Department of Mathematics and Systems Analysis

Mathematical Models for Longevity Risk Management

Helena Aro

Aalto University

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Mathematical Models for Longevity Risk Management

Helena Aro

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Abstract

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Abstract

This thesis presents mathematical models for longevity risk management. An overall objective is to develop methods for hedging the cash flows of longevity-linked liabilities on financial markets. This is obtained by modelling the law of a multivariate stochastic process describing mortality and asset returns, with particular emphasis on the long-term development of mortality and its connections with asset returns. The resulting stochastic model is then applied to study the roles of systematic and non-systematic risks in pension portfolios and, ultimately, to investigate optimal investment from the viewpoint of an investor with longevity-linked liabilities. We show how the hedge of a longevity-linked cash flow can be improved by taking the liabilities into account in investment decisions.

Keywords Longevity risk, Stochastic modelling, Stochastic optimization, Systematic mortality risk, Non-systematic mortality risk, Market risk, Hedging

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Tiivistelmä

Väitöskirjassa esitellään matemaattisia malleja pitkäikäisyysriskin hallintaan. Tavoitteena on kehittää menetelmiä kuolevuudesta riippuvien kassavirtojen suojaamiseen rahoitusmarkkinoilla. Tätä tarkoitusta varten kuolevuuden ja sijoitustuottojen yhteisjakaumaa mallinnetaan moniuloitteisena stokastisena prosessina. Pääpaino mallinnuksessa on kuolevuuden pitkän aikavälin kehityksellä sekä yhteyksillä sijoitustuottoihin. Näin saatavaa stokastista mallia sovelletaan eläkevakuutusportfolioiden systemaattisen ja epäsystemaattisen riskin tarkastelussa sekä optimaalisen sijoitusstrategian valinnassa kuolevuudesta riippuvien vaateiden tapauksessa.Työssä näytetään, kuinka kuolevuudesta riippuvien vaateiden suojausta voidaan parantaa huomioimalla nämä vaateet suojausstrategiaan liittyviä investointipäätöksiä tehtäessä.

Avainsanat Pitkäikäisyysriski, stokastinen mallinnus, stokastinen optimointi, systemaattinen kuolevuusriski, ei-systemaattinen kuolevuusriski, markkinariski, suojaus

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Preface

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I thank prof. Enrico Biffis, my opponent, for taking the time from his many duties to prepare for and attend the defense in Helsinki. I also thank the pre-examiners, Dr. Katja Hanewald and Dr. Daniel Kuhn, for their valuable comments on my thesis.

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supplier of general advice, Matlab consultation and laughter. I also want to thank my fellow researchers Suski, Jaska, Riku, and Priikka for cathartic lunch gatherings and tea evenings over the years. Aleksi deserves to be mentioned here for constant support in all matters scientific and non-scientific.

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Finally, I thank Äiti, Isi, and Maarit for always being there for me. I am especially grateful to Äiti for stressing the importance of studying mathematics during my school years. Thank you Markku for making me very happy.

Helsinki, August 1, 2013,

Helena Aro

Contents

List of Publications

This thesis consists of an overview and of the following publications which are referred to in the text by their Roman numerals.

I H. Aro, T. Pennanen. A user-friendly approach to stochastic mortality modelling. *European Actuarial Journal*, 1:151–167, 2011.

II H. Aro, T. Pennanen. Stochastic modelling of mortality and financial markets. *Scandinavian Actuarial Journal*, doi:10.1080/03461238.2012.724442, Published online: 3 December 2012.

III H. Aro. Systematic and non-systematic mortality risk in pension portfolios. Accepted for publication in *North American Actuarial Journal*, arXiv:1307.8020v2, 2013.

IV H. Aro, T. Pennanen. Liability-driven investment in longevity risk management. Accepted for publication in *Springer volume on financial optimization*, arXiv:1307.8261v1, 2013.

Author's Contribution

Publication I: "A user-friendly approach to stochastic mortality modelling"

The author participated in the analysis, performed all numerical experiments, and contributed to a major part of the writing.

Publication II: "Stochastic modelling of mortality and financial markets"

Major parts of the analysis and all of the numerical experiments are work by the author. The article is mainly written by the author.

Publication III: "Systematic and non-systematic mortality risk in pension portfolios"

The analysis and the numerical experiments were designed in collaboration with the thesis advisor. The article was written by the author.

Publication IV: "Liability-driven investment in longevity risk management"

Major parts of the analysis and all of the numerical experiments are work by the author. The article is mainly written by the author.

1. Introduction

Continuous but uncertain improvements in general longevity pose significant challenges to the insurance industry as well as social security systems. As the impacts of medical advances, environmental changes or lifestyle issues on longevity remain unpredictable, the need for effective tools of quantitative risk management is acute. The requirement has recently been accentuated by the financial crisis, as well as the new Solvency II regulation that leans heavily on risk assessment of individual financial institutions in calculation of capital requirements.

Various longevity-linked instruments have been proposed for the management of longevity risk; see e.g. [10, 13, 9, 34, 55]. However, a major challenge facing the development of a longevity market is hedging the risk that stems from issuing a longevity-linked instrument. The supply for longevity-linked instruments might increase if their cash-flows could be hedged by appropriately trading in assets for which liquid markets already exist. Such development has been seen e.g. in options markets, which flourished after the publication of the seminal Black–Scholes–Merton model. As the cash-flows of mortality-linked instruments have less to do with existing financial markets than those of simple stock options, their cash flows cannot be perfectly hedged, and the seller of a mortality-linked instrument always retains some risk. However, it may be possible to diminish the residual risk by an appropriate choice of an investment strategy. Such hedging strategies would also benefit pension providers and life insurers who need to hedge their longevity exposures.

This thesis presents mathematical models for longevity risk management. An overall objective was to develop methods for hedging cash flows of longevity-linked liabilities on financial markets. To this end, the following issues are addressed:

1. Modelling the law of a multivariate stochastic process consisting of mortality and asset returns, with particular emphasis on

   - Long-term development of mortality

   - Connections between mortality and asset returns

2. Utilizing these connections in designing hedging strategies for mortality-linked cash flows, using numerical methods.

Modelling of the phenomenon, as described by the first item, is addressed in Sections 2 and 3. The resulting models are then employed to analyse the hedging problem in Sections 4 and 5, covering the second item.

Section 2 outlines a user-friendly stochastic framework for mortality modelling. The logistic transforms of survival probabilities in different age groups of a given population are modelled by linear combinations of *basis functions* across the cohorts. The weights assigned to each basis function serve as *risk factors* that vary over time. Survivor numbers are assumed to be binomially distributed, which, under very lenient assumptions about the basis functions, results in a strictly convex log-likelihood function when calibrating the model.

Based on statistical analysis presented in Publication II, Section 3 proposes a stochastic model for the joint long-term development of mortality and financial markets, employing the risk factors of a three-parameter version of the mortality model presented in the previous section. The model incorporates such features as the eventual stabilization of mortality rates, long-term link of old-age mortality to GDP, short-term connection between mortality and GDP, and connection of GDP to interest rates.

The non-systematic component of mortality risk is associated with the number of members in a population and is theoretically diversifiable, while systematic mortality risk stems from uncertainty in the future survival probabilities, and hence does not depend on the number of participants in the scheme. Section 4 assesses the effects of non-systematic and systematic mortality risks on the required initial capital in a pension plan, again utilizing the mortality model of Section 2. We discover that for a pension plan with few members the impact of pooling on the required capital per person is strong, but non-systematic risk diminishes rapidly as the number of members increases. Systematic mortality risk, on the other hand, remains a significant source of risk is a pension portfolio.

Building on the work presented in earlier sections, Section 5 investigates optimal investment from the point of view of an investor with longevity-linked liabilities. We employ a computational procedure that constructs diversified strategies from parametric *basis strategies*, and suggest trading strategies that are motivated by connections between mortality and financial markets, as observed in Section 3. We notice that the risk associated with the diversified strategy decreases as these liability-driven basis strategies are included, as opposed to employing non-liability-driven strategies only. This hedging approach can be applied to the pricing and hedging of longevity-linked instruments, as well as the asset-liability management of pension plans and life insurers.

2. A user-friendly approach to stochastic mortality modelling

2.1 Background

A wealth of stochastic discrete-time models have been proposed to capture the uncertainty in the future development of mortality, see [22] for a review. The seminal work of Lee and Carter [51] introduced in 1992 is still widely popular and has subsequently inspired several related works [17, 52, 15, 32, 31]. More refined models with multiple stochastic factors were subsequently proposed by Renshaw and Haberman [68] and Cairns et al. [19], with extensions incorporating cohort effects by Renshaw and Haberman [69] and Cairns et al. [23]. A slightly different vein of reseach [28] has applied penalized splines to mortality modelling. Some of the most recent works utilize Bayesian methods [24, 62].

Publication I complements the previous literature by presenting a stochastic mortality modelling framework which, courtesy of its flexible construction and the tangible interpretation of its risk factors, is particularly convenient in subsequent studies on the relationships between mortality and economic variables.

2.2 The framework

Let $E_{x.t}$ be the number of individuals aged $[x, x+1)$ years at the beginning of year t in a given population. Our aim is to model the values of $E_{x,t}$ over time $t = 0, 1, 2, \ldots$ for a given set $X \subset \mathbb{N}$ of ages. We assume that the conditional distribution of $E_{x+1,t+1}$ given $E_{x,t}$ is binomial:

$$E_{x+1,t+1} \sim \mathrm{Bin}(E_{x,t}, p_{x,t}), \tag{2.1}$$

where $p_{x,t}$ is the *survival probability*, the probability that an individual aged x and alive at the beginning of year t is still alive at the end of that year. A stochastic mortality model is obtained by modelling the logistic transforms of survival probabilities $p_{x,t}$ as stochastic processes, by means of a linear combination

$$\operatorname{logit} p_{x,t} := \ln\left(\frac{p_{x,t}}{1-p_{x,t}}\right) = \sum_{i=1}^{n} v_i(t)\phi_i(x), \tag{2.2}$$

where ϕ_i are *basis functions* across the cohorts defined by the user, and v_i are stochastic *risk factors* that vary over time. When modelling the vector of risk factors $v = (v_1, \ldots, v_n)$ as a $\mathbb{R}^n$-valued stochastic process, this formulation implies that $p_{x,t} \in (0,1)$.

The future values of $E_{x+1,t+1}$ are obtained by sampling from $\mathrm{Bin}(E_{x,t}, p_{x,t})$. The uncertainty in the future values of $p_{x,t}$ represents the *systematic mortality risk*. Even if the 'true' survival probabilities were known, future population sizes would still be random, which accounts for the *non-systematic mortality risk*. However, as the population grows, the fraction $E_{x+1,t+1}/[E_{x,t}p_{x+t,t}]$ converges in distribution to constant 1. In large enough pools the main uncertainty comes from unpredictable variations in the future values of $p_{x,t}$, and the population dynamics are well described by $E_{x+1,t+1} = E_{x,t}p_{x,t}$. The roles of systematic and non-systematic risk are discussed further in Section 4.

With appropriate choices of the basis functions $\phi_i(\cdot)$ one can incorporate chosen properties of $p_{\cdot,t}$ in the model. To illustrate, one may wish to construct a model where the probabilities $p_{x,t}$ behave continuously or smoothly across ages, as in the classic Gompertz model for mortality. This can be simply achieved by choosing continuous or smooth basis functions, respectively. The choice of the basis functions also determines the interpretation of the risk factors. If, for example, the basis functions are such that $\phi_k(x) = 1$ but $\phi_i(x) = 0$ for $i \neq k$ for a certain age x, then the risk factor $v_k(t)$ equals the logistic survival probability at age x in year t. Concrete interpretations facilitate the modelling of future values of the risk factors. One may, for instance, be able to deduce dependencies between v and certain economic factors such as investment returns.

Once the basis functions ϕ_i have been chosen, the vector $v = (v_1, \ldots, v_n)$ of risk factors is modelled as a multivariate stochastic process in discrete time. The model specification can be based solely on the user's views about the future development of survival probabilities, historical data, or both. The historical values of the risk factors $v(t) = (v_1(t), \ldots, v_n(t))$ can be easily constructed by maximum likelihood estimation. It can be shown that the resulting log-likelihood function $l_t : \mathbb{R}^n \to \mathbb{R}$ is concave under very mild assumptions on the basis functions (see Publication I). Concavity implies that local maxima of l_v are true maximum likelihood estimators. Strict concavity, in turn, implies that the estimators are unique; see e.g. [70, Theorem 2.6]. Besides guaranteeing well defined estimators, convexity facilitates the numerical maximization of l_t.

2.3 A three-parameter model for adult mortality

We implement a three-parameter version of the mortality model for adult (ages 18–100) mortality with three piecewise linear basis functions given by

$$\phi^1(x) = \begin{cases} 1 - \frac{x-18}{32} & \text{for } x \leq 50 \\ 0 & \text{for } x \geq 50, \end{cases}$$

$$\phi^2(x) = \begin{cases} \frac{1}{32}(x-18) & \text{for } x \leq 50 \\ 2 - \frac{x}{50} & \text{for } x \geq 50, \end{cases}$$

$$\phi^3(x) = \begin{cases} 0 & \text{for } x \leq 50 \\ \frac{x}{50} - 1 & \text{for } x \geq 50. \end{cases}$$

The linear combination $\sum_{i=1}^{3} v_t^i \phi^i(x)$ will then be piecewise linear and continuous as a function of the age x. The risk factors v_t^i represent points on the logistic survival probability curve:

$$v_t^1 = \operatorname{logit} p_{18,t},\ v_t^2 = \operatorname{logit} p_{50,t},\ v_t^3 = \operatorname{logit} p_{100,t}.$$

It is worth noting that when the historical parameter risk factor values are obtained by maximum likelihood techniques, v^1 is determined by the mortality of 18–50 year olds, v^2 by all ages, and v^3 by the ages between 50 and 100.

As an illustration, Figure 2.1 shows the historical values of parameter v^1 that result from fitting the model into female mortality data from six OECD countries. An interesting feature is that the parameter values of several of the countries appear to stabilize, in particular during the last 20 years. This phenomenon is examined more closely in next section.

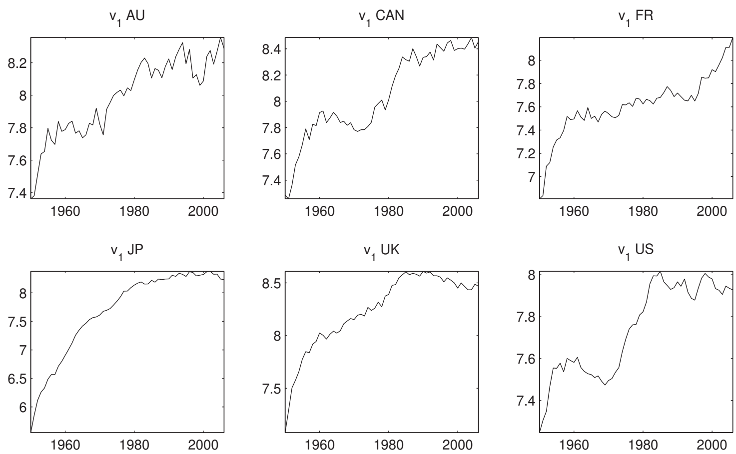

Figure 2.1. Historical values for risk factor v^1, females. Note the different scales.

3. Stochastic modelling of mortality and financial markets

3.1 Background

A pertinent question in longevity risk management is whether mortality will continue to decline, and for how long. It was conjectured already in [82] that, in the long run, mortality rates will tend to stabilize. Some experts suggest that lifestyle factors, such as obesity, may soon hinder further mortality improvements; see [63]. There is also some indication that the decline in coronary heart disease mortality is levelling out in some age groups in the UK and the Netherlands [1, 76]. A recent survey of mortality trends in Europe can be found in [53].

When analysing mortality from the perspective of longevity-linked liabilities, one is also interested in the joint development of mortality and financial markets, particularly in the long term. Earlier studies provide evidence of connections between mortality and economic cycles, usually represented by GDP or unemployment. Some studies suggest that in the long run, higher economic output results in lower mortality [66, 67]. Others report a more immediate link between the phases of the economic cycle and mortality: Ruhm [71] and Tapia Granados [73, 74] discovered that mortality rates increase during economic expansions. The connection of GDP to both short-term and long-term mortality has recently been studied in [43] and [42]. These studies indicate that short-term mortality and macroeconomic fluctuations are closely linked.

GDP, in turn, is connected with various sectors of financial markets; see [72] for an extensive review. For instance, there is strong indication of a link between economic activity and the term spread of interest rates [38, 47, 45, 46, 37, 30, 65], although the connection may have weakened since the mid-80s [81]. Another connection exists between credit spreads and GDP, as discussed e.g. in [41], [8] and [36]. Although the connection between stock markets and economic cycle is not unambiguous, there is evidence that such a link also exists [59, 39, 5, 16, 25, 38].

In Publication II we studied the possible stabilization of mortality rates, and corroborated earlier findings on the link between mortality and GDP as well as the connection of GDP to interest rates. A stochastic model based on statistical analysis was presented.

3.2 The model

Based on the broad statistical analysis of adult (ages 18–100) mortality and economic factors in Publication II, we propose a model that describes the long-term joint development of mortality and financial markets. It incorporates the following features also suggested by others:

- eventual stabilization of mortality rates of the young [82]
- long-term link of old-age mortality to GDP [42]
- short-term connection between mortality and GDP [38, 73, 74, 42]
- connection of GDP to interest rates [81, 36, 72].

To model the the long-term development of mortality and financial markets, we propose a linear multivariate stochastic difference equation

$$\Delta x_t = Ax_{t-1} + b + \varepsilon_t,$$

where $x = [v_t^{f,1}, v_t^{f,2}, v_t^{f,3}, v_t^{m,1}, v_t^{m,2}, v_t^{m,3}, g_t, s_t^T, s_t^C]$, $A \in \mathbb{R}^{9\times 9}$, $b \in \mathbb{R}^9$, and ε_t are $\mathbb{R}^9$-valued random vectors describing the unexpected fluctuations in the risk factors. The variables $v_t^{f,i}$ and $v_t^{m,i}$ are the mortality risk factors of females and males, respectively, g_t denotes the logarithm of GDP per capita, s_t^T represents the term spread of interest rates, and s_t^C is the credit spread. This compact formulation is straightforward to study, both analytically and numerically.

The equations for risk factors $v^{f,1}$ and $v^{m,1}$, corresponding with the survival probabilities of young ages as described in the previous section, depict mean-reverting behaviour. The risk factors $v^{f,2}$ and $v^{f,2}$, corresponding with the survival probabilities of the middle-aged, follow a random walk with a drift. The equations for v^3 for both genders describe a direct link between mortality of ages between 50 and 100 (as v^3 depends on these ages by definition) and GDP. The interpretation is that the drift of v_t^3 depends on its relation to g_t. If v_t^3 lags behind g_t, the drift increases.

The per capita log-GDP g_t depends on the developments of the term spread s_t^T and credit spread s_t^C. Large term spreads anticipate high GDP growth rates. On the other hand, large credit spreads precede small GDP growth rates. This effect is illustrated in Figure 3.1 on US interest rate and GDP data. The developments of the term spread s_t^T and the credit spread s_t^C are both described with a mean reverting equation, reflecting the Vasicek interest rate model [77].

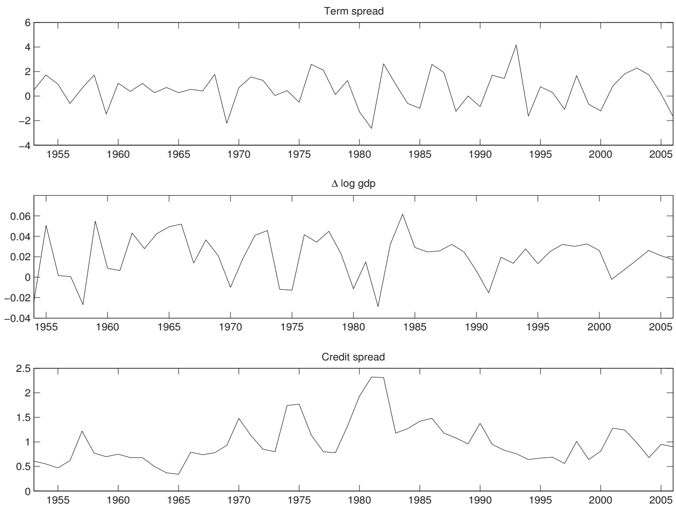

Figure 3.1. Term spread, differenced log-GDP, and credit spread.

4. Systematic and non-systematic mortality risk in pension portfolios

4.1 Background

The mortality risk of a population can be decomposed into two components: systematic and non-systematic or idiosyncratic mortality risk, as elaborated in Section 2. Their roles in longevity risk management have been studied in previous literature. Coppola at al. [26] consider the contributions of mortality and investment risks to the variability in the present value of liabilities, given annuity portfolios of different sizes. Olivieri [61] also considers the impact of systematic and random fluctuations on the present value of future benefit payments under a deterministic financial structure. Milevsky et al. [58] show how the standard deviation of payoffs per policy diminishes to a constant as the number of policies increases. They discovered that when there are dozens of policies, the contribution of non-systematic risk is still notable, but for portfolios larger than a thousand members it reduces to negligible. Hári et al. [44] have examined the impact of non-systematic risk on a capital reserve, described as a proportion of the present value of the liabilities, required to reduce the probability of underfunding to an acceptable level. Donnelly [33] considers the role of non-systematic risk in a pension plan by studying how the coefficient of variation for the liabilities of the scheme varies with its number of participants.

In Publication III, we show how the least amount of initial capital required to cover the liabilities of pension portfolio varies with the size of the portfolio. We consider a multi-period model of stochastic asset returns and liabilities, and determine the minimum initial capital needed to cover the liabilities in terms of a convex risk measure, given a degree of risk aversion.

4.2 Valuation of defined-benefit pension liabilities

Consider a defined-benefit pension plan, where the number of members aged x at time t is denoted by $E_{x,t}$. We assume that each alive member receives an index-linked annual unit benefit at times $t = 1, 2, \ldots, T$, until termination of the scheme at $t = T$. The yearly pension claims amount to

$$c_t = \frac{I_t}{I_0} \sum_{x \in X} d_x E_{x,t},$$

where I_t is the index value, $X \subset \mathbb{N}$ is the set of age groups in the pension plan, and the constant d_x depends on the value of the index and accrued pension benefit at time $t = 0$. We will look for the least amount of capital w_0 that suffices to cover the liabilities until the termination of the scheme.

In order to study the effects of non-systematic and systematic risks on capital requirements, we apply the valuation approach described in [49]. At each t, the insurer pays out c_t and invests the remaining wealth w_t in financial markets. The investment returns are modelled as a stochastic process, which is dependent on the chosen investment strategy used by the insurer. As in [49], we define the value of liabilities as the least initial capital that enables the investor to hedge the cash flows with given risk tolerance.

The liabilities $(c_t)_{t=0}^T$ and returns $(R_t)_{t=0}^T$ are modelled as stochastic processes, and the problem can be formulated as

$$\begin{aligned} &\text{min} && w_0 \quad \text{over } w \in \mathcal{N} \\ &\text{subject to} && w_t \le R_t w_{t-1} - c_t \quad t = 1, \ldots, T \\ & && \rho(w_T) \le 0, \end{aligned} \tag{4.1}$$

where $\mathcal{N}$ are stochastic processes adapted to a given filtration $(\mathcal{F}_t)_{t=0}^T$. The variable $R_t = \sum_{j=1}^J R_t^j \pi_t^j$ is the return over period $[t-1, t]$ per unit amount of cash invested, π_t^j is the proportion of wealth invested each of the J assets, and ρ is the *entropic risk measure* defined for a random variable X as

$$\rho(X) = \frac{1}{\gamma} \log E[e^{-\gamma X}]. \tag{4.2}$$

In the risk-neutral case where $\rho(X) = E[X]$, it can be shown that the required initial wealth is

$$w_0 = \frac{\sum_{t=1}^T E(\Pi_{s=t+1}^T R_s c_t)}{E(\Pi_{s=1}^T R_s)}.$$

Further, in the special case where R_t is independent of both its past values and liabilities c_t, we obtain the actuarial best estimate, the expected value of discounted claims. This is the valuation method used in Solvency II.

4.3 Numerical illustrations

In the following simulation study all members in the pension scheme are females aged 65 at $t = 0$, and the term of the scheme is $T = 35$. Each member receives a unit benefit per year. The risk aversion parameter value was set to $\gamma = 0.05$. We generated $N = 500000$ scenarios, computed the final wealth w_T in each scenario for a given initial wealth w_0, and approximated the expectation in (4.2) as a Monte Carlo estimate. The smallest w_0 to yield a nonnegative risk for terminal wealths was obtained with a simple line search.

Investment returns depend both on the returns on individual assets and the chosen investment strategy. *Fixed proportions* (FP) is a strategy where, in the presence of J assets, the allocation is rebalanced at the beginning of each holding period into set proportions given by a vector $\pi \in \mathbb{R}^J$, the components of which sum up to one. In our example we consider two fixed proportions strategies on bonds and equities, namely

$$\pi^S = [\pi_{bond}, \pi_{stock}] = [0.75, 0.25]$$

and

$$\pi^R = [\pi_{bond}, \pi_{stock}] = [0.5, 0.5]$$

In the first, safer strategy π^S a 75% weight is placed on bonds and a 25% weight on equities, whereas in the other, riskier strategy π^R the weights are 50% and 50%.

Figures 4.1 and 4.2 plot the initial capitals per individual for various numbers of participants E_0 for each strategy, and the two different investment strategies π^S and π^R. The dotted line indicates the level of initial capital required in the presence of systematic risk only, that is when the numbers of survivors are not sampled from binomial distribution but approximated by their expectation as described in Section 2. Initially the required capital drops sharply. With a few dozen members, the effect of nonsystematic risk on the initial capital is already comparatively small.

Levels of initial capital required in the risk-neutral case and the actuarial best estimate, along with capital required in the presence of systematic risk only for both investment strategies, are presented in Table 4.1. For the risk-neutral case the required capital is slightly smaller than for the actuarial best estimate. This difference arises from the fact that the risk-neutral risk measure takes into account the dependencies in asset returns and liabilities.

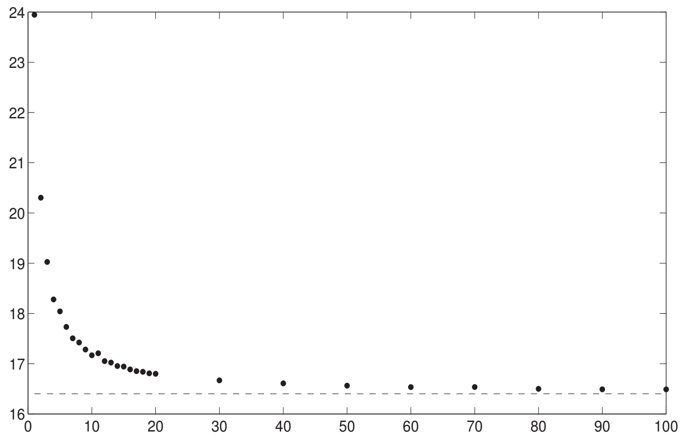

Figure 4.1. Initial capital requirement per individual, investment strategy π^S. Dotted line indicates the level of initial capital required in the presence of systematic risk only.

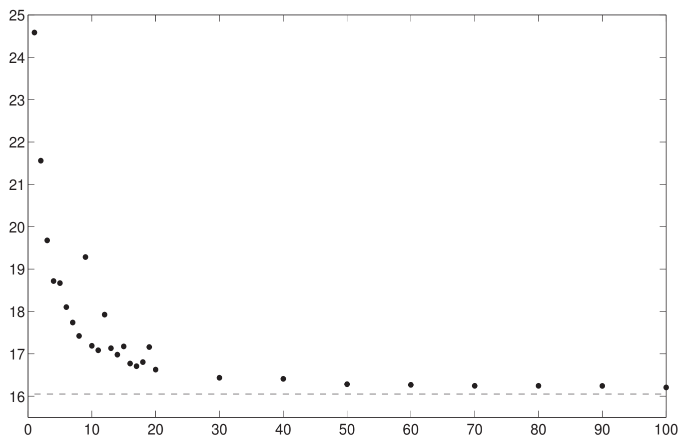

Figure 4.2. Initial capital requirement per individual, investment strategy π^R. Dotted line indicates the level of initial capital required in the presence of systematic risk only.

Table 4.1. Systematic risk, initial capital

Risk measure	Entropic, $\gamma = 0.05$	$\rho = E[X]$	Actuarial best estimate
π^S	16.40	15.45	15.50
π^R	16.05	14.09	14.12

5. Liability-driven investment in longevity risk management

5.1 Background

Earlier research on hedging longevity risk has studied the valuation of various types of annuities and other life insurance products, see [64] for an overview of various annuities. Frees et al. [40] have investigated the valuation of annuities with dependent mortality models. Marceau et al. [56] have considered calculation of reserves for life insurance policies in a stochastic mortality and interest rate environment through simulations of the prospective loss random variable. Brown et al. [18] have explored the range of practices in the pricing of various annuity products. Wilkie et al. [83] applied both quantile and conditional tail expectation reserving, as well as option pricing methodology to hedge a guaranteed annuity option (GAO). Option valuation theory was also recently applied to GAOs by e.g. Ballotta and Haberman [3]. Another approach is natural hedging, where insurers can hedge longevity risks internally between their own business products [27, 78]. On the other hand, Cairns et al. [21] draw from similarities between force of mortality and interest rates to construct frameworks for valuation of mortality-linked instruments.

Further, the pricing of mortality-linked instruments has attracted attention, and several approaches have been suggested, see [6] for a recent review. A popular approach is risk-neutral pricing, which is based on the theory stating that in an incomplete but arbitrage-free market, there exists at least one risk-free measure. In risk-neutral pricing one identifies such a measure, and calculates the corresponding price [12, 35]. However, this methodology has been questioned on the basis of the illiquidity of the markets [4, 6]. Milevsky et al. [57] and Bayraktar et al. [7] have developed a theory where mortality risk assumed by the issuer of a longevity-linked contingent claim is compensated for by a predetermined instantaneous Sharpe ratio. Another vein of research employs the Wang transform [79, 80], distorting the distribution of the survivor index to create risk-adjusted expected values for the longevity-linked cash flow [55]. Recent studies on longevity-linked instruments also investigate how such instruments, once in existence, can be used to hedge mortality risk exposures in pensions or insurance liabilities [12, 11, 20, 54, 29].

Publication IV approaches the hedging problem by studying optimal investment from the point of view of an insurer with longevity-linked liabilities. We demonstrate how the hedge of a longevity-linked cash flow can be improved by taking the liabilities into account in investment decisions. This is achieved by optimally diversifying a given initial capital amongst several investment strategies, some of which employ statistical connections between assets and liabilities.

5.2 The asset-liability management problem

Consider an insurer with given initial capital w_0 and longevity-linked liabilities with claims c_t over time $t = 1, 2, \ldots, T$. After paying out c_t at time t, the insurer invests the remaining wealth in financial markets. We look for investment strategies whose proceeds fit the liabilities as well as possible, in the sense of a given risk measure ρ on the remaining wealth at time T. Assume a finite set J of liquid assets (bonds, equities, ...) that can be traded at times $t = 0, \ldots, T$. The return on asset j over period $[t-1, t]$ will be denoted by $R_{t,j}$, and the amount of cash invested in asset j over period $(t, t+1]$ by $h_{t,j}$. Then, the asset-liability management problem of the insurer can then be written as

$$\begin{aligned}
&\text{minimize} \quad && \rho\Big(\sum_{j\in J} h_{T,j}\Big) \quad \text{over} \quad h \in \mathcal{N} \\
&\text{subject to} \quad && \sum_{j\in J} h_{0,j} \le w_0 \\
& && \sum_{j\in J} h_{t,j} \le \sum_{j\in J} R_{t,j} h_{t-1,j} - c_t \quad t = 1, \ldots, T \\
& && h_t \in D_t, \quad t = 1, \ldots, T
\end{aligned} \tag{ALM}$$

The liabilities $(c_t)_{t=0}^T$ and the investment returns $(R_{t,j})_{t=0}^T$ are modelled as stochastic processes. The set $\mathcal{N}$ denotes the $\mathbb{R}^J$-valued adapted investment strategies $(h_t)_{t=0}^T$. Being adapted means that the portfolio h_t chosen at time t may only depend on information observed by time t. The last constraint describes portfolio constraints. The set D_t is allowed to be random, but it is known at time t. The risk measure ρ is a convex function on the space of real-valued random variables. It describes the insurer's preferences over random terminal wealth distributions.

5.3 Investment strategies

The trading strategies employed in subsequent simulations can be divided into two categories. In non-liability-driven strategies the proportions of wealth invested in different assets are independent of the values of the liabilities, whereas liability-driven strategies the proportions depend, directly or indirectly, on the longevity-linked liabilities. The objective is to compare how liability-driven strategies affect the optimal value of the ALM problem.

The non-liability-driven strategies included well-know parametric investment strategies: Buy and Hold, Fixed Proportions, and Target Date Fund. The set of liability-driven strategies comprises Constant Proportion Portfolio Insurance as well as Spread strategies, Survival Index strategies and Wealth strategies, where the proportions invested in certain assets depend on term and credit spreads, the survival index of a reference population, or remaining wealth, respectively.

5.4 Computational results

In general, analytical solutions to the problem (ALM) are not available. We will employ the numerical procedure presented in [48, 50], which is a computational method for constructing a diversified aggregate investment strategy out of a set of simple parametric strategies called *basis strategies*. Initial choices of basis strategies are utilized by adjusting their convex combination to the given objective and the risk factors of the problem. It is to be noted that a convex combination of feasible basis strategies is always feasible, since the optimization problem is convex.

In the following numerical illustrations, the termination date was set to $T = 30$, and the cash flows c_t were defined as the survival index S_t of a cohort of US females aged 65 at time $t = 0$. The structure of this instrument is basically the same as in the first longevity bond issued in 2004 by the European Investment Bank (for a more detailed description see e.g. [9]). The asset returns R_t and liabilities c_t were modelled as a multivariate stochastic process.

The objective was to investigate if liability-driven investment strategies can lead to reductions in the risk associated with a cash flow of longevity-linked liabilities. To this end, we used two sets of basis strategies. The first set consisted of non-liability-driven basis strategies, and the second set encompassed both the non-liability-driven and additional liability-driven basis strategies. We computed the optimal aggregate investment strategy and the corresponding value of the risk measure function ρ for each set, using the computational procedure. We then proceeded to compare the optimal values of the objective ρ associated with each set. In order to discern to which extent a possible reduction in risk can be attributed to considering the liabilities, as opposed to merely having a larger number of strategies, we also regarded a portfolio optimization problem without liabilities for both sets of basis strategies. The optimal allocations were computed for different values of risk aversion parameters γ. The larger the parameter, the more risk averse the investor.

Table 5.1 summarizes the resulting values of the objective function. We observe that as the risk aversion grows, so does the reduction in risk of the ALM problem with liabilities when the liability-driven strategies are included. This is plausible since the higher the risk aversion, the more the risk measure places importance to the fact that the asset returns conform to the liabilities. As for the optimization problem with zero liabilities, the effect of adding the liability-driven strategies was negligible and independent of the level of risk aversion.

Table 5.1. Objective function values.

	$\gamma = 0.05$		$\gamma = 0.1$		$\gamma = 0.3$		$\gamma = 0.5$	
	$c_t = S_t$	$c_t = 0$	$c_t = S_t$	$c_t = 0$	$c_t = S_t$	$c_t = 0$	$c_t = S_t$	$c_t = 0$
Basis strategies								
Non-LDI	-27.46	-75.14	-18.64	-60.82	-11.16	-46.73	-9.17	-41.81
All	-27.90	-75.14	-19.84	-60.84	-12.40	-46.87	-10.16	-42.14
reduction (%)	1.6	0.006	6.47	0.04	11.14	0.3	10.71	0.8

6. Concluding remarks

This thesis suggestes quantitative techniques for longevity risk management. Our aim is to develop such models and methods that would be particularly suited for the asset-liability management of longevity-linked cash flows.

Publication I proposes a framework for stochastic mortality modelling. The flexible construction is not only easy to accommodate to the preferences of the user, but also enables the assignment of tangible interpretations to the risk factors of the model, which in turn facilitates modelling their future behaviour. The wide applicability of the framework is reflected by the fact that it has served as a basis for modelling other insurance-related phenomena [2].

Publication II proposes stochastic models for the joint development of mortality and financial markets, utilizing the risk factors of a three-factor version of the mortality modelling framework of Publication I. Particular emphasis is placed on the long-term patterns in longevity, and their connections with the economy. Due to its simplicity, the model is easy to study both analytically and numerically. Since the underlying risk factors in our model have natural interpretations, its behaviour is easily judged by the user, which renders the model suitable for the analysis of mortality-linked cash-flows and associated investment strategies. The model is easy to calibrate to both historical data and user's expectations about the future development of mortality and the economy.

Publication III studies the effects of non-systematic and systematic mortality risks on the required capital reserves for a defined-benefit pension plan. We compute the required initial capital per person for different numbers of members in the pension scheme. Our main finding is that for pension plans with few members the impact of pooling on the capital requirement per capita is strong, but non-systematic risk is offset rapidly in pension schemes as the number of members grows. Systematic mortality risk, on the other hand, remains a significant source of risk is a pension portfolio.

Publication IV suggestes several liability-driven investment strategies for longevity-linked liabilities. We are able to show numerically that liability-driven investment can outperform common strategies that do not take into account the liabilities. These strategies may help pension insurers and issuers of longevity-linked instruments in asset-liability management, reserving, and in underwriting new insurance contracts. Although promising, the findings of Publication IV still leave substantial room for further research. The basis strategies employed in the numerical simulations are only an example of liability-driven strategies. Discovering and employing new connections between longevity-linked cash flows and asset returns would further improve the diversification strategy.

It remains to be seen if the so far continuous improvements in general longevity will persist, or if they will slow down or even cease altogether. Some experts predict that the steady decline in overall mortality during past decades will continue for the foreseeable future [60, 75, 14], while others [63] suggest that lifestyle factors, such as obesity, may soon hinder further mortality improvements. Combined with the changing regulatory environment, this possible future transition continues to offer unexplored avenues for novel research.

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