum field theory is based on functional field integrals [11, 12, 13] which allow one to incorporate non-perturbative methods and handle topologically non-trivial structures. Moreover, the functional integration method often leads to a significantly more transparent and simple way of treating interacting systems perturbatively compared to the classification and evaluation of various Feynman diagrams and dealing with the tensorial character of interaction vertices.

2.1 Generating functionals

Before going to the topic of temperature fluctuation statistics, it is instructive to go through the underlying theory in some detail. This section follows loosely the treatment given in Ref. [11]. The state of a quantum many-body system is characterized by its density matrix, $\hat{\rho}$. Given the Hamiltonian operator $\hat{\mathrm{H}}$ corresponding to this system, the time evolution of the density matrix is given by the von Neumann equation[1]

$$\partial_t \hat{\rho} = -\mathrm{i}[\hat{\mathrm{H}}, \hat{\rho}]. \tag{2.1}$$

This is formally solved by

$$\hat{\rho}(t) = \hat{\mathrm{U}}(t, t_0)\hat{\rho}(t_0)\hat{\mathrm{U}}^\dagger(t, t_0), \tag{2.2}$$

[1] Throughout this overview I use units such that $\hbar = k_B = 1$.

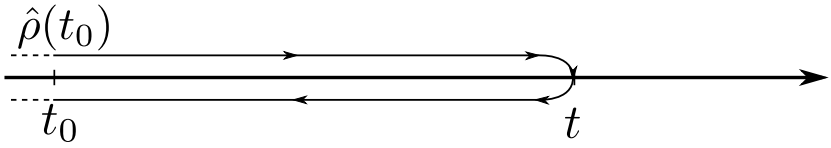

Figure 2.1. Forward–backward time contour along which the density matrix evolves.

where $\hat{\rho}(t_0)$ is the density matrix at some initial time t_0, and the time evolution operator is given by the time-ordered exponent

$$\hat{\mathrm{U}}(t,t_0) = \mathbb{T}\exp\left[-\mathrm{i}\int_{t_0}^{t}\mathrm{d}t'\hat{\mathrm{H}}(t')\right]. \tag{2.3}$$

Its Hermitian conjugate is equivalent to time evolution backwards in time, $\hat{\mathrm{U}}^\dagger(t,t_0) = \hat{\mathrm{U}}(t_0,t)$.

Often our goal is to calculate the expectation value of some observable, represented by an operator $\hat{\mathrm{X}}$, at time t. This is given by

$$\langle X(t)\rangle = \frac{\mathrm{tr}\Big[\hat{\mathrm{X}}\hat{\rho}(t)\Big]}{\mathrm{tr}[\hat{\rho}(t)]} = \frac{\mathrm{tr}\Big[\hat{\mathrm{X}}\hat{\mathrm{U}}(t,t_0)\hat{\rho}(t_0)\hat{\mathrm{U}}(t_0,t)\Big]}{\mathrm{tr}\Big[\hat{\mathrm{U}}(t,t_0)\hat{\rho}(t_0)\hat{\mathrm{U}}(t_0,t)\Big]} = \frac{\mathrm{tr}\Big[\hat{\mathrm{U}}(t_0,t)\hat{\mathrm{X}}\hat{\mathrm{U}}(t,t_0)\hat{\rho}(t_0)\Big]}{\mathrm{tr}[\hat{\rho}(t_0)]}, \tag{2.4}$$

where the trace is over the many-body Hilbert space. In the last equality I have used the cyclic invariance of the trace, and the identity $\hat{1} = \hat{\mathrm{U}}(t_0,t)\hat{\mathrm{U}}(t,t_0)$. This form affords a pictorial interpretation in terms of a time contour shown in Fig. 2.1: The density matrix first evolves in time from t_0 to t, the observable X is calculated, and finally the density matrix evolves backwards in time to t_0.

The expectation values can also be obtained from a generating functional. The Hamiltonian is generalized to include a *source term* $\hat{\mathrm{H}}_X = \hat{\mathrm{H}} \pm \eta(t)\hat{\mathrm{X}}/2$, where $\eta(t)$ is a complex-valued function, and the plus (minus) sign corresponds to the forward (backward) part of the contour. The expectation value is given by the functional derivative

$$\langle X(t)\rangle = \mathrm{i}\frac{\delta\mathcal{Z}[\eta]}{\delta\eta(t)}\bigg|_{\eta=0}, \tag{2.5}$$

where

$$\mathcal{Z}[\eta] = \frac{\mathrm{tr}\left[\hat{\mathrm{U}}_{C,X}(t,t_0)\hat{\rho}(t_0)\right]}{\mathrm{tr}\left[\hat{\rho}(t_0)\right]} \tag{2.6}$$

is the generating functional, and $\hat{\mathrm{U}}_{C,X}(t,t_0) = \hat{\mathrm{U}}_X(t_0,t)\hat{\mathrm{U}}_X(t,t_0)$ is the contour evolution operator including the source field. Since the source field breaks the symmetry between the forward and backward branch, this is not equal to the identity operator. The generating functional can be

straightforwardly used to generate higher moments and correlators,[2]

$$\langle X(t_1)X(t_2)\rangle = -\frac{\delta^2\mathcal{Z}[\eta]}{\delta\eta(t_1)\delta\eta(t_2)}\bigg|_{\eta=0}, \tag{2.7}$$

or even the complete probability distribution via the functional integral

$$\mathcal{P}[X] = \int \mathcal{D}\eta e^{\mathrm{i}\int \mathrm{d}\tau\eta(\tau)X(\tau)}\mathcal{Z}[\eta]. \tag{2.8}$$

The Keldysh partition function is obtained by setting the source term to zero

$$\mathcal{Z}[0] = \frac{\mathrm{tr}\left[\hat{\mathrm{U}}_C(t,t_0)\hat{\rho}(t_0)\right]}{\mathrm{tr}\left[\hat{\rho}(t_0)\right]} = 1, \tag{2.9}$$

being equal to unity since the symmetry-breaking source term is absent.

2.2 Keldysh formalism

To use the functional formulation with a many-body quantum system, the quantum fields have to be introduced to the theory. This is done in the standard language of second quantization with creation and annihilation operators acting in Fock space. The procedures for bosons and fermions differ slightly from each other. Since we are interested in electronic systems, I only consider the fermion case here.

2.2.1 Coherent states

The functional field integrals are naturally represented in the coherent state basis. The coherent fermion state $|\psi\rangle$ parameterized by a *Grassmann number* ψ is defined as the eigenstate of the annihilation operator, $\hat{\mathrm{c}}|\psi\rangle = \psi|\psi\rangle$. Similarly, $\langle\psi|\hat{\mathrm{c}}^\dagger = \langle\psi|\bar{\psi}$, where $\bar{\psi}$ is another Grassmann number, *unrelated* to ψ. The anti-commutativity of the fermion operators imply that also the Grassmann numbers anti-commute. For this reason Grassmann numbers obey a peculiar but well-defined algebra, details of which can be found from Ref. [12]. For completeness, I list some of their properties below.

The matrix elements of a normally ordered operator[3] can be evaluated with $\langle\psi|\hat{\mathrm{O}}(\hat{\mathrm{c}}^\dagger,\hat{\mathrm{c}})|\psi'\rangle = O(\bar{\psi},\psi')\langle\psi|\psi'\rangle$, where $\langle\psi|\psi'\rangle = \exp\left(\bar{\psi}\psi'\right)$. The trace of an operator is

$$\mathrm{tr}\left(\hat{\mathrm{O}}\right) = \int \mathrm{d}\bar{\psi}\mathrm{d}\psi e^{-\bar{\psi}\psi}\langle -\psi|\hat{\mathrm{O}}(\hat{\mathrm{c}}^\dagger,\hat{\mathrm{c}})|\psi\rangle, \tag{2.10}$$

[2]The correct ordering of the operators is built-in to the functional field theory.
[3]In a normally ordered operator all the creation operators are to the left of all the annihilation operators.

where the integration over Grassmann variables is defined by

$$\int \mathrm{d}\psi = 0, \quad \int \mathrm{d}\psi\psi = 1. \tag{2.11}$$

The Gaussian integrals in the following can be calculated by using the formula

$$\begin{aligned}\mathcal{Z}[\bar{\eta},\eta] &= \int \prod_j \mathrm{d}\bar{\psi}_j \mathrm{d}\psi_j \exp\left\{-\sum_{ij}\bar{\psi}_i A_{ij}\psi_j + \sum_j \left(\bar{\psi}_j\eta_j + \bar{\eta}_j\psi_j\right)\right\}, \\ &= \det(\mathbf{A}) \exp\left\{\sum_{ij}\bar{\eta}_i (A^{-1})_{ij}\eta_j\right\},\end{aligned} \tag{2.12}$$

where $\mathbf{A}$ is any invertible complex matrix.

2.2.2 Action for free fermions

Consider a system of free electrons, described by the Hamiltonian

$$\hat{\mathrm{H}} = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} \hat{\mathrm{c}}^\dagger_{\mathbf{k}\sigma}\hat{\mathrm{c}}_{\mathbf{k}\sigma}. \tag{2.13}$$

In terms of the coherent states the Keldysh partition function takes the form

$$\begin{aligned}\mathcal{Z} &= \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left\{\mathrm{i}\int_C \mathrm{d}t\mathrm{d}t' \sum_{\mathbf{k}\mathbf{k}'}\sum_{\sigma\sigma'}\left[\bar{\psi}_{\mathbf{k}\sigma}(t) G^{-1}_{\mathbf{k}\mathbf{k}'\sigma\sigma'}(t,t')\psi_{\mathbf{k}'\sigma'}(t')\right]\right\}, \\ &\equiv \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left\{\mathrm{i}S[\bar{\psi},\psi]\right\},\end{aligned} \tag{2.14}$$

where $G^{-1}_{\mathbf{k}\mathbf{k}'\sigma\sigma'}(t,t') = (\mathrm{i}\partial_t - \epsilon_{\mathbf{k}})\,\delta_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'}$ is the inverse Green function, and the time integration goes over the contour shown in Fig. 2.1. The last equality defines the *action* S of the system. The initial density matrix $\hat{\rho}(t_0)$ is seemingly absent from the partition function [c.f. Eq. (2.9)]. It is actually hidden in the boundary conditions for ψ and $\bar{\psi}$ at t_0.

It is somewhat inconvenient to carry along the integration contour. For this reason, it is customary to split the contour into a forward branch and a backward branch, and define the variables separately on the two contours:

$$\begin{aligned}S[\bar{\psi},\psi] = \int_{-\infty}^{\infty} \mathrm{d}t\mathrm{d}t' \sum_{\mathbf{k}\mathbf{k}'}\sum_{\sigma\sigma'} \big[&\bar{\psi}_{+,\mathbf{k}\sigma}(t) G^{-1}_{\mathbf{k}\mathbf{k}'\sigma\sigma'}(t,t')\psi_{+,\mathbf{k}'\sigma'}(t') \\ &-\bar{\psi}_{-,\mathbf{k}\sigma}(t) G^{-1}_{\mathbf{k}\mathbf{k}'\sigma\sigma'}(t,t')\psi_{-,\mathbf{k}'\sigma'}(t')\big],\end{aligned} \tag{2.15}$$

where ψ_+ resides on the forward branch and ψ_- on the backward branch. Here we have also taken $t_0 \to -\infty$ and extended the integration all the

way to ∞, since the forward and backward branches cancel each other out for times larger than t.

Next we perform a Keldysh rotation according to

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \tag{2.16}$$

$$\begin{pmatrix} \bar{\psi}_1 & \bar{\psi}_2 \end{pmatrix} = \begin{pmatrix} \bar{\psi}_+ & \bar{\psi}_- \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \tag{2.17}$$

and the rotated Green function matrix obtains the form

$$\mathbf{G} = \begin{pmatrix} G^R & G^K \\ 0 & G^A \end{pmatrix}. \tag{2.18}$$

The superscripts R, A, and K denote the retarded, advanced, and Keldysh components of the Green's function matrix, respectively. R- and A-components contain information about the energies and lifetimes of the states of the system, whereas the Keldysh component keeps track of their occupation. Collecting the wave vector, spin, and Keldysh indices into a matrix structure, we can write

$$S[\bar{\mathbf{\Psi}}, \mathbf{\Psi}] = \int_{-\infty}^{\infty} \mathrm{d}t\mathrm{d}t' \bar{\mathbf{\Psi}}(t) \mathbf{G}^{-1}(t, t') \mathbf{\Psi}(t'). \tag{2.19}$$

Sometimes it is convenient not to do the Keldysh rotation and instead work in the $\{+, -\}$-basis. The advantage of the Keldysh formalism is the simplicity of matrix operations by virtue of the triangular form of the Green function matrix.

2.2.3 Effective action

In Eq. (2.19) the visible degrees of freedom are the microscopic electron fields, described by the vectors $\bar{\mathbf{\Psi}}$ and $\mathbf{\Psi}$. In many cases, the important and interesting variables are macroscopic, e.g., total energy or total charge. In other cases we are interested in the calculation of expectation values or statistics of observables; in this case the important variable is the source field. To make these facts visible, the microscopic fields should be integrated out. We do this by first making the action quadratic in the electron fields by using a Hubbard–Stratonovich transformation [14, 15]

$$\frac{\exp\left(\sum_{ij} \rho_i A_{ij} \rho_j\right)}{\sqrt{\det \mathbf{A}^{-1}}} = \int \prod_j \frac{\mathrm{d}\phi_j}{\sqrt{\pi}} \exp\left\{ -\sum_{ij} \phi_i (A^{-1})_{ij} \phi_j + 2 \sum_j \phi_j \rho_j \right\}, \tag{2.20}$$

where $\rho = \bar{\psi}\psi$, and then performing the Gaussian integration over these fields. This transformation is of course not required, if the action already

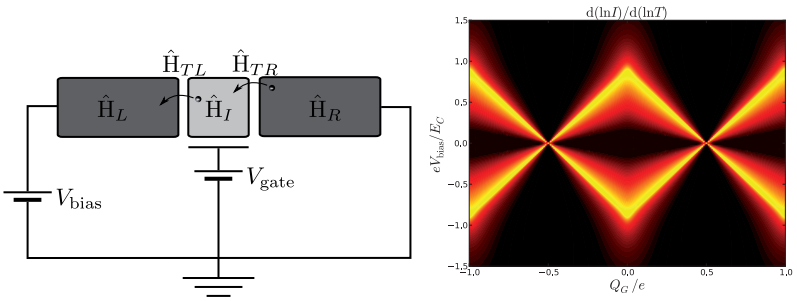

Figure 2.2. Left: Schematic diagram of a single-electron transistor, biased with voltage V_{bias}. The preferred charge on the island can be tuned with the gate voltage V_{gate}. Left and right leads are described by noninteracting electrons. Electrons on the island feel a potential generated by all the other electrons via Coulomb interaction. Electrons can tunnel through the insulating barriers separating the island from the leads. Right: Temperature sensitivity $\mathrm{d}(\ln I)/\mathrm{d}(\ln T)$ of the current through the transistor. The sensitivity is highest (bright yellow regions) at voltages where the simultaneous tunneling of two electrons competes with sequential tunneling of single electrons.

is quadratic, like in the noninteracting theory. The cost of the Hubbard–Stratonovich transformation is the appearance of a new (bosonic) field, which has to be integrated over, but which usually represents the physically relevant variable. As a result, the generating functional is transformed into the form

$$\mathcal{Z}[\eta] = \int \mathcal{D}\phi_1 \ldots \mathcal{D}\phi_N \exp\left\{\mathrm{i}S_{\mathrm{eff}}[\phi_i, \eta]\right\}, \tag{2.21}$$

where $S_{\mathrm{eff}}[\phi_i, \eta]$ is the *effective action*, depending on the source field η, and the fields ϕ_i arising from N Hubbard–Stratonovich transformations. This also makes the practical calculations more tractable, not having to worry about the peculiar algebra of the Grassmann numbers.

2.3 Effective action for a single-electron transistor

Single-electron transistor [16] could be called the archetype of a nanoelectronic device. It consists of a small metallic island with strong Coulomb interaction contacted with tunnel junctions to two leads and coupled capacitively to a gate electrode (see Fig. 2.2). It has been thoroughly studied both theoretically [17, 18, 19, 20] and experimentally [21, 22, 23], and is also the focus of Publications I and II. Moreover, it serves as a nice example for the application of functional field integrals [24, 25].

The Hamiltonian for the single-electron transistor is [26]

$$\hat{\mathrm{H}} = \hat{\mathrm{H}}_L + \hat{\mathrm{H}}_R + \hat{\mathrm{H}}_I + \hat{\mathrm{H}}_{TL} + \hat{\mathrm{H}}_{TR}, \tag{2.22}$$

where the noninteracting leads ($\alpha \in \{L, R\}$) are described by

$$\hat{\mathrm{H}}_\alpha = \sum_\nu \epsilon_\nu \hat{\mathrm{c}}^\dagger_{\alpha\nu} \hat{\mathrm{c}}_{\alpha\nu}, \tag{2.23}$$

and the island Hamiltonian is

$$\hat{\mathrm{H}}_I = \sum_\mu \epsilon_\mu \hat{\mathrm{d}}^\dagger_\mu \hat{\mathrm{d}}_\mu + E_C \left(\sum_\mu \hat{\mathrm{d}}^\dagger_\mu \hat{\mathrm{d}}_\mu - N_0 \right)^2 . \tag{2.24}$$

Here, $E_C = e^2/(2C)$ is the charging energy of the island and N_0 is the electrostatically preferred charge, set by the gate voltage. The tunneling between the leads and the island is described by

$$\hat{\mathrm{H}}_{T\alpha} = \sum_{\nu\mu} t_{\alpha\nu\mu} \hat{\mathrm{c}}^\dagger_{\alpha\nu} \hat{\mathrm{d}}_\mu + \mathrm{H.c.}, \tag{2.25}$$

where $t_{\nu\mu}$ is the tunneling matrix element between states ν and μ.

The interaction part of the action, defined along the Keldysh contour, is

$$S_C[\bar\psi, \psi] = \int_C \mathrm{d}t \left(\sum_\mu \bar\psi_\mu \psi_\mu - N_0 \right) E_C \left(\sum_{\mu'} \bar\psi_{\mu'} \psi_{\mu'} - N_0 \right). \tag{2.26}$$

Hubbard–Stratonovich decoupling of the interaction results in

$$\begin{aligned} S_C[\bar\psi, \psi, V] =& \frac{1}{4E_C} \int_C \mathrm{d}t V^2 - \int_C \mathrm{d}t V \left(\sum_\mu \bar\psi_\mu \psi_\mu - N_0 \right), \\ =& \int \mathrm{d}t \left(\frac{1}{2E_C} V_c V_q + V_q N_0 \right) - \int \mathrm{d}t \bar{\mathbf{\Psi}} \begin{pmatrix} V_c & \frac{1}{2} V_q \\ \frac{1}{2} V_q & V_c \end{pmatrix} \mathbf{\Psi}, \end{aligned} \tag{2.27}$$

where $V_c = \frac{1}{2}(V_+ + V_-)$ and $V_q = V_+ - V_-$ are the two Keldysh-rotated components of a real bosonic field, to be integrated over.

The total action can then be written as a sum of

$$\begin{aligned} S_\alpha[\bar{\mathbf{\Psi}}_\alpha, \mathbf{\Psi}_\alpha] &= \int \mathrm{d}t \mathrm{d}t' \bar{\mathbf{\Psi}}_\alpha(t) \begin{pmatrix} \mathrm{i}\partial_t + \mathrm{i}0^+ - \boldsymbol{\epsilon}_\alpha & 2\mathrm{i}0^+ \mathbf{F}_\alpha \\ 0 & \mathrm{i}\partial_t - \mathrm{i}0^+ - \boldsymbol{\epsilon}_\alpha \end{pmatrix} \mathbf{\Psi}_\alpha(t'), \\ S_I[\bar{\mathbf{\Psi}}_I, \mathbf{\Psi}_I, V] &= \int \mathrm{d}t \mathrm{d}t' \bar{\mathbf{\Psi}}_I(t) \begin{pmatrix} \mathrm{i}\partial_t + \mathrm{i}0^+ - \boldsymbol{\epsilon}_I - V_c & -\frac{1}{2} V_q + 2\mathrm{i}0^+ \mathbf{F}_I \\ -\frac{1}{2} V_q & \mathrm{i}\partial_t - \mathrm{i}0^+ - \boldsymbol{\epsilon}_I - V_c \end{pmatrix} \mathbf{\Psi}_I(t'), \\ S_{T\alpha}[\bar{\mathbf{\Psi}}_\alpha, \mathbf{\Psi}_\alpha, \bar{\mathbf{\Psi}}_I, \mathbf{\Psi}_I] &= \int \mathrm{d}t \mathrm{d}t' \left[\bar{\mathbf{\Psi}}_\alpha(t) \begin{pmatrix} \mathbf{T}_\alpha & 0 \\ 0 & \mathbf{T}_\alpha \end{pmatrix} \mathbf{\Psi}_I(t') \right. \\ &\quad \left. + \bar{\mathbf{\Psi}}_I(t) \begin{pmatrix} \mathbf{T}^\dagger_\alpha & 0 \\ 0 & \mathbf{T}^\dagger_\alpha \end{pmatrix} \mathbf{\Psi}_\alpha(t') \right], \\ S_C[V] &= \int \mathrm{d}t \left(\frac{1}{2E_C} V_c V_q + V_q N_0 \right). \end{aligned} \tag{2.28}$$

The $\mathbf{F}$ matrices store the information about the *initial* distribution functions, including possible biasing by a chemical potential or temperature

difference. From here it is easy to see that the electrons on the island effectively feel a fluctuating potential V_c due to other electrons. Finally, for the calculation of the statistics of energy on the island, we need a suitable source term. It is given by

$$S_{\text{source}}[\dot{\xi}] = \int \mathrm{d}t \bar{\mathbf{\Psi}}_I(t) \begin{pmatrix} 0 & \dot{\xi}\boldsymbol{\epsilon}_I \\ \dot{\xi}\boldsymbol{\epsilon}_I & 0 \end{pmatrix} \mathbf{\Psi}_I(t). \tag{2.29}$$

We include the time derivative in the definition of the source field for convenience, simplifying the next step.

It is convenient to remove the fluctuating potential and the source field by a gauge transformation

$$\mathbf{\Psi}_I \mapsto e^{-\mathrm{i}\mathbf{\Phi}+\mathrm{i}\boldsymbol{\xi}\otimes\boldsymbol{\epsilon}_I}\mathbf{\Psi}_I, \quad \bar{\mathbf{\Psi}}_I \mapsto \bar{\mathbf{\Psi}}_I e^{\mathrm{i}\mathbf{\Phi}-\mathrm{i}\boldsymbol{\xi}\otimes\boldsymbol{\epsilon}_I}, \tag{2.30}$$

where $\mathbf{\Phi} = \phi_c + \sigma_1\phi_q/2$, $\boldsymbol{\xi} = \sigma_1\xi$, $\partial_t\phi_{c,q} = V_{c,q}$, and σ_1 is the first Pauli spin matrix. As a result, V vanishes from S_I,[4] and S_{source} vanishes altogether, but the tunneling matrices are modified according to

$$\begin{pmatrix} \mathbf{T}_\alpha & 0 \\ 0 & \mathbf{T}_\alpha \end{pmatrix} \mapsto \begin{pmatrix} \mathbf{T}_\alpha & 0 \\ 0 & \mathbf{T}_\alpha \end{pmatrix} e^{-\mathrm{i}\mathbf{\Phi}+\mathrm{i}\boldsymbol{\xi}\otimes\boldsymbol{\epsilon}_I}. \tag{2.31}$$

To obtain the effective action we perform the Gaussian integrals over the fermion fields $\bar{\mathbf{\Psi}}$ and $\mathbf{\Psi}$. The result is

$$\begin{aligned} S_{\text{eff}}[\phi,\xi] =& \operatorname{tr}\ln\left(1+\mathbf{G}_0\mathbf{T}\right) + \int \mathrm{d}t \left(\frac{1}{2E_C}\partial_t\phi_c\partial_t\phi_q + \partial_t\phi_q N_0\right), \\ \equiv& S_T[\phi,\xi] + S_C[\phi], \end{aligned} \tag{2.32}$$

where the trace includes integrations over time,

$$\begin{aligned} \mathbf{G}_0 &= \begin{pmatrix} \mathbf{G}_{0,L} & 0 & 0 \\ 0 & \mathbf{G}_{0,I} & 0 \\ 0 & 0 & \mathbf{G}_{0,R} \end{pmatrix}, \\ \mathbf{T} &= \begin{pmatrix} 0 & \mathbf{T}_L e^{-\mathrm{i}\mathbf{\Phi}+\mathrm{i}\boldsymbol{\xi}\otimes\boldsymbol{\epsilon}_I} & 0 \\ \mathbf{T}_L^\dagger e^{\mathrm{i}\mathbf{\Phi}-\mathrm{i}\boldsymbol{\xi}\otimes\boldsymbol{\epsilon}_I} & 0 & \mathbf{T}_R^\dagger e^{\mathrm{i}\mathbf{\Phi}-\mathrm{i}\boldsymbol{\xi}\otimes\boldsymbol{\epsilon}_I} \\ 0 & \mathbf{T}_R e^{-\mathrm{i}\mathbf{\Phi}+\mathrm{i}\boldsymbol{\xi}\otimes\boldsymbol{\epsilon}_I} & 0 \end{pmatrix}, \end{aligned} \tag{2.33}$$

and $\mathbf{G}_{0,L/I/R}$ are the free Green functions of the left lead, island, and right lead, respectively. Assuming weak coupling to the leads and that the tunneling quasiparticles lose their phase coherence before reaching

[4] The gauge transformation also introduces terms proportional to $\mathrm{i}0^+$ due to the Keldysh structure of S_I. These can be safely neglected since the tunneling terms dominate over them.

the other tunnel barrier (in other words, the out-tunneling quasiparticle is always different from the in-tunneling one), we can expand the trace of the logarithm to leading order in t_α:

$$S_T[\phi,\xi] = \sum_\alpha \mathrm{tr}\left(\mathbf{G}_{0,\alpha}\mathbf{T}_\alpha e^{-\mathrm{i}\mathbf{\Phi}+\mathrm{i}\boldsymbol{\xi}\otimes\boldsymbol{\epsilon}_I}\mathbf{G}_{0,I}\mathbf{T}_\alpha^\dagger e^{\mathrm{i}\mathbf{\Phi}-\mathrm{i}\boldsymbol{\xi}\otimes\boldsymbol{\epsilon}_I}\right). \tag{2.34}$$

It is possible to cast this into a form [12]

$$S_T[\phi,\xi] = \frac{g}{2}\int \mathrm{d}t\mathrm{d}t' \begin{pmatrix} e^{-\mathrm{i}\phi_+(t)} & e^{-\mathrm{i}\phi_-(t)} \end{pmatrix} \mathbf{L}^{(\xi)}(t,t') \begin{pmatrix} e^{\mathrm{i}\phi_+(t')} \\ e^{\mathrm{i}\phi_-(t')} \end{pmatrix}, \tag{2.35}$$

where we have transformed back to fields defined on the forward and backward branch, and where the matrix $\mathbf{L}^{(\xi)}$ depends on ξ but not ϕ. Furthermore, $g_\alpha = 4\pi^2\nu_I\nu_\alpha|t_\alpha|^2$ is the dimensionless conductance of contact α, and ν is the density of states. We have also assumed tunneling matrix elements t to be independent of the lead and island Hilbert space indices, and a left–right symmetric structure, $g_L = g_R = g$.

In nearly isolated islands, $g \ll 1$, the phases fluctuate rapidly, and no longer represent suitable degrees of freedom. On the other hand, the charge on the island fluctuates only a little. The charge degree of freedom is brought in by another Hubbard–Stratonovich transformation. Using Eq. (2.20) with $\phi = n$ and $\rho = \dot{\phi}$ brings the charging action to the form

$$\begin{aligned} S_C[\phi,n] &= \int \mathrm{d}t\left[n_c\partial_t\phi_q + n_q\partial_t\phi_c - 2E_C n_q(n_c+N_0)\right], \\ &= \int \mathrm{d}t\left[n_+\partial_t\phi_+ - n_-\partial_t\phi_- - E_C(n_+^2-n_-^2) - 2E_C N_0(n_+-n_-)\right], \end{aligned} \tag{2.36}$$

again at a cost of an additional functional integration over n.

Using the form of Eq. (2.35), the exponentiated action can be expanded in a Taylor series

$$\begin{aligned} e^{\mathrm{i}S_T[\phi,\xi]} = \sum_{m=0}^\infty \frac{1}{m!}\left(\frac{\mathrm{i}g}{2}\right)^m \prod_{k=1}^m \left(\int \mathrm{d}t_k\mathrm{d}t_k' \sum_{\sigma_k\sigma_k'}\right) e^{-\mathrm{i}\sum_k\left[\phi_{\sigma_k}(t_k)-\phi_{\sigma_k'}(t_k')\right]} \\ \times \prod_{k=1}^m L^{(\xi)}_{\sigma_k\sigma_k'}(t_k,t_k'), \end{aligned} \tag{2.37}$$

where $\sigma,\sigma' \in \{+,-\}$. Performing now the ϕ integral in the partition function gives a delta functional which fixes n to

$$n_\sigma(t) = n(-\infty) - \sigma\sum_k \theta(t-t_k)\delta_{\sigma\sigma_k} + \sigma\sum_k \theta(t-t_k')\delta_{\sigma\sigma_k'}, \tag{2.38}$$

where the initial charge state $n(-\infty) \in \mathbb{Z}$. The charge on the island jumps at $t = t_k'$ and $t = t_k$ in steps of one electron. By inserting this to the

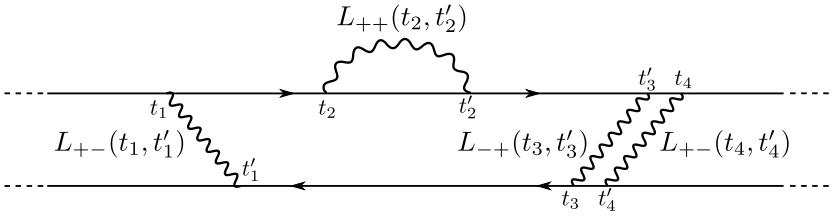

Figure 2.3. Example contributions to the expansion in Eq. (2.39). The two first diagrams correspond to sequential events, i.e., the tunneling lines do not overlap. The last two lines overlap, describing a cotunneling event.

charging action, the generating functional assumes the form

$$\mathcal{Z}[\xi] = \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{\mathrm{i}g}{2}\right)^m \prod_{k=1}^{m} \left(\int \mathrm{d}t_k \mathrm{d}t_k' \sum_{\sigma_k \sigma_k'} \right) e^{-\mathrm{i}E_C \int \mathrm{d}t \left[n_+^2(t) - n_-^2(t) + 2N_0 (n_+(t) - n_-(t)) \right]} \times \prod_{k=1}^{m} L_{\sigma_k \sigma_k'}^{(\xi)}(t_k, t_k'). \tag{2.39}$$

The generating functional is expressed as a sum over quasiparticle in- and out-tunneling events, connected by the elements of the kernel $L_{\sigma_k \sigma_k'}(t_k, t_k')$. This is illustrated in Fig. 2.3. Elements of the type L_{+-} and L_{-+} correspond to actual tunneling processes which change the charge state of the island. L_{++} and L_{--} describe virtual processes which leave the charge state unchanged, but renormalize the dimensionless conductance of the junctions and the energy gap of the adjacent charge states [27, 28].

The temporal extension of $L_{\sigma_k \sigma_k'}(t_k, t_k')$ is related to the typical duration of the tunneling events, $\delta t \sim E_C^{-1}$. On the other hand, the typical time between tunneling events is of the order of $(gE_C)^{-1}$ [12]. Thus, for $g \ll 1$ tunneling events happen mostly sequentially, not overlapping with each other in time. In this limit the calculation of the generating functional is a lot simpler. If we furthermore are only interested in the low-frequency correlations, we can neglect the time dependence of the source field ξ, and the generating functional reduces to [29]

$$\mathcal{Z}[\xi] = \exp\left\{ -\Lambda_{\min}(\xi) t_0 \right\}, \tag{2.40}$$

where t_0 is the measurement time during which ξ has a non-zero value, and $\Lambda_{\min}(\xi)$ is the smallest eigenvalue of a matrix constructed from the elements $L_{\sigma_k \sigma_k'}^{(\xi)}$. For the purposes of calculating the statistics of heat currents, it is necessary to include also processes where one quasiparticle tunnels in and another quasiparticle tunnels out in a single *inelastic cotunneling* event [17], described by a diagram where two tunneling lines overlap. This is done in Publications I and II for the case when the single-

electron events always appear as back-to-back in- and out-tunneling pairs, not interrupted by cotunneling events.

3. Statistics of temperature fluctuations

The theory of full counting statistics (FCS) [30, 31] is a powerful tool for the study of fluctuations of electric currents. The theory of temperature fluctuation statistics formulated here is closely related to FCS. The source term for the electric current, i.e., "counting field," is accompanied by a corresponding source term for the heat current [c.f. Eq. (2.29)]. A calculation along the lines of FCS produces a probability distribution for the transferred energy in some period of time. In order to obtain the probability distribution for temperature or internal energy, additional steps have to be taken. These steps are detailed below, after a rigorous definition of the concept of temperature fluctuations.

3.1 Equilibrium fluctuations

In equilibrium statistical mechanics, open systems, i.e., systems which can exchange energy and particles with their surroundings, have by definition temperature and chemical potential which are fixed to those of their surroundings, and which do not fluctuate [32]. These in turn determine the expectation values for energy and particle number, which, on the other hand, can fluctuate [3]. In equilibrium, however, these fluctuations are small in the thermodynamic limit where the number of degrees of freedom N tends to infinity:

$$\frac{\sqrt{\mathrm{var}(E)}}{\langle E\rangle} \propto N^{-1/2}, \quad \frac{\sqrt{\mathrm{var}(N)}}{\langle N\rangle} \propto N^{-1/2}. \tag{3.1}$$

Out of equilibrium these considerations do not necessarily apply: Temperature and chemical potential may fluctuate,[1] and these fluctuations may be large.

[1] Strictly speaking, temperature and chemical potential are not even defined out of equilibrium. Many observables still depend on some *effective* temperature and chemical potential, which play an identical role to the real thermodynamic variables. It is in this sense that I use these terms in the following.

3.2 Quasi-equilibrium fluctuations

Theory of quantum transport always includes the concept of *reservoirs*, large contacts with an infinite number of particles in local equilibrium. When these particles are electrons, their energy distribution is described by a Fermi–Dirac function with a well-defined chemical potential and temperature,

$$f(E) = \frac{1}{e^{(E-\mu)/T} + 1}. \tag{3.2}$$

The object of interest (*island* in the following), e.g., a quantum dot, a metallic granule, a carbon nanotube, or a sheet of graphene, is usually connected between two or more reservoirs biased at different chemical potentials and/or temperatures. Given the Hamiltonian of the system the occupation numbers of various quantum states on the island could in principle be solved and the observable of interest, e.g., electric current, then calculated with the methods described in Chapter 2. In the case of a non-interacting island the occupation numbers are simply determined by the injection and extraction rates to the specific state, and the distribution function can deviate a lot from the Fermi function [33, 34]. This case is studied in Publication VI in an all-superconducting tunnel structure, where a small superconducting island is driven out of equilibrium by a rapid injection of quasiparticles from superconducting reservoirs.

In contrast, in the case of a strong electron–electron and electron–phonon interaction, the injection and extraction rates due to driving are small compared to the electron–electron and electron–phonon relaxation rates, and the island is in equilibrium. The electron energy distribution is described with a Fermi function at the phonon temperature and with a chemical potential determined by electrostatics.

In between these regimes lies *quasi-equilibrium* [35], where the interaction between electrons is strong enough for them to assume an energy distribution of the Fermi form, but with a temperature and chemical potential to be determined separately. The physical prescription is to impose the conservation of energy current and electric current, giving two equations fixing the temperature and chemical potential. Depending on the environment of the island and its contacts to the reservoirs, the electric and thermal RC times can be large compared to other time scales of the system, implying the possibility of charge and energy buildup or depletion on the island. In this case the assumption of a fixed temperature and chemical potential is, of course, wrong. The conserved quantities are

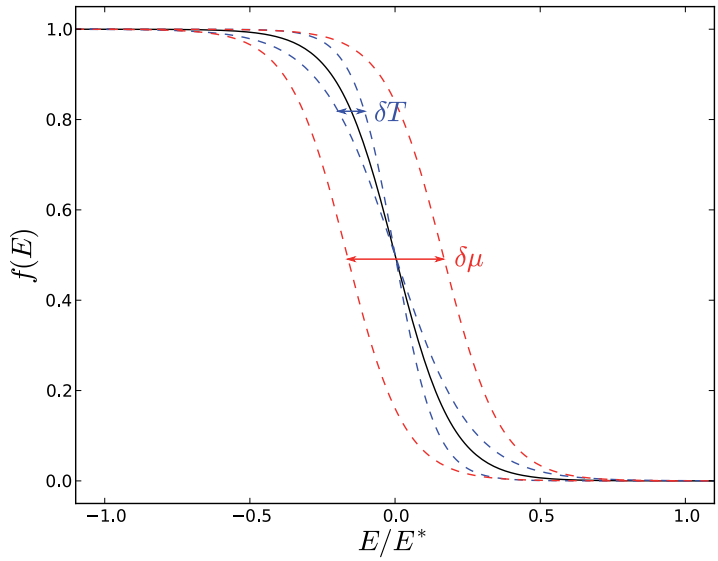

Figure 3.1. Equilibrium Fermi–Dirac distribution function with $T/E^* = 0.1$, E^* being some characteristic energy scale, and $\mu = 0$ (black). Temperature fluctuates up (down) when energy is deposited on (removed from) the island (dashed blue lines). Chemical potential fluctuates up (down) when negative charge is deposited on (removed from) the island (dashed red lines).

now charge and energy, with a possibility for them to accumulate on the island [36]. In quasi-equilibrium the electron–electron relaxation time is the shortest time scale and the energy distribution of electrons on the island adapts quickly to the changes in the total energy and charge by a change in the temperature or chemical potential. As a result, temperature and chemical potential fluctuate on the thermal and electric RC time scales, respectively. This is exemplified in Fig. 3.1.

To determine whether a given system is in quasi-equilibrium, we have to calculate the electron–electron relaxation rate and compare it with the driving rate. For low-energy electrons the relaxation is mainly due to the interaction of an electron with the potential fluctuations due to all other electrons [37][see also the discussion below Eq. (2.28)], varying slowly both in space and time. This interaction can be described by a collision integral [37, 38]

$$\frac{\partial f(E)}{\partial t} = \int \mathrm{d}\omega \left[-S_V(\omega) f(E)(1 - f(E+\omega)) + S_V(-\omega) f(E+\omega)(1 - f(E)) \right], \tag{3.3}$$

where $S_V(\omega)$ is the spectral density of the potential fluctuations. First term of the integrand describes scattering from energy E to energy $E+\omega$,

the second term describes the inverse process. In the case of only a few excited electrons at energy E in an otherwise zero-temperature metal in equilibrium, this can be solved explicitly. Putting $f(E+\omega) = \theta(-E-\omega)$ and using the fluctuation–dissipation relation $S_V(\omega) = 2\chi(\omega)\theta(-\omega)$ we obtain

$$f(E,t) = f(E,0)\exp\left(-2t\int_0^E \mathrm{d}\omega\chi(\omega)\right), \tag{3.4}$$

from which we identify

$$\frac{1}{\tau_{e-e}(E)} = 2\int_0^E \mathrm{d}\omega\chi(\omega) \tag{3.5}$$

as the relaxation rate. Here, $\chi(\omega)$ is the electric susceptibility. For a diffusive d-dimensional system it is given by [37]

$$2\chi(\omega) = G_Q\int\frac{\mathrm{d}\mathbf{q}}{(2\pi)^d}\mathrm{Re}(Z_d(\omega,\mathbf{q}))\frac{\omega D\mathbf{q}^2}{|D\mathbf{q}^2 - \mathrm{i}\omega|^2}, \tag{3.6}$$

where $G_Q = 2e^2/h$ is the conductance quantum, D is the diffusion coefficient, and $Z_d(\omega,\mathbf{q})$ is the impedance for a d-dimensional conductor:

$$\mathrm{Re}(Z_d(\omega,\mathbf{q})) = \begin{cases} \frac{\rho}{\mathbf{q}^2}, & d=3, \\ \frac{\mathbf{q}^2/\rho}{(\mathbf{q}^2/\rho)^2+\omega^2\tilde{C}^2}, & d=2 \text{ and } 1. \end{cases} \tag{3.7}$$

Here, ρ is the d-dimensional resistivity and $\tilde{C}$ the capacitance per unit area (2D) or length (1D). In three dimensions the impedance does not depend on capacitance since the accumulation of surface charge can be neglected. Performing the integrals we obtain

$$\frac{1}{\tau_{e-e}(E)} = \begin{cases} \frac{1}{6\sqrt{2}\pi}G_Q\rho E^{3/2}/D^{1/2}, & d=3, \\ \frac{1}{8}G_Q\frac{\rho}{1+\rho\tilde{C}D}E, & d=2, \\ \frac{\pi}{\sqrt{2}}G_Q\frac{\rho}{(1+\rho\tilde{C}D)(1+\sqrt{\rho\tilde{C}D})}E^{1/2}D^{1/2}, & d=1. \end{cases} \tag{3.8}$$

What should be used for E, and how do we determine the dimensionality? Typical energy for the excited electrons is given by the bias voltage, $E = eV$. The dimensionality is determined by comparing the distance the electron travels during the relaxation, $l_{e-e} = \sqrt{D\tau_{e-e}}$, to the dimensions of the island. If l_{e-e} is larger than, say, two of the dimensions of the island, the island is effectively a 1D system.

The electron–electron relaxation time should be compared to the dwell time, i.e., the time an electron spends traveling through the structure. It is related to the Thouless energy

$$\frac{1}{\tau_d} = E_{\mathrm{Th}} = \frac{G}{G_Q}\delta, \tag{3.9}$$

where G is the total conductance of the structure, including the contacts, and δ is the single-particle level spacing on the island. Whether a given structure is in quasi-equilibrium is then determined by the condition

$$\frac{\tau_{e-e}}{\tau_d} = \frac{G}{G_Q}\frac{G(L=\sqrt{D/eV})}{G_Q}\frac{\delta}{eV} \ll 1, \tag{3.10}$$

where $G(L = \sqrt{D/eV})$ is the conductance of a piece of island of size $(\sqrt{D/eV})^d$. In tunnel structures this is typically very well satisfied. Structures with good contacts are easier to drive out of equilibrium: In Ref. [34] $\tau_{e-e}/\tau_d \approx 35$ for one of the measured wires.

Going beyond quasi-equilibrium, it is not so clear what is meant by temperature fluctuations. As shown in Publication VI, the non-equilibrium electron distribution functions in general cannot be characterized by a simple temperature. Sometimes it is possible to unambiguously define some effective temperature which fluctuates as a result of fluctuating energy flows. In the research on temperature fluctuations reported in this Dissertation it is always assumed that the island is in quasi-equilibrium, and thus the concept of temperature fluctuations is clearly defined.

3.2.1 Role of the electron–phonon interaction

Another source of relaxation, driving the electron system towards equilibrium, is the electron–phonon interaction. The electron–phonon scattering rate is given by [37]

$$\frac{1}{\tau_{e-\mathrm{ph}}(E)} \simeq \lambda E \left(\frac{E}{\omega_D}\right)^2, \tag{3.11}$$

where λ is a dimensionless electron–phonon coupling constant and ω_D the Debye frequency, i.e., maximum phonon energy, being of the order of 100 K. Therefore, at cryogenic temperatures of the order of 1 K the electron–phonon scattering rate is typically negligible compared to the electron–electron scattering rate.

The heat current from electrons to phonons may still be comparable to the heat current to reservoirs. The electron–phonon heat current is proportional to T^5 [39], diminishing quickly at very low temperatures. In the case when the system can be considered completely decoupled from phonons, it is said to be *overheated*. Temperature fluctuations are especially important in overheated systems.

3.3 Effective action approach to fluctuation statistics

To describe the statistics of temperature fluctuations we start with the generating functional for charge and heat currents from the island to reservoirs $j = 1, 2, 3 \ldots$.[2] It is given by [40]

$$\begin{aligned}\mathcal{Z}[\{\chi_j\},\{\xi_j\}] &= \int \mathcal{D}\mathbf{\Psi}\mathcal{D}\bar{\mathbf{\Psi}} \exp\left\{ \mathrm{i}S[\mathbf{\Psi},\bar{\mathbf{\Psi}}] + \mathrm{i}\sum_i \int \mathrm{d}t \chi_i I_i(\mathbf{\Psi},\bar{\mathbf{\Psi}}) \right. \\ &\qquad \left. + \mathrm{i}\sum_i \int \mathrm{d}t \xi_i \dot{H}_i(\mathbf{\Psi},\bar{\mathbf{\Psi}}) \right\}, \\ &= \exp\left\{ \mathrm{i}\sum_i S_{\mathrm{eff},i}\left[\chi_i,\xi_i\right] \right\}, \end{aligned} \tag{3.12}$$

where $S[\mathbf{\Psi},\bar{\mathbf{\Psi}}]$ is the action of the entire system (reservoirs and island), and the rest of the terms in the exponent are source terms for electric and energy currents. In the last line I have integrated over the microscopic degrees of freedom $\mathbf{\Psi},\bar{\mathbf{\Psi}}$, to obtain the effective action for *connector* i between the island and reservoir i, $S_{\mathrm{eff},i}$. This separation between the reservoirs is possible only when electrons arriving from one reservoir lose all of their phase coherence before entering another reservoir. The connector action depends only on the temperature and chemical potential of the reservoir and the island, and the properties of the contact between them. For electron reservoirs, it has the general form [41]

$$S_{\mathrm{eff},i} = \frac{1}{2}\sum_n \mathrm{tr}\ln\left[1 + T_n^i \frac{\{\mathbf{G}_i(\chi_i,\xi_i),\mathbf{G}_I\} - 2}{4}\right], \tag{3.13}$$

where T_n^i are the transmission eigenvalues of the contact [5] and $\mathbf{G}$ are the quasiclassical Keldysh Green functions [42, 43] of the reservoir and the island.

All moments of the currents can be calculated by functional differentiation of the generating functional. For example, the electron–phonon heat current noise in a proximity Josephson sensor is obtained from this expression in Publication V. In the case of a separable effective action, like above, the currents to different reservoirs are uncorrelated, and the probability distribution for the realization of a certain set of currents

[2] Reservoirs are not limited to electron reservoirs. One of them could be, for example, the phonon bath. In this case only energy transfer is possible.

$\{I_j\}, \{\dot{H}_j\}$ factorizes to [44]

$$\prod_i P[I_i, \dot{H}_i] = \int \prod_i \mathcal{D}\chi_i \mathcal{D}\xi_i \times \exp\left\{ \sum_i \left(-\mathrm{i} \int \mathrm{d}t \chi_i I_i - \mathrm{i} \int \mathrm{d}t \xi_i \dot{H}_i + \mathrm{i} S_{\mathrm{eff},i}\left[\chi_i, \xi_i\right] \right) \right\}. \tag{3.14}$$

This expression describes an island where an arbitrary amount of heat and charge can be deposited or extracted, i.e., it seems that they are not conserved. In physical systems continuity equations connect the currents in different reservoirs. These are incorporated into the probability distribution as delta functionals, resulting in [44, 45]

$$\begin{aligned} P[\{I_j\}, \{\dot{H}_j\}] &= \int \mathcal{D}Q_I \mathcal{D}E_I \delta\left(\sum_i I_i - \dot{Q}_I \right) \delta\left(\sum_i \dot{H}_i - \dot{E}_I \right) \prod_i P[I_i, \dot{H}_i], \\ &= \int \mathcal{D}Q_I \mathcal{D}E_I \mathcal{D}\chi_I \mathcal{D}\xi_I \exp\left\{ \mathrm{i} \int \mathrm{d}t \chi_I \left(\sum_i I_i - \dot{Q}_I \right) \right\} \\ &\quad \times \exp\left\{ \mathrm{i} \int \mathrm{d}t \xi_I \left(\sum_i \dot{H}_i - \dot{E}_I \right) \right\} \prod_i P[I_i, \dot{H}_i], \end{aligned} \tag{3.15}$$

where Q_I is the charge and E_I the energy on the island.[3] Integrating over all the realizations of currents must yield a probability of one. Therefore we get a representation for the Keldysh partition function in the form

$$\begin{aligned} \mathcal{Z} &= \int \prod_i \mathcal{D}I_i \mathcal{D}\dot{H}_i P[\{I_j\}, \{\dot{H}_j\}], \\ &= \int \mathcal{D}Q_I \mathcal{D}E_I \mathcal{D}\chi_I \mathcal{D}\xi_I \exp\left\{ -\mathrm{i} \int \mathrm{d}t \chi_I \dot{Q}_I - \mathrm{i} \int \mathrm{d}t \xi_I \dot{E}_I + \mathrm{i} \sum_i S_{\mathrm{eff},i}\left[\chi_I, \xi_I\right] \right\}, \end{aligned} \tag{3.16}$$

which is a convenient starting point for further analysis.

Another interesting topic is the study of fluctuation theorems [46, 47], which relate the probability to observe a certain realization of physical process to the probability to observe its time-reversed counterpart. Fluctuation relations for electric and heat currents can be obtained from the generating functional in a relatively straightforward manner [48].

3.3.1 Dynamics of charge and energy

In Eq. (3.16) the dependence of $S_{\mathrm{eff},i}$ on Q_I and E_I is left implicit. As discussed above, these dictate the temperature and chemical potential in

[3] In some cases, e.g., single-electron transistor, the effective action already includes charge conservation, and the delta functional for currents is not needed. In this case the dependence on Q_I and χ_I drops out of the expressions in this section.

the Fermi distribution function of the island. The model used to connect these together is described by [44, 49]

$$E_I = \frac{\pi^2 T_I^2}{6\delta} + \frac{1}{2} C \mu_I^2, \quad Q_I = C\mu_I, \tag{3.17}$$

where C is the total capacitance of the island. Using these equations the dependence on Q_I and E_I is brought in to the Keldysh component of the effective action.

We can compare the time scale for charge transport $\tau_c = C/G$ to the time scale for energy transport $\tau_E = C_h/G_{\rm th} = \pi^2 T/(3\delta G_{\rm th})$, where $G_{\rm th}$ is the thermal conductance. Using the Wiedemann–Franz law, $G_{\rm th} = \pi^2/(3e^2)TG$,[4] we get $\tau_c/\tau_E = \delta/(2E_C) \ll 1$. This implies that the rapid potential fluctuations average out on the time scale of the energy and temperature fluctuations and can be neglected. This also means that whereas the fluctuations of charge have an essentially flat frequency spectrum up to a frequency of $1/\tau_c$, the bandwidth of the temperature fluctuations only extends up to a lower frequency of $1/\tau_E$. This can serve as a tell-tale signal that the observed fluctuations are related to the temperature fluctuations.

3.3.2 Saddle point approximation

When the fluctuations of charge and energy on the island are small compared to their average values, the functional integral in Eq. (3.16) can be evaluated in the saddle point approximation. The total effective action

$$S_{\rm eff}\left[\chi_I, \xi_I, Q_I, E_I\right] = \sum_i S_{{\rm eff},i}\left[\chi_I, \xi_I\right] - \int {\rm d}t \chi_I \dot{Q}_I - \int {\rm d}t \xi_I \dot{E}_I, \tag{3.18}$$

has four saddle point equations: [50]

$$\begin{aligned} \frac{\delta S_{\rm eff}}{\delta \chi_I} &= 0, \quad \frac{\delta S_{\rm eff}}{\delta \xi_I} = 0, \\ \frac{\delta S_{\rm eff}}{\delta Q_I} &= 0, \quad \frac{\delta S_{\rm eff}}{\delta E_I} = 0. \end{aligned} \tag{3.19}$$

These have always a classical solution in which $\chi_I = \xi_I = 0$, describing the relaxation of Q_I and E_I to their stationary configuration. The other solution with finite χ_I and ξ_I describes "anti-relaxation," i.e., fluctuations where Q_I and/or E_I deviate quickly from their stationary values.

Since we are mainly interested in the temperature fluctuations, we can use the saddle point solutions for $\chi_I(t)$ and $\mu_I(t)$ in the limit of fast potential fluctuations, $\tau_c \ll \tau_E$. In this limit there is no charge depletion

[4] This is not an exact law in the nanostructures considered here, but a good estimate nonetheless.

or accumulation on the island, and the chemical potential is given by the current conservation law and appears static. The charge counting field follows the energy counting field adiabatically, $\chi_I = -\mu_I \xi_I$.

Using the saddle point solutions it is possible to calculate the classical action corresponding to the fluctuation trajectories, and from that calculate the probability density of finding the island at some temperature to an exponential accuracy [50].

3.3.3 Fokker–Planck equation

In the saddle point approximation the mechanism with which the system jumps between relaxation and anti-relaxation trajectories is left undefined. This is caused by fluctuations around the saddle point solutions. To take these into account, we can convert the functional integral in Eq. (3.16) to a Fokker–Planck equation. This method is outlined in Publication II. Neglecting the statistics of charge for now (in the limit $\tau_c \ll \tau_E$ we can just use the saddle point solutions from the previous section), the connector part of the effective action, $\sum_i S_{\mathrm{eff},i}\left[\xi_I\right]$, is converted into an operator $\hat{\mathrm{S}}$ by first ordering it so that all ξ_I are to the left of all E_I, and then replacing the fields ξ_I with energy derivatives, $\xi_I \mapsto -\partial_{E_I}$. The probability distribution for energy then obeys the Fokker–Planck equation[5]

$$\partial_t \mathcal{P}(E,t) = \hat{\mathrm{S}}\mathcal{P}(E,t). \tag{3.20}$$

By setting the left hand side to zero, we obtain a differential equation for the *stationary* probability distribution,

$$\hat{\mathrm{S}}\mathcal{P}_{\mathrm{st}}(E) = 0, \tag{3.21}$$

calculation of which is often our ultimate goal.

3.3.4 Langevin equation

A complementary approach is to convert the Fokker–Planck equation for the stationary probability distribution of energy into a stochastic time evolution equation, i.e., Langevin equation, for energy. Let us assume that the total effective action in Eq. (3.18) is quadratic in ξ_I (again, neglecting

[5]Traditionally, the nomenclature Fokker–Planck is reserved for equations which are of second order in derivatives ∂_{E_I}. This is also the case in Publication II. In Publications III and IV the relevant equation is of infinite order in ∂_{E_I}, but we have decided to use the term Fokker–Planck equation also in this case.

statistics of charge):

$$S_{\text{eff}}\left[\xi_I, E_I\right] = \int \mathrm{d}t \left\{ \xi_I \dot{H}_I(E_I) + \frac{1}{2}\xi_I^2 S_{\dot{H}_I}(E_I) - \xi_I \dot{E}_I \right\}. \tag{3.22}$$

The notation here suggests that the coefficients of the first and second order terms are given by the heat current and its noise, respectively. Making a Hubbard–Stratonovich transformation we transform the action into the form

$$S_{\text{eff}}\left[\eta, \xi_I, E_I\right] = \int \mathrm{d}t \left\{ \xi_I \left[\dot{H}_I(E_I) - \dot{E}_I + \eta \right] - \frac{1}{2}\eta^2 S^{-1}_{\dot{H}_I}(E_I) \right\}, \tag{3.23}$$

with an additional functional integration over η. The action is now linear in ξ_I, and performing the functional integral over ξ_I gives a delta functional of the quantity in square brackets. This means that only paths which satisfy the Langevin equation

$$\partial_t E_I(t) = \dot{H}_I(E_I(t)) + \eta(t), \tag{3.24}$$

contribute to the partition function. Here, $\eta(t)$ is a random fluctuating "force" with statistics given by the weight factor in Eq. (3.23).

When the action is not quadratic in ξ_I, the derivation of the corresponding Langevin equation is not as straightforward as in the quadratic case. Indeed, it might not even be possible. For an action with only even orders of ξ_I one can make multiple Hubbard–Stratonovich transformations, each one bringing about an additional random force. For an action with an exponential dependence on ξ_I it can be done by expanding the exponentiated action in a Taylor series, as shown in Publication IV.

Using the Langevin equation we can simulate timelines of energy, or some other observable, which sometimes are more informative than the full distributions obtained from the Fokker–Planck equation.

3.4 Non-Gaussian fluctuations

As discussed in Sec. 3.1, fluctuations of energy (and thus, temperature) are small in equilibrium in the limit of a large number of degrees of freedom. This includes, for example, systems with nanometer-size metallic islands described above. Since the relative variance of the temperature $\propto \delta/\langle T\rangle$ is small, the distribution can be regarded as Gaussian. For temperature [50]

$$P(T) \propto \exp\left[-\frac{\pi^2 (T - \langle T\rangle)^2}{6\langle T\rangle \delta} \right]. \tag{3.25}$$

It is not enough to just drive the system out of equilibrium in order to make the fluctuations non-Gaussian. In the simplest example of a noninteracting island coupled to two reservoirs and driven by a potential or temperature difference the probability distribution for temperature is close to Gaussian [50]. The weight in the non-Gaussian tails of the distribution is so small that the observation of these rare non-Gaussian events is well nigh impossible.

The single-electron transistor (SET) studied in Publications I and II and in Section 2.3 has the interesting property that below a certain bias voltage V_C the single-electron tunneling processes can only cool the island regardless of the island temperature, whereas above V_C also heating is possible. Up to V_C the second order co-tunneling processes stabilize the temperature of the island to a very small value. This suggests that small fluctuations in the heat current can give rise to large relative fluctuations in the temperature. In Publication II we show that the probability distribution near V_C is indeed non-Gaussian, provided that the transistor island is not too large, the single particle level spacing satisfying $\delta \gtrsim (T_C/V_C)^5 V_C$, where

$$\frac{T_C}{V_C} \approx \frac{1}{\sqrt{2}\ln(1/g)} \ll 1. \tag{3.26}$$

The magnitude of the fluctuations remain small, though: $\delta T/T_C \approx T_C/V_C$. This combined with the requirement $g \lesssim 10^{-3}$ for ultra high resistance tunnel barriers of the order of $1\ \mathrm{M}\Omega$ makes the experimental detection of these fluctuations challenging.

A somewhat similar property is found in a superconductor–insulator–normal metal–insulator–superconductor (SINIS) tunnel structure studied in Publications III and IV. The single-particle tunneling processes below a bias voltage of 2Δ, Δ being the energy gap in the superconducting leads, can only cool the island. In contrast to SET the second order processes deposit energy on the island in discrete units of 2Δ. The average energy on the island biased close to 2Δ depends on the rate γ of these second order processes, $\langle E\rangle \simeq 2\Delta\gamma^{4/3}$. When $\gamma \ll 1$ the relative fluctuations become extremely large: $\delta E/\langle E\rangle \simeq \gamma^{-4/3}$. This leads to a strongly non-Gaussian distribution of energy (and hence, temperature) when the second order tunneling rate is small, but still well within reach of the state-of-the-art experiments.

3.5 Induced fluctuations in other observables

When temperature fluctuations are not taken into account, the temperature of the island is reduced into a parameter of the theory just like the temperature of the reservoirs. When observables are calculated, they usually depend on these parameters. For example, the expectation value of the electric current is given by some function of temperature,

$$\langle I \rangle \equiv I(T). \tag{3.27}$$

In the presence of temperature fluctuations, to lowest order in the temperature dependence of the current, this is modified to

$$\langle I \rangle = I(\langle T \rangle). \tag{3.28}$$

Let us next consider current noise:

$$\langle I^2 \rangle - \langle I \rangle^2 \equiv S_I(T), \tag{3.29}$$

which is modified by the temperature fluctuations to

$$\langle I^2 \rangle - \langle I \rangle^2 = S_I(\langle T \rangle) + [\partial_T I(\langle T \rangle)]^2 \operatorname{var}(T). \tag{3.30}$$

If the temperature fluctuations are appreciable, and if the temperature sensitivity of the current is high, the intrinsic current noise can be dominated by the noise induced by the temperature fluctuations. This is the case, for example, in SET, where the electric current depends exponentially on temperature. Characterized with the Fano factor, $F = S_I/(2eI)$, the temperature-fluctuation induced noise figure is given by $F = 1/48(V_C/T_C)^4 \propto \ln^4(1/g)$, which exceeds 100 for $g = 10^{-3}$, as shown in Publication I. In comparison, the Fano factor for the intrinsic noise is $F = 1$, the induced noise being possibly several orders of magnitude larger. In Publication II we also show that the statistics of these huge current fluctuations are strongly non-Gaussian, and that the noise is dominated by large current spikes at low frequency.

Maxwell's demon [51] is a thought experiment of an entity which is able to open and close a door between two containers of gas, allowing molecules moving faster than average to pass only in one direction and molecules moving slower than average to pass only in the other direction. As a result, one container cools down and the other heats up, and entropy seems to decrease without doing any work. A more diligent analysis shows that the demon generates entropy in the process and the second law of thermodynamics is not violated. Still, devices which mimic this concept are called Maxwell's demons [52, 53, 54].

In Publication III we also describe how the measurement of current fluctuations could allow us to trap the device in a low-temperature state, similar to the idea of a Maxwell's demon. When the current measurement shows that the device has fluctuated to a very low temperature, the current through the device is switched off. The system then stays at this temperature for the duration of the electron–phonon scattering time, which is typically long compared to the other time scales.

3.6 Detection of temperature fluctuations

Induced fluctuations can also provide a way to detect the temperature fluctuations indirectly by detecting the induced fluctuations in some other observable. As shown in Publication III, the statistics of temperature fluctuations can be gathered by monitoring the current through the device in real time, provided that the intrinsic shot noise of the device is small compared to the temperature-fluctuation induced current fluctuations. The measurement scheme for a generic nanostructure is shown in Fig. 3.2. The temperature dependent current through the device, $I(T)$, generates a fluctuating voltage V_x across a shunt resistor $R_{\rm sh}$, which is then amplified and detected with a fast oscilloscope.

What should be the value of the shunt resistor? In order for the temperature-fluctuation induced noise to be detectable, the additional noise generated by the resistor and the amplifier must be smaller than the actual signal to be measured. This implies that $\sqrt{\mathrm{var}(\bar{V}_x)} \ll \langle \bar{V}_x \rangle$, where the overbar denotes a time average [see Eq. (3.31) below]. To make further progress we decompose the fluctuating voltage into a signal part and a noise part, $V_x = V + \delta V$, and average over a short measurement period $\tau \ll \tau_E$ during which the signal stays constant so that

$$\langle \bar{V}_x \rangle = \frac{1}{\tau} \int_0^\tau \mathrm{d}t \langle V_x(t) \rangle = V. \tag{3.31}$$

Variance of the measured signal is given by

$$\begin{aligned} \langle \bar{V}_x^2 \rangle - \langle \bar{V}_x \rangle^2 =& \frac{1}{\tau^2} \int_0^\tau \int_0^\tau \mathrm{d}t \mathrm{d}t' \langle \delta V(t) \delta V(t') \rangle, \\ =& \frac{1}{\tau^2} \int_0^\tau \int_0^\tau \mathrm{d}t \mathrm{d}t' \int \frac{\mathrm{d}\omega}{2\pi} e^{-\mathrm{i}\omega(t-t')} S_{V_x}(\omega), \end{aligned} \tag{3.32}$$

where $S_{V_x}(\omega)$ is the spectral noise power. In the case of thermal noise due to the shunt resistor, S_{V_x} is independent of frequency [5] and the variance becomes

$$\mathrm{var}(\bar{V}_x) = \frac{S_{V_x}}{\tau} = \frac{2TR_{\rm sh}R_{\rm dyn}^2}{\tau(R_{\rm sh}+R_{\rm dyn})^2}, \tag{3.33}$$

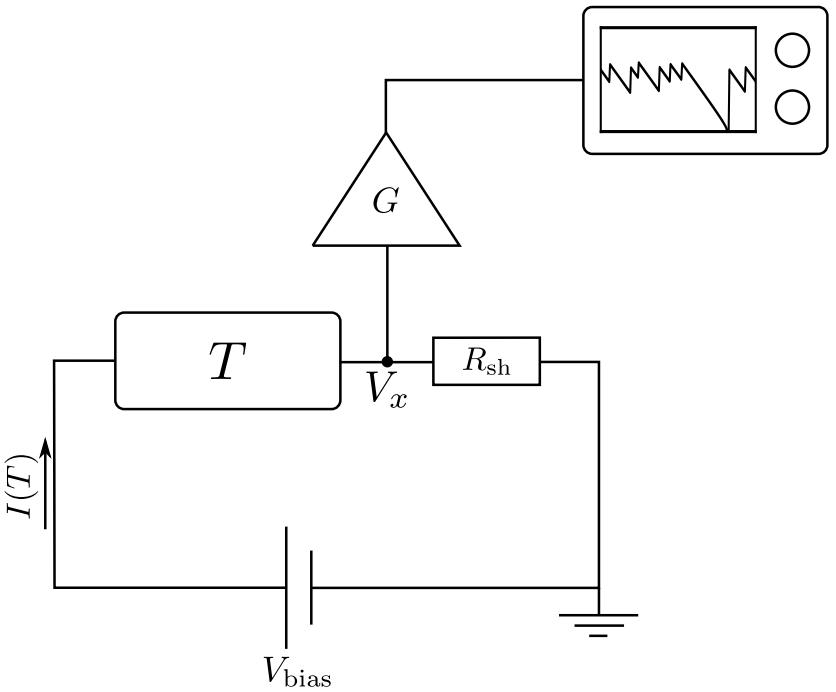

Figure 3.2. Example measurement scheme for the detection of temperature fluctuations via current fluctuations. The current through the nanostructure at a fluctuating temperature T, $I(T)$, generates voltage fluctuations across the shunt resistance R_{sh}. These are amplified and recorded with an oscilloscope.

where $R_{\rm dyn} \equiv (V_{\rm bias} - V)/\bar{I}$ is the dynamical resistance of the nanostructure.

For quantum fluctuations in the shunt resistor we have [5]

$$\begin{aligned} S_{V_x}(\omega) =& \frac{R_{\rm sh}R_{\rm dyn}^2}{(R_{\rm sh}+R_{\rm dyn})^2}\omega\theta(\omega), \\ {\rm var}(\bar{V}_x) =& \frac{R_{\rm sh}R_{\rm dyn}^2}{\tau^2(R_{\rm sh}+R_{\rm dyn})^2}\frac{\gamma - {\rm Ci}(\omega_c\tau) + \ln(\omega_c\tau)}{2\pi}, \\ \approx& 0.04\frac{R_{\rm sh}R_{\rm dyn}^2}{\tau^2(R_{\rm sh}+R_{\rm dyn})^2}, \end{aligned} \tag{3.34}$$

where γ is the Euler–Mascheroni constant, ${\rm Ci}(z)$ the cosine integral, and $\omega_c \approx 1/\tau$ a high-frequency cutoff for the fluctuation spectrum. Amplifier noise spectrum has the same form as the quantum fluctuations in the shunt resistor, but with a prefactor n, which depends on the number of noise quanta added by the amplifier. The lower limit $n = 1/2$ can be achieved with a quantum limited amplifier, adding half a quantum of noise [55].

The signal-to-noise ratio becomes

$$\frac{\langle\bar{V}_x\rangle}{\sqrt{{\rm var}(\bar{V}_x)}} = \sqrt{r_{\rm sh}}\sqrt{\left(\frac{V_{\rm bias}}{eR_{\rm dyn}}\tau\right)\left(\frac{eV_{\rm bias}}{\max(2T, 0.04/\tau)}\right)}, \tag{3.35}$$

where $r_{\rm sh} = R_{\rm sh}/R_{\rm dyn}$. It is high when the number of particles transmitted through the structure during the measurement period is large, and when the bias voltage is large compared to the temperature and the measurement bandwidth. The signal-to-noise ratio is maximized when the value of the shunt resistor is maximized. However, for a large $R_{\rm sh}$ the current fluctuations in the nanostructure cause large fluctuations in the bias voltage across the nanostructure. This is not consistent with the assumption of pure voltage bias in the theory. Pure voltage bias is reached in the *opposite* limit, $R_{\rm sh} \to 0$. Therefore, it is necessary to make a compromise between the "purity" of voltage bias and the signal-to-noise ratio on a case-by-case basis.

As an example, using the numbers from Publication III we can check what value for the shunt resistor in series with the SINIS structure would give a signal-to-noise ratio of, say, 10. Taking $T = 10$ mK, $V_{\rm bias} = 2\Delta/e = 2$ K $\times\, k_B/e$, $R_{\rm dyn} = 110$ kΩ, $\tau = 1$ ns, and a quantum limited amplifier, we get

$$\frac{V_{\rm bias}}{eR_{\rm dyn}}\tau = 9.8, \ \frac{eV_{\rm bias}}{\max(2T, 0.04/\tau)} = 100, \text{ and } R_{\rm sh} = 11\ \text{k}\Omega.$$

Comparing this value to the dynamical resistance implies that the structure is quite well voltage biased. By halving the shunt resistance the bi-

asing can be improved further with the signal-to-noise ratio still staying above 7 due to the square root dependence on $R_{\rm sh}$.

For an overheated single-electron transistor the numbers from Publication II are more favorable. Taking $T = 10$ mK, $V_{\rm bias} = V_C = 1$ mV, $R_{\rm dyn} = 1.6 \times 10^9\ \Omega$, $\tau = 3.8 \times 10^{-2}$ s, we get

$$\frac{V_{\rm bias}}{eR_{\rm dyn}}\tau = 1.5 \times 10^5, \ \frac{eV_{\rm bias}}{\max\left(2T, 0.04/\tau\right)} = 0.6 \times 10^3, \text{ and } R_{\rm sh} = 1.8\ \text{k}\Omega.$$

Here the challenge is the ultra-high resistance of the order of 1 MΩ of the tunnel barriers.

4. Summary and conclusions

In this Dissertation I study the theory of electronic heat transport in nanometer scale metallic structures using non-equilibrium quantum transport theory. The main focus is on the random fluctuations of heat currents between the nanoelectronic system and its surroundings, and the associated fluctuations in the internal energy and temperature of the system. Fluctuations in general play an important role in nanoelectronics. In addition to their detrimental effect on the accuracy of measurement instruments and other devices, they can also be used to extract information about the nature of the physical processes in the system.

Typically, temperature fluctuations are relevant in small systems where the single-particle level spacing is relatively large compared to some other energy scale of the system. Moreover, the fluctuations are large when energy is mostly transported by processes which are sensitive to temperature. Examples of these are the heat currents carried by electrons in single-electron transistors and superconductor–insulator–normal metal structures — two common devices in the field of nanoelectronics. In these systems the temperature dependence of the heat current is exponential in certain bias voltage regimes. Significant electron–phonon coupling tends to diminish these effects.

The temperature-fluctuation induced effects on other observables can be significant, sometimes even dominating. Again, the relevance of this is determined by the temperature sensitivity of the other observable. Especially the current noise in a single-electron transistor with highly resistive tunnel barriers is strongly affected by the temperature fluctuations. In many systems it should be possible to detect the temperature fluctuations in this way: By measuring the fluctuations of some other observable it is possible to infer the statistics of temperature fluctuations. Experimental realization is certainly challenging, but it should not be impossible.

Better understanding of the measurement process and its limitations requires the description of the measurement circuit (or at least part of it) in the same effective action formalism together with the nanostructure. By virtue of the flexibility of the theory this should not be too difficult to achieve in the near future.

The functional integral approach to Keldysh formalism is a solid foundation upon which the theory of temperature fluctuation statistics is built. Its formulation in terms of the Fokker–Planck and Langevin equations is developed and used in the majority of publications constituting this Dissertation. These formulations can be used to study the fluctuations of temperature and other relevant quantities in a wide variety of systems, not limited to those mentioned in this overview. Examples of such systems are diffusive wires, nanoelectromechanical systems (NEMS), circuit quantum electrodynamics, and even some chemical or biological systems. Some of the most interesting phenomena appear in systems driven strongly out of equilibrium, possibly not admitting a definition of any kind for an effective temperature. One such example is the supercurrent enhancement or reversal in non-equilibrium superconducting structures, which strongly depend on the shape of the electron distribution function. Statistics of energy fluctuations in these systems could be one of the near-term directions for the development of this formalism.

It is difficult to imagine applications for temperature fluctuations. Their observation, however, might present the chance to drive a system into a state of exceptionally low (or high) temperature by using the feedback from the detector to control the physical mechanism causing the fluctuations. Similarly, it could be possible to "engineer" rare states of some other fluctuating quantities.

In the field of nanoelectronics temperature fluctuations should in general be viewed as a harmful source of noise accompanying the inevitable miniaturization of electronics. It is my hope that the methods and results presented in this Dissertation enable a critical evaluation of the relevance of temperature fluctuations in other fields of physics — even other fields of science — and provide ideas for the minimization of their harmful effects.

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