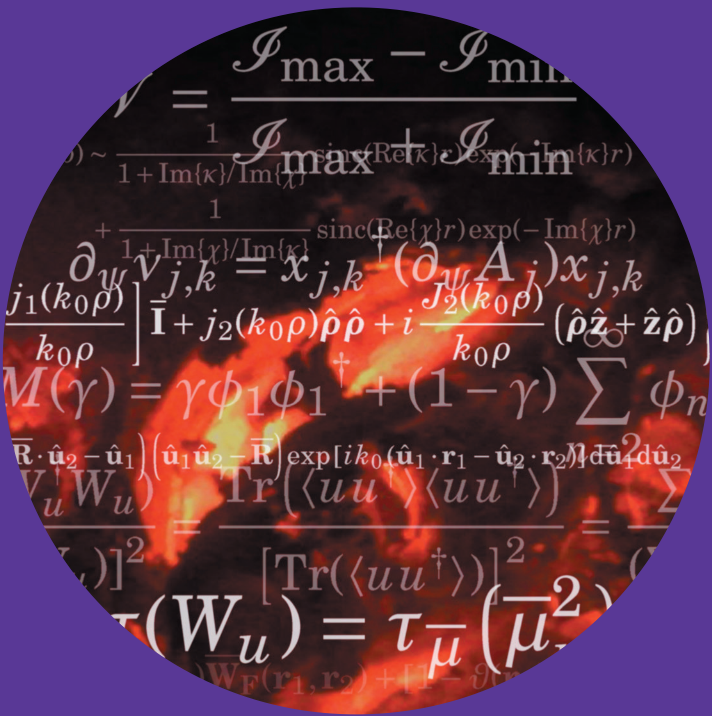
Department of Applied Physics

Electromagnetic Coherence Theory, Universality Results, and Effective Degree of Coherence

Kasimir Blomstedt



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Electromagnetic Coherence Theory, Universality Results, and Effective Degree of Coherence

Kasimir Blomstedt

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Aalto University
School of Science
Department of Applied Physics
Optics and Photonics

Supervising professor
Professor Matti Kaivola

Thesis advisors
Professor Ari T. Friberg
Doctor Tero Setälä

Preliminary examiners
Professor Kalle-Antti Suominen
University of Turku, Finland
Associate Professor Gregory Gbur
The University of North Carolina at Charlotte, USA

Opponents
Research Professor Manuel Nieto-Vesperinas
Universidad Autónoma de Madrid, Spain

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Abstract

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Abstract

Electromagnetic coherence is central to modern optics and photonics. Three topics from classical coherence theory are considered: 1) coherence of blackbody radiation within a cavity and at a cavity aperture, 2) universality of spatial coherence of fields created by homogeneous and isotropic sources, and 3) the effective degree of coherence.

Blackbody sources and fields have played a pivotal role in the development of quantum physics. The cross-spectral density (CSD) and degree-of-coherence of the blackbody field at an opening in a blackbody cavity wall and in the far field are computed, and it is shown that the aperture CSD obtained in previous works is, in fact, erroneous. The effect of the cavity wall on the CSD of the field is also studied. It is found that the wall indeed has an appreciable influence, but that this does not affect the CSD of the field in the aperture. The coherent-mode representation of the CSD function of the vector-valued blackbody field is derived for a spherical volume, and the importance of the constituent coherent modes is assessed. This expansion marks the first time that the coherent-mode representation is determined within a three-dimensional region. The blackbody results can be used to model thermal sources and propagation of natural light.

In prior works it has been shown that the degree-of-coherence functions of scalar optical fields, which are produced by stochastically homogeneous and isotropic sources, all have the same universal form in lossless infinite systems. Here that result is extended to electromagnetic fields. Additionally, it is proven that for actual systems, which necessarily are finite and lossy, the universal character disappears, regardless of how negligible the losses are or how large the source region is. These considerations apply to optical media and sources of all sizes, from nanoscopic to macroscopic.

It is shown that the effective degree of coherence can be extended from field representations in volumes to arbitrary (Hilbert) spaces. In addition, it is demonstrated that the effective degree of coherence is an intrinsic property of the electromagnetic field, and that it can be computed from almost any field representation. In fact, what is proven is that the effective degree of coherence is invariant to scaled unitary mappings. Finally, it is shown that of all Hilbert space functionals with this property, only the effective degree of coherence is additive.

Keywords electromagnetic coherence theory, blackbody field, universality of spatial coherence, effective degree of coherence

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Tiivistelmä

Tekijä
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Tiivistelmä

Sähkömagneettisella koherenssiteorialla on keskeinen merkitys optiikassa ja fotoniikassa. Työssä tutkitaan kolmea aihetta klassisen koherenssiteorian piiristä: 1) mustan kappaleen säteilyn koherenssiominaisuuksia kaviteetissa ja siihen tehdyssä aukossa, 2) homogeenisten ja isotrooppisten lähteiden synnyttämien kenttien ristispektritiheyden universaalisuutta lähdealueessa, sekä 3) efektiviistä koherenssiastetta.

Mustan kappaleen lähteiden ja kenttien tutkimuksella on ollut ratkaiseva rooli kvanttifysiikan kehityksessä. Työssä lasketaan mustan kappaleen säteilyn ristispektritiheys ja koherenssiaste mustan kappaleen kaviteetin seinämässä olevassa aukossa sekä kaukokentässä, ja osoitetaan, että aikaisemmissa julkaisuissa esitetty ristispektritiheyden lauseke aukossa on virheellinen. Lisäksi tutkitaan, miten seinämä, jossa aukko sijaitsee, vaikuttaa ristispektritiheyteen. Mustan kappaleen osalta lasketaan myös ristispektritiheyden koherenttimoodikehitelmä sen kentälle pallomaisessa tilavuudessa. Tämä on ensimmäinen kerta, kun sähkömagneettisen kentän koherenttimoodikehitelmä on esitetty kolmiulotteisessa alueessa. Saatuja tuloksia voidaan hyödyntää mallinnettaessa lämpölähteitä sekä luonnollisen valon etenemistä.

Aikaisemmissa tutkimuksissa on osoitettu, että tilastollisesti homogeenisten ja isotrooppisten äärettömän isojen lähteiden tuottamien skalaaristen kenttien koherenssifunktioilla on kaikilla sama universaali muoto. Työssä tulos yleistetään vektoriarvoisille kentille, mutta toisaalta todistetaan, että todellisille, välttämättä äärellisille ja häviöllisille systeemeille, universaalia muotoa ei esiinny, vaikka häviöt niissä olisivat kuinka mitättömiä ja tarkasteltavat lähdealueet kuinka suuria tahansa. Saadut tulokset pätevät mielivaltaisen kokoisille optisille materiaaleille ja lähteille, nanoskooppisista makroskooppisiin.

Tässä työssä näytetään myös, että efektiivisen koherenssiasteen määritelmä voidaan laajentaa kentän paikkaesityksistä koskemaan mielivaltaisia Hilbertin avaruuksia. Lisäksi osoitetaan, että efektiivinen koherenssiaste on sähkömagneettisen kentän 'luontainen' ominaisuus, joka on laskettavissa melkein kaikista kentän esityksestä. Tämä on seurausta siitä, että efektiivinen koherenssiaste ei muutu skaalatuissa unitaarimuunnoksissa. Viimein todistetaan, että kaikista Hilbert-avaruuden funktionaaleista, joilla on tämä ominaisuus, efektiivinen koherenssiaste on ainoa, joka on myös summautuva.

Avainsanat sähkömagneettinen koherenssiteoria, mustan kappaleen kenttä, paikkakoherenssin universaalisuus, efektiivinen koherenssiaste

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Sammandrag

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Författare
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Sammandrag

Elektromagnetisk koherens har en central position i den moderna optiken och fotoniken. I denna avhandling studeras tre ämnen ur klassisk koherensteori: 1) svartkroppsstrålningens koherensegenskaper i ett hålrum och i en öppning till hålrummet, 2) universalitet av spatial koherens hos radiationsfält utstrålade från homogena och isotropiska källor, samt 3) den effektiva koherensgraden.

Svartkroppskällor och tillhörande fält har haft en betydande roll i utvecklandet av kvantfysik. Här studeras svartkroppsstrålningens korsspektrala densitet och koherensgrad i en öppning i en svartkropp samt i öppningens fjärrfält. Ett tidigare presenterat uttryck för den korsspektrala densiteten i aperturen visas vara felaktigt. Vidare undersöks hur hålrummets vägg influerar den korsspektrala densiteten. Effekten visar sig vara märkbar, men sådan att den korsspektrala densiteten i aperturen ej påverkas. En expansion i koherenta moder av den elektromagnetiska svartkroppstrålningen framställs i en sfärisk volym. Detta är första gången en expansion i koherenta moder tagits fram i ett tredimensionellt rum. Resultaten gällande svartkroppar kan utnyttjas i modeller av termiska källor och när man studerar hur naturligt ljus färdas.

Tidigare forskningsresultat påvisar att koherensgraden av skalära optiska fält - vars källor är statistiskt homogena och isotropiska - alltid antar en specifik universell form i oändligt stora system där färdförluster inte förekommer. Detta resultat utvidgas här till att omfatta även vektoriella elektromagnetiska fält. Därtill bevisas det att den universella karaktären försvinner för verkliga system, som nödvändigtvis är ändliga och som dämpar de optiska fälten, oberoende av förlusternas obefintlighet eller källområdets storlek. Dessa resultat kan tillämpas för optiska medier och källor av alla storlekar, från nanoskopiska till makroskopiska.

I denna avhandlingen utvidgar man den effektiva koherensgradens definition från att omfatta fält i tredimensionella rum till godtyckliga funktioner ur Hilbertrum. Därtill demonstreras att den effektiva koherensgraden är en 'inre' egenskap av det elektromagnetiska fältet. Detta innebär att den kan beräknas från nästan alla representationer av ett elektromagnetiskt fält. Vad som egentligen bevisas är att den effektiva koherensgraden är invariant gentemot unitära transformationer och skalär multiplikation av fältet. Till sist bevisas det att av alla funktionaler i Hilbertrummet med denna egenskap är endast den effektiva koherensgraden additiv.

Nyckelord elektromagnetisk koherensteori, svartkroppsfält, universalitet av spatial koherens, effektiv koherensgrad

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Preface

This thesis represents the end of a long and winding project that started over ten years ago, in 2002, when I began my research in optical coherence theory at the Optics and Molecular Materials laboratory at Helsinki University of Technology (HUT). Before this I had been concentrating on monochromatic electromagnetic fields, which I studied in the context of diffractive optics and optical and electromagnetic scattering theory, and which provided the topics of my Master's thesis and my Licentiate's thesis, respectively. I stayed at the Optics and Molecular Materials laboratory till the end of the year 2005. From the beginning of the year 2006, I worked as a lecturer in the Biophysics laboratory at the University of Turku (UTU), where I was able to continue my research into second-order coherence theory, but where the distance to my advisors caused me to take on research goals, which proved to be extremely hard to achieve — so hard, in fact, that this is work still in progress, although promising results have at last emerged. Still I endured with those goals also after my lectureship at UTU in Turku ended and my employment no longer provided me with a university context, in which to pursue my research.

Therefore, I was delighted when my advisors contacted me in the spring of 2013 and offered me a chance to complete my PhD thesis in a three-month stint at the University of Eastern Finland (UEF) in Joensuu, where they are positioned at the moment of this writing. Basically, I have thus worked on the research topics presented in this thesis between the years 2002 and 2006, and again in 2013, which is also reflected in the lopsided time frame in which the included papers have been published. Most of the research has been done in the Optics and Molecular Materials laboratory at HUT, but one paper was finished at the Royal Institute of Technology (KTH) in Sweden, and the last paper as well as this thesis have been authored while I have been visiting the Institute of Photonics at UEF, a visit during which my salary was paid by the Optics and Photonics laboratory at Aalto University (AU).

I am thankful for the efforts of my advisors, Prof. Ari T. Friberg (ATF) and Dr. Tero Setälä, without whom this thesis would never have been. I also want to extend my gratitude to Prof. Matti Kaivola (MK) as the head of the former Optics and Molecular Materials laboratory HUT, now the laboratory of Optics and Photonics at AU, for providing me a place at which to do research work and for arranging financial support for my work in 2005 and again in the spring of 2013. Furthermore, I am indebted to the people with whom I have had a chance to collaborate when doing the research presented here. In addition to the aforementioned this list includes Dr. Jari Lindberg, Dr. Jani Tervo, and Prof. Jari Turunen.

During my PhD years at HUT and at AU I have also had the pleasure to work with a number of interesting individuals. For valuable and intriguing discussions on physics and other topics I want to thank, in alphabetical order, Dr. Markus Hautakorpi, Dr. Klas Lindfors, Dr. Thomas Lindvall, Dr. Arri Priimägi, and Dr. Andriy Shevchenko. For efficiency at organizing all things bureaucratic Orvokki Nyberg deserves my sincerest gratitude. At UTU I was fortunate to work with Prof. Pekka Hänninen, M.Sc. Pilvi Ylander, and M.Sc. Sami Koho. When visiting UEF I had the pleasure to make the acquaintance with M.Sc. Andreas Norrman, M.Sc. Timo Voipio, and M.Sc. Henri Kellock. I also got to enjoy exhilarating sessions of squash and pool with ATF.

In addition to the arrangements made by MK, my research has been funded by a grant from HUT and my accommodation in Joensuu has been paid jointly by the Optics and Photonics laboratory at AU and the Institute of Photonics at UEF.

At last, but definitely not least, I wish to thank my parents for their support during all the years I have been doing research, and my beloved Elina, who has heroically endured through my spouts of research madness and who has encouraged me in times of desperation.

Helsinki, October 23, 2013,

Kasimir Blomstedt

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List of publications

This thesis consists of an overview and of the following publications which are referred to in the text by their Roman numerals.

I T. Setälä, J. Lindberg, K. Blomstedt, J. Tervo, and A. T. Friberg, "Coherent-mode representation of a statistically homogeneous and isotropic electromagnetic field in spherical volume", Phys. Rev. E **71**, 036618 (2005).

II K. Blomstedt, T. Setälä, J. Tervo, J. Turunen, and A. T. Friberg, "Partial polarization and electromagnetic spatial coherence of blackbody radiation emanating from an aperture", Phys. Rev. A **88**, 013824 (2013).

III T. Setälä, K. Blomstedt, M. Kaivola, and A. T. Friberg, "Universality of electromagnetic-field correlations within homogeneous and isotropic sources", Phys. Rev. E **67**, 026613 (2003).

IV K. Blomstedt, T. Setälä, and A. T. Friberg, "Effect of absorption on the spatial coherence in scalar fields generated by statistically homogeneous and isotropic sources", Phys. Rev. E **72**, 056604 (2005).

V K. Blomstedt, T. Setälä, and A. T. Friberg, "Arbitrarily short coherence length in wave fields within finite lossless source regions", Phys. Rev. E **75**, 026610 (2007).

VI K. Blomstedt, T. Setälä, and A. T. Friberg, "Effective degree of coherence: general theory and application to electromagnetic fields", J. Opt. A: Pure Appl. Opt. **9**, 907–919 (2007).

Conference presentations

T. Setälä, K. Blomstedt, M. Kaivola, and A. T. Friberg, “Universality of electromagnetic-field correlations within homogeneous and isotropic sources”, Northern Optics 2003 — Proceedings, June 16–18, 2003, Espoo, Finland, p. 66.

K. Blomstedt, T. Setälä, M. Kaivola, and A. T. Friberg, “Universal structure of electromagnetic-field correlations produced by homogeneous and isotropic sources”, ICO Topical meeting on Polarization Optics, Polvijärvi, Finland, Selected Papers **8** (University of Joensuu, Department of Physics, Joensuu, 2003), pp. 156–157.

Other publications to which the author has contributed (not included in this thesis)

I. Kallioniemi, J. Saarinen, K. Blomstedt, and J. Turunen, "Polygon approximation of the fringes of diffractive elements", Appl. Opt. **36**, 7217–7223 (1997).

K. Blomstedt, E. Noponen, and J. Turunen, "Surface-profile optimization of diffractive 1:1 imaging lenses", J. Opt. Soc. Am. A **18**, 521–525 (2001).

E. Hernesniemi, K. Blomstedt, and M. Fortelius, "Multi-view stereo 3D reconstruction of lower molars of Recent and Pleistocene rhinoceroses for mesowear analysis", Palaeontologica Electronica **14**, 2T:1–15 (2011).

Author's contribution

The author has had a key role in all aspects of the research work summarized in this thesis and reported in Publications I–VI.

Publications I–III

The author has contributed to the topics and contents considered in these papers and he has carried out the mathematical derivations (in some cases in parallel with other authors). He has also contributed to the interpretation of the results and to the writing of the papers.

Publications IV–VI

These papers are based on the author's ideas, and the author has carried out the mathematical derivations and numerical computations. He has also been the driving force behind the interpretation of the results, and he has written the initial versions of all the papers.

1. Introduction

Optics has been studied systematically at least from antiquity [1, 2], and its basic principles are well known, but new insights and applications of optics are still emerging to this day[§] [3, 4]. Specifically, with nanoscale fabrication technology maturing, the field of nanophotonics has been rapidly developing and, in particular, near-field optical effects have found important applications for example in observation and manipulation of microscopic or nanoscopic structures [5, 6]. The new field of electromagnetic or optical metamaterials [7, 8], including invisibility cloaks [9,10], also bears mentioning. Electromagnetic theory is also important for the modeling and development of applications where heat (and light) radiation, both in the near and the far field, needs to be controlled [11], such as in the capture of solar radiation for energy and in the cooling of microchips. It is furthermore central in the understanding of wireless transmission of signals, and more recently of electric energy, not to mention fiber optics technology and communications [12], on which the internet as we know it is based. Optical theory of course also has a vital role to play in the development of 3D image transmission technology (for both entertainment and scientific purposes), which appears to finally mature beyond static holograms.

Although the 20th century saw electromagnetic theory evolve into quantum optics [13], and finally into quantum electrodynamics [14], with applications such as the laser as a consequence [15], there still is a place for, and much to understand about classical optics and electrodynamics, whose foundations lie in the Maxwell equations. Indeed, whereas quantum electrodynamics is useful for representing the interactions between fundamental particles, and quantum optics can be used to describe the detailed interactions between electromag-

[§]Of the last fifteen Nobel Prizes in Physics, six have been fully or partly awarded to research done in optics, with the list of recipients being as follows: 1997 (Chu, Cohen-Tannoudji, and Phillips), 2000 (Alferov and Kroemer), 2001 (Cornell, Ketterle, and Wieman), 2005 (Glauber, Hall, and Hänsch), 2009 (Kao), and 2012 (Haroche and Wineland).

netic fields and matter at the molecular level, these theories are usually too involved to be useful when macroscopic applications are considered. Nevertheless, there is an intimate relationship between quantum optics and advanced classical electromagnetic theory [16]. In particular, rigorous electromagnetic theory finds its simplest form when it is applied to monochromatic radiation, which in nature is well approximated only by laser light, a quantum optics phenomenon. At the other end of the spectrum of complexity lies natural or thermal light, whose behavior could be explained by Planck only when he introduced the idea of quantized energy levels [17], a concept which was pivotal in the development of quantum mechanics [18]. For a classically rigorous electromagnetic treatment of natural light, or chaotic radiation in general, the field of coherence theory emerged in the latter part of the 19th century [19], and has been evolving ever since [20, 21]. Interestingly, the behavior of laser light caused ideas from classical coherence theory to be applied in quantum optics, whereby the quantum theory of optical coherence [22–24] was born, mainly through the efforts of Glauber [25, 26].

In this thesis we will be concerned with classical second-order coherence theory. Classical coherence theory, which is based on Maxwell's equations, is a statistical description of electromagnetic fields, much like thermodynamics is a statistical description of molecular dynamics. The reasons for a stochastic treatment are the same in both cases, that is, the detailed behavior of the fields respective molecules is unavailable or too complicated to analyze. For electromagnetic fields it is their rapid fluctuations in time, or phases, that typically cannot be determined. Second-order classical coherence theory studies the correlations between electromagnetic field amplitude at pairs of (space–time, space–frequency, etc.) points. Because electromagnetic fields and light are classically measured in terms of their intensity, which is a quadratic property of the field, second-order coherence theory provides the theoretical framework needed for a complete modeling and analysis of electromagnetic fields in terms of intensity measurements. When the correlations between more than two field amplitudes are considered, the order of the coherence theory increases accordingly. Hence there is actually a hierarchy of classical coherence theories, with higher order theories encompassing and extending the lower order theories.

Although classical coherence theories of higher orders than second find uses when analyzing for example nonlinear optical systems [27], the higher order correlation properties of electromagnetic fields are most conveniently studied in terms of quantum coherence theory. In quantum coherence theory the correlations that are considered are those between photon counts [16]. Since a single photon count can directly be related to the intensity of the electromagnetic field, which classically is a second-order property, it follows that the nomenclatures in classical coherence theory and quantum coherence theory differ by a factor of two. Hence classical second-order coherence theory corresponds to quantum first-order coherence theory, classical fourth-order coherence theory is equivalent to quantum second-order coherence theory, and so on. A well-known application of quantum second-order coherence theory, or intensity interferometry, is provided by the ground-breaking experiment in stellar interferometry conducted by Hanbury Brown and Twiss in 1956 [28]. In this experiment it was shown that stellar diameters could be determined by considering the correlations between the signals of two photodetectors onto which the light of a star (Sirius) was reflected by mirrors placed some distance apart. Because atmospheric conditions affect the intensity of light much less than the phase of light, the Hanbury Brown–Twiss interferometer is more accurate than Michelson's stellar interferometer [29], which has a similar design, but is based on amplitude correlations. The Michelson stellar interferometer constitutes one of the first applications of electromagnetic (classical second-order) coherence theory.

Classical second-order coherence theory was developed mainly in terms of scalar fields during the last part of the 19th century and the first part of the 20th century. This development was mostly application driven, where the applications included for example Michelson's stellar interferometer discussed above, studying the illumination in microscopes, and modeling the radiation from blackbody or thermal sources. The latter part of the 20th century saw an amalgamation of previously somewhat separate treatments into a complete theory of second-order coherence for optical fields as mediated primarily by Wolf [30]. Since then second-order coherence theory has been extended to include electromagnetic fields and hence partially polarized fields, which were, however, already much earlier recognized to be well described by the same statistical methods as partially coherent fields. The theoretical focus of second-order coherence theory has shifted from a time-dependent description of electromagnetic fields to an angular-frequency description of the fields, and the theoretical framework is still continually explored and extended, with numerous new applications of the theory having emerged in recent years. Specifically, second-order coherence theory has been successfully applied to examine focused light fields [31] and their effects on atoms and molecules, to study the transport of optical coherence properties in plasmonic structures [32], to analyze and design ghost imaging apparatuses [33], and to extend radiometric theory from a scalar realm to electromagnetic fields [34–36], in particular with respect to blackbody radiation.

The motivations for the choice of subject matters studied in this thesis can roughly be divided into three categories: 1) to study blackbody radiation in the electromagnetic framework, in part in terms of the universal-form cavity field, and in part by modeling the field radiated by a blackbody in an electromagnetically rigorous way, 2) to extend the reported universality results concerning fields sourced by stochastically homogeneous and isotropic sources from scalar fields to electromagnetic fields, and to study how losses affect the universality results (in the scalar case), and 3) to gain a better theoretical insight into the so-called effective degree of coherence and its reported invariance to certain transformations. In the chapters that follow we give more detailed backgrounds and historical accounts of these topics.

Our main results in the subjects covered are as follows. For blackbody radiation we present the first electromagnetic coherent-mode expansion of the vector-valued cavity field, and its scalar analog, in a three-dimensional (spherical) volume, and we have shown that previous results concerning the field at the cavity aperture lacks an important term. We have extended the universality results concerning fields sourced by stochastically homogeneous and isotropic sources to electromagnetic fields, but at the same time we have also shown that calling such results universal is somewhat questionable as the universal character vanishes in the presence of even the tiniest amount of loss or if the source region is finite, which of course is the case in any actual system. Our results furthermore disprove the commonly held belief that the field coherence length cannot be shorter than that of blackbody radiation, or about half a (vacuum) wavelength. In fact, we present a construction, which can be used to obtain fields with arbitrarily short coherence lengths inside their source regions. Finally, we have proven that the effective degree of coherence is invariant to so-called scaled unitary mappings, which in practice means that the effective degree of coherence of an electromagnetic field has the same value when computed from any reasonable representation of the field. We have also shown that of all functionals of the field with this property, the effective degree of coherence is the only one which is additive, that is, which can be constructed as the sum of its parts.

The presentation in this thesis is organized so that initially we review the theoretical foundations of electromagnetic theory and second-order coherence theory to the extent used in our papers. This is done in Chaps. 2 and 3. After that we present the results of our researches in the following three chapters, so that in Chap. 4 we consider the blackbody radiation at an aperture in a blackbody cavity and in the far field (the cavity field is already considered in Chap. 3), in Chap. 5 we discuss the universality results, and in Chap. 6 we delve into the mathematical theory of the effective degree-of-coherence functional. Finally, in Chap. 7 we summarize the main conclusions.

2. Electromagnetic theory

In this chapter we introduce the mathematical framework that accurately describes the physics of classical non-relativistic macroscopic electromagnetic fields. This framework was essentially introduced by James Clerk Maxwell [37–41] (Maxwell's papers are reproduced in Ref. 42), who based it on the works of among others Ampère, Felici, Faraday, Helmholtz, Monsotti, and Thomson, and supplemented it by what is now known as the electric displacement. This addition allowed Maxwell to conclude that electric and magnetic fields not only couple, but can actually be interpreted as two sides of the same thing, the electromagnetic field, which he showed behaves like a wave. He also proved that in vacuum the electromagnetic field propagates with the speed $(\varepsilon_0\mu_0)^{-1/2}$ (modern notation), where ε_0 and μ_0 are the electric permittivity and the magnetic permeability of vacuum, respectively. This result, together with the then known value $(\varepsilon_0\mu_0)^{-1/2} = 310,740,000\,\mathrm{m/s}$ obtained by Weber and Kohlrausch in 1855, and the then known values for the speed of light, $314,858,000\,\mathrm{m/s}$ (Fizeau, 1848), $298,000,000\,\mathrm{m/s}$ (Foucault, 1850) and $308,000,000\,\mathrm{m/s}$ (astronomical observations), led Maxwell to draw the conclusion that light is an electromagnetic field [39]. The experiments commenced by Hertz in 1886 proved this assertion and the existence of electromagnetic waves that travel at the speed of light [43].

Although the field equations presented in the following describe the same physics as the equations in Maxwell's seminal paper [41], they are nowadays given in a much more streamlined form. The Maxwell equations first appeared in print in such a form around the middle of the 1880s as presented by Oliver Heaviside in terms of the grad, curl and div operators he introduced, but it seems that J. Willard Gibbs, who, in turn, introduced the symbols $\cdot$ and $\times$, had privately pursued a similar approach already a few years earlier [44]. The electromagnetic field equations, together with concepts from statistics, form the foundations of the electromagnetic second-order coherence theory, which is presented in the next chapter.

2.1 Maxwell's equations and time-harmonic fields

The behavior of the electromagnetic field is in classical physics completely described by the macroscopic Maxwell equations. In Gaussian or CGS[§] units these equations are given by

$$\nabla\times\mathbf{H}(\mathbf{r},t)-\frac{1}{c}\partial_t\mathbf{D}(\mathbf{r},t)=\frac{4\pi}{c}\mathbf{J}'_{\mathrm{e}}(\mathbf{r},t), \tag{2.1}$$

$$\nabla\times\mathbf{E}(\mathbf{r},t)+\frac{1}{c}\partial_t\mathbf{B}(\mathbf{r},t)=-\frac{4\pi}{c}\mathbf{J}'_{\mathrm{h}}(\mathbf{r},t), \tag{2.2}$$

$$\nabla\cdot\mathbf{D}(\mathbf{r},t)=4\pi\rho_{\mathrm{e}}(\mathbf{r},t), \tag{2.3}$$

$$\nabla\cdot\mathbf{B}(\mathbf{r},t)=4\pi\rho_{\mathrm{h}}(\mathbf{r},t), \tag{2.4}$$

where $\mathbf{E}$ denotes the electric field, $\mathbf{D}$ denotes the electric displacement, $\mathbf{H}$ denotes the magnetic field, $\mathbf{B}$ denotes the magnetic induction, $\mathbf{J}'_{\mathrm{e}}$ denotes the (total) electric current density, ρ_{e} denotes the (total) electric charge density, $\mathbf{J}'_{\mathrm{h}}$ denotes the (total) magnetic current density, and ρ_{h} denotes the (total) magnetic charge density. In addition $\mathbf{r}$ is the spatial position vector, t denotes time, and c is the speed of light in vacuum. Here the term 'total' in reference to charges and currents means the sum of primary (collections and movements of charges which are not responses to external forces) and secondary (collections and movements of charges in response to external forces) charges and currents, respectively.

The description of how the fields $\mathbf{D}$ and $\mathbf{B}$, and the total current densities $\mathbf{J}'_{\mathrm{e}}$ and $\mathbf{J}'_{\mathrm{h}}$ are connected to the fields $\mathbf{E}$ and $\mathbf{H}$, and the primary current densities $\mathbf{J}_{\mathrm{e}}$ and $\mathbf{J}_{\mathrm{h}}$ is provided by the so-called constitutive relations. In their most general form these relations can be written as

$$\mathbf{D}(\mathbf{r},t)=\mathbf{E}(\mathbf{r},t)+\boldsymbol{\chi}_{\mathrm{e}}[E,H](\mathbf{r},t), \tag{2.5}$$

$$\mathbf{B}(\mathbf{r},t)=\mathbf{H}(\mathbf{r},t)+\boldsymbol{\chi}_{\mathrm{h}}[E,H](\mathbf{r},t), \tag{2.6}$$

$$\mathbf{J}'_{\mathrm{e}}(\mathbf{r},t)=\mathbf{J}_{\mathrm{e}}(\mathbf{r},t)+\boldsymbol{\sigma}_{\mathrm{e}}[E,H](\mathbf{r},t), \tag{2.7}$$

$$\mathbf{J}'_{\mathrm{h}}(\mathbf{r},t)=\mathbf{J}_{\mathrm{h}}(\mathbf{r},t)+\boldsymbol{\sigma}_{\mathrm{h}}[E,H](\mathbf{r},t), \tag{2.8}$$

where $\boldsymbol{\chi}_{\mathrm{e,h}}$ are susceptibility operators and $\boldsymbol{\sigma}_{\mathrm{e,h}}$ are conductivity operators. These operators are nonlinear in general, and causality implies that their value at a space–time point $(\mathbf{r},t)$ can depend on the values of $\mathbf{E}(\mathbf{r}',t')$ and $\mathbf{H}(\mathbf{r}',t')$ at only those space–time points $(\mathbf{r}',t')$ that satisfy $|\mathbf{r}-\mathbf{r}'|\leq c(t-t')$.

It is convenient to represent the fields in Maxwell's equations, here denoted collectively by $\mathbf{F}(\mathbf{r},t)$, in terms of their complex analytic signal [16, 45] Fourier

[§]Although present convention implies that SI units should be used, we apply Gaussian units since the equations for the electric and magnetic fields are then completely symmetric in form, making the mathematical treatment more transparent.

transforms $\mathbf{F}(\mathbf{r},\omega)$ as

$$\mathbf{F}(\mathbf{r},t) = 2\,\mathrm{Re}\left\{\int_0^\infty \mathbf{F}(\mathbf{r},\omega)\exp(-i\omega t)\mathrm{d}\omega\right\}, \tag{2.9}$$

where $\mathbf{F} \in \{\mathbf{E},\mathbf{H},\mathbf{D},\mathbf{B},\mathbf{J}'_\mathrm{e},\mathbf{J}'_\mathrm{h},\rho_\mathrm{e},\rho_\mathrm{h}\}$, ω is the angular frequency, and $\mathbf{F}(\mathbf{r},\omega)$ denotes the time-harmonic component of the field at angular frequency ω.

In this thesis we consider electromagnetic fields only in media, which are (optically) linear, isotropic, spatially nondispersive, and stationary in time (i.e., independent of absolute time). Furthermore, we assume that there are no magnetic monopoles and neither primary nor secondary magnetic currents. Then the susceptibility and conductivity operators take on a particularly simple form, and the constitutive relations (2.5)–(2.8) can be written for time-harmonic fields as

$$\mathbf{D}(\mathbf{r},\omega) = \varepsilon_\mathrm{r}(\mathbf{r},\omega)\mathbf{E}(\mathbf{r},\omega), \tag{2.10}$$

$$\mathbf{B}(\mathbf{r},\omega) = \mu(\mathbf{r},\omega)\mathbf{H}(\mathbf{r},\omega), \tag{2.11}$$

$$\mathbf{J}'_\mathrm{e}(\mathbf{r},\omega) = \sigma(\mathbf{r},\omega)\mathbf{E}(\mathbf{r},\omega) + \mathbf{J}(\mathbf{r},\omega), \tag{2.12}$$

$$\mathbf{J}'_\mathrm{h}(\mathbf{r},\omega) = \mathbf{0}, \tag{2.13}$$

where ε_r denotes the dielectric permittivity, μ denotes the magnetic permeability, and σ denotes the conductivity.

By applying the time-harmonic constitutive relations, we can write the time-harmonic versions of the Maxwell equations (2.1)–(2.4) in the form

$$\nabla\times\mathbf{H}(\mathbf{r},\omega) + ik_0\varepsilon(\mathbf{r},\omega)\mathbf{E}(\mathbf{r},\omega) = \frac{4\pi}{c}\mathbf{J}(\mathbf{r},\omega), \tag{2.14}$$

$$\nabla\times\mathbf{E}(\mathbf{r},\omega) - ik_0\mu(\mathbf{r},\omega)\mathbf{H}(\mathbf{r},\omega) = \mathbf{0}, \tag{2.15}$$

$$\nabla\cdot[\varepsilon(\mathbf{r},\omega)\mathbf{E}(\mathbf{r},\omega)] = 4\pi\rho_\mathrm{e}(\mathbf{r},\omega) = -i\frac{4\pi}{\omega}\nabla\cdot\mathbf{J}(\mathbf{r},\omega), \tag{2.16}$$

$$\nabla\cdot\left[\mu(\mathbf{r},\omega)\mathbf{H}(\mathbf{r},\omega)\right] = 0, \tag{2.17}$$

where $k_0 = \omega/c = 2\pi/\lambda$ is the vacuum wave number and λ is the vacuum wavelength of the field. The latter form of Eq. (2.16) follows when we operate on Eq. (2.14) by $\nabla\cdot$ and use the result to define ρ_e. In these equations we have, as is customary, included the conduction properties of the material into the complex dielectric permittivity ε, which is defined as

$$\varepsilon(\mathbf{r},\omega) = \varepsilon_\mathrm{r}(\mathbf{r},\omega) + i\frac{4\pi}{\omega}\sigma(\mathbf{r},\omega). \tag{2.18}$$

Observe, however, that causality implies that $\varepsilon_\mathrm{r}(\mathbf{r},\omega)$ is in general already complex-valued.

We now obtain from Eqs. (2.14) and (2.15) for the electric and magnetic fields the expressions

$$\mathbf{E}(\mathbf{r},\omega) = \frac{i}{k_0\varepsilon(\mathbf{r},\omega)}\left[\nabla\times\mathbf{H}(\mathbf{r},\omega) - \frac{4\pi}{c}\mathbf{J}(\mathbf{r},\omega)\right], \tag{2.19}$$

$$\mathbf{H}(\mathbf{r},\omega) = -\frac{i}{k_0\mu(\mathbf{r},\omega)}\nabla\times\mathbf{E}(\mathbf{r},\omega), \tag{2.20}$$

which show that once either field is known, the other field is completely determined. When we introduce the expression (2.20) into Eq. (2.14), we get after some rearrangements the wave equation

$$\mu(\mathbf{r},\omega)\nabla\times\left[\frac{1}{\mu(\mathbf{r},\omega)}\nabla\times\mathbf{E}(\mathbf{r},\omega)\right] - \kappa^2(\mathbf{r},\omega)\mathbf{E}(\mathbf{r},\omega) = i4\pi\frac{k_0}{c}\mu(\mathbf{r},\omega)\mathbf{J}(\mathbf{r},\omega), \tag{2.21}$$

where the complex wave number κ is defined as

$$\kappa^2(\mathbf{r},\omega) = k_0^2\varepsilon(\mathbf{r},\omega)\mu(\mathbf{r},\omega). \tag{2.22}$$

If we apply the identity $\nabla\times(\nabla\times\mathbf{F}) = \nabla(\nabla\cdot\mathbf{F}) - \nabla^2\mathbf{F}$, and the relation (2.16), we can rewrite the wave equation (2.21) as

$$\nabla^2\mathbf{E}(\mathbf{r},\omega) + \kappa^2(\mathbf{r},\omega)\mathbf{E}(\mathbf{r},\omega) = \mathbf{Q}(\mathbf{r},\omega), \tag{2.23}$$

where

$$\begin{aligned}\mathbf{Q}(\mathbf{r},\omega) = {} & i\frac{k_0\mu(\mathbf{r},\omega)}{c}\mathbf{J}(\mathbf{r},\omega) + i\frac{1}{\omega}\nabla\left[\frac{1}{\varepsilon(\mathbf{r},\omega)}\nabla\cdot\mathbf{J}(\mathbf{r},\omega)\right] \\ & + \frac{1}{4\pi}\nabla\{\nabla\ln[\varepsilon(\mathbf{r},\omega)]\}\cdot\mathbf{E}(\mathbf{r},\omega) + \frac{1}{4\pi}\nabla\ln[\mu(\mathbf{r},\omega)]\times[\nabla\times\mathbf{E}(\mathbf{r},\omega)].\end{aligned} \tag{2.24}$$

At optical frequencies (natural) materials are normally nonmagnetic, so that typically $\mu(\mathbf{r},\omega)\approx 1$.

For vacuum regions, where $\varepsilon(\mathbf{r},\omega) = 1$, $\mu(\mathbf{r},\omega) = 1$, and $\mathbf{J}(\mathbf{r},\omega) = \mathbf{0}$, the wave equations (2.21) and (2.23) simplify into the reduced (vacuum) wave equations

$$\nabla\times[\nabla\times\mathbf{E}(\mathbf{r},\omega)] - k_0^2\mathbf{E}(\mathbf{r},\omega) = \mathbf{0}, \tag{2.25}$$

and

$$\nabla^2\mathbf{E}(\mathbf{r},\omega) + k_0^2\mathbf{E}(\mathbf{r},\omega) = \mathbf{0}, \tag{2.26}$$

respectively.

In the vacuum wave equation (2.26) the different components of the electric field vector $\mathbf{E}(\mathbf{r},\omega)$ separately satisfy the Helmholtz equation

$$\nabla^2U(\mathbf{r},\omega) + k_0^2U(\mathbf{r},\omega) = 0. \tag{2.27}$$

Here $U = \hat{\mathbf{u}}\cdot\mathbf{F}$ and $\hat{\mathbf{u}}\in\{\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}}\}$, where $\{\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}}\}$ is a triplet of Cartesian unit vectors in $\mathbf{R}^3$. The separation of the vector components does not happen in general inside material media as can be seen from Eqs. (2.23) and (2.24). In special cases, however, such as when the properties of the material change slowly with position, so that $k_0|\nabla\{\nabla \ln[\varepsilon(\mathbf{r},\omega)]\}|$ and $k_0|\nabla \ln[\mu(\mathbf{r},\omega)]|$ are small enough to make the sum of the latter two terms on the right-hand side of Eq. (2.24) negligible with respect to the sum of the first two terms, the components (approximately) separate [3, 46] and we get for each component an equation of the form

$$\nabla^2 U(\mathbf{r},\omega) + \kappa^2(\mathbf{r},\omega)U(\mathbf{r},\omega) = -4\pi Q(\mathbf{r},\omega), \tag{2.28}$$

where the source term $Q(\mathbf{r},\omega)$ is determined by the specific approximation used and what component is considered.

In view of the approximations made when deriving the scalar Helmholtz equation (2.28), that expression, in particular, completely (exactly) describes the behavior of the electromagnetic field in regions where the material properties are constant. Thereby, if the material is approximated as a collection of uniform regions separated by discontinuity boundaries, the behavior of the electromagnetic field can be described in full by scalar fields that are coupled only at the discontinuity boundaries (by Maxwell's equations). In some geometries, such as in systems that are invariant in one spatial coordinate, the electromagnetic fields can be represented by scalar fields that do not couple at the discontinuity boundaries and thus the scalar description applies throughout space. Because optical systems are typically such that the above mentioned approximate or special conditions are met, optical fields are often described by scalar fields, with their vectorial nature ignored on account that it seldom alters the results to a degree that would warrant a more complicated treatment. For our purposes here, we need the theory of scalar fields on one hand since many of our results apply directly only to such fields, and on the other hand since we want to use the simpler equations of scalar fields as a basis on which to develop the more complicated equations of vector-valued electromagnetic fields.

2.2 Electromagnetic energy relations and intensity measurements

If we consider a region $\Omega \subset \mathbb{R}^3$, it can be shown [47] that the time-average (over all time) of the electromagnetic power that radiates out of this region, as carried by an electromagnetic field $[E,H]$ at angular frequency ω, is given by the energy flux functional

$$\mathscr{F}_\Omega[E,H](\omega) = \int_{\partial\Omega} \mathbf{S}(\mathbf{r},\omega)\cdot\hat{\mathbf{n}}(\mathbf{r})\mathrm{d}\mathbf{r}, \tag{2.29}$$

where

$$\mathbf{S}(\mathbf{r},\omega) = \frac{c}{2}\,\mathrm{Re}\left\{\mathbf{E}(\mathbf{r},\omega)\times\mathbf{H}^*(\mathbf{r},\omega)\right\} \tag{2.30}$$

is the Poynting vector at frequency ω, $\partial\Omega$ is the boundary of the region Ω, and $\hat{\mathbf{n}}(\mathbf{r})$ denotes the outward surface normal of $\partial\Omega$ at $\mathbf{r} \in \partial\Omega$. The energy relation (2.29) was first formulated by Poynting [48] for time-dependent electromagnetic fields.

The time-averaged electromagnetic (potential) energy at frequency ω stored inside the region Ω is given by the expression

$$\mathscr{W}_\Omega[E,H](\omega) = \frac{1}{4}\int_\Omega \left[|\mathbf{E}(\mathbf{r},\omega)|^2 + |\mathbf{H}(\mathbf{r},\omega)|^2\right] \mathrm{d}\mathbf{r} = \mathscr{W}_\Omega[E](\omega) + \mathscr{W}_\Omega[H](\omega), \qquad (2.31)$$

which also introduces the electric energy $\mathscr{W}_\Omega[E](\omega)$ and the magnetic energy $\mathscr{W}_\Omega[H](\omega)$ in the obvious way.

It is beyond the scope of this thesis to show how the scalar field energy flux functional follows from the electromagnetic energy flux functional when the appropriate assumptions are made. However, since the scalar field is extensively used to describe both optical and acoustical waves, the appropriate expression has been presented elsewhere [3, 49]. Indeed, we have the representation

$$\mathscr{F}_\Omega[U](\omega) = -\frac{c}{2k_0}\int_{\partial\Omega} \mathrm{Im}\left\{\left[U(\mathbf{r},\omega)\nabla U^*(\mathbf{r},\omega)\right]\cdot\hat{\mathbf{n}}(\mathbf{r})\right\}\mathrm{d}\mathbf{r} \qquad (2.32)$$

for the energy flux at frequency ω radiated out of a region Ω by a scalar field U.

When the direction of the electromagnetic radiation, as given by the sign of the energy flux functional $\mathscr{F}_\Omega[E,H](\omega)$ or $\mathscr{F}_\Omega[U](\omega)$, with respect to a region Ω with no primary sources [$\mathbf{J}(\mathbf{r},\omega) = \mathbf{0}$ or $Q(\mathbf{r},\omega) = 0$], is the same (into, $-$, or out from, $+$, the region Ω) for all electromagnetic or scalar fields, it is convenient to use this direction of energy flux to characterize Ω. Hence we call a region Ω *lossy* if $\mathscr{F}_\Omega[\cdot](\omega) \le 0$, *amplifying* if $\mathscr{F}_\Omega[\cdot](\omega) \ge 0$, and *lossless* if $\mathscr{F}_\Omega[\cdot](\omega) = 0$ for every electromagnetic field $[E,H]$ or scalar field U. The terms lossy and lossless are standard terminology, whereas the term amplifying is here introduced for symmetry. The archetypal lossless region (and indeed the only true lossless region) is vacuum, whereas any conductor provides an example of a lossy region (energy is lost through Joule heating). Laser gain materials are examples of typical amplifying regions.

Let us now study how the region characterizations are reflected in the material parameters of the regions. To begin with we look at scalar fields, for which the energy flux out of Ω, when $\partial\Omega$ lies in vacuum and Ω contains no primary sources, attains the form

$$\mathscr{F}_\Omega[U](\omega) = -\frac{c}{2k_0}\int_\Omega \mathrm{Im}\{\kappa^2(\mathbf{r},\omega)\}\,|U(\mathbf{r},\omega)|^2\,\mathrm{d}\mathbf{r}, \qquad (2.33)$$

when the expression (2.32) is developed by applying the divergence theorem and the Helmholtz equation (2.28). From the expression (2.33) it follows that for scalar fields a region Ω is lossy if $\mathrm{Im}\{\kappa^2(\mathbf{r},\omega)\} \geq 0$, amplifying if $\mathrm{Im}\{\kappa^2(\mathbf{r},\omega)\} \leq 0$, and lossless if $\mathrm{Im}\{\kappa^2(\mathbf{r},\omega)\} = 0$ for all $\mathbf{r}$ in Ω. In other cases the region Ω has a different character for different fields U.

For electromagnetic fields, with the same assumptions on the region Ω as in the scalar case, the expression (2.29) can be developed into

$$\begin{aligned}&\mathscr{F}_\Omega[E,H](\omega)\\&= -\frac{c}{4\pi k_0}\int_\Omega \left[\mathrm{Im}\left\{k_0^2\varepsilon(\mathbf{r},\omega)\right\}|\mathbf{E}(\mathbf{r},\omega)|^2 + \mathrm{Im}\left\{k_0^2\mu(\mathbf{r},\omega)\right\}|\mathbf{H}(\mathbf{r},\omega)|^2\right]\mathrm{d}\mathbf{r},\end{aligned} \tag{2.34}$$

when the divergence theorem as well as relations (2.19) and (2.20) are used. From the expression (2.34) it follows that a region Ω is for vector-valued electromagnetic fields lossy when $\mathrm{Im}\{k_0^2\mu(\mathbf{r},\omega)\} \geq 0$ and $\mathrm{Im}\{k_0^2\varepsilon(\mathbf{r},\omega)\} \geq 0$, amplifying when $\mathrm{Im}\{k_0^2\mu(\mathbf{r},\omega)\} \leq 0$ and $\mathrm{Im}\{k_0^2\varepsilon(\mathbf{r},\omega)\} \leq 0$, and lossless when $\mathrm{Im}\{k_0^2\mu(\mathbf{r},\omega)\} = 0$ and $\mathrm{Im}\{k_0^2\varepsilon(\mathbf{r},\omega)\} = 0$, with no specific character in other cases. We note that this characterization differs from the characterization obtained in the case of scalar fields. In particular, we observe that for vector-valued electromagnetic fields the electric and magnetic contributions can in general have a different character, which of course is not possible for scalar fields. Accordingly the characterization in the electromagnetic case is in general not expressible in terms of the complex wave number $\kappa(\mathbf{r},\omega)$. However, when $\mathrm{Im}\{\mu(\mathbf{r},\omega)\} = 0$ and $\mathrm{Re}\{\mu(\mathbf{r},\omega)\} \geq 0$, which, to a good degree of accuracy, is usually the case in optics, we can further develop the expression (2.34), to get

$$\mathscr{F}_\Omega[E,H](\omega) = -\frac{c}{4\pi k_0}\int_\Omega \mathrm{Im}\left\{\kappa^2(\mathbf{r},\omega)\right\}\frac{1}{\mu(\mathbf{r},\omega)}|\mathbf{E}(\mathbf{r},\omega)|^2\,\mathrm{d}\mathbf{r}, \tag{2.35}$$

which provides us with the same characterization of regions as in the scalar case. A similar result is available if we make assumptions on ε instead of on μ, or if either is slowly changing [see comment before Eq. (2.28)]. For our purposes here, it is sufficient that we can use the same characterization for regions in the electromagnetic case as in the scalar case. Hence, we will from here on assume that the scalar characterization of regions in terms of the complex wave number is applicable also for electromagnetic fields.

We note that for both scalar fields and electromagnetic fields the energy flux functional vanishes when the region Ω is not only devoid of primary sources, but vacuum-filled as well. Indeed, then $\varepsilon,\mu \in \mathbb{R}$, so that $\kappa \in \mathbb{R}$, whereby the expressions (2.33) and (2.34) yield the results $\mathscr{F}_\Omega[U](\omega) = 0$ and $\mathscr{F}_\Omega[E,H](\omega) = 0$, respectively.

Finally, let us consider measurements of the electromagnetic or scalar field. A measurement is a transfer of energy from the object that is observed to a detector. Here we take the detector to be represented by a lossy region Ω, where at least a part of the loss is in terms of the measurement signal we want to retrieve. Furthermore, we assume that the signal is proportional to the loss, so that without losing generality we can take the signal to be equal to the negative of the energy flux out of the region Ω as carried by the field that is measured, divided by the surface area $|\partial\Omega| = \int_{\partial\Omega} \mathrm{d}\mathbf{r}$. Specifically, we have in view of the energy flux functionals (2.29) and (2.32), for an electromagnetic field $[E,H]$ and a scalar field U with respect to the region Ω, the measurement outcomes

$$\mathscr{I}_\Omega[E,H](\omega) = -|\partial\Omega|^{-1}\mathscr{F}_\Omega[E,H](\omega) = -|\partial\Omega|^{-1}\int_{\partial\Omega}\mathbf{S}(\mathbf{r},\omega)\cdot\hat{\mathbf{n}}(\mathbf{r})\mathrm{d}\mathbf{r} \tag{2.36}$$

and

$$\begin{aligned}\mathscr{I}_\Omega[U](\omega) &= -|\partial\Omega|^{-1}\mathscr{F}_\Omega[U](\omega)\\ &= |\partial\Omega|^{-1}\frac{c}{2k_0}\int_{\partial\Omega}\mathrm{Im}\left\{\left[U(\mathbf{r},\omega)\nabla U^*(\mathbf{r},\omega)\right]\cdot\hat{\mathbf{n}}(\mathbf{r})\right\}\mathrm{d}\mathbf{r},\end{aligned} \tag{2.37}$$

respectively. Although these representations are strictly valid only when the surface $\partial\Omega$ is closed, it is customary in optics to assume that the flux is appreciable, say only across the front face $\partial\Pi^+$ of a photodetector Π. When this face is (nearly) planar and the Poynting vector of the field varies only slightly across it, it follows from Eq. (2.36) that the energy flux of the field into the photodetector divided by the surface area is to good accuracy given by

$$\mathscr{I}_\Pi([E,H],\omega) \approx |\mathbf{S}(\mathbf{r},\omega)\cdot\hat{\mathbf{n}}(\mathbf{r})|, \tag{2.38}$$

where $\mathbf{r}\in\partial\Pi^+$ is a typical point on the front surface $\partial\Pi^+$. The right-hand side of the expression (2.38) is the intensity of the electromagnetic field [3], and we extend this terminology to cover the measurement outcomes in Eqs. (2.36) and (2.37) as well. Thus, what we mean by an intensity measurement, is the physical realization of these equations.

2.3 Uniqueness results and fundamental solutions

It can be shown that the scalar Helmholtz equation (2.28) has a unique solution when it is accompanied by the Sommerfeld radiation (absorption) condition [50]

$$\lim_{r\to\infty} r\left[\partial_r U(\mathbf{r},\omega) \mp ik_0 U(\mathbf{r},\omega)\right] = 0, \tag{2.39}$$

where the limit holds uniformly in all directions $\hat{\mathbf{r}}$ and where $-$ corresponds to outgoing fields (field carrying energy to infinity) and $+$ to incoming fields (field carrying energy from infinity). Here $r = |\mathbf{r}|$ and $\hat{\mathbf{r}} = \mathbf{r}/r$ denote the length of the vector $\mathbf{r}$ and its direction (unit vector), respectively. The same convention will be used henceforth for all vectors.

For electromagnetic electric fields the solution to either the wave equation (2.21) or the wave equation (2.23) is unique when these equations are augmented with the Silver–Müller radiation (absorption) condition [51–53]

$$\lim_{r\to\infty} r\left[\nabla\times\mathbf{E}(\mathbf{r},\omega)\mp ik_0\hat{\mathbf{r}}\times\mathbf{E}(\mathbf{r},\omega)\right]=\mathbf{0}, \tag{2.40}$$

where the limit holds uniformly in all directions $\hat{\mathbf{r}}$ and where the signs are as in the Sommerfeld condition. We note that when a scalar field is equal to a component of (or otherwise sensibly derived from) an electromagnetic field that satisfies the Silver–Müller condition, the scalar field itself satisfies the corresponding Sommerfeld condition.

Let us now consider the special case, where $\kappa(\mathbf{r},\omega)=\kappa$, with $\mathrm{Im}\{\kappa^2\}\geq 0$, viz., the case where, except for the primary sources, all space consists of the same non-amplifying medium. Then the unique solution to the scalar Helmholtz equation (2.28) with the Sommerfeld radiation condition (2.39) for outgoing fields is given by

$$U(\mathbf{r},\omega)=\int_\Omega G_\kappa(\mathbf{r},\mathbf{r}')Q(\mathbf{r}',\omega)\mathrm{d}\mathbf{r}', \tag{2.41}$$

where

$$G_\kappa(\mathbf{r},\mathbf{r}')=\frac{\exp(i\kappa|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|} \tag{2.42}$$

is the Green's function of the system and Ω is the support of the source distribution Q. This Green's function is the fundamental solution of the scalar Helmholtz equation (2.28) that satisfies the Sommerfeld radiation condition (2.39) for outgoing fields. Formally, we have

$$\nabla^{\bullet 2}G_\kappa(\mathbf{r},\mathbf{r}')+\kappa^2G_\kappa(\mathbf{r},\mathbf{r}')=-4\pi\delta(\mathbf{r}-\mathbf{r}'), \tag{2.43}$$

where $\nabla^\bullet$ denotes ∇ or ∇'. We use this notation in what follows without further comment.

Quite analogously to the scalar case, the electromagnetic wave equations have fundamental solutions. We consider these solutions for a system where $\varepsilon(\mathbf{r},\omega)=\varepsilon$ and where $\mu(\mathbf{r},\omega)=\mu$ is real and positive, so that $\kappa(\mathbf{r},\omega)=\kappa$, and the system is lossy when $\mathrm{Im}\{\kappa^2\}>0$. In this system the electromagnetic field satisfies the wave equation (2.23), where the source term expression (2.24) simplifies into

$$\mathbf{Q}(\mathbf{r},\omega)=\frac{i\mu}{\omega}\left\{\mathbf{J}(\mathbf{r},\omega)+\frac{1}{\kappa^2}\nabla[\nabla\cdot\mathbf{J}(\mathbf{r},\omega)]\right\}. \tag{2.44}$$

It thereby follows that each component of the electric field satisfies a scalar Helmholtz equation of the form (2.28), and we can apply the (unique) scalar solution (2.41) to obtain the unique vectorial solution in the form

$$\mathbf{E}(\mathbf{r},\omega)=\int_\Omega G_\kappa(\mathbf{r},\mathbf{r}')\mathbf{Q}(\mathbf{r}',\omega)\mathrm{d}\mathbf{r}', \tag{2.45}$$

where Ω denotes the support of the source distribution $\mathbf{J}$. When $\mathbf{r} \notin \Omega$, the integrand is everywhere regular and we can use, twice in succession, the divergence theorem together with the fact that the source distribution vanishes by definition on $\partial\Omega$, to rewrite the solution (2.45) in the form

$$\mathbf{E}(\mathbf{r},\omega) = \frac{i\mu}{\omega}\int_{\Omega} \overline{\mathbf{G}}_{\kappa}(\mathbf{r},\mathbf{r}') \cdot \mathbf{J}(\mathbf{r}',\omega)\mathrm{d}\mathbf{r}', \quad \mathbf{r} \notin \Omega, \tag{2.46}$$

where

$$\overline{\mathbf{G}}_{\kappa}(\mathbf{r},\mathbf{r}') = \left(\overline{\mathbf{I}} + \frac{1}{\kappa^2}\nabla^{\bullet}\nabla^{\bullet}\right) G_{\kappa}(\mathbf{r},\mathbf{r}') \tag{2.47}$$

is the dyadic Green's tensor and $\overline{\mathbf{I}}$ denotes the unit dyad. The dyadic Green's tensor formally satisfies the wave equations

$$\nabla^{\bullet} \times \left[\nabla^{\bullet} \times \overline{\mathbf{G}}_{\kappa}(\mathbf{r},\mathbf{r}')\right] - \kappa^2 \overline{\mathbf{G}}_{\kappa}(\mathbf{r},\mathbf{r}') = 4\pi\delta(\mathbf{r}-\mathbf{r}')\overline{\mathbf{I}}, \tag{2.48}$$

together with the divergence conditions

$$\nabla^{\bullet} \cdot \overline{\mathbf{G}}_{\kappa}(\mathbf{r},\mathbf{r}') = \mathbf{0}. \tag{2.49}$$

Finally, we note that the assumption $\mathbf{r} \notin \Omega$ is necessary for the representation (2.46) to hold, since otherwise the proper application of the divergence theorem introduces non-vanishing surface terms that must be included. This subtlety in the use of the dyadic Green's tensor representation is discussed by Yaghjian [54], among others.

3. Second-order coherence theory

As was indicated in the introduction, second-order coherence theory is closely related to intensity measurements. Indeed, knowledge of the mutual coherence operator or, equivalently, the cross-spectral density operator in an optical system is sufficient to determine the outcome of all intensity measurements therein. The relationship is, in fact, reversible in that the mutual coherence operator and the cross-spectral density operator can, in principle, be completely determined from intensity measurements. In practice, all required measurements may not, however, be realizable.

We note that even though electromagnetic coherence theory is usually described in terms of random fields, which is both convenient and which finds motivation in quantum optics, the physical fields are not actually random, but randomness encompasses the lack of detailed knowledge of their behavior. Indeed, the second-order coherence theory of stationary fields as outlined from a stochastic point of view in the following, can actually also be derived by assuming that all media are stationary in the sense conveyed by Eqs. (2.10)–(2.13) and by assuming that all intensity measurements of electromagnetic fields are (approximately) unchanging over a time interval which is large with respect to the electromagnetic fluctuations. These two assumptions immediately lead to the full second-order description of stationary electromagnetic fields, including the Wiener–Khintchine theorem [16], as well as the concepts of the mutual coherence function and quasi-monochromatic fields.

Throughout the development of (second-order) coherence theory, the so-called Young's double-slit (or double pinhole) interference experiment [55] has had a central role to play [56]. In this experiment (see Fig. 3.1) an electromagnetic (light) field (L_1) passes through two pinholes in a screen (S_1) that otherwise blocks the field. Behind this screen lies another screen (S_2) onto which the thus disturbed light field (L_2) impinges. Because of the interference of the monochromatic components of the field, interference fringes are generally ob-

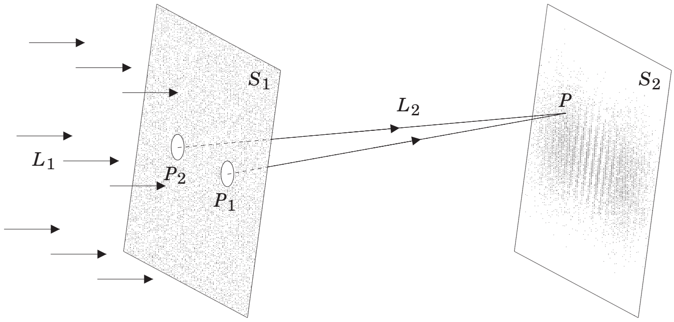


Figure 3.1. Young's interference experiment. A light field L_1 impinges on a screen S_1 with two pinhole apertures, P_1 and P_2. The light fields L_2 passing through the two pinholes interfere at a later screen S_2, where an interference pattern can be observed. This process is illustrated in terms of two light rays that converge at an observation point P on the screen S_2.

served on this latter screen. For fully monochromatic light, the intensity of these fringes go from zero to their maximum, whereas for polychromatic light, the total intensity of all the light does not vanish completely. In fact, in the extreme case where the total intensity is approximately uniformly distributed among the frequencies, the light field produced by interference is of nearly constant intensity. How well the minima and maxima of the intensity distribution are discernible is called the *visibility* of the interference pattern and it was mathematically defined by Michelson [29] as

$$\mathscr{V} = \frac{\mathscr{I}_{\max} - \mathscr{I}_{\min}}{\mathscr{I}_{\max} + \mathscr{I}_{\min}}, \tag{3.1}$$

where $\mathscr{I}_{\max}$ and $\mathscr{I}_{\min}$ denote the maximum and minimum values of the intensity of the field in the vicinity of the observation point P. Defined in this way the visibility is proportional to the absolute value of the so-called complex degree of coherence of the field, when it is modeled as a scalar field. We discuss the degree of coherence in more detail in Sec. 3.2.

Starting with the experiments done by Young [55], the pre-1960s development of (second-order) electromagnetic coherence theory is mainly built on the works by Verdet, Michelson, von Laue, Berek, van Cittert, Zernike, Hopkins, Wolf, and Blanc-Lapierre and Dumontet [3, 19, 30]. Other important contributors to the theory of partial coherence and partial polarization include Stokes, Wiener, Perrin, Pancharatnam, Hurwitz, Gabor, Gamo, Parrent, and Bourret [3, 19, 30]. Of these early contributors Michelson applied techniques, which today would be classified as part of coherence theory, to extract information about the diameters of astronomical objects (stars) from the coherence properties of the light observed by an interferometer placed in conjunction with a telescope. Berek, in turn, used his form of coherence theory to study image formation in microscopes. The researches of van Cittert and Zernike led to the formulation of what now is known as the van Cittert–Zernike theorem [3, 16, 19] and, as will be discussed below, Zernike gave a specific definition of the degree of coherence of a light field [57]. The coherence properties of electromagnetic fields were studied from an information, or entropy, perspective by von Laue, Gamo, and Gabor. Gabor is of course famous for his role in the theory of holography to which his researches contributed [58]. Reflections of the investigations of von Laue and Gamo can on the other hand be found for example when considering effective or overall degrees of coherence of the electromagnetic field, as is done in Chap. 6. Finally, the part played by Wolf in the formalization and advancement of (second-order) coherence theory cannot be overstated. His important contribution to this development is highlighted by the large number of groundbreaking papers he has authored, a selection of which is given in Ref. 30.

3.1 Cross-spectral density of scalar and electromagnetic fields

The close relationship between second-order correlations and intensity measurements means that a full description of all (intensity) measurements of an electromagnetic field can be made within the framework of second-order coherence theory. In particular, it follows that no detailed knowledge beyond the second-order correlations of the field need to be considered or retained. Let us now assume that the field is described by a random process. Furthermore, if all intensity measurements of the field are (nearly) unchanged in time and all material operators of the system are stationary [that is, constitutive relations of the form (2.10)–(2.13) hold throughout], the random process describing the field is stationary in the wide sense [59]. It then follows that the so-called mutual coherence operator (function) between two space–time points $(\mathbf{r}_1, t_1)$ and $(\mathbf{r}_2, t_2)$ of such a random process only depends on the time-difference $\tau = t_2 - t_1$ and is given by the time-average

$$\Gamma_F(\mathbf{r}_1, \mathbf{r}_2, \tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} F(\mathbf{r}_1, t) F^*(\mathbf{r}_2, t+\tau) \mathrm{d}t \tag{3.2}$$

for a scalar process F, and by

$$\Gamma_{\mathbf{F}}(\mathbf{r}_1, \mathbf{r}_2, \tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathbf{F}(\mathbf{r}_1, t) \mathbf{F}^\dagger(\mathbf{r}_2, t+\tau) \mathrm{d}t \tag{3.3}$$

for a vector-valued process $\mathbf{F}$, where $\dagger$ denotes the adjoint (complex conjugate transpose) of a vector. Here we attach the complex conjugation on the second field rather than on the first, contrary to what is customary in coherence theory, because with this choice our notation is consistent with that used in functional analysis, from which many of the mathematical results applied in second-order coherence theory derive. Because of this choice, the expressions containing the mutual coherence operator, the cross-spectral density operator or degree-of-coherence functions are complex conjugates of those found in many of the referenced papers, including our own.

It is convenient to combine the expressions (3.2) and (3.3) into one by using a common notation. For that purpose we will represent the scalar or vector-valued field by a random process u, with the property that at all times t we have $u(t) \in H$, where H is a Hilbert space. In Eqs. (3.2) and (3.3), the corresponding Hilbert spaces could for example be $L^2(\Omega)$ and $L^2(\Omega \times \{x,y,z\})$, where $\Omega \subset \mathbb{R}^3$ is some region of interest. We now severe the connection to explicit coordinates and use a *coordinate-free* notation, where only the temporal dependence is explicitly shown, to replace Eqs. (3.2) and (3.3) by the expression

$$\Gamma_u(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} u(t)u^\dagger(t+\tau)\mathrm{d}t \tag{3.4}$$

for the mutual coherence operator of u.

Since the operator formalism used in this work is not commonly employed in electromagnetic coherence theory, we briefly recall here some of the basic properties of (complex) Hilbert spaces. First of all, every Hilbert space H is a linear vector space, which means that if $u,v \in H$ then $\alpha u + \beta v \in H$ for any (finite) $\alpha, \beta \in \mathbb{C}$. Associated with each Hilbert space is its inner product $\{u,v\}_H \in \mathbb{C}$ for $u,v \in H$, where the subscript is dropped if there is no risk of confusion. The inner product is linear in its first argument and it satisfies the property $\{v,u\}_H = \{u,v\}_H^*$. In addition, it generates the norm of the Hilbert space, given by $\|u\|_H = \sqrt{\{u,u\}_H}$ for every $u \in H$, and by definition $u \in H$ implies that $\|u\|_H < \infty$. A Hilbert space is a complete inner product space, whereby the limit of any convergent (in the norm) sequence $\{u_n\}_n$, with $u_n \in H$, will also be in H. Thus for example a convergent infinite sum of functions from a Hilbert space H defines a function in that space. For notational clarity and to mimic the notation used in matrix analysis, we here employ the Hilbert space adjoint $u^\dagger$ of a function $u \in H$. This adjoint is defined by the relation

$$u^\dagger v = \{v,u\}_H, \quad \forall v \in H. \tag{3.5}$$

An operator A in a Hilbert space H_1 is a (linear) mapping $A : H_1 \to H_2$ that maps every function $u \in H_1$ to a function $v = Au \in H_2$. Here $H_2 = H_1$ is acceptable and often the case in practice. When $\|Au\|_{H_2}/\|u\|_{H_1} \le C < \infty$, $\forall u \in H_1$, the operator or mapping A is bounded, and its norm $\|A\|$ is the smallest real number C, which satisfies the inequality. With this norm the set of bounded linear operators on a Hilbert space H is complete, which means that the limit of any convergent (in the norm) sequence, such as a series, of such operators converges to a bounded linear operator. A particular type of operator of interest in this thesis is the outer product $A = vu^\dagger$, where $u \in H_1$ and $v \in H_2$ are any two functions. This operator, which is a rank-1 operator, maps the one-dimensional subspace $\{u\} \subset H_1$ into the Hilbert space H_2 according to the explicit representation $Aw = vu^\dagger w = \{w, u\}_{H_1} v$, where $w \in H_1$. As indicated above, any convergent combination of such operators is also a bounded linear operator. Hence, in particular, this holds for the mutual coherence operator Γ_u defined in Eq. (3.4), provided that the representation converges. Since the functions $u(t)$ in that definition belong to the Hilbert space H, it follows that $\Gamma_u(\tau) : H \to H$ and if we assume that the representation converges, we get when we operate with $\Gamma_u(\tau)$ on $v \in H$ the explicit result

$$\Gamma_u(\tau)v = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \{v, u(t+\tau)\} u(t) \mathrm{d}t, \tag{3.6}$$

where we have dropped the subscript H for convenience. This result illustrates the role of $\Gamma_u(\tau)$ as a Hilbert space operator. We now return to the main subject of coherence theory.

When the random process u in Eq. (3.4) is not only wide-sense stationary, but also ergodic [59], that is, when the fluctuations of the process are such that their correlations persist only for finite time differences, the values the process attains at sufficiently separated times essentially represent different realizations of one and the same random process. Hence the time-average for such a process is well approximated by the ensemble average, so that the mutual coherence operator (function) (3.4) can be expressed as

$$\Gamma_u(\tau) = \left\langle u(t) u^\dagger(t+\tau) \right\rangle = \left\langle u(0) u^\dagger(\tau) \right\rangle, \tag{3.7}$$

where the angle brackets denote ensemble averaging and the latter expression follows since t is arbitrary. Here we will assume that all electromagnetic fields and sources of interest are at least wide-sense stationary and ergodic, so that their mutual coherence operators are of the form (3.7).

The shift from temporal averaging to ensemble averaging over a set of realizations of the process makes it sensible to consider the correlation properties of the angular frequency representations of the realizations. However, since stationary fields are ever present, their energy is infinite and as a consequence their Fourier transforms are only formal and must be interpreted in terms of distributions. Thereby, we can formally use the inverse of the representation (2.9) and the definition (3.7) to compute

$$\begin{aligned}\left\langle u(\omega_1)u^\dagger(\omega_2)\right\rangle &= \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}\left\langle u(t_1)u^\dagger(t_2)\right\rangle \exp[i(\omega_1 t_1 - \omega_2 t_2)]\,\mathrm{d}t_1\mathrm{d}t_2 \\ &= \delta(\omega_2-\omega_1)W_u(\omega_1),\end{aligned} \tag{3.8}$$

where the operator

$$W_u(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma_u(\tau)\exp(-i\omega\tau)\mathrm{d}\tau \tag{3.9}$$

is the cross-spectral density operator of the process u. Equation (3.9) is a variant of the Wiener–Khintchine theorem [16, 59]. We note that the Wiener–Khintchine relation (3.9) can be used to *define* the cross-spectral density operator, as we do here, but then the direct connection to the complex analytic signal representation (2.9) of the process is lost. We show later that this connection can be re-established.

It is straightforward, but somewhat tedious, to prove that the cross-spectral density operator as given in Eq. (3.9) is a non-negative, self-adjoint, Hilbert–Schmidt operator [16]. In particular, it is a compact operator, which can be represented in terms of its eigenvalues $\lambda_n(\omega)$ and eigenfunctions $\phi_n(\omega)$ (in the Hilbert space H) by its so-called Mercer series (which is a consequence of the Hilbert–Schmidt theorem) [16, 60, 61] as

$$W_u(\omega) = \sum_{n=1}^{\infty}\lambda_n(\omega)\phi_n(\omega)\phi_n^\dagger(\omega), \tag{3.10}$$

where we assume that the eigenvalues are numbered in decreasing order so that $\lambda_1(\omega) \geq \lambda_2(\omega) \geq \cdots \geq 0$, and we take the set $\{\phi_n(\omega)\}_{n=1}^{\infty}$ to be a complete orthonormal basis for H. We thus include in this set the orthonormal basis of the (possibly trivial) null-space of $W_u(\omega)$. For those functions $\phi_n(\omega)$ not in the null-space, we have from Eq. (3.10) the representation

$$\phi_n(\omega) = \frac{1}{\lambda_n(\omega)}W_u(\omega)\phi_n(\omega), \tag{3.11}$$

where $W_u(\omega)\phi_n(\omega)$ denotes the function, which is obtained when $W_u(\omega)$ operates on the function $\phi_n(\omega)$. As we demonstrate below, the Mercer series can be used to prove that the cross-spectral density operator may, in fact, be expressed as an ensemble expectation of the form

$$W_u(\omega) = \left\langle v(\omega)v^\dagger(\omega)\right\rangle, \tag{3.12}$$

for some ensemble of harmonic processes $v(\omega) \in H$, with $\langle\|v(\omega)\|^2\rangle < \infty$. Indeed, the Mercer series can be interpreted as such a representation. However, it is important to note that the realizations $u(\omega)$, which follow from $u(t)$ via the inverse of the Fourier representation (2.9), cannot be used here, since the assumption of stationarity implies that the functions $u(\omega)$ do not exist as ordinary functions (processes). With this restriction in mind we will, however, from now on, as is customary, denote the realizations in Eq. (3.12) by $u(\omega)$ to uphold notational consistency.

Let us now consider random scalar or electromagnetic fields in a setting where the material parameters are non-stochastic. Then every scalar or electromagnetic field u in such a system satisfies an equation of the form given for example by Eq. (2.28) for scalar fields, and by Eq. (2.21) or Eq. (2.23) for vector-valued fields, where the operator (denoted here by $\mathscr{L}_\omega$) operating on the field on the left-hand side is deterministic. If we use $q(\omega)$ to collectively denote the source terms on the right-hand sides of these equations, we can write the equations in the unified form

$$\mathscr{L}_\omega u(\omega) = -4\pi q(\omega). \tag{3.13}$$

It can be shown, for example by an excursion to temporal representations, that the eigenfunctions of the cross-spectral density operator $W_u(\omega)$ then satisfy a differential equation of the same form, but with an eigenfunction-dependent source term $\sigma_n(\omega)$, viz.,

$$\mathscr{L}_\omega \phi_n(\omega) = \frac{1}{\lambda_n(\omega)} \mathscr{L}_\omega W_u(\omega)\phi_n(\omega) = -4\pi\sigma_n(\omega), \tag{3.14}$$

where the supports of the source terms $\sigma_n(\omega)$ lie in the union of the supports of the source term realizations $q(\omega)$. Hence the eigenfunctions $\phi_n(\omega)$ are monochromatic fields at angular frequency ω and the Mercer series (3.10) then implies that the cross-spectral density operator of angular frequency ω, as given by the Fourier transform (3.9) is, in fact, expressible in terms of fields at the same frequency. Thus for such an operator the random fields $v(\omega)$ in the representation (3.12) can be taken as monochromatic fields of the systems in which the operator is considered. Usually this conclusion is derived only for fields in vacuum [16, 62], but we need the more general representation established here in what follows.

Let us now consider a stationary random source distribution $q(\omega)$ and the associated stationary random field $u(\omega)$, which satisfy a differential equation of the form (3.13). By reversing that equation, we can develop the source cross-spectral density operator as

$$\begin{aligned} 16\pi^2 W_q(\omega) &= \left\langle [-4\pi q(\omega)][-4\pi q(\omega)]^\dagger \right\rangle = \left\langle [\mathscr{L}_\omega u(\omega)][\mathscr{L}_\omega u(\omega)]^\dagger \right\rangle \\ &= \mathscr{L}_\omega \mathscr{L}_\omega^\dagger \left\langle u(\omega) u^\dagger(\omega) \right\rangle = \mathscr{L}_\omega \mathscr{L}_\omega^\dagger W_u(\omega), \end{aligned} \tag{3.15}$$

which also defines the operator $\mathscr{L}_\omega^\dagger$. For scalar fields that are given by the differential equation (2.28) or electromagnetic fields that are given by Eq. (2.23), we have $\mathscr{L}_\omega = \nabla^2 + \kappa^2$ and Eq. (3.15) is explicitly given by

$$\left[\nabla_1^2 + \kappa^2(\mathbf{r}_1,\omega)\right]\left[\nabla_2^2 + \kappa^{*2}(\mathbf{r}_2,\omega)\right] W_u(\mathbf{r}_1,\mathbf{r}_2,\omega) = 16\pi^2 W_q(\mathbf{r}_1,\mathbf{r}_2,\omega), \tag{3.16}$$

where $W_u(\mathbf{r}_1,\mathbf{r}_2,\omega) = \langle u(\mathbf{r}_1,\omega)u^\dagger(\mathbf{r}_2,\omega)\rangle$ and $W_q(\mathbf{r}_1,\mathbf{r}_2,\omega) = \langle q(\mathbf{r}_1,\omega)q^\dagger(\mathbf{r}_2,\omega)\rangle$.

When the wave number is constant throughout space $[\kappa(\mathbf{r},\omega) = \kappa]$, the solution to the inhomogeneous double Helmholtz equation (3.16) can be represented in terms of the scalar Helmholtz equation Green's function given in Eq. (2.42) as

$$W_u(\mathbf{r}_1,\mathbf{r}_2,\omega) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G_\kappa(\mathbf{r}_1,\mathbf{r}_2')G_\kappa^*(\mathbf{r}_2,\mathbf{r}_2')W_q(\mathbf{r}_1',\mathbf{r}_2',\omega)\mathrm{d}\mathbf{r}_1'\mathrm{d}\mathbf{r}_2', \tag{3.17}$$

where $\mathbb{R}^3$ denotes all space and we have taken $\mathbb{R}^3 \times \mathbb{R}^3$ as the support of $W_q(\omega)$. This is the general representation of the solution to the double Helmholtz equation (3.16) when outgoing (Sommerfeld, Silver–Müller) boundary conditions are assumed.

3.2 Degree of coherence

Let us go back to Young's double pinhole interference experiment as illustrated in Fig. 3.1, where two pinholes are illuminated by a field and the resulting interference pattern is observed on a screen behind the pinholes. To begin with we assume that the field is scalar and quasi-monochromatic, that is, its spectrum is tightly concentrated around a central angular frequency ω_0. We also take the pinholes to be so small that the field behind each pinhole is approximately described by a complex random variable, say A_1 and A_2, for the two pinholes P_1 and P_2, respectively. Following the analysis by Zernike [57], we note that the (average) intensity of the field at each pinhole is proportional to

$$\mathscr{I}_k = \langle |A_k|^2 \rangle, \tag{3.18}$$

where $k \in \{1,2\}$. When the spectral bandwidth of the field is sufficiently small, the propagation of the field from the pinholes to the observation screen (S_2) will, apart from a here inessential common constant factor, essentially induce a deterministic phase-difference β, depending on the observation point P, between the contributions from each pinhole. Specifically, for this model to be valid, the inverse of the spectral bandwidth of the field, the coherence time, must exceed the differences in times of flight from the two pinholes to every observation point of interest. The field at a point P can (apart from a constant factor) then be written as $A = A_1\exp(i\beta) + A_2$, whereby the corresponding intensity $\mathscr{I}$ is given by

$$\mathscr{I} = \langle |A|^2 \rangle = \mathscr{I}_1 + \mathscr{I}_2 + 2\,\mathrm{Re}\left[\langle A_1 A_2^* \rangle \exp(i\beta)\right]. \tag{3.19}$$

When P is slightly shifted, so that to a good approximation only β changes, the maximum and minimum intensities observed are given by $\mathscr{I}_{\max} = \mathscr{I}_1 + \mathscr{I}_2 + 2\left|\langle A_1 A_2^* \rangle\right|$ and $\mathscr{I}_{\min} = \mathscr{I}_1 + \mathscr{I}_2 - 2\left|\langle A_1 A_2^* \rangle\right|$, respectively. We introduce these values into the Michelson formula (3.1) to obtain for the visibility of the interference pattern on S_2 the value

$$\mathscr{V} = \frac{2\left|\langle A_1 A_2^* \rangle\right|}{\mathscr{I}_1 + \mathscr{I}_2} = 2\left(\eta + \eta^{-1}\right)^{-1} \frac{\left|\langle A_1 A_2^* \rangle\right|}{\sqrt{\mathscr{I}_1 \mathscr{I}_2}}, \tag{3.20}$$

where $\eta = \sqrt{\mathscr{I}_1 / \mathscr{I}_2}$.

Before the development of the mathematical techniques for assessing partially coherent fields, it was of paramount importance to know, in the design and analysis of optical systems for example, whether the system can be treated as incoherent (intensities add) or as coherent (amplitudes add), or if its behavior lies between theses two extremes (partial coherence). An early problem in coherence theory was how to characterize a field (mathematically), so that this characterization matches the physical (appearance of interference fringes) notion of coherence. Zernike [57] solved this problem by connecting the definition of coherence to the concept of visibility as put forward by Michelson [29]. He defined the degree of coherence as follows:

> By definition *the "degree of coherence" of two light-vibrations shall be equal to the visibility of the interference fringes that may be obtained from them under the best of circumstances*, that is, when both intensities are made equal and only small path differences are introduced. Frits Zernike, 1938.

As the path differences in our example above are assumed small (only a phase difference introduced), it follows that the degree of coherence can be obtained from Eq. (3.20) by putting $\eta = 1$. However, following Zernike and the later developments in coherence theory, it is actually more useful to define the complex degree of coherence, which is given by

$$\gamma = \frac{\langle A_1 A_2^* \rangle}{\sqrt{\mathscr{I}_1 \mathscr{I}_2}}. \tag{3.21}$$

Extending this definition to any two points of a scalar field, we can thereby, in the case of a stationary field U, define the complex degree of coherence between the field values at two points $\mathbf{r}_1$ and $\mathbf{r}_2$, and separated by time τ as

$$\gamma_U(\mathbf{r}_1, \mathbf{r}_2, \tau) = \frac{\Gamma_U(\mathbf{r}_1, \mathbf{r}_2, \tau)}{\sqrt{\Gamma_U(\mathbf{r}_1, \mathbf{r}_1, 0)\Gamma_U(\mathbf{r}_2, \mathbf{r}_2, 0)}}, \tag{3.22}$$

where Γ_U is the mutual coherence function of U as given by Eq. (3.2). We note that γ_U is the complex correlation coefficient of the random variables describing the field at the two points. Therefore the purely mathematical concept of correlation and the physical notion of coherence and visibility are actually one and the same! Because of this, or directly from the definition (3.22), it follows that

$$0 \leq |\gamma_U(\mathbf{r}_1, \mathbf{r}_2, \tau)| \leq 1, \tag{3.23}$$

with 0 representing complete incoherence (no correlation) and 1 representing complete coherence (total correlation), respectively. Although our derivation and the direct connection of the complex degree of coherence to the fringe visibility in Young's interference experiment is valid only for quasi-monochromatic fields, the definition (3.22) makes sense and is commonly used for general scalar fields. In fact, it has recently been shown [63] that if an achromatic Fourier transform element is added to Young's setup, the quantity $\gamma_U(\mathbf{r}_1, \mathbf{r}_2, 0)$ can directly be measured for a field with an arbitrary spectral distribution. An achromatic delay (e.g., by a path through mirrors) in front of either pinhole can then be used to fully determine $\gamma_U(\mathbf{r}_1, \mathbf{r}_2, \tau)$, for all τ.

Entirely analogously to the definition (3.22) we can mathematically define the complex spectral degree of coherence μ_U for a stationary field U by

$$\mu_U(\mathbf{r}_1, \mathbf{r}_2, \omega) = \frac{W_U(\mathbf{r}_1, \mathbf{r}_2, \omega)}{\sqrt{W_U(\mathbf{r}_1, \mathbf{r}_1, \omega) W_U(\mathbf{r}_2, \mathbf{r}_2, \omega)}}, \tag{3.24}$$

where W_U is the cross-spectral density operator of the field U. Since this definition is purely formal, it is valid for fields with any spectral distribution. On account of the representation (3.12), μ_U is, like γ_U, a complex correlation coefficient with the property that

$$0 \leq |\mu_U(\mathbf{r}_1, \mathbf{r}_2, \omega)| \leq 1. \tag{3.25}$$

Here again, 0 denotes complete incoherence and 1 denotes complete coherence. We note that even though the definition (3.24) was here introduced only formally, Wolf [64] has shown that when the degree of coherence γ_U is measured for a field (light) that has passed through a narrowband filter around the angular frequency ω_0, whereby it is quasi-monochromatic, it follows that

$$\mu_U(\mathbf{r}_1, \mathbf{r}_2, \omega_0) = \gamma_U(\mathbf{r}_1, \mathbf{r}_2, 0). \tag{3.26}$$

Hence the spectral degree of coherence has a direct connection to measurable quantities of the field and it can be as easily determined for a given field as the (temporal) degree of coherence. In the following we will only be interested in the spectral degree of coherence, so that in accordance with common practice and with no fear of misinterpretations we will henceforth use the term (complex) degree of coherence to mean the spectral degree of coherence μ_U.

Whereas the degree-of-coherence function (μ_U) has a straightforward definition for scalar fields, which connects it nicely to both visibility and correlation, the situation is much more complex for vector-valued electromagnetic fields [65]. However, such an extension of the degree-of-coherence function to vectorial fields is important since all electromagnetic (light) fields are vector-valued. The problems encountered in extending the scalar concept of the degree of coherence to the vectorial case are basically all due to the fact that at each spatial point the electromagnetic field is represented by three (independent) scalar fields. The question is how the degree of coherence should be defined between two three-component functions and if the correlations between the functions at each point should be accounted for or not. From the work done on the subject and the number of alternative definitions put forward [66–73], it seems that it may be impossible to retain all salient features of the scalar degree-of-coherence function in a vectorial extension. Thereby it is mostly application dependent, which of the different functional forms given by the suggested degree-of-coherence functions best describes any particular situation. The most central definitions will be presented below. Here we will only consider the spectral degree-of-coherence functions even though all have a temporal counterpart, which furthermore is the one that has usually been defined first. In particular, in some of the references below, the temporal definition is exclusively considered, but since the spectral definition follows readily, it is not useful to distinguish between the two on this (definition, basic properties) level.

Historically the first function to be called the vector degree of coherence was

$$\mu_{\mathrm{KW}}(\mathbf{r}_1,\mathbf{r}_2,\omega) = \frac{\mathrm{Tr}[\overline{\mathbf{W}}_{\mathbf{E}}(\mathbf{r}_1,\mathbf{r}_2,\omega)]}{\left\{\mathrm{Tr}[\overline{\mathbf{W}}_{\mathbf{E}}(\mathbf{r}_1,\mathbf{r}_1,\omega)]\right\}^{1/2}\left\{\mathrm{Tr}[\overline{\mathbf{W}}_{\mathbf{E}}(\mathbf{r}_2,\mathbf{r}_2,\omega)]\right\}^{1/2}}, \tag{3.27}$$

where $\overline{\mathbf{W}}_{\mathbf{E}}(\mathbf{r}_1,\mathbf{r}_2,\omega) = \langle \mathbf{E}(\mathbf{r}_1,\omega)\mathbf{E}^{\dagger}(\mathbf{r}_2,\omega)\rangle$. The degree-of-coherence function μ_{KW} was originally introduced in the time-domain in 1963 by Karczewski [66, 67] and later re-introduced in the frequency-domain in 2003 by Wolf [68]. The definition (3.27) of the degree of coherence follows, when the ideas used to obtain the scalar degree of coherence in the context of Young's interference experiment are applied in a straightforward manner to electromagnetic fields.

The appeal of the degree-of-coherence function μ_{KW} is that its modulus, $|\mu_{\mathrm{KW}}|$, equals the fringe visibility in Young's experiment for full electromagnetic fields with the same intensity at both pinholes, and thereby closely mimics the scalar degree-of-coherence function. The function μ_{KW} does not, however, account for the polarization (vector) properties of the field, and hence it predicts for example zero coherence between orthogonal vector components even when they are proportional to the same scalar signal [74], and that completely coherent fields can be fully unpolarized [68]. Furthermore, the function μ_{KW} is not invariant to local rotations of the coordinate system, which occur for instance when curvilinear coordinates are used [69]. To remedy this shortcoming, Tervo et al. [69] introduced the function μ_{STF} defined by

$$\mu_{\mathrm{STF}}(\mathbf{r}_1,\mathbf{r}_2,\omega) = \frac{\|\overline{\mathbf{W}}_{\mathbf{E}}(\mathbf{r}_1,\mathbf{r}_2,\omega)\|_F}{\{\mathrm{Tr}[\overline{\mathbf{W}}_{\mathbf{E}}(\mathbf{r}_1,\mathbf{r}_1,\omega)]\}^{1/2}\{\mathrm{Tr}[\overline{\mathbf{W}}_{\mathbf{E}}(\mathbf{r}_2,\mathbf{r}_2,\omega)]\}^{1/2}}, \tag{3.28}$$

where $\|\cdot\|_F$ is the matrix Frobenius norm [75]. This definition has also many other invariance properties that could be expected of a degree-of-coherence function [62,69,76–78], including, in contrast to μ_{KW}, that it is compatible with the notion of coherent modes (see Sec. 3.3).

The quote by Zernike cited earlier can be interpreted (if only the emphasized part is considered, as is appropriate for vector-valued fields) so that the degree of coherence is given by the maximum visibility that is obtained when the field values at the two pinholes are altered (attenuated, amplified, delayed, polarization rotated, etc.) freely and independently. Considerations of this nature are the basis on which Refrégiér and Goudail [70], and Gori et al. [71] have based their degree-of-coherence functions

$$\mu_{\mathrm{RG}}(\mathbf{r}_1,\mathbf{r}_2,\omega) = \|[\overline{\mathbf{W}}_{\mathbf{E}}(\mathbf{r}_1,\mathbf{r}_1,\omega)]^{-1/2}\overline{\mathbf{W}}_{\mathbf{E}}(\mathbf{r}_1,\mathbf{r}_2,\omega)[\overline{\mathbf{W}}_{\mathbf{E}}(\mathbf{r}_2,\mathbf{r}_2,\omega)]^{-1/2}\|_2 \tag{3.29}$$

and

$$\mu_{\mathrm{G}}(\mathbf{r}_1,\mathbf{r}_2,\omega) = \frac{\|\overline{\mathbf{W}}_{\mathbf{E}}(\mathbf{r}_1,\mathbf{r}_2,\omega)\|_T}{\{\mathrm{Tr}[\overline{\mathbf{W}}_{\mathbf{E}}(\mathbf{r}_1,\mathbf{r}_1,\omega)]\}^{1/2}\{\mathrm{Tr}[\overline{\mathbf{W}}_{\mathbf{E}}(\mathbf{r}_2,\mathbf{r}_2,\omega)]\}^{1/2}}, \tag{3.30}$$

respectively. Here $\|\cdot\|_2$ is the matrix spectral norm and $\|\cdot\|_T$ denotes the matrix trace norm [75]. It can be shown [70] that the function μ_{RG} corresponds to the maximum value that the fringe visibility ($|\mu_{\mathrm{KW}}|$) can attain for a specific pair of points of an electromagnetic field when the field vectors are adjusted in accordance with what is listed above. On the other hand, the function μ_{G} is obtained when non-uniform attenuations or amplifications of the polarization components at either location are excluded from the considerations. This restriction was introduced by the authors because such mappings correspond to irreversible changes of the field [71]. The two degree-of-coherence functions μ_{RG} and μ_{G} do not, however, single out coherent modes as would be desirable for a degree-of-coherence function. This point is discussed in the next section.

Luis [72] has introduced the degree-of-coherence function

$$\begin{aligned}\mu_{\mathrm{L}}(\mathbf{r}_1,\mathbf{r}_2,\omega) = \Bigg\{-\frac{1}{3}+\frac{4}{3}\Big\{&\mathrm{Tr}[\overline{\mathbf{W}}_{\mathbf{E}}(\mathbf{r}_1,\mathbf{r}_1,\omega)]+\mathrm{Tr}[\overline{\mathbf{W}}_{\mathbf{E}}(\mathbf{r}_2,\mathbf{r}_2,\omega)]\Big\}^{-2}\\ &\times\Big\{\mathrm{Tr}[\overline{\mathbf{W}}^2_{\mathbf{E}}(\mathbf{r}_1,\mathbf{r}_1,\omega)]+\mathrm{Tr}[\overline{\mathbf{W}}^2_{\mathbf{E}}(\mathbf{r}_2,\mathbf{r}_2,\omega)]\\ &+2\,\mathrm{Tr}[\overline{\mathbf{W}}^{\dagger}_{\mathbf{E}}(\mathbf{r}_1,\mathbf{r}_2,\omega)\overline{\mathbf{W}}_{\mathbf{E}}(\mathbf{r}_1,\mathbf{r}_2,\omega)]\Big\}\Bigg\}^{1/2},\end{aligned} \tag{3.31}$$

which applies as such to two-dimensional vector fields but can equally well be defined for three- or higher-dimensional vector fields (the numerical constants in the expression depend on the dimensionality of the vectors). This function is essentially a measure of how far from the unit matrix (corresponding to full incoherence) the 4×4 matrix, consisting of the pairwise correlations of all four components of the field at the two points, lies. Hence μ_{L} is in spirit closely related to the degree-of-polarization functions of two- and three-component fields [79]. The degree-of-polarization function P_3 of three-component fields occuring in this context was (re-)introduced into optics by Setälä et al. [80, 81], and it is closely linked to the degree-of-coherence function μ_{STF} evaluated at a single point, as is evidenced by the relation [69]

$$\mu_{\mathrm{STF}}(\mathbf{r},\mathbf{r},\omega) = \sqrt{\frac{2P_3(\mathbf{r},\omega)+1}{3}}. \tag{3.32}$$

The function P_3 attains the value 0 for fully unpolarized fields, and the value 1 for fully polarized fields, respectively. As with degree-of-coherence functions of electromagnetic fields, there is a debate about the correct way to extend the well-established concept of degree of polarization of two-component fields to three-component fields, and the function P_3 is just one of the suggested quantities, which include the quantities presented by Setälä et al. [80, 81], Ellis et al. [82], Luis [79], Réfrégier [83], and Dennis [84]. Overviews of this subject, which still is in constant development [85–87], can be obtained for example from Refs. 88 and 89.

We observe that the function μ_{L} is also obtained when the so-called effective degree of coherence is applied to the double pinhole system in Young's experiment. The effective degree of coherence is considered in depth in Chap. 6, where we also look at some specific properties of the function μ_{L}.

Finally, we note that the characters of the degree-of-coherence functions are not altered if they are mapped by (monotonic) functions. Specifically, although for example scalings and shifts, as found in the definition (3.31), may be used, e.g., to make the degree of coherence of coherent fields 1 and that of incoherent fields 0, such a mapping has no deep significance beyond cosmetics. Therefore it is not useful to treat degree-of-coherence functions, that can be mapped to each other, as distinct.

3.3 Coherent-mode representation of cross-spectral density operators

Let us consider the random process or field (scalar or vector-valued) u in a Hilbert space H. Furthermore, let us assume that this is a zero-mean process§, i.e., $\langle u \rangle = 0$. This assumption does not restrict the generality of our exposition, since if u is not a zero-mean process, we can consider $u - \langle u \rangle$ instead, and add $\langle u \rangle$ and $\langle u \rangle \langle u^\dagger \rangle$ to the results where appropriate. With the zero-mean assumption, the cross-spectral density operator W_u is the autocovariance function of u. It then follows that u has, in terms of the eigenfunctions ϕ_n of W_u as defined by (3.11), a so-called Karhunen–Lòeve expansion [16, 60]

$$u = \sum_{n=1}^{\infty} \tilde{u}_n \phi_n, \tag{3.33}$$

where the stochastic properties of u are contained in the pairwise uncorrelated random variables $\tilde{u}_n$ and its Hilbert-space properties are represented by the deterministic orthonormal basis functions ϕ_n [60].

The non-stochasticity of the basis functions ϕ_n and the uncorrelatedness of the coefficients $\tilde{u}_n$ mean that the terms in the Mercer series (3.10) represent mutually uncorrelated random processes or fields. This is manifested by the fact that the corresponding operators are rank-1 operators, that is, eigenvalue-scaled outer products of the eigenfunctions, $\lambda_n \phi_n \phi_n^\dagger$. Now since correlation and coherence are the same thing for scalar fields, it is not surprising that the scalar degree-of-coherence function (3.24) is unimodular for rank-1 operators (operators that factor), which thus correspond to fully coherent fields. Accordingly the Mercer series (3.10) is also called the *coherent-mode expansion* of the cross-spectral density operator W_u.

The situation is unfortunately not as clear for the vectorial degree-of-coherence functions considered in the last section. Again, the reason is related to the question of how correlation between multicomponent signals should be represented in terms of one scalar quantity. However, since the interpretation of the Mercer series as an expansion in coherent modes is both theoretically and practically extremely useful, it follows that we would like rank-1 operators (factored or outer product operators) and only those, to represent coherent fields also in the vectorial case. This litmus-test of degree-of-coherence functions is passed by the degree-of-coherence functions μ_{STF} [76] and μ_{L} (this follows from a similar property of the effective degree of coherence, which is considered in

§All optical fields are zero-mean, since they do not include constant electric or magnetic fields.

Chap. 6). It is, however, not passed by the degree-of-coherence functions μ_{KW}, μ_{RG}, or μ_{G}. The squared magnitude of the first of these can take on any value between 0 and 1 for a rank-1 operator, when the function is evaluated for different pairs of points $(\mathbf{r}_1, \mathbf{r}_2)$. For the two latter degree-of-coherence functions any rank-2 operator, where the two eigenfunctions are pointwise orthonormal throughout the domain of interest, serves as a counterexample since both functions are unimodular for such fields.

With the eigenvalues λ_n in the Mercer series (3.10) being numbered in decreasing order, a truncation of the Karhunen–Lòeve expansion (3.33) to its first N terms yields the best possible N-term representation of u with respect to the mean-square error [60]. In fact, the truncation error is given by the sum $\sum_{n=N+1}^{\infty} \lambda_n$, which suggests that the eigenvalues in the Mercer series (3.10) correspond to the weight or importance of the corresponding eigenfunctions ϕ_n in representing the field or process u. Thus the complexity of the field is described by the number of these eigenfunctions needed for a reasonably accurate representation of the field. It is then of interest to determine the Mercer series or coherent-mode expansions of model fields, to gauge of what complexity these fields actually are. Without resorting to numerical estimates for the eigenvalues, this can, however, only be done when the Mercer expansion is known in closed form. Such is the case for the universal forms of the cross-spectral density operator as considered in the following.

As will be shown in Chap. 5, stochastically homogeneous and isotropic source distributions will in quite general circumstances give rise to a universal form of the field cross-spectral density operator. In the vectorial case this form is also found for fields inside blackbody cavities. The universal form of the cross-spectral density function (apart from irrelevant scaling factors) is given for a scalar field U and an electromagnetic electric field $\mathbf{E}$ by

$$W_U(\mathbf{r}_1, \mathbf{r}_2, \omega) = \mathrm{Im}\{G_\kappa(\mathbf{r}_1, \mathbf{r}_2)\} \tag{3.34}$$

and

$$\overline{\mathbf{W}}_{\mathbf{E}}(\mathbf{r}_1, \mathbf{r}_2, \omega) = \mathrm{Im}\left\{\overline{\mathbf{G}}_\kappa(\mathbf{r}_1, \mathbf{r}_2)\right\}, \tag{3.35}$$

respectively (see Chap. 5). Here G_κ and $\overline{\mathbf{G}}_\kappa$ are the free-space Green's function and the free-space dyadic Green's function corresponding to an unbounded lossless region where the wave number is κ, as given by Eqs. (2.42) and (2.47), respectively. In Publication I, we show that the coherent-mode expansions of the universal forms (3.34) and (3.35) can be written for a ball of radius R centered at $\mathbf{0}$, $B(\mathbf{0}, R)$, as

$$W_U(\mathbf{r}_1, \mathbf{r}_2, \omega) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \kappa c_{n,m} \psi_{n,m}(\mathbf{r}_1, \kappa) \psi_{n,m}^*(\mathbf{r}_2, \kappa) \tag{3.36}$$

and

$$\overline{\mathbf{W}}_{\mathbf{E}}(\mathbf{r}_1,\mathbf{r}_2,\omega) = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} \kappa c_{n,m}\mathbf{M}_{n,m}(\mathbf{r}_1,\kappa)\mathbf{M}^{\dagger}_{n,m}(\mathbf{r}_2,\kappa) + \sum_{n=1}^{\infty}\sum_{m=-n}^{n} \kappa d_{n,m}\mathbf{N}_{n,m}(\mathbf{r}_1,\kappa)\mathbf{N}^{\dagger}_{n,m}(\mathbf{r}_2,\kappa), \tag{3.37}$$

where the functions $\psi_{n,m}$, and the vector wave functions $\mathbf{M}_{n,m}$ and $\mathbf{N}_{n,m}$ are given by

$$\psi_{n,m}(\mathbf{r},\kappa) = c_{n,m}^{-1/2} j_n(\kappa r) Y_n^m(\hat{\mathbf{r}}) \tag{3.38}$$

and

$$\mathbf{M}_{n,m}(\mathbf{r},\kappa) = \frac{1}{\sqrt{n(n+1)}}\nabla\times\left[\psi_{n,m}(\mathbf{r},\kappa)\mathbf{r}\right], \tag{3.39}$$

$$\mathbf{N}_{n,m}(\mathbf{r},\kappa) = \frac{1}{\sqrt{n(n+1)}}\kappa^{-1}\nabla\times\left\{\nabla\times\left[\psi_{n,m}(\mathbf{r},\kappa)\mathbf{r}\right]\right\}, \tag{3.40}$$

respectively. In Eq. (3.38) j_n denote the spherical Bessel functions of order n, and Y_n^m are the spherical harmonics of order n and index m [90]. The eigenvalues κc_n and κd_n can be determined from

$$c_{n,m} = \int_0^R r^2 [j_n(\kappa r)]^2\,\mathrm{d}r = \frac{R^3}{2}\left\{[j_n(\kappa R)]^2 - j_{n-1}(\kappa R)j_{n+1}(\kappa R)\right] \tag{3.41}$$

and

$$d_{n,m} = \frac{n+1}{2n+1}c_{n-1,m} + \frac{n}{2n+1}c_{n+1,m}. \tag{3.42}$$

We note that the index m does not affect the eigenvalues $c_{n,m}$ or the eigenvalues $d_{n,m}$, which thus both are $(2n+1)$ times degenerate.

To study how the number of eigenvalues needed for a good approximation depends on the radius R of the ball, we take $\kappa = k_0$ and construct from the eigenvalues $k_0 c_{n,m}$ the sequence $C_l = \sum_{m=-n_l}^{n_l} k_0 c_{n_l,m}$, where the index subsequence n_l is so chosen that $C_l \geq C_{l+1}$, for $l = 0,1,\ldots$. In this sequence a degenerate eigenvalue is represented by one value C_l, which is equal to the degenerate eigenvalue times its degeneracy, and these values C_l are arranged in decreasing order. The idea here is to capture the importance of a degenerate eigenvalue to the expansion (3.36), when an all-or-none strategy is used for retaining the corresponding eigenfunctions. We use the index subsequence n_l to approximate the representation (3.36) by the truncated sum

$$W_U(\mathbf{r}_1,\mathbf{r}_2,\omega) \approx \sum_{l=0}^{L-1}\sum_{m=-n_l}^{n_l} \kappa k_0 c_{n_l,m}\psi_{n_l,m}(\mathbf{r}_1,\kappa)\psi^{*}_{n_l,m}(\mathbf{r}_2,\kappa), \tag{3.43}$$

where the definition of C_l yields for the relative truncation error the expression

$$\eta(L,R) = \frac{\sum_{l=L}^{\infty} C_l}{\sum_{l=0}^{\infty} C_l}. \tag{3.44}$$

Since $C_l \geq 0$, this expression is monotonically decreasing with L and hence the (inverse) function $L(\eta,R)$ exists and is well defined. This function returns, for a given R, the number of coefficients C_l needed to achieve a truncation error that does not exceed η.

In Fig. 3.2 we have plotted $L(\eta,R)$ for $\eta \in [0.01, 1.00]$ and $R/\lambda \in [0, 640]$. We observe that when η decreases, L at first grows rapidly, but then settles into near linear growth, before abruptly stopping at the required value. This cut-off -like

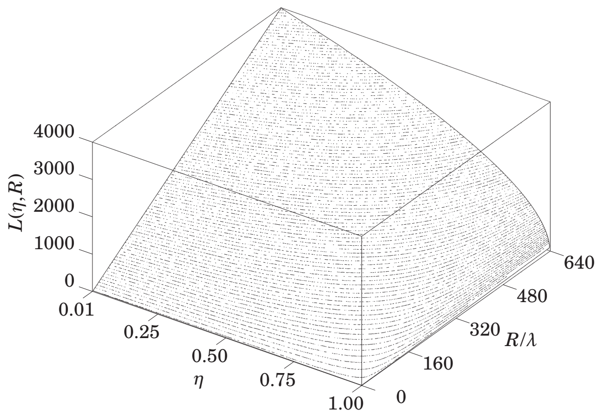


Figure 3.2. Number (L) of groups of degenerate eigenvalues needed for the representation of the scalar universal cross-spectral density operator with relative error η in a ball of radius R.

behavior stems from the fact that the eigenvalues $c_{n,m}$ as defined in Eq. (3.41), decrease exponentially (to 0) once a threshold value of n, which depends on R, is exceeded. On the other hand, we also note that L grows (almost) linearly with the radius R, in fact $L \approx k_0 R$ for small η. Thereby, since $n_l \approx l$ for large l, the degeneracies of the eigenvalues $k_0 c_{n,m}$ imply that on the order of $(k_0 R)^2$ terms are required in the truncated sum (3.43) for a fixed relative error to be attained. The number of eigenfunctions needed for the representation is consequently proportional to the surface area of the spherical shell of the ball. This result is no surprise, since the eigenfunctions $\psi_{n,m}$ are orthogonal over any spherical surface, so that the radial coordinate does not add to the complexity of the representation.

We note that since the values $d_{n,m}$ as given by Eq. (3.42) approximately interlace the values $c_{n,m}$, the number of terms required to represent the vectorial cross-spectral density operator $\overline{\mathbf{W}}_{\mathbf{E}}(\mathbf{r}_1, \mathbf{r}_2, \omega)$ of Eq. (3.37) is about twice as large as for the scalar case, reflecting the fact that an electromagnetic free field is fully described by two scalar fields, with the three vector components being coupled by the divergence condition (2.16).

In Sec. 5.5 it is shown that the universal form of the cross-spectral density operator, Eq. (3.36) or Eq. (3.37), also corresponds to an incoherent collection of scalar or vectorial plane waves. The discussion above thus suggests that in vacuum ($\kappa = k_0$) such an ensemble requires about $(k_0R)^2$ terms in a Karhunen–Lòeve expansion of the form (3.33) for scalar fields and twice that number for vector fields. Furthermore, as was noted earlier, no other set of basis functions can yield a smaller mean-square error with fewer terms in the corresponding representation. Hence, we may conclude that the random behaviors of the scalar and electromagnetic fields in these cases have intrinsic complexities that cannot be represented with less than about $(k_0R)^2$ and $2(k_0R)^2$ uncorrelated, linearly independent terms, respectively.

3.4 Coherence length

The *coherence length* of a stochastic field is a quantity, which has no rigorous definition, but in practice the coherence length is taken to be the largest distance between two points in the field over which there is significant coherence or correlation. Since coherence is measured in terms of the degree-of-coherence functions considered in the previous section, it follows that the coherence length is deduced from these and can, in the vector case, be different for different choices of the degree-of-coherence function. In this thesis the coherence length of a vector-valued field is, however, exclusively determined from the degree-of-coherence function μ_{STF}. Below we specify the coherence length for typical functional forms of the degree-of-coherence functions and also consider examples of functional forms for which it is difficult to specify the coherence length in an unambiguous way.

In Fig. 3.3 we have displayed the functional form of the magnitudes of four degree-of-coherence functions that appear in this thesis. The functions are all presented in terms of the wavelength-normalized distance r/λ between the two observation points $\mathbf{r}_1$ and $\mathbf{r}_2$, and thus they correspond to homogeneous and isotropic fields (see Secs. 5.3 and 5.4), for which the degree-of-coherence functions are of the simple form $\mu(\mathbf{r}_1, \mathbf{r}_2, \omega) = \mu(r, \omega)$. Although the scalar degree-of-coherence function always reaches the maximum value 1 at $r = 0\lambda$, the degree-of-coherence functions of vector fields may not attain the value 1 at all. Indeed, for all the functions considered here the maximum occurs at $r = 0\lambda$ where, according to Eq. (3.32), the degree-of-coherence function μ_{STF} of an unpolarized field has the value $1/\sqrt{3}$. Therefore, to simplify comparisons of different functional forms, all of the functions have been normalized with respect to the maximum value of their modulus, $\mu_{\max}$. The functions used in the plots, together with their maximum values, are listed in Tab. 3.1, where we have also indicated

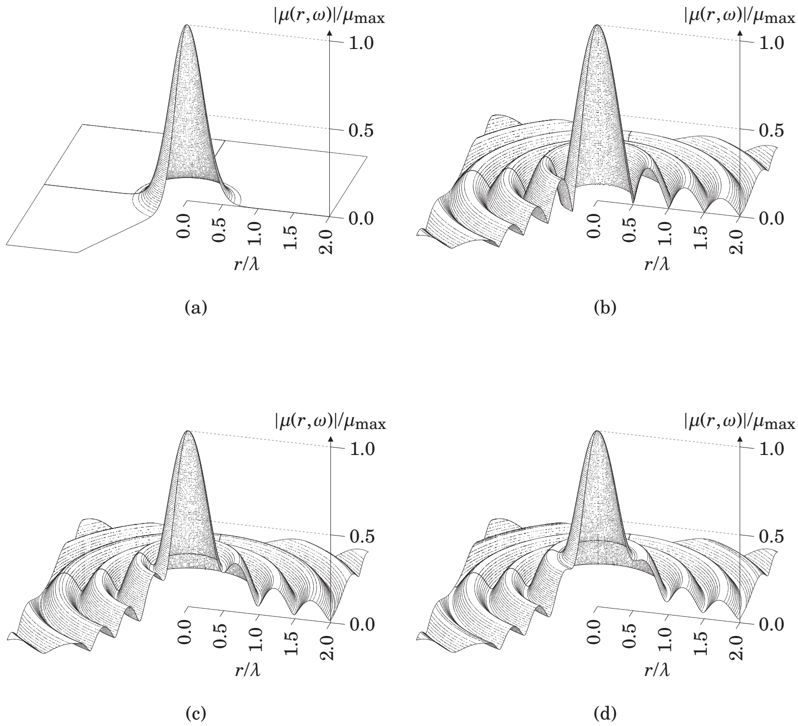


Figure 3.3. Examples of degree-of-coherence functions encountered in second-order coherence theory: (a) Gaussian form, (b) scalar sinc-form (universal form), (c) vector universal form, and (d) vector form in blackbody aperture.

	$\lvert\mu(r,\omega)\rvert$	$\mu_{\max}$	Same func. form in Eqs.
(a)	$\left\lvert \exp\left[-\frac{(k_0 r)^2}{4}\right]\right\rvert$	1	(5.46)[§]
(b)	$\lvert \mathrm{sinc}(k_0 r)\rvert$	1	(5.8), (5.9), (5.32), (5.36)
(c)	$\sqrt{\frac{1}{3}j_0^2(k_0 r)+\frac{1}{6}j_2^2(k_0 r)}$	$\frac{1}{\sqrt{3}}$	(5.37)
(d)	$\sqrt{\frac{1}{3}j_0^2(k_0 r)+\frac{1}{6}j_2^2(k_0 r)+\frac{1}{2}\left[\frac{J_2(k_0 r)}{k_0 r}\right]^2}$	$\frac{1}{\sqrt{3}}$	(4.7)

[§]In this case the degree-of-coherence function is also of the form (5.46).

Table 3.1. Functional forms of degree-of-coherence functions considered in this thesis, with corresponding equation numbers and maximum values.

which equations in later sections are of each particular form.

With respect to the plots in Fig. 3.3, it is clear that coherence is approximately limited to the neighborhood of the central peak in each case, and that these peaks are almost all of equal width. For the functions in Figs. 3.3 (b)–(d) this correspondence follows naturally from their functional forms, whereas for the Gaussian function plotted in Fig. 3.3 (a) we have on purpose chosen the value $2/k_0^2$ for its variance. The same idea is behind the choices we make in Sec. 5.6, where we consider a cross-spectral density operator with a Gaussian degree-of-coherence function. Incidentally, the coherence length can be defined most easily in the Gaussian case. Indeed, we note from Fig. 3.3 (a) that there are practically no correlations present for $r > 0.5\lambda$. Specifically, in other applications, such as in laser beam diameter estimation (of approximately Gaussian beams) [4, 15], the cutoff height of a Gaussian is taken to be $1/e^2 \approx 0.135$, which here would translate into a coherence length $r = \lambda/\pi \approx 0.318\lambda$. However, as the following discussion suggests, it is useful to set the coherence length of this Gaussian to exactly $r = \lambda/2$.

Assigning a coherence length for the sinc-function displayed in Fig. 3.3 (b) is not as straightforward as for the Gaussian function, because of the presence of side lobes in the former. However, the convention for example in radiometric and coherence theoretic considerations of (planar) blackbody sources (see Ref. 91), where the degree-of-coherence function of the representative scalar field at the emitter surface has the same sinc-form, is to use $r = \lambda/2$ as the coherence length. We observe that $r = \lambda/2$ is the location of the first zero of the sinc-function, so this definition includes the central peak in full. Although the functions in Figs. 3.3 (c) and (d) do not vanish at or before $r = \lambda/2$, and the function in Fig. 3.3 (d) actually plateaus at this location, the relative heights of the functions in (c) and (d) outside their central peaks are not much larger than the relative height of the first sidelobe of the function in (b). Because all three functions are related to blackbody radiation (see Chap. 4), it is then useful to take the coherence length in all these cases to be exactly $r = \lambda/2$. The same coherence length was chosen for the Gaussian function, since its central peak has nearly the same width as the sinc-function central peak in Fig. 3.3 (b), and since it is practical to apply the commonly used blackbody coherence length as a yardstick when comparing coherence lengths.

To illustrate the ambiguous nature of the coherence length for general functional forms, we consider the functions plotted in Fig. 3.4. For these functions it is challenging to define the coherence length in a useful way, and the whole concept of a coherence length is questionable in these cases. The same kind of chal-

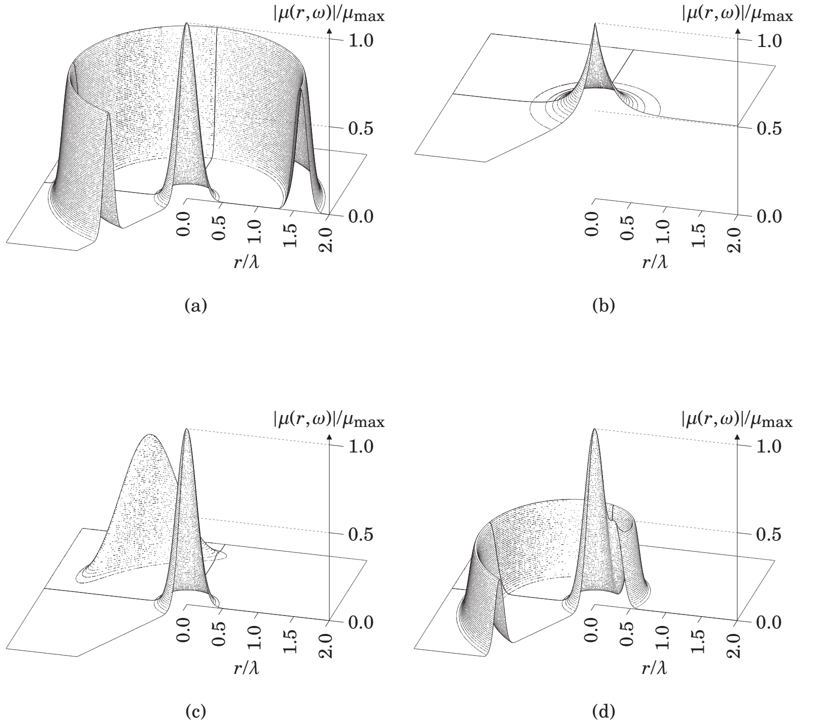


Figure 3.4. Examples of functional forms for which it is difficult to unambiguously define a coherence length in a useful way.

lenge is present for example when one tries to apply the concept of bandwidth to arbitrary signals or define the spectral range of multimode radiation [16,45]. It is because of this difficulty that we have in Publications IV and V (see also Chap. 5), where the coherence lengths of fields and sources play an important role, chosen to limit our considerations to fields and sources with either Gaussian or sinc-type degree-of-coherence functions, which, as discussed above, have well-defined or agreed-upon coherence lengths.

4. Blackbody radiation

Blackbody radiation is the radiation emitted by a perfectly absorbing body that is in thermal equilibrium with its surroundings. Although such blackbodies are only theoretical constructs, the radiation emanating from actual systems that are near thermal equilibrium can be modeled on blackbody radiation. Indeed, thermal (chaotic) radiation is *defined* by Mandel and Wolf [16] to be radiation which can be obtained from blackbody radiation by (spatial and/or temporal) linear filtering. In practice sources of thermal radiation are best described in thermodynamical terms, because their microscopic (chaotic) behavior is assumed to be known only stochastically. Such sources include most 'natural' sources of light. For example stars and incandescent lamps or other heated filaments are thermal emitters, and the detected cosmic background radiation is well described as blackbody radiation.

The importance and abundance of thermal emitters imply that a good understanding of blackbody radiation is essential to the analysis and modeling of radiation phenomena. It was the desire to explain the early measurements of the spectrum of light (and radiated heat) obtained from heated bodies that led Planck to formulate his celebrated expression for the spectral distribution of blackbody radiation, given by [4, 16, 17]

$$4a_0(\omega) = \frac{2\hbar\omega^3}{\pi c^3}\frac{1}{\exp(\hbar\omega/k_\mathrm{B}T)-1}, \tag{4.1}$$

where T is the absolute (equilibrium) temperature of the body, k_B is the Boltzmann constant, and $\hbar$ is the reduced Planck constant. Here the left-hand side is displayed in a form, which is consistent with the notation used in Publication II. Planck later managed to explain the expression (4.1) theoretically by invoking the then ad hoc assumption of quantized energy levels (see for example Ref. 17), an assumption that contributed to the birth of quantum mechanics.

In his derivation of the spectral distribution law (4.1), Planck studied the blackbody radiation inside a cavity with walls impermeable to radiation. It can be shown that as long as a cavity has this property and when it contains an arbitrarily sized (small) blackbody either inside the cavity or as part of the cavity wall, a blackbody electromagnetic field will eventually be induced within the cavity at thermal equilibrium, regardless of what other properties the cavity walls may have [17]. A small aperture introduced into the cavity wall is a good approximation of a blackbody source to the outside. Accordingly, a blackbody cavity with an aperture, as illustrated in Fig. 4.1, can be used as both a theoretical and an experimental analog for blackbody sources, as was originally suggested by Wien and Lummer [92].

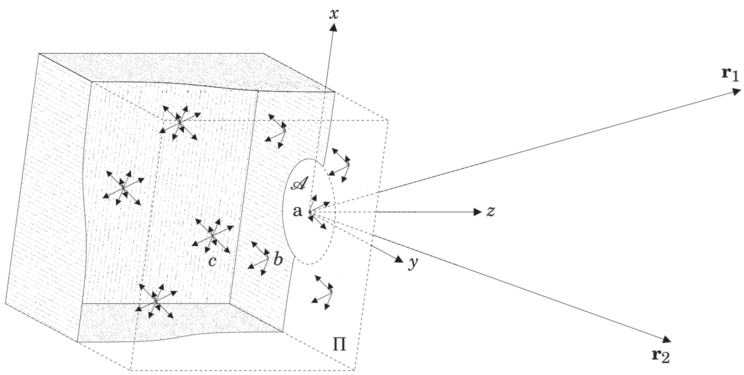


Figure 4.1. Illustrating the coordinate axes and observation-point vectors relating to a blackbody cavity with an aperture $\mathscr{A}$. The cavity is assumed to be asymptotically large with the aperture wall Π fixed in the plane $z = 0$. Also shown are the plane wave propagation directions away from points (a) in the aperture, (b) on the cavity boundary (wall) and (c) at interior regions of the cavity. Not shown are the complementary plane waves that propagate toward these points.

Although Planck in places applied rigorous electromagnetic theory in his analysis [17], Planck's description of the spatial behavior of the blackbody radiation field was mainly based on geometrical or ray optics considerations [3]. It was only later that a full electromagnetic treatment of blackbody radiation in closed (infinitely large) cavities was established [93–97]. The far-field patterns of blackbody apertures and other surface emitters have been extensively studied in radiometry [98], but there the treatment is in terms of scalar fields and for blackbody radiation, in particular, the planar source distribution is deduced from the far-field properties of the radiation rather than vice versa (see for example Refs. 99 and 100). It seems that the electromagnetic cross-spectral density of a blackbody field in the aperture was first determined by James [101]. He also showed that the radiation emanating from the aperture is unpolarized in every direction of the far field, and his results were extended in the paraxial case to the cross-spectral density of the far field by Lahiri and Wolf [102]. As is shown by us in Publication II, the influence of an opening in the cavity wall on the field inside the aperture is, however, treated erroneously in these papers. We also derive the correct expressions for the cross-spectral density in the aperture as well as in the far field. These results are presented in the following, where we also discuss how the cavity wall, in which the aperture is located, affects the field. Surprisingly, its contribution has so far been completely neglected (by us and others) in this context, but fortunately it turns out that although the wall changes the field near it, it does not influence the cross-spectral density of the field in the aperture or outside the cavity.

4.1 Blackbody radiation in and from aperture

We begin by considering a large (at least with respect to the wavelengths of interest) vacuum-filled cavity, which is at thermodynamic equilibrium and which does not radiate to the outside world. For an infinitely large cavity, the cross-spectral density operator of the blackbody field inside the cavity is given by [96, 97]

$$\begin{aligned}\overline{\mathbf{W}}_{\mathbf{E}}(\mathbf{r}_1,\mathbf{r}_2,\omega) &= 4\pi a_0(\omega)\left(\bar{\mathbf{I}}+\frac{1}{k_0^2}\nabla_1\nabla_1\right)\frac{\sin(k_0|\mathbf{r}_1-\mathbf{r}_2|)}{k_0|\mathbf{r}_1-\mathbf{r}_2|}\\ &= \frac{4\pi a_0(\omega)}{k_0}\,\mathrm{Im}\{\overline{\mathbf{G}}_{k_0}(\mathbf{r}_1,\mathbf{r}_2)\}.\end{aligned} \tag{4.2}$$

This cross-spectral density operator also describes a uniform distribution of uncorrelated, unpolarized plane waves [103, 104] and hence it can also be expressed in the form (see Publication II and Sec. 5.5)

$$\begin{aligned}&\overline{\mathbf{W}}_{\mathbf{E}}(\mathbf{r}_1,\mathbf{r}_2,\omega)\\ &= a_0(\omega)\int_\alpha\int_\alpha \delta(\hat{\mathbf{u}}_2-\hat{\mathbf{u}}_1)\left(\bar{\mathbf{I}}-\hat{\mathbf{u}}_1\hat{\mathbf{u}}_1\right)\exp[ik_0(\hat{\mathbf{u}}_1\cdot\mathbf{r}_1-\hat{\mathbf{u}}_2\cdot\mathbf{r}_2)]\,\mathrm{d}\hat{\mathbf{u}}_1\mathrm{d}\hat{\mathbf{u}}_2\\ &= a_0(\omega)\int_\alpha\left(\bar{\mathbf{I}}-\hat{\mathbf{u}}\hat{\mathbf{u}}\right)\exp[ik_0\hat{\mathbf{u}}\cdot(\mathbf{r}_1-\mathbf{r}_2)]\,\mathrm{d}\hat{\mathbf{u}},\end{aligned} \tag{4.3}$$

where α denotes the region of solid angles of interest, which here is the complete spherical shell $\mathbb{S}$. In line with the argumentation of Planck [17], we take the field to be homogeneous and isotropic everywhere inside the cavity and, in particular, arbitrarily close to the cavity walls. Hence we assume that Eqs. (4.2) and (4.3) can be used to represent the field just inside the cavity walls. As is shown in Sec. 4.2, this assumption is not actually true, but the necessary corrections do not affect the conclusions or the results presented below for the field inside the aperture and the field outside the cavity.

We now consider the particular geometry shown in Fig. 4.1, where the cavity resides in the half-space $z < 0$ with a part (Π) of its wall lying in the plane $z = 0$. This part of the wall furthermore has an aperture ($\mathscr{A}$), whose dimensions are much larger than the wavelengths of interest, but at the same time sufficiently small so that the blackbody field reaching the aperture from the inside of the cavity is not disturbed. Specifically, we assume that the field component propagating toward the wall is not altered when the aperture is introduced. On the other hand, since there is no wall at the aperture, there is no source for the field component propagating inward away from the aperture and thus the blackbody field at the aperture consists only of waves that (eventually) propagate into the half-space $z > 0$. This means that the cross-spectral density operator of the field at the aperture is given by Eq. (4.3), where the solid angle α is restricted to the hemisphere, which corresponds to plane waves that propagate into the half-space $z > 0$. In Publication II we have evaluated the integral in Eq. (4.3) at $z = 0$ for this specific case, obtaining for the cross-spectral density operator of the blackbody radiation inside the aperture the expression

$$\begin{aligned}&\overline{\overline{\mathbf{W}}}_{\mathbf{E}}^{(\mathscr{A})}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)\\&\quad=2\pi a_0(\omega)\left\{\left[j_0(k_0\rho)-\frac{j_1(k_0\rho)}{k_0\rho}\right]\overline{\overline{\mathbf{I}}}+j_2(k_0\rho)\hat{\boldsymbol{\rho}}\hat{\boldsymbol{\rho}}+i\frac{J_2(k_0\rho)}{k_0\rho}\left(\hat{\boldsymbol{\rho}}\hat{\mathbf{z}}+\hat{\mathbf{z}}\hat{\boldsymbol{\rho}}\right)\right\},\end{aligned}\tag{4.4}$$

where $\boldsymbol{\rho}_j = \mathbf{r}_j(z=0)$, $j = 1,2$, $\boldsymbol{\rho} = \boldsymbol{\rho}_2 - \boldsymbol{\rho}_1$ and $\hat{\mathbf{z}}$ denotes the unit vector along the z-axis. Here and henceforth J_n denotes the Bessel function of order n [90].

The first two terms in the representation (4.4) are equal to half the result obtained from Eq. (4.3) when $\alpha = \mathbb{S}$. These terms represent a homogeneous and isotropic contribution to the cross-spectral density operator in the aperture. This is in contrast to the last term, which is anisotropic and particular in form to the plane $z = 0$. It is this third term that is missing in the representation considered by James [101], and by Lahiri and Wolf [102]. In fact, in both papers it is for some reason explicitly stated without proof that the cross-spectral density operator corresponding to the plane waves traveling only in the $z > 0$ direction is precisely half of the cross-spectral density operator corresponding to all plane waves. Omitting the third term from the expression (4.4) actually means that the cross-spectral density operator in the aperture is not compatible with a field that satisfies the divergence condition (2.16) in the half-space $z > 0$.

Because we take the blackbody cavity to be radiating only through the aperture, whose dimensions are large with respect to the wavelength, it follows that we can use the Rayleigh diffraction formula of the first kind [3,16] together with the assumption that the field vanishes outside the aperture (Kirchhoff boundary condition), to determine the far-field pattern to a high degree of accuracy. When we use this approach for each of the three scalar components of the electric field for a circular aperture (radius b), as we have done in Publication II (where the radius was denoted by ε), we obtain for the cross-spectral density operator of the far-field pattern the expression

$$\overline{\mathbf{W}}_{\mathbf{E}}^{(\infty)}(\boldsymbol{r}_1,\boldsymbol{r}_2,\omega) = (2\pi k_0)^2(\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}_1)(\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}_2)\frac{\exp[-ik_0(r_2-r_1)]}{r_1r_2}\overline{\mathbf{T}}(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2), \tag{4.5}$$

where

$$\begin{aligned}\overline{\mathbf{T}}(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) = {} & \frac{a_0(\omega)b^2}{2\pi k_0^2}\frac{J_1(k_0b\sigma)}{k_0b\sigma} \\ & \times(1-\overline{\sigma}^2)^{-1/2}\left[\overline{\mathbf{I}}-(1-\overline{\sigma}^2)\hat{\mathbf{z}}\hat{\mathbf{z}}-\overline{\boldsymbol{\sigma}}\overline{\boldsymbol{\sigma}}-(1-\overline{\sigma}^2)^{1/2}(\hat{\mathbf{z}}\overline{\boldsymbol{\sigma}}+\overline{\boldsymbol{\sigma}}\hat{\mathbf{z}})\right],\end{aligned} \tag{4.6}$$

with $\sigma_j = (\overline{\mathbf{I}}-\hat{\mathbf{z}}\hat{\mathbf{z}})\cdot\hat{\mathbf{r}}_j$, $j = 1,2$, $\boldsymbol{\sigma} = \boldsymbol{\sigma}_2-\boldsymbol{\sigma}_1$ and $\overline{\boldsymbol{\sigma}} = (\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)/2$. It is straightforward to show that the far-field pattern corresponding to this cross-spectral density operator is completely unpolarized. We have proven this fact in Publication II.

From the cross-spectral densities given by Eq. (4.4), and Eqs. (4.5) and (4.6), we get for the corresponding degree-of-coherence functions μ_{STF}, the expressions

$$\mu_{\mathrm{STF}}^{(\mathscr{A})}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega) = \frac{1}{\sqrt{3}}\left[j_0^2(k_0\rho)+\frac{1}{2}j_2^2(k_0\rho)+\frac{3}{2}\frac{J_2^2(k_0\rho)}{(k_0\rho)^2}\right]^{1/2} \tag{4.7}$$

and

$$\mu_{\mathrm{STF}}^{(\infty)}(\mathbf{r}_1,\mathbf{r}_2,\omega) = \sqrt{2}\left|\frac{J_1(k_0b\sigma)}{k_0b\sigma}\right|\left[\frac{(\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}_1)(\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}_2)}{1-\overline{\boldsymbol{\sigma}}^2}\right]^{1/2} \tag{4.8}$$

in the aperture and in the far field, respectively. In the paraxial regime the expression (4.8) reduces to the result for unpolarized fields obtained from the electromagnetic van Cittert–Zernike theorem [105]. The degree-of-coherence function of the aperture is plotted in Fig. 3.3 (d), where it can be compared to the degree-of-coherence functions of the scalar universal form [Fig. 3.3 (b)] and to the blackbody degree-of-coherence function for fields inside the cavity [Fig. 3.3 (d)]. The degree-of-coherence function of the aperture field, as for the two other degree-of-coherence functions, the coherence length of the field is of the order of half a wavelength.

The far-field degree-of-coherence function is, in turn, plotted in Fig. 4.2 for a range of aperture radii b in terms of the spherical polar angles θ_1 and θ_2 of the unit vectors $\hat{\mathbf{r}}_1 = (\theta_1,\phi_1)$ and $\hat{\mathbf{r}}_2 = (\theta_2,\phi_2)$, where the azimuthal angles have the fixed values $\phi_1 = 0$ and $\phi_2 = 0$. We note that for large radii the angular coherence length in paraxial directions is of the order of 0.01 rad, and that it is longer for larger angles. The coherence length is also longer for smaller apertures, which is explained by the fact that the extent of the point spread function of the radiation is inversely proportional to the aperture size. The longer coherence length at oblique angles can, in turn, be attributed to the cosine-law -like behavior of the degree-of-coherence function in Eq. (4.8) for such angles.

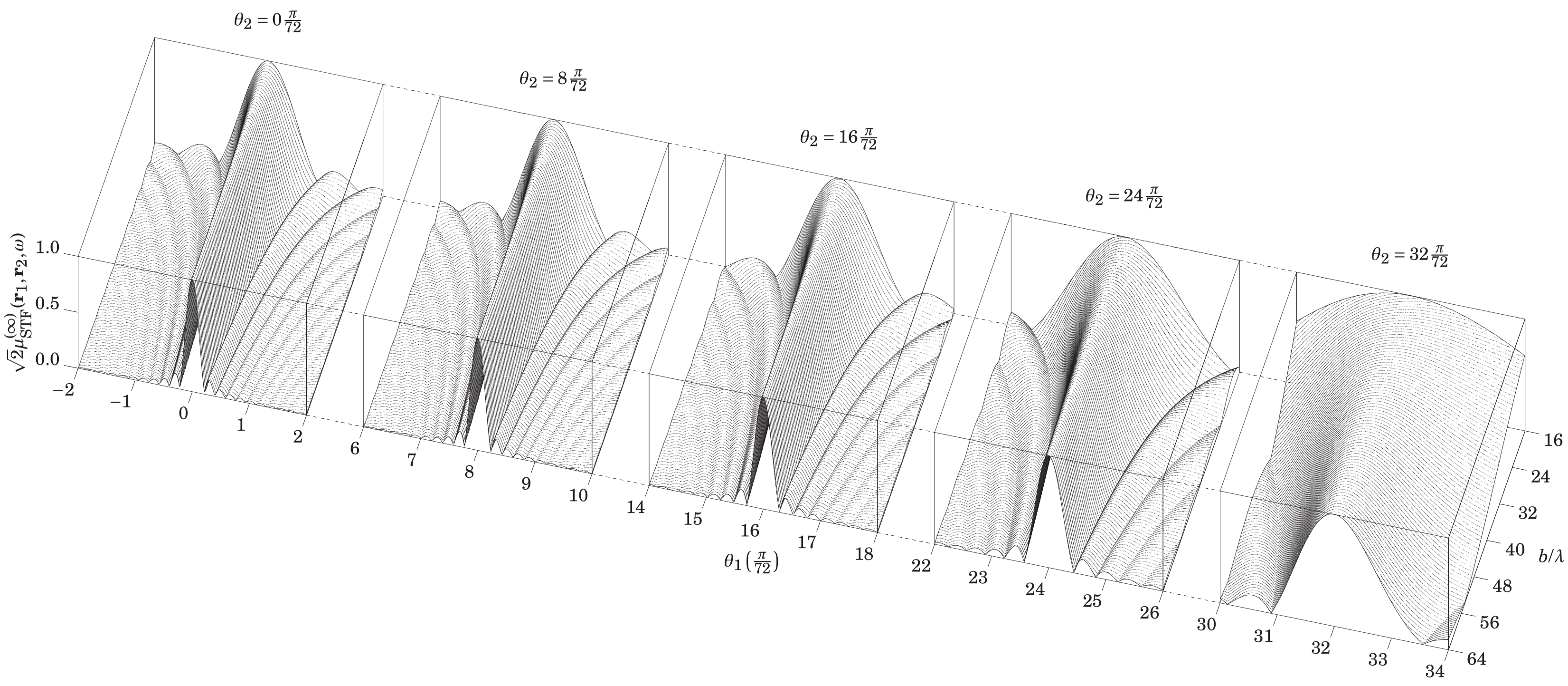


Figure 4.2. Far-field degree-of-coherence function $\mu_{\mathrm{STF}}^{(\infty)}(\mathbf{r}_1, \mathbf{r}_2, \omega)$ of blackbody radiation as a function of the source aperture radius b, the spherical polar angles θ_1 and θ_2, and with fixed azimuthal angles $\phi_1 = 0$ and $\phi_2 = 0$, representing the two observation directions $\hat{\mathbf{r}}_1 = (\theta_1, \phi_1)$ and $\hat{\mathbf{r}}_2 = (\theta_2, \phi_2)$, respectively. The coherence length in the far field increases when the source aperture radius b decreases or when the polar angles increase. For aperture radii that are much larger than the wavelength, the far-field pattern is essentially δ-correlated for all but the most oblique polar angles.

It is also of interest to note that the radiant intensity or power per solid angle of the far-field pattern corresponding to the cross-spectral density operator (4.5) in the direction $\hat{\mathbf{r}}$ is given by

$$J(\hat{\mathbf{r}},\omega) = 2a_0(\omega)\pi b^2(\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}), \tag{4.9}$$

which shows that the blackbody radiation satisfies the so-called Lambert cosine law, that is, the radiant intensity of the radiation is directly proportional to the cosine of the angle between the direction of propagation of the radiation and the surface normal of the source.

The expression (4.4), derived by us in Publication II, represents to the best of our knowledge the first time the full electromagnetic cross-spectral density operator of blackbody radiation has been (correctly) determined inside the aperture. The agreement between the results from our rigorous derivation and the results corresponding to the planar models of blackbody radiation used for example in the context of radiative transfer, validate the latter as models of blackbody or thermal radiation on theoretical grounds. Agreement with experimental results has of course long been known and has, in fact, formed the basis of most of the models. Our results now provide a theoretical confirmation of these models and furthermore form a foundation on which also rigorous electromagnetic analyses of blackbody radiation can be built.

4.2 Effect of the boundary at $z = 0$

Because the main result in Publication II is the introduction of the third term in the aperture cross-spectral density expression (4.4), and since this term follows from properly accounting for the aperture on the cavity wall, which had not been done previously, it is prudent to note that our derivations are based on an assumption that obviously is suspect. Indeed, although it turns out not to ultimately affect our results, we have not considered how the boundary or wall at the finite coordinate position $z = 0$, where the aperture is eventually introduced, might influence the form of the cross-spectral density operator of the field in the cavity. According to the Planckian analysis of blackbody radiation [17], the boundary should have no effect, but in view of the fact that rigorous electromagnetic analysis reveals that the finiteness of the cavity actually changes the blackbody spectrum itself (see for example Ref. 106), care must be taken when drawing conclusions about the electromagnetic details.

To analyze the effect of the boundary or wall at $z = 0$ on the cavity radiation near it, we must know the material and build of this wall. As discussed previously, the only requirement that needs to be placed on the wall material is that the wall is impermeable to radiation. Here we follow Planck and take the cavity walls to be perfect conductors, which are also perfect reflectors, whereby they cannot be permeable. Actually, it seems that the perfect conductor is the only 'simple' material that fulfills the nonpermeability assumption. Furthermore, the cross-spectral density of the radiation in a blackbody cavity occupying the half-space $z < 0$, with a perfectly conducting wall at $z = 0$, has already been determined by Agarwal [97], who obtained the additive correction (which is here represented in our notation)

$$\begin{aligned}\Delta\overline{\mathbf{W}}_{\mathbf{E}}(\mathbf{r}_1,\mathbf{r}_2,\omega) &= -4\pi a_0(\omega)\left[\left(\overline{\mathbf{I}}+\frac{1}{k_0^2}\nabla_1\nabla_1\right)\frac{\sin(k_0|\mathbf{r}_1-\overline{\mathbf{R}}\cdot\mathbf{r}_2|)}{k_0|\mathbf{r}_1-\overline{\mathbf{R}}\cdot\mathbf{r}_2|}\right]\cdot\overline{\mathbf{R}} \\ &= -\overline{\mathbf{W}}_{\mathbf{E}}\left(\mathbf{r}_1,\overline{\mathbf{R}}\cdot\mathbf{r}_2,\omega\right)\cdot\overline{\mathbf{R}}\end{aligned} \tag{4.10}$$

to the cross-spectral density function of Eq. (4.2). Here $\overline{\mathbf{R}} = \overline{\mathbf{I}} - 2\hat{\mathbf{z}}\hat{\mathbf{z}}$ is an operator that reflects the vector it operates on in the plane $z = 0$. When $\alpha = \mathbb{S}$, we can use the first part of the representation (4.3) in Eq. (4.10) to obtain the expression

$$\begin{aligned}&\Delta\overline{\mathbf{W}}_{\mathbf{E}}(\mathbf{r}_1,\mathbf{r}_2,\omega) \\ &= a_0(\omega)\int_\alpha\int_\alpha \delta\left(\overline{\mathbf{R}}\cdot\hat{\mathbf{u}}_2-\hat{\mathbf{u}}_1\right)\left(\hat{\mathbf{u}}_1\hat{\mathbf{u}}_2-\overline{\mathbf{R}}\right)\exp[ik_0(\hat{\mathbf{u}}_1\cdot\mathbf{r}_1-\hat{\mathbf{u}}_2\cdot\mathbf{r}_2)]\,\mathrm{d}\hat{\mathbf{u}}_1\mathrm{d}\hat{\mathbf{u}}_2.\end{aligned} \tag{4.11}$$

We note that the assumption of a half-space cavity is tantamount to considering a cavity, whose dimensions are large with respect to the wavelength of the radiation, and is thereby already covered by previous specifications.

Let us now introduce an aperture into the perfect conductor as we did in Sec. 4.1, and again consider the cavity field at the aperture, but now augmenting the cross-spectral density operator by the additional term (4.11). When we apply the restriction of propagation only into the half-space $z > 0$ in this expression, so that α represents the corresponding hemisphere, we observe that the unit vector $\hat{\mathbf{u}}_1$ then always points into the half-space $z > 0$ and the unit vector $\overline{\mathbf{R}}\cdot\hat{\mathbf{u}}_2$ always points into the half-space $z < 0$, whereby the argument to the δ-function in Eq. (4.11) never vanishes. Thereby it follows from that equation, that in this case the additive correction $\Delta\overline{\mathbf{W}}_{\mathbf{E}}(\mathbf{r}_1,\mathbf{r}_2,\omega)$ vanishes identically. Physically this result implies that the additive term (only) describes the cross-correlations between plane waves propagating toward the half-space $z < 0$ and plane waves propagating toward the half-space $z > 0$, and hence this term does not influence the correlations between the field components that propagate into the $z > 0$ half-space [fields (a) in Fig. 4.1]. Thus, as only the latter kind of plane waves are present in the aperture, our previous assumption that these components of the field alone determine the far-field pattern in the half-space $z > 0$ turns out to be valid. Furthermore, we note that since the perfect conductor reflects the completely uncorrelated cavity field 'onto' itself [fields (b) in Fig. 4.1], the correlations between the field components propagating towards different half-spaces are, in fact, expected. Nevertheless, this effect is not present in the Planckian analysis. For two points, $\mathbf{r}_1$ and $\mathbf{r}_2$ inside the cavity, the additive correction given by Eq. (4.10) drops off at a rate, which is roughly inversely proportional to $|z_1 + z_2|$, and hence deep inside the cavity [fields (c) in Fig. 4.1] the effect of this correction becomes negligible with respect to the main term (4.2).

Although we have here considered only a perfectly reflecting boundary, the thermal equilibrium of the cavity walls and the radiation within the cavity, together with the fact that no radiation escapes the cavity, suggest, as discussed previously, that the material of the nonpermeable cavity walls does not affect the equilibrium form of the blackbody radiation emitted from the aperture. Therefore the conclusion that only the 'free-space' form of the cross-spectral density operator, as given by Eq. (4.2), needs to be considered when determining the cross-spectral density operator of the fields inside and escaping through the aperture, holds for every blackbody cavity, whatever material its walls are made of.

5. Universality of the degree of coherence

The works presented in Refs. 107–110 suggest that for scalar fields produced by homogeneous and isotropic sources in unbounded regions, the degree-of-coherence function (3.24) inside the source region always attains the same, universal, sinc-form [see Eq. (5.9)], which is also equal to the imaginary part of the scalar free-space Green's function, given in Eq. (3.34). This universality result was later extended by us in Publication III to vector-valued electromagnetic fields.

Although the early results [107, 109] suggested that the universal behavior found for asymptotically unbounded lossless regions should be a good model for what happens in bounded and slightly lossy regions, this is actually not the case. Indeed, we prove in Publication IV that a finite amount of loss, however small, is sufficient to invalidate universality in (asymptotically) unbounded regions. Furthermore, in Publication V we show that the universality result is, in general, not a good model for fields sourced by arbitrary finite-sized source regions, whether these regions are lossy or not.

A specific property of the universality result is that it implies that the coherence lengths of fields produced by homogeneous and isotropic sources are equal to, or at least of the order of, the blackbody (transversal) coherence length $\lambda/2$ [98, 110]. Since completely incoherent, or δ-correlated, sources produce fields that exhibit the universal form, this seems to suggest that fields sourced by homogeneous and isotropic sources, with otherwise arbitrary coherence properties, cannot have coherence lengths that are appreciably shorter than the blackbody coherence length $\lambda/2$. As our results show that the universal form is not as universal as was first thought, they also remove the lower bound of the field coherence length. Indeed, we have shown in Publications IV and V that for both lossy unbounded source regions and finite-sized source regions with arbitrary properties, the coherence length of the generated field can be made as short as desired by selecting the homogeneous and isotropic source distribution appropriately.

In the following we present the results concerning the coherence properties of fields inside their source regions in more detail. We cover the results that point toward a universal character for the degree-of-coherence function for both scalar and vector-valued electromagnetic fields, but in particular concentrate on the results, which show that the universal character is undermined by (arbitrarily) small losses and region boundaries that lie at finite distances. Accordingly, we conclude by showing that for finite regions there is actually no universality.

5.1 Degree-of-coherence functions for homogeneous fields

For stochastically homogeneous scalar or vector-valued fields u, that is, for fields whose cross-spectral density operator satisfies

$$W_u(\mathbf{r}_1, \mathbf{r}_2, \omega) = W_u(\mathbf{r}_1 - \mathbf{r}_2, \omega), \tag{5.1}$$

it follows that $W_u(\mathbf{r}, \mathbf{r}, \omega) = W_u(\mathbf{0}, \omega)$, whereby the dependence of every degree-of-coherence function considered in Sec. 3.2 on the difference $\mathbf{r}_1 - \mathbf{r}_2$, is completely mediated by the cross-spectral density operator $W_u(\mathbf{r}_1 - \mathbf{r}_2, \omega)$. Of course, these functions also depend on the constant operator or value $W_u(\mathbf{0}, \omega)$, but apart from the degree-of-coherence function μ_{RG} this simply amounts to one or two numerical constants in the expression. For the function μ_{RG} the situation is slightly more complicated, but in this case too, the qualitative behavior of the degree of coherence can be understood by studying $W_u(\mathbf{r}_1 - \mathbf{r}_2, \omega)$. To avoid the unnecessary clutter that would result if we considered each of the degree-of-coherence functions separately, we will in the following investigate their properties in terms of the cross-spectral density operators. The important results are, however, also displayed in terms of selected degree-of-coherence functions.

5.2 Field cross-spectral density corresponding to a δ-correlated source within a bounded spherical region

In this section we concentrate on systems, which can be described in scalar terms, because the scalar model sources are simpler than vectorial ones. This is, however, not an essential restriction since the results are readily extendable to vectorial systems if so desired. Here we also assume that all space is lossless and of one medium, whereby it is mathematically equivalent to vacuum and we take $\kappa = k_0$ in what follows.

To begin with we consider a source, which is confined inside a finite ball $B(\mathbf{0},R)$ with radius $R<\infty$, and which is completely uncorrelated therein. Mathematically we thus have for the source correlation function W_Q the expression

$$W_Q(\mathbf{r}_1,\mathbf{r}_2,\omega)=Q_0\Theta(r_1\le R)\Theta(r_2\le R)\delta(\mathbf{r}_1-\mathbf{r}_2), \tag{5.2}$$

where Q_0 is an arbitrary constant and the function Θ equals 1 when its argument is true and 0 when its argument is false. With the source correlation function given by Eq. (5.2), we get when we introduce this function into the expression (3.17), for the corresponding cross-spectral density operator of the scalar field U, the representation

$$\begin{aligned}W_U(\mathbf{r}_1,\mathbf{r}_2,\omega)&=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}G_{k_0}(\mathbf{r}_1,\mathbf{r}_1')G^*_{k_0}(\mathbf{r}_2,\mathbf{r}_2')W_Q(\mathbf{r}_1',\mathbf{r}_2')\mathrm{d}\mathbf{r}_1'\mathrm{d}\mathbf{r}_2'\\&=Q_0\int_{B(\mathbf{0},R)}G_{k_0}(\mathbf{r}_1,\mathbf{r}')G^*_{k_0}(\mathbf{r}_2,\mathbf{r}')\mathrm{d}\mathbf{r}'.\end{aligned} \tag{5.3}$$

We then consider the cross-spectral density operator and the corresponding scalar degree-of-coherence function as given by Eq. (3.24) for the case when $\mathbf{r}_1=\mathbf{r}$ and $\mathbf{r}_2=\mathbf{0}$. Specifically, we fix one of the observation points at the center of the spherical source region. For the cross-spectral density operator we then have straightforwardly from Eq. (5.3) the expression

$$\begin{aligned}W_U(\mathbf{r},\mathbf{0},\omega)&=Q_0\int_{B(\mathbf{0},R)}G_{k_0}(\mathbf{r},\mathbf{r}')G^*_{k_0}(\mathbf{0},\mathbf{r}')\mathrm{d}\mathbf{r}'\\&=Q_0\int_{B(\mathbf{0},R)}\frac{\exp(ik_0|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}\frac{\exp(-ik_0r')}{r'}\mathrm{d}\mathbf{r}',\end{aligned} \tag{5.4}$$

where the last step follows from the representation (2.42). This expression is further developed in Ref. 107 into the form

$$W_U(\mathbf{r},\mathbf{0},\omega)=4\pi RQ_0\left[\left(1-\frac{1}{2}\frac{r}{R}\right)j_0(k_0r)+\frac{i}{2}\frac{r}{R}j_1(k_0r)\right], \tag{5.5}$$

which holds for $r<R$, and the (transient) degree-of-coherence function is shown to have the expression

$$\mu'(\mathbf{r},\mathbf{0},\omega)=\frac{\left(1-\frac{1}{2}\frac{r}{R}\right)j_0(k_0r)+\frac{i}{2}\frac{r}{R}j_1(k_0r)}{\left[1-S\left(\frac{r}{R}\right)\right]^{1/2}},\quad r<R, \tag{5.6}$$

where the auxiliary function

$$S(z)=\frac{1}{2}+\frac{1}{4}\frac{z^2-1}{z}\ln\left(\frac{1+z}{1-z}\right) \tag{5.7}$$

has been introduced for notational convenience.

When $r/R\ll 1$, the transient degree-of-coherence function (5.6) attains the form

$$\mu'(\mathbf{r},\mathbf{0},\omega)\sim j_0(k_0r)=\frac{\sin(k_0r)}{k_0r}=\mu(\mathbf{r},\mathbf{0},\omega),\quad \frac{r}{R}\ll 1. \tag{5.8}$$

Furthermore, if the radius of the system becomes asymptotically large, that is $R \to \infty$, the degree-of-coherence function becomes

$$\mu(\mathbf{r}_1,\mathbf{r}_2,\omega) \sim \mu'(\mathbf{r}_1-\mathbf{r}_2,\mathbf{0},\omega) \sim j_0(k_0|\mathbf{r}_1-\mathbf{r}_2|) = \frac{\sin(k_0|\mathbf{r}_1-\mathbf{r}_2|)}{|\mathbf{r}_1-\mathbf{r}_2|}, \qquad (5.9)$$

irrespectively of whether $\mathbf{r}_2 = \mathbf{0}$ or not.

The behavior of the transient degree-of-coherence function (5.6) is studied in Fig. 5.1, where the difference $|\mu'(\mathbf{r},\mathbf{0},\omega)-\mu(\mathbf{r},\mathbf{0},\omega)|$ between it and its asymptotic form (5.8) is plotted for different values of $|\mathbf{r}|$ and R. We observe that when R exceeds about 5λ, the difference between the transient degree-of-coherence function and its asymptotic form is almost negligible and becomes more so when R grows larger.

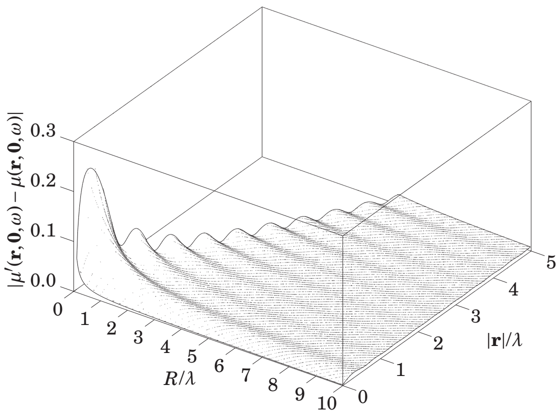


Figure 5.1. Absolute difference between the transient degree-of-coherence function $\mu'(\mathbf{r},\mathbf{0},\omega)$ of Eq. (5.6) and its asymptotic (universal) form $\mu(\mathbf{r},\mathbf{0},\omega)$, given by Eq. (5.8) as a function of the distance $|\mathbf{r}|$ and the ball radius R.

From these results we can draw a few important conclusions. First of all, the degree-of-coherence function of a field corresponding to a completely uncorrelated source distribution in a finite ball rapidly converges to the form (5.9), when the radius of the ball exceeds about five wavelengths. Hence this form is a good representant for the degree-of-coherence function in such situations. Secondly, although the source distribution is completely uncorrelated (δ-correlated), the corresponding field distribution degree-of-coherence function is not peaked. In fact, the corresponding coherence length is $\lambda/2$, which is of the same order of magnitude as the wavelength of the field. In the next section, we see that the form (5.9) is actually common to many fields produced by large (unbounded) homogeneous and isotropic source distributions.

5.3 Homogeneous sources in unbounded regions

The results of the previous section, which showed that the degree-of-coherence function converges rapidly with the source region radius R to its form for large source regions, suggest that it should be possible to draw conclusions about the relations between source and field coherence functions by studying unbounded regions instead of actual finite regions. Such an approach is attractive since it markedly simplifies the mathematical expressions and allows for simple relations to be derived. The downside to this approach is, however, that the mathematical expressions simplify mainly because effects at the source region boundaries become insignificant, which may not be true in actual systems. In this section we consider (negligibly) lossy unbounded regions and in the following two sections we look at what effects are missed by making these assumptions.

To begin with, we note that if the source region is unbounded and the source term $W_q(\mathbf{r}_1,\mathbf{r}_2,\omega)$ in the expression (3.17) does not drop off rapidly enough when $|\mathbf{r}_1|,|\mathbf{r}_2| \to \infty$, that expression will in general diverge if $\mathrm{Im}\{\kappa^2\} \le 0$, that is, if the source region medium is not lossy (see Sec. 2.2). Therefore, as a renormalization procedure, we assume that $\mathrm{Im}\{\kappa^2\} > 0$, or which is the same for $\mathrm{Re}\{\kappa\} > 0$, that $\mathrm{Im}\{\kappa\} > 0$, and for lossless regions we take the limit $\mathrm{Im}\{\kappa\} \to 0^+$ only at the very end of all calculations, when we compute the degree-of-coherence function. This approach can be motivated for example if one observes that any real medium, except the classical vacuum, is lossy. Suppose then that the source correlation function W_q is homogeneous or of the form

$$W_q(\mathbf{r}_1,\mathbf{r}_2,\omega) = W_q(\mathbf{r}_1-\mathbf{r}_2,\omega) = W_q(\mathbf{r},\omega), \tag{5.10}$$

where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. It can be shown [108] that when this source distribution is used in the expression (3.17), where $\mathrm{Im}\{\kappa\} > 0$, we get for the cross-spectral density function W_u the expression

$$\begin{aligned} W_u(\mathbf{r}_1,\mathbf{r}_2,\omega) &= \frac{2\pi}{\mathrm{Im}\,\kappa}\int_{\mathbb{R}^3} j_0(\mathrm{Re}\{\kappa\}|\mathbf{r}-\mathbf{r}'|)\exp(-\mathrm{Im}\{\kappa\}|\mathbf{r}-\mathbf{r}'|)W_q(\mathbf{r}',\omega)\mathrm{d}\mathbf{r}' \\ &= W_u(\mathbf{r},\omega), \end{aligned} \tag{5.11}$$

where $\mathbf{r}' = \mathbf{r}'_1 - \mathbf{r}'_2$ and the last step follows directly from the expression itself. Specifically, the field produced by a homogeneous source distribution is itself homogeneous.

The derivation of Eq. (5.11) as presented in Ref. 108 only applies as such for three-dimensional systems. In later papers a Fourier-transform -based approach is used instead. This alternative approach was introduced in an effort to obtain results for other than three-dimensional systems [110]. Here we will only consider three-dimensional systems, but since we apply the Fourier-transform -based approach in our own papers, we outline it here for completeness.

When the source distribution in the double Helmholtz equation (3.16) is homogeneous, the right-hand side of that equation remains unchanged if we replace $\mathbf{r}_1$ and $\mathbf{r}_2$ by $\mathbf{r}_1+\mathbf{x}$ and $\mathbf{r}_2+\mathbf{x}$, respectively, where the shift $\mathbf{x}$ is arbitrary. The left-hand side remains unchanged in form, but $W_u(\mathbf{r}_1,\mathbf{r}_2,\omega)$ is replaced by $W_u(\mathbf{r}_1+\mathbf{x},\mathbf{r}_2+\mathbf{x},\omega)$. Because the fields satisfy the Sommerfeld or Silver–Müller radiation conditions, the solution to the double Helmholtz equation is unique, and hence it follows from the arbitrariness of $\mathbf{x}$ that the field cross-spectral density function must be homogeneous as well. Thereby, if we set $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$, we can rewrite the double Helmholtz equation (3.16) in the homogeneous case and when the wave number is constant throughout space $[\kappa(\mathbf{r},\omega)=\kappa(\omega)]$ as

$$\left(\nabla^2+\kappa^2\right)\left(\nabla^2+\kappa^{*2}\right)W_u(\mathbf{r},\omega)=16\pi^2 W_q(\mathbf{r},\omega). \tag{5.12}$$

Let us now introduce for the homogeneous source and field cross-spectral coherence and correlation functions the Fourier representations given by the transform pair

$$W_a(\mathbf{r},\omega)=\int_{\mathbb{R}^3}\widehat{W}_a(\mathbf{k},\omega)\exp(-i\mathbf{k}\cdot\mathbf{r})\mathrm{d}\mathbf{k}, \tag{5.13}$$

and

$$\widehat{W}_a(\mathbf{k},\omega)=\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}W_a(\mathbf{r},\omega)\exp(i\mathbf{k}\cdot\mathbf{r})\mathrm{d}\mathbf{r}, \tag{5.14}$$

where $a\in\{u,q\}$. When the representation (5.13) is introduced into the homogeneous double Helmholtz equation (5.12) and the linear independence of the trigonometric exponential functions is used, we arrive at

$$(-k^2+\kappa^2)\left(-k^2+\kappa^{*2}\right)\widehat{W}_u(\mathbf{k},\omega)=16\pi^2\widehat{W}_q(\mathbf{k},\omega), \tag{5.15}$$

which is an algebraic equation.

It is straightforward to write the solution to this algebraic equation. Indeed, since $\mathrm{Im}\{\kappa\}>0$, the function

$$\widehat{g}(\mathbf{k})=\frac{1}{|k^2-\kappa^2|}, \tag{5.16}$$

is regular for real k and the solution to Eq. (5.15) can be written in the form

$$\widehat{W}_u(\mathbf{k},\omega)=16\pi^2\widehat{g}(\mathbf{k})\widehat{W}_q(\mathbf{k},\omega). \tag{5.17}$$

From the Fourier-transform convolution theorem it now follows that the solution to the homogeneous double Helmholtz equation (5.12) is given by

$$W_u(\mathbf{r},\omega)=\frac{2}{\pi}\int_{\mathbb{R}^3}g(\mathbf{R})W_q(\mathbf{r}-\mathbf{R},\omega)\mathrm{d}\mathbf{R}=\frac{2}{\pi}\int_{\mathbb{R}^3}g(\mathbf{r}-\mathbf{R})W_q(\mathbf{R},\omega)\mathrm{d}\mathbf{R}. \tag{5.18}$$

Since the inverse Fourier transform of the function $\widehat{g}(\mathbf{k})$ defined in Eq. (5.16) is given by [110]

$$g(\mathbf{r}) = \frac{\pi^2}{\mathrm{Im}\{\kappa\}} j_0(\mathrm{Re}\{\kappa\}r)\exp(-\mathrm{Im}\{\kappa\}r), \tag{5.19}$$

we immediately see that the solution (5.18) agrees with the solution (5.11), as it should. The derivation of this solution has been formal in that we have not here addressed questions relating to the convergence of the involved integrals. That we have done in Publication IV, where we have shown that for a continuous and absolutely integrable scalar source correlation function $W_Q(\mathbf{r},\omega)$, with $W_Q(\mathbf{0},\omega) < \infty$, the expression (5.18) [and hence, by association Eq. (5.11)] provides the unique solution to the homogeneous double Helmholtz equation (5.12).

Since we are interested in computing the degree-of-coherence function in lossless regions, where the representation (5.11) diverges, we consider the (renormalization) limit

$$\lim_{\mathrm{Im}\{\kappa\}\to 0^+} \mathrm{Im}\{\kappa\} W_u(\mathbf{r},\omega) = 2\pi \int_{\mathbb{R}^3} j_0(\kappa|\mathbf{r}-\mathbf{r}'|) W_q(\mathbf{r}',\omega)\mathrm{d}\mathbf{r}', \tag{5.20}$$

from which we have, in particular,

$$\lim_{\mathrm{Im}\{\kappa\}\to 0^+} \mathrm{Im}\{\kappa\} W_u(\mathbf{0},\omega) = 2\pi \int_{\mathbb{R}^3} j_0(\kappa|\mathbf{r}'|) W_q(\mathbf{r}',\omega)\mathrm{d}\mathbf{r}'. \tag{5.21}$$

Although these equations hold for both scalar and vector-valued electromagnetic fields, it is useful to consider the electromagnetic case explicitly in terms of the electric field and the source current coherence function $W_\mathbf{J}$. Since the Helmholtz equation source $\mathbf{Q}$ is given in terms of the current distribution $\mathbf{J}$ by Eq. (2.44), we can follow the same steps as in going from Eq. (2.45) to Eq. (2.46), to rewrite the results (5.20) and (5.21) as

$$\begin{aligned}
&\lim_{\mathrm{Im}\{\kappa\}\to 0^+} \mathrm{Im}\{\kappa\} \overline{\mathbf{W}}_\mathbf{E}(\mathbf{r},\omega) \\
&= 2\pi \int_{\mathbb{R}^3} \left[\left(\overline{\mathbf{I}} + \frac{1}{\kappa^2}\nabla'\nabla'\right)\cdot\left(\overline{\mathbf{I}} + \frac{1}{\kappa^2}\nabla'\nabla'\right) j_0(\kappa|\mathbf{r}-\mathbf{r}'|)\right]\cdot \overline{\mathbf{W}}_\mathbf{J}(\mathbf{r}',\omega)\mathrm{d}\mathbf{r}' \\
&= 2\pi \int_{\mathbb{R}^3} \left[\left(\overline{\mathbf{I}} + \frac{1}{\kappa^2}\nabla'\nabla'\right) j_0(\kappa|\mathbf{r}-\mathbf{r}'|)\right]\cdot \overline{\mathbf{W}}_\mathbf{J}(\mathbf{r}',\omega)\mathrm{d}\mathbf{r}' \\
&= 2\pi \int_{\mathbb{R}^3} \mathrm{Im}\left\{\overline{\mathbf{G}}_\kappa(\mathbf{r},\mathbf{r}')\right\}\cdot \overline{\mathbf{W}}_\mathbf{J}(\mathbf{r}',\omega)\mathrm{d}\mathbf{r}'
\end{aligned} \tag{5.22}$$

and

$$\lim_{\mathrm{Im}\{\kappa\}\to 0^+} \mathrm{Im}\{\kappa\} \overline{\mathbf{W}}_\mathbf{E}(\mathbf{0},\omega) = 2\pi \int_{\mathbb{R}^3} \mathrm{Im}\left\{\overline{\mathbf{G}}_\kappa(\mathbf{0},\mathbf{r}')\right\}\cdot \overline{\mathbf{W}}_\mathbf{J}(\mathbf{r}',\omega)\mathrm{d}\mathbf{r}', \tag{5.23}$$

respectively. Here the second step in Eq. (5.22) follows from the properties of the spherical Bessel functions, and the last step follows from the definitions (2.42) and (2.47).

When the expressions for $W_u(\mathbf{0},\omega)$ and $W_{\mathbf{E}}(\mathbf{0},\omega)$ do not vanish, i.e., when

$$\int_{\mathbb{R}^3} j_0(\kappa|\mathbf{r}'|)W_q(\mathbf{r}',\omega)\mathrm{d}\mathbf{r}' \neq 0 \tag{5.24}$$

and

$$\int_{\mathbb{R}^3} \mathrm{Im}\left\{\overline{\mathbf{G}}_\kappa(\mathbf{0},\mathbf{r}')\right\}\cdot\overline{\mathbf{W}}_{\mathbf{J}}(\mathbf{r}',\omega)\mathrm{d}\mathbf{r}' \neq \mathbf{0}, \tag{5.25}$$

we can use the representations (5.20) and (5.21), and the representations (5.22) and (5.23) in the degree-of-coherence functions considered in Sec. 3.2. In the scalar case, with the function (3.24), we then obtain the expression

$$\mu(\mathbf{r},\omega) = \lim_{\mathrm{Im}\{\kappa\}\to 0^+} \frac{W_U(\mathbf{r},\omega)}{W_U(\mathbf{0},\omega)} = \frac{\int_{\mathbb{R}^3} j_0(\kappa|\mathbf{r}-\mathbf{r}'|)W_Q(\mathbf{r}',\omega)\mathrm{d}\mathbf{r}'}{\int_{\mathbb{R}^3} j_0(\kappa|\mathbf{r}'|)W_Q(\mathbf{r}',\omega)\mathrm{d}\mathbf{r}'}. \tag{5.26}$$

In the vector case we consider the functions μ_{KW} and μ_{STF}, for which we get the expressions

$$\mu_{\mathrm{KW}}(\mathbf{r},\omega) = \lim_{\mathrm{Im}\{\kappa\}\to 0^+} \frac{\mathrm{Tr}\left[\int_{\mathbb{R}^3} \mathrm{Im}\left\{\overline{\mathbf{G}}_\kappa(\mathbf{r},\mathbf{r}')\right\}\cdot\overline{\mathbf{W}}_{\mathbf{J}}(\mathbf{r}',\omega)\mathrm{d}\mathbf{r}'\right]}{\mathrm{Tr}\left[\int_{\mathbb{R}^3} \mathrm{Im}\left\{\overline{\mathbf{G}}_\kappa(\mathbf{0},\mathbf{r}')\right\}\cdot\overline{\mathbf{W}}_{\mathbf{J}}(\mathbf{r}',\omega)\mathrm{d}\mathbf{r}'\right]} \tag{5.27}$$

and

$$\mu_{\mathrm{STF}}(\mathbf{r},\omega) = \lim_{\mathrm{Im}\{\kappa\}\to 0^+} \frac{\left\|\int_{\mathbb{R}^3} \mathrm{Im}\left\{\overline{\mathbf{G}}_\kappa(\mathbf{r},\mathbf{r}')\right\}\cdot\overline{\mathbf{W}}_{\mathbf{J}}(\mathbf{r}',\omega)\mathrm{d}\mathbf{r}'\right\|_F}{\mathrm{Tr}\left[\int_{\mathbb{R}^3} \mathrm{Im}\left\{\overline{\mathbf{G}}_\kappa(\mathbf{0},\mathbf{r}')\right\}\cdot\overline{\mathbf{W}}_{\mathbf{J}}(\mathbf{r}',\omega)\mathrm{d}\mathbf{r}'\right]}. \tag{5.28}$$

These expressions represent scalar and vector degree-of-coherence functions for fields whose sources satisfy the condition (5.24) or the condition (5.25). When these conditions are not met, a more careful analysis of how the cross-spectral density functions diverge when $\mathrm{Im}\{\kappa\} \to 0^+$ is necessary to obtain an explicit representation of the corresponding degree-of-coherence function.

5.4 Homogeneous and isotropic sources in unbounded regions, universality of field degree-of-coherence

Next we consider fields with homogeneous and isotropic source distributions, starting with scalar fields, for which the correlation functions of such source distributions satisfy

$$W_Q(\mathbf{r},\omega) = W_Q(|\mathbf{r}|,\omega). \tag{5.29}$$

To evaluate the expression (5.20) in this case, we use the spherical Bessel function addition theorem [90, 111]

$$j_0(\kappa|\mathbf{r}-\mathbf{r}'|) = 4\pi\sum_{n,m} j_n(\kappa|\mathbf{r}|)j_n(\kappa|\mathbf{r}'|)Y_n^m(\hat{\mathbf{r}})Y_n^{m\,*}(\hat{\mathbf{r}}'), \tag{5.30}$$

so that we get

$$
\begin{aligned}
&\lim_{\mathrm{Im}\{\kappa\}\to 0^+} \mathrm{Im}\{\kappa\} W_U(\mathbf{r},\omega) \\
&\quad = 8\pi^2 \sum_{n,m} \left[\int_{\mathbb{R}^3} j_n(\kappa r') Y_n^{m*}(\hat{\mathbf{r}}') W_Q(r',\omega) \mathrm{d}\mathbf{r}' \right] j_n(\kappa r) Y_n^m(\hat{\mathbf{r}}) \\
&\quad = 8\pi^2 \left[\int_0^\infty r'^2 j_0(\kappa r') W_Q(r',\omega) \mathrm{d}r' \right] j_0(\kappa r) = D_S j_0(\kappa r),
\end{aligned}
\tag{5.31}
$$

where D_S depends on W_Q, but is a constant in $\mathbf{r}$.

Suppose then that the condition (5.24) holds, so that we get for the corresponding degree-of-coherence function from Eq. (5.26) the expression

$$
\mu(\mathbf{r},\omega) = j_0(\kappa r) = \mathrm{sinc}(\kappa r) = \mathrm{Im}\{G_\kappa(\mathbf{r},\mathbf{0})\}, \tag{5.32}
$$

which holds irrespective of the detailed structure of $W_Q(|\mathbf{r}|,\omega)$. Hence the degree-of-coherence function takes on the *universal* form given by Eq. (5.32) with respect to all (lossless) homogeneous and isotropic sources that satisfy the condition (5.24). This universality result encompasses the blackbody and δ-correlated source expression (5.9). Remarkably, Eq. (5.32) suggests that the field coherence properties do not depend on the source correlation properties, as long as the sources are stochastically homogeneous and isotropic.

In Ref. 109 it was studied how the universality result emerges for a field sourced by a Bessel-correlated model source of the form

$$
W_Q(\mathbf{r}_1,\mathbf{r}_2,\omega) = Q_0 j_0(\chi|\mathbf{r}_1-\mathbf{r}_2|)\Theta(r_1 \le R)\Theta(r_2 \le R), \tag{5.33}
$$

when $R \to \infty$. Here Q_0 is an arbitrary scalar constant. As in Ref. 107 or Sec. 5.2, it is observed that for this source distribution the field degree-of-coherence function has closely converged on the universality result when the source region radius R exceeds about 5λ. The universality result is extended in Ref. 110 explicitly to two-dimensional and three-dimensional, and by inference to all D-dimensional systems with $D \ge 2$, where the universal form is dimension dependent. It is also proven that no universality emerges in one-dimensional systems.

The corresponding universality result for vector-valued electromagnetic fields was obtained by us in Publication III. For the vector-valued fields isotropicity limits the form of the cross-spectral density function of the source more than in the scalar case [112]. Specifically, for the cross-spectral density operator of a homogeneous and isotropic source current distribution we have the general form

$$
\overline{\mathbf{W}}_{\mathbf{J}}(\mathbf{r},\omega) = W_1(r,\omega)\hat{\mathbf{r}}\hat{\mathbf{r}} + W_2(r,\omega)\bar{\mathbf{I}}, \tag{5.34}
$$

where W_1 and W_2 are arbitrary scalar functions. When this form is used in the representation (5.22), we have shown in Publication III that we the corresponding cross-spectral density operator becomes

$$\begin{aligned}\lim_{\mathrm{Im}\{\kappa\}\to 0^+} \mathrm{Im}\{\kappa\}\overline{\mathbf{W}}_{\mathbf{E}}(\mathbf{r},\omega) &= D_V \,\mathrm{Im}\left\{\overline{\mathbf{G}}_\kappa(\mathbf{r},\mathbf{0})\right\} \\ &= D'_V\left[\frac{1}{3}j_0(\kappa r)\bar{\mathbf{I}} - \frac{1}{6}j_2(\kappa r)\left(\bar{\mathbf{I}} - 3\hat{\mathbf{r}}\hat{\mathbf{r}}\right)\right],\end{aligned} \tag{5.35}$$

where D_V and D'_V are constants with respect to $\mathbf{r}$. With this result, we get for the degree-of-coherence functions in Eqs. (5.27) and (5.28) the expressions

$$\mu_{\mathrm{KW}}(\mathbf{r},\omega) = j_0(\kappa r) = \mathrm{sinc}(\kappa r) \tag{5.36}$$

and

$$\mu_{\mathrm{STF}}(\mathbf{r},\omega) = \sqrt{\frac{1}{3}[j_0(\kappa r)]^2 + \frac{1}{6}[j_2(\kappa r)]^2}, \tag{5.37}$$

which show that the universality result holds for vector-valued fields as well. It is, however, useful to note that the exact form is different for different degree-of-coherence functions. Indeed, as discussed in Sec. 3.2, these functions differ in how they treat polarization effects, and from the universality results (5.36) and (5.37) we have for example $\mu_{\mathrm{KW}}(\mathbf{0},\omega) = 1$ and $\mu_{\mathrm{STF}}(\mathbf{0},\omega) = 1/\sqrt{3}$. Since there is no single agreed-upon degree-of-coherence function for vector-valued fields, it is actually the fixed functional form of the cross-spectral density operator (5.35) that best represents the universal character.

5.5 Universality results and free fields

To understand why the homogeneous and isotropic source distributions give rise to fields, whose cross-spectral densities attain one and the same universal form, it is helpful to consider the free-field plane wave model, which was put forward in Ref. 103 for scalar fields, and which was later extended to vector-valued fields in Ref. 104.

A free field is by definition a field that has no sources at finite distances and hence satisfies the homogeneous Helmholtz equation (2.27) in the scalar case and the vacuum wave equation (2.25) or (2.26) in the full electromagnetic case. It can be shown that a free field can be represented as a collection of plane waves propagating in all directions [35]. In the scalar and vector cases this representation takes the forms [103, 104]

$$U(\mathbf{r},\omega) = \int_{\mathbb{S}} A(\hat{\mathbf{k}},\omega)\exp(ik_0\hat{\mathbf{k}}\cdot\mathbf{r})\mathrm{d}\hat{\mathbf{k}} \tag{5.38}$$

and

$$\mathbf{E}(\mathbf{r},\omega) = \int_{\mathbb{S}} \mathbf{A}(\hat{\mathbf{k}},\omega)\exp(ik_0\hat{\mathbf{k}}\cdot\mathbf{r})\mathrm{d}\hat{\mathbf{k}}, \tag{5.39}$$

respectively. Here, as before, $\mathbb{S}$ denotes the complete spherical shell.

When the free field is such that all the plane waves have the same intensity and are uncorrelated, and in the case of electromagnetic plane waves, unpolarized, so that the cross-spectral density operators of the plane waves become [103, 104]

$$W_A(\hat{\mathbf{k}}_1, \hat{\mathbf{k}}_2, \omega) = \left\langle A(\hat{\mathbf{k}}_1, \omega) A^*(\hat{\mathbf{k}}_2, \omega) \right\rangle = A(\omega)\delta(\hat{\mathbf{k}}_2 - \hat{\mathbf{k}}_1) \tag{5.40}$$

and

$$\overline{\mathbf{W}}_{\mathbf{A}}(\hat{\mathbf{k}}_1, \hat{\mathbf{k}}_2, \omega) = \left\langle \mathbf{A}(\hat{\mathbf{k}}_1, \omega) \mathbf{A}^\dagger(\hat{\mathbf{k}}_2, \omega) \right\rangle = A(\omega)\delta(\hat{\mathbf{k}}_2 - \hat{\mathbf{k}}_1)\left(\overline{\mathbf{I}} - \hat{\mathbf{k}}_1\hat{\mathbf{k}}_1\right), \tag{5.41}$$

it can be shown that the corresponding cross-spectral density operators are proportional to the universal forms given by the last expressions in Eqs. (5.31) and (5.35), respectively [103, 104]. In the vectorial case the cross-spectral density operator also matches the representations in Eqs. (4.2) and (4.3).

A collection of free fields consisting of mutually uncorrelated (and unpolarized) plane waves then produces the same universal field cross-spectral density operators as obtained for fields sourced by homogeneous and isotropic sources. Since plane waves can be taken to be sourced at infinity, we can then interpret such free fields as consisting of contributions from mutually uncorrelated sources that lie asymptotically far away from the observation point. If we contrast this with the renormalization procedure used to obtain the universality results in Secs. 5.3 and 5.4, it is apparent that the universality results are due to the contributions of source regions that are at asymptotic distances from every observation point, and thereby mutually uncorrelated, but which in a (nearly) lossless system collectively produce fields that in amplitude completely overshadow the fields produced by local sources. This leads us to draw the conclusion that the losses, or rather the lack of losses, plays an important part in the universality results. We consider this connection in detail in the next section.

5.6 Effect of losses on the universality results

In this section we consider how losses affect the relationship between source and field coherence functions. To avoid unnecessarily complicating the expressions, we concentrate here on scalar sources and scalar fields. Analogous expressions can be found for vectorial sources and fields.

Let us begin by considering asymptotically large losses in the context of the expression (5.11), valid for fields generated by homogeneous sources. We make the change of variables $\mathbf{S} = \mathrm{Im}\{\kappa\}\mathbf{r}'$ in that expression to get for a scalar field U with a source Q the result

$$W_U(\mathbf{r},\omega) = \frac{2\pi}{(\mathrm{Im}\{\kappa\})^4}\int_{\mathbb{R}^3} \mathrm{sinc}(\mathrm{Re}\{\kappa\}S/\mathrm{Im}\{\kappa\})\exp(-S)W_Q(\mathbf{r}-\mathbf{S}/\mathrm{Im}\{\kappa\},\omega)\mathrm{d}\mathbf{S}. \tag{5.42}$$

We now assume that W_Q is absolutely integrable and continuous, and that $W_Q(\mathbf{0},\omega) < \infty$, so that the integral in Eq. (5.42) converges uniformly to the integral of the limit of its integrand when $\mathrm{Im}\{\kappa\} \to \infty$. By this result, which we have proven in Publication IV, we get from Eq. (5.42) the limiting form

$$\lim_{\mathrm{Im}\{\kappa\}\to\infty} (\mathrm{Im}\{\kappa\})^4 W_U(\mathbf{r},\omega) = 2\pi\int_{\mathbb{R}^3} \exp(-S)W_Q(\mathbf{r},\omega)\mathrm{d}\mathbf{S} = 16\pi^2 W_Q(\mathbf{r},\omega). \tag{5.43}$$

In view of Eq. (5.43) we consequently have

$$W_U(\mathbf{r},\omega) \sim \frac{1}{(\mathrm{Im}\{\kappa\})^4}16\pi^2 W_Q(\mathbf{r},\omega), \tag{5.44}$$

so that the cross-spectral density operator of the field is asymptotically proportional to that of the source, when the losses become infinite. We now use the result (5.44) to compute for the degree-of-coherence function (3.24) the expression

$$\mu(\mathbf{r},\omega) = \frac{W_U(\mathbf{r},\omega)}{W_U(\mathbf{0},\omega)} \sim \frac{16\pi^2 W_Q(\mathbf{r},\omega)/(\mathrm{Im}\{\kappa\})^4}{16\pi^2 W_Q(\mathbf{0},\omega)/(\mathrm{Im}\{\kappa\})^4} = \frac{W_Q(\mathbf{r},\omega)}{W_Q(\mathbf{0},\omega)}, \tag{5.45}$$

where the convergence is uniform as is shown in Publication IV. Accordingly, the degree-of-coherence function of a field produced by a statistically homogeneous source converges uniformly to the degree-of-coherence function of the source, when the losses become asymptotically large. This is, of course, what one might also intuitively expect.

From the considerations in the previous section we know that for asymptotically small losses the degree-of-coherence function of the field approaches the universal form (5.32) if the field is sourced by a homogeneous and isotropic source. In order to gain a better insight into the transition from asymptotically large to asymptotically small losses, we in the following study two homogeneous and isotropic model sources. Specifically, we consider a Gaussian source and a damped-sinc source, given by

$$W_Q(\mathbf{R},\omega) = \exp(-R^2/\gamma^2) \tag{5.46}$$

and

$$W_Q(\mathbf{R},\omega) = \mathrm{sinc}(\chi R) = \frac{1}{i2\mathrm{Re}\{\chi\}R}\left[\exp(i\chi R) - \exp(-i\chi^* R)\right], \tag{5.47}$$

respectively. Here 2γ represents the $1/e$ width of the Gaussian and we take $\mathrm{Im}\{\chi\} > 0$.

When the source (5.46) is used in the expression (5.11), we have shown in Publication IV that the corresponding scalar degree-of-coherence function is given by

$$\begin{aligned}\mu(\mathbf{r},\omega) = {} & \left[2\,\mathrm{Im}\left\{i\kappa\exp(-\kappa^2\gamma^2/4)\,\mathrm{erfc}(-i\kappa\gamma/2)\right\}r\right]^{-1} \\ & \times \mathrm{Im}\Big\{\exp(-\kappa^2\gamma^2/4)\Big[\exp(i\kappa r)\,\mathrm{erfc}(-i\kappa\gamma/2 - r/\gamma) \\ & \qquad - \exp(-i\kappa r)\,\mathrm{erfc}(-i\kappa\gamma/2 + r/\gamma)\Big]\Big\},\end{aligned} \tag{5.48}$$

where erfc is the complementary error function [111].

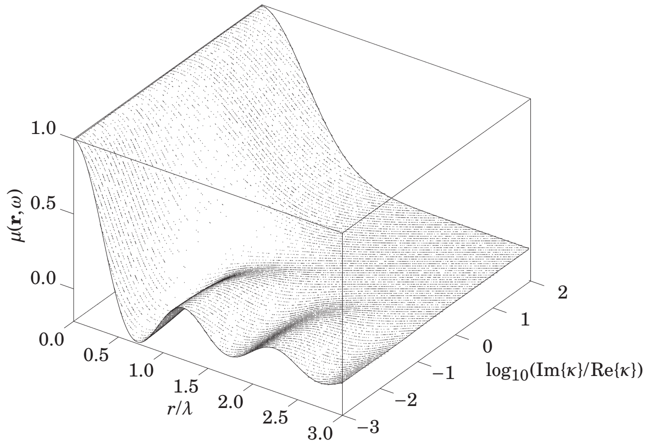
The graphs in Fig. 5.2 display the behavior of the degree-of-coherence function given by Eq. (5.48), as a function of the normalized separation r/λ and the relative loss $\mathrm{Im}\{\kappa\}/\mathrm{Re}\{\kappa\}$. These plots correspond to the parameter values $\mathrm{Re}\{\kappa\}\gamma/2 = 9/4$ and $\mathrm{Re}\{\kappa\}\gamma/2 = 4/9$. Here $\mathrm{Re}\{\kappa\}\gamma/2$ is roughly the ratio of the width of $\exp(-r^2/\gamma^2)$ to the width of the main lobe of $\mathrm{sinc}(\mathrm{Re}\{\kappa\}r)$. From the figures we can see that the limiting behavior corresponding to asymptotically small losses is attained at slightly different amounts of loss in the two cases. Indeed, for the parameter value 9/4 the asymptotic behavior is reached only for relative losses smaller than 10^{-2}, whereas for the parameter value 4/9 the relative losses can be as high as 10^{-1} without affecting the asymptotic form appreciably. These examples then confirm the asymptotic behavior, but also show that the rate, at which the degree-of-coherence function converges to the asymptotic form when the losses become smaller, is dependent on the exact functional form of μ.

Next we introduce the source (5.47) into to expression (5.11), whereby we get after lengthy computations shown in Publication IV, for the corresponding degree-of-coherence function the expression

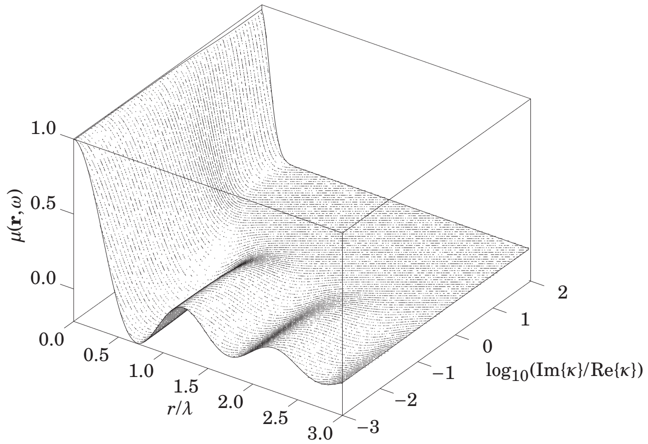
$$\begin{aligned}\mu(\mathbf{r},\omega) = {} & \left|\frac{\chi+\kappa}{\chi+\kappa^*}\right|^2 \Big[\frac{\mathrm{Im}\{\chi\}}{\mathrm{Im}\{\chi\}+\mathrm{Im}\{\kappa\}}\,\mathrm{sinc}(\mathrm{Re}\{\kappa\}r)\exp(-\mathrm{Im}\{\kappa\}r) \\ & \qquad + \frac{\mathrm{Im}\{\kappa\}}{\mathrm{Im}\{\chi\}+\mathrm{Im}\{\kappa\}}\,\mathrm{sinc}(\mathrm{Re}\{\chi\}r)\exp(-\mathrm{Im}\{\chi\}r)\Big] \\ & + 4\frac{\sqrt{\mathrm{Im}\{\chi\}\mathrm{Im}\{\kappa\}}}{\mathrm{Im}\{\chi\}+\mathrm{Im}\{\kappa\}} \\ & \qquad \times \mathrm{Im}\left\{\frac{\sqrt{\mathrm{Im}\{\chi\}\mathrm{Im}\{\kappa\}}(\chi^*+\kappa^*)}{|\chi+\kappa^*|^2}\,\mathrm{sinc}\left(\frac{\chi-\kappa}{2}r\right)\exp\left(i\frac{\chi+\kappa}{2}r\right)\right\}.\end{aligned} \tag{5.49}$$

When $\mathrm{Im}\{\chi\},\mathrm{Im}\{\kappa\} \ll \mathrm{Re}\{\chi\},\mathrm{Re}\{\kappa\}$, we get from Eq. (5.49) the asymptotic expression

$$\begin{aligned}\mu(\mathbf{r},\omega) \sim {} & \frac{1}{1+\mathrm{Im}\{\kappa\}/\mathrm{Im}\{\chi\}}\,\mathrm{sinc}(\mathrm{Re}\{\kappa\}r)\exp(-\mathrm{Im}\{\kappa\}r) \\ & + \frac{1}{1+\mathrm{Im}\{\chi\}/\mathrm{Im}\{\kappa\}}\,\mathrm{sinc}(\mathrm{Re}\{\chi\}r)\exp(-\mathrm{Im}\{\chi\}r).\end{aligned} \tag{5.50}$$



(a)



(b)

Figure 5.2. Behavior of the degree-of-coherence function in Eq. (5.48) with the separation r and the relative loss $\mathrm{Im}\{\kappa\}/\mathrm{Re}\{\kappa\}$, plotted at parameter values (a) $\mathrm{Re}\{\kappa\}\gamma/2 = 9/4$ and (b) $\mathrm{Re}\{\kappa\}\gamma/2 = 4/9$.

We note that when $\mathrm{Im}\{\chi\}/\mathrm{Im}\{\kappa\} \gg 1$, the first term in Eq. (5.50) dominates and the degree-of-coherence function deviates only slightly from the universal form (5.32) since $\mathrm{Im}\{\kappa\} \ll \mathrm{Re}\{\kappa\}$, by assumption. On the other hand, when $\mathrm{Im}\{\chi\}/\mathrm{Im}\{\kappa\} \ll 1$, the second term dominates and the degree-of-coherence function of the field is nearly equal to that of the source, for which $\mathrm{Re}\{\chi\}$ is arbitrary.

We have plotted the degree-of-coherence function given by Eq. (5.49) in Fig. 5.3, with $\mathrm{Re}\{\chi\} = 4\,\mathrm{Re}\{\kappa\}$ and the relative loss $\mathrm{Im}\{\kappa\}/\mathrm{Re}\{\kappa\} = 10^{-16}$. These parameter values imply that the plot is in the regime of the asymptotic form (5.50). From the figure we note that when the parameter $\mathrm{Im}\{\chi\}$ decreases, the degree-of-coherence function of the field converges toward that of

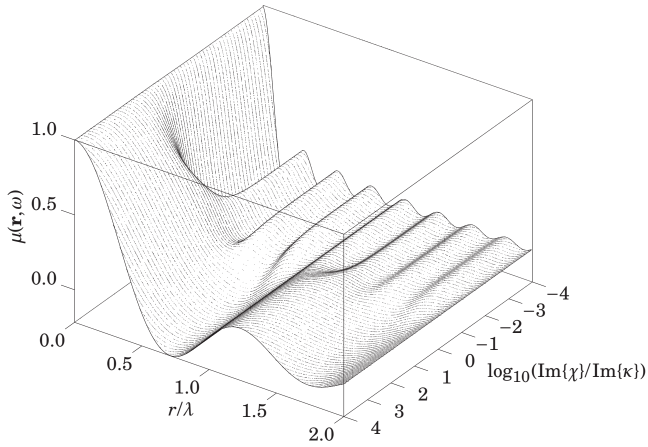


Figure 5.3. Transition of the degree-of-coherence function of the field from its universal form to a form with a shorter coherence length, when the loss is kept constant at $\mathrm{Im}\{\kappa\} = 10^{-16}\,\mathrm{Re}\{\kappa\}$, and the parameter $\mathrm{Im}\{\chi\}/\mathrm{Im}\{\kappa\}$ is varied. This parameter controls the functional form of the source cross-spectral density.

the source, whereas when the parameter decreases, the function converges to the universal form. This example then explicitly illustrates the fact that a fixed amount of relative loss (here 10^{-16}) can correspond both to negligibly small losses (universal form) or large losses (degree of coherence of the source), which was already hinted at by the previous example. Thereby the assertion that the universal form of the degree-of-coherence function is reached for ‘negligibly’ small losses [110] is not unambiguous, even when the criterion (5.24) holds. Accordingly, the smallness of the losses is relative to the functional form of the degree-of-coherence function, and since all actual systems are lossy, this means that care must be taken when assuming that the degree-of-coherence function of a particular low-loss system is of the universal form (5.32).

5.7 Effect of source boundaries on the universality

In the previous section we showed that the universality result (5.32) should not be taken too literally when the system under study is lossy, as real systems are. We now consider, again referring to actual systems, what happens to universality if the system is not unbounded. As discussed before, this question was in part answered in Ref. 109, where the properties of a field corresponding to a Bessel correlated source are studied when the source region radius R grows toward infinity from a value below the wavelength of the field. The results of that investigation suggest that ‘unbounded’ behavior starts to dominate when

R exceeds about 5λ, which is in agreement with the results for δ-correlated sources obtained in Ref. 107. Below we, however, show that for any bounded region, lossy or not, the (restrictedly homogeneous and isotropic) source distribution can be so chosen that the field has any prescribed cross-spectral density inside the source region.

Because we have considered the scalar case in detail in Publication V, we concentrate here on the vector-valued electromagnetic case. Now, let $\Omega' \subset \mathbb{R}^3$ denote a source region, and let $\mathbf{F}$ be any vector-valued random function with twice continuously differentiable realizations, defined over a larger region $\Omega \supset \Omega'$, and let $\overline{\mathbf{W}}_{\mathbf{F}}$ be its cross-spectral density operator. Here we furthermore assume that the regions satisfy $\Omega' \cap \partial\Omega = \emptyset$. We then consider a random electromagnetic field $\mathbf{E}'$, which satisfies a wave equation of the form (2.21), where the material properties and the random source current $\mathbf{J}'$ are sufficiently regular, as well as the Silver–Müller radiation condition (2.40). In addition, we assume that the random field $\mathbf{E}'$ and the random function $\mathbf{F}$ are not correlated anywhere, so that

$$\left\langle \mathbf{E}'(\mathbf{r},\omega)\mathbf{F}^{\dagger}(\mathbf{r}')\right\rangle = \overline{\mathbf{0}}, \quad \forall \mathbf{r},\mathbf{r}'. \tag{5.51}$$

We then put

$$\mathbf{E}(\mathbf{r},\omega) = \vartheta(\mathbf{r})\mathbf{F}(\mathbf{r}) + [1-\vartheta(\mathbf{r})]\mathbf{E}'(\mathbf{r},\omega), \tag{5.52}$$

where ϑ is any twice continuously differentiable function, such that

$$\vartheta(\mathbf{r}) = \begin{cases} 1, & \mathbf{r} \in \Omega' \\ 0, & \mathbf{r} \notin \Omega. \end{cases} \tag{5.53}$$

Next we define the source current $\mathbf{J}$ by the equation

$$\mu(\mathbf{r},\omega)\nabla \times \left[\frac{1}{\mu(\mathbf{r},\omega)}\nabla \times \mathbf{E}(\mathbf{r},\omega)\right] - \kappa^2(\mathbf{r},\omega)\mathbf{E}(\mathbf{r},\omega) = i4\pi\frac{k_0}{c}\mu(\mathbf{r},\omega)\mathbf{J}(\mathbf{r},\omega). \tag{5.54}$$

It then follows, in particular, that $\mathbf{E}$ satisfies this equation and the Silver–Müller radiation condition (2.40). Accordingly, the source current distribution $\mathbf{J}$ uniquely specifies $\mathbf{E}$ and, thus, once $\mathbf{J}$ is specified we can determine $\mathbf{E}$. Because $\mathbf{E}'$ and $\mathbf{F}$ are uncorrelated, it follows that the cross-spectral density operator $\overline{\mathbf{W}}_{\mathbf{E}}$ of the field $\mathbf{E}$ is given by

$$\overline{\mathbf{W}}_{\mathbf{E}}(\mathbf{r}_1,\mathbf{r}_2,\omega) = \vartheta(\mathbf{r}_1)\vartheta(\mathbf{r}_2)\overline{\mathbf{W}}_{\mathbf{F}}(\mathbf{r}_1,\mathbf{r}_2) + [1-\vartheta(\mathbf{r}_1)][1-\vartheta(\mathbf{r}_2)]\overline{\mathbf{W}}_{\mathbf{E}'}(\mathbf{r}_1,\mathbf{r}_2,\omega), \tag{5.55}$$

where $\overline{\mathbf{W}}_{\mathbf{E}'}$ denotes the cross-spectral density operator of the field $\mathbf{E}'$. This relation shows that the cross-spectral density $\overline{\mathbf{W}}_{\mathbf{E}}(\mathbf{r}_1,\mathbf{r}_2,\omega)$ equals the cross-spectral density $\overline{\mathbf{W}}_{\mathbf{F}}(\mathbf{r}_1,\mathbf{r}_2)$ for all pairs of points $\mathbf{r}_1,\mathbf{r}_2 \in \Omega'$. It then follows that any of the vector degree-of-coherence functions defined in Sec. 3.2 will agree for the electromagnetic field $\mathbf{E}$ and the arbitrary vector-valued function $\mathbf{F}$ at such points. We have thus shown that the coherence properties of the field **E**, within its source region Ω minus the boundary region $\Omega\backslash\Omega'$, are equal to the coherence properties of the random function **F**.

Since this result holds for an arbitrary random function **F**, the corresponding cross-spectral density operator $\overline{\overline{\mathbf{W}}}_{\mathbf{F}}$ can, in particular, be assumed to be homogeneous and isotropic within the source region Ω. Because the coherence properties of the random electromagnetic field **E**, as constructed above, equal the coherence properties of **F** in Ω', and since we can take Ω' to be arbitrarily close to the (full) source region Ω, it then follows that no universal form exists for the cross-spectral density operators of electromagnetic fields sourced by homogeneous and isotropic sources inside the source regions. As was mentioned above, we presented the corresponding results for scalar fields in Publication V. In view of the definitions of the degree-of-coherence functions in Sec. 3.2 these results then also imply a lack of universality for scalar and electromagnetic degree-of-coherence functions. Specifically, we observe that there is no lower limit on the coherence length of a field within its source region. Our results thus overthrow the widely held belief that no field can have a coherence length that is shorter than $\lambda/2$, which, as is discussed in Sec. 3.4 is the coherence length of for example fields produced by blackbody sources.

We illustrate the electromagnetic construction by a concrete example, where $\Omega = [-R/2, R/2]^3$ and $\Omega' = [-R'/2, R'/2]^3$ are two nested cubes with $R' < R$, as shown in Fig. 5.4. We also assume that apart from the source current **J**, all

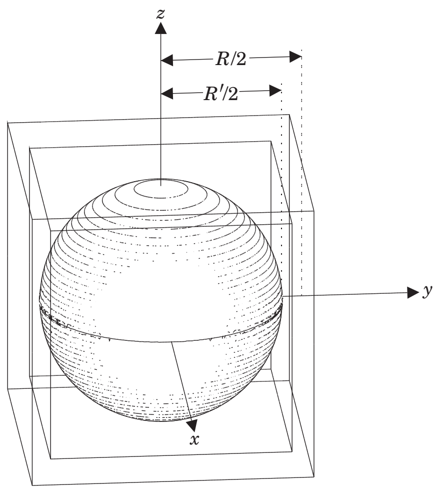


Figure 5.4. Illustration of the regions $\Omega = [-R/2, R/2]^3$ and $\Omega' = [-R'/2, R'/2]^3$, and the spherical source region $B(\mathbf{0}, R'/2)$ used in the example.

space is vacuum, viz., $\kappa(\mathbf{r}, \omega) = k_0$ and for the magnetic permeability we have

$\mu(\mathbf{r},\omega) = 1$ in Eq. (5.54) for all $\mathbf{r}$. In addition, we take the fields $\mathbf{E}'$ in the ensemble $\{\mathbf{E}'\}$ to be produced by an ensemble $\{\mathbf{J}'\}$ of spherical source currents (see Fig. 5.4) of the form

$$\mathbf{J}'(\mathbf{r},\omega) = \begin{cases} Q_0 \nabla \times \left[j_1(k_0 r) Y_1^0(\hat{\mathbf{r}})\mathbf{r} \right], & |\mathbf{r}| < R'/2, \\ \mathbf{0}, & |\mathbf{r}| \geq R'/2, \end{cases} \tag{5.56}$$

where Q_0 is a random variable, which is uncorrelated with $\mathbf{F}$, so that $\langle \mathbf{F}(\mathbf{r}) Q_0^* \rangle = \mathbf{0}$ for all $\mathbf{r}$. It is straightforward to show, e.g., by using the results in Publication VI, that the fields $\mathbf{E}'$ corresponding to the source currents $\mathbf{J}'$ in Eq. (5.56) are explicitly given outside the ball $B(\mathbf{0}, R'/2)$ by

$$\mathbf{E}'(\mathbf{r}) = C Q_0 \nabla \times \left[h_1^{(1)}(k_0 r) Y_1^0(\hat{\mathbf{r}})\mathbf{r} \right], \quad |\mathbf{r}| > R'/2, \tag{5.57}$$

where C is a constant that depends on the radius R' and $h_1^{(1)}$ is the spherical Hankel function of type 1 and order 1. For the cross-spectral density operator $\overline{\mathbf{W}}_{\mathbf{F}}$, we take the blackbody or universal form Eq. (5.35)

$$\overline{\mathbf{W}}_{\mathbf{F}}(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{3} j_0(\chi r) \bar{\mathbf{I}} - \frac{1}{6} j_2(\chi r) \left(\bar{\mathbf{I}} - 3\hat{\mathbf{r}}\hat{\mathbf{r}} \right), \tag{5.58}$$

where $\chi \in \mathbb{R}$ and $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. In this example we define the function ϑ by

$$\vartheta(\mathbf{r}) = P(|\hat{\mathbf{x}} \cdot \mathbf{r}|) P(|\hat{\mathbf{y}} \cdot \mathbf{r}|) P(|\hat{\mathbf{z}} \cdot \mathbf{r}|), \tag{5.59}$$

where

$$P(s) = \begin{cases} 1, & 0 \leq s < R'/2, \\ p\left(\dfrac{2s - R'}{R - R'} \right), & R'/2 \leq s \leq R/2, \\ 0, & s > R/2, \end{cases} \tag{5.60}$$

and $p(t) = -6t^5 + 15t^4 - 10t^3 + 1$, which makes ϑ twice continuously differentiable, as required.

Let us now consider the cross-spectral density operator differential equation (3.15), when the corresponding field differential equation (3.13) is given by the wave equation (5.54). We then introduce the representations (5.57) and (5.58) into that differential equation (where $\kappa = k_0$), and note that it follows from Eq. (2.48) (where $\kappa = \chi$) that the imaginary part of the dyadic Green's tensor satisfies the corresponding homogeneous wave equation, whereby we get for $\mathbf{r}_1, \mathbf{r}_2 \in \Omega'$ for the cross-spectral density of the source the expression $\overline{\mathbf{W}}_{\mathbf{J}}(\mathbf{r}_1, \mathbf{r}_2, \omega) = (\chi^2 - k_0^2)^2 \overline{\mathbf{W}}_{\mathbf{F}}(\mathbf{r}_1, \mathbf{r}_2, \omega)$. Hence the source distribution $\mathbf{J}$ has inside Ω' the same stochastic properties as the function $\mathbf{F}$. Specifically, from the definition (5.58) it follows that $\overline{\mathbf{W}}_{\mathbf{J}}$ is homogeneous and isotropic within Ω'.

In Fig. 5.5 we have used $R = \lambda$, $R' = 6\lambda/7$ and $\chi = 7k_0$ to plot the degree-of-coherence function $\mu_{\mathrm{STF}}(\mathbf{r}_1, \mathbf{r}_2, \omega)$ of $\mathbf{E}'$ on two different planes Π dissecting

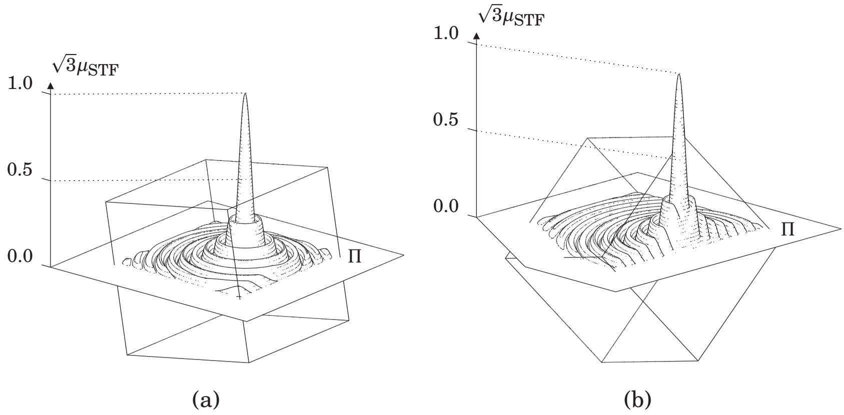


Figure 5.5. Two examples of the degree-of-coherence function μ_{STF} of a field inside a cube that contains the source distribution. The degree-of-coherence function is in both cases displayed between a fixed point and all other points in a plane Π that dissects the cube. The side of the cube is equal to the wavelength λ.

the cube $[-R/2, R/2]^3$. To obtain a clearer illustration of the situation, we have thereby not here made use of the possibility to let R' be almost equal to R. In the plots the point $\mathbf{r}_1 \in \Pi$ is kept fixed and $\mathbf{r}_2 \in \Pi$ is swept across Π. The location of $\mathbf{r}_1$ is revealed by the peak where the degree-of-coherence function reaches its maximum value $1/\sqrt{3}$. Since the side of the cube is equal to λ, it is clear from these plots that the coherence length of the field inside the source region is much less than $\lambda/2$, both far from [Fig. 5.5(a)] and near [Fig. 5.5(b)] the boundary. Indeed, since the degree-of-coherence function of the field follows the degree-of-coherence function of $\mathbf{F}$, Eq. (5.58) with $\chi = 7k_0$ implies that the coherence length is, in fact, $\lambda/14$. We note that although the source region diameter in this example does not exceed 5λ, which has been established as the limit when the universal character of the field is expected for spherical source regions, and homogeneous and isotropic source distributions [107,109] (see also Secs. 5.2 and 5.4), our construction remains valid for any source region diameter. This is shown explicitly for scalar fields in spherical regions in Publication V, where we have obtained a similar result for a larger source region diameter. This example then serves to prove the assertion that the coherence length of (electromagnetic) fields can be arbitrarily small in the source region, even when the source is lossless, and (locally) homogeneous and isotropic.

6. Effective degree of coherence

In this chapter we will discuss the so-called effective degree of coherence, denoted by $\overline{\mu}$, which is also known as the overall or global degree of coherence [62, 72, 113–125]. This quantity was originally introduced in terms of the degree of incoherence $h = 1 - \overline{\mu}^2$ long ago [126, 127], but it has attracted new interest in the last few years. In particular, its invariance to different transformations and representations of the electromagnetic field have been explored [113, 122–124]. Furthermore, it provides a fresh perspective to the discussion of how to define the degree-of-coherence function for electromagnetic fields, as shown for example in Ref. 72.

Our interest in the effective degree-of-coherence functional was to explore how far the invariance of this function extends. In the following we present an overview of the results we obtained in Publication VI, which show that the effective degree of coherence can, in fact, be taken as an intrinsic property of the electromagnetic field. To begin with we consider its definition as an energy or intensity weighted average of the squared modulus of the scalar degree-of-coherence function and then go on to show that it can be represented in an abstract Hilbert-space setting. This point of view gives us an important insight into the nature of the effective degree of coherence, lets us extend its use beyond scalar fields in a natural manner, and provides us with the tools to prove the intrinsic nature of the effective degree of coherence of an electromagnetic field. The abstract Hilbert-space representation suggests that there are other possible definitions of an effective degree of coherence, which are equally intrinsic. These other definitions, however, lack an important additional property which the original effective degree of coherence possesses, that is, the effective degree of coherence when applied to a sum of orthogonal fields can be expressed in terms of the pairwise effective degrees of coherence of these fields. We also discuss practical applications of the effective degree of coherence and how to measure it in actual systems.

6.1 Effective degree of coherence: definition and explicit expressions

The effective degree of coherence, $\overline{\mu}_U$, of a random scalar field U over a region Ω, which for scalar fields is usually taken to be two- (surface) or three-dimensional (volume), is defined as the root-mean-square average of the intensity-weighted two-point degree-of-coherence function (3.24) [115, 119, 121, 122]. Hence its square can be written in terms of the covariance function $W_U(\mathbf{r}_1,\mathbf{r}_2,\omega)$ as

$$\begin{aligned}\overline{\mu}_U^2 &= \frac{\int_\Omega\int_\Omega W_U(\mathbf{r}_1,\mathbf{r}_1,\omega)W_U(\mathbf{r}_2,\mathbf{r}_2,\omega)|\mu_U(\mathbf{r}_1,\mathbf{r}_2,\omega)|^2 \mathrm{d}\mathbf{r}_1\mathrm{d}\mathbf{r}_2}{\left[\int_\Omega W_U(\mathbf{r},\mathbf{r},\omega)\mathrm{d}\mathbf{r}\right]^2} \\ &= \frac{\int_\Omega\int_\Omega |W_U(\mathbf{r}_1,\mathbf{r}_2,\omega)|^2 \mathrm{d}\mathbf{r}_1\mathrm{d}\mathbf{r}_2}{\left[\int_\Omega W_U(\mathbf{r},\mathbf{r})\mathrm{d}\mathbf{r}\right]^2}.\end{aligned} \tag{6.1}$$

As the discussion in Sec. 3.1 suggests, the covariance function $W_U(\mathbf{r}_1,\mathbf{r}_2,\omega)$ is also an operator in a Hilbert space, which for scalar fields typically is $L^2(\Omega)$, for some region Ω. This allows us to rewrite the definition (6.1) in a coordinate-free form, given by [114, 115, 117]

$$\overline{\mu}_u^2 = \frac{\mathrm{Tr}(W_u^\dagger W_u)}{[\mathrm{Tr}(W_u)]^2} = \frac{\mathrm{Tr}\left(\langle uu^\dagger\rangle\langle uu^\dagger\rangle\right)}{\left[\mathrm{Tr}(\langle uu^\dagger\rangle)\right]^2} = \frac{\sum_n \lambda_n^2}{(\sum_n \lambda_n)^2}, \tag{6.2}$$

where we have dropped the explicit dependence on the angular frequency ω for notational clarity, and where each realization of the random field u belongs to a Hilbert space H and $\langle \|u\|^2\rangle < \infty$ is assumed. In Eq. (6.2) the last representation follows from an eigenvalue expansion of W_u of the form (3.10), where the existence of such an expansion for all covariance operators was discussed in Sec. 3.1. It is clear from the definition (6.2) that $\overline{\mu}_u^2 \geq 0$, and if we apply Schwartz' inequality to the numerator of the last expression, we get the upper bound $\overline{\mu}_u^2 \leq 1$, so that

$$0 \leq \overline{\mu}_u^2 \leq 1, \tag{6.3}$$

where the values 1 and 0 correspond to a coherent and a completely incoherent field, respectively, just like with ordinary degree-of-coherence functions. In particular, since equality is obtained in Schwartz' inequality when at most one term is nonzero, we furthermore can characterize complete coherence ($\overline{\mu}_u^2 = 1$) by

$$\lambda_1 \neq 0, \quad \lambda_n = 0,\ n > 1, \tag{6.4}$$

that is, the covariance operator W_u corresponds to a completely coherent function precisely when its Mercer series only consists of one coherent mode, whereby the covariance operator is a rank-1 operator. This of course exactly mirrors the property required from ordinary degree-of-coherence functions as discussed in Sec. 3.3.

We note that the coordinate-free expression (6.2) does not depend on the Hilbert space H in which the covariance operator W_u is defined. Thus we can use the coordinate-free expression to define the effective degree of coherence for a covariance operator in any Hilbert space. In particular, we can then consider for example the Hilbert space $L^2(\Omega \times \{x,y,z\})$, where $\Omega \subset \mathbb{R}^3$ and $\{x,y,z\}$ represents the discrete space of vector components. Suppose now that $\{E\}$ is an ensemble of electric fields $E \in L^2(\Omega \times \{x,y,z\})$, with $\langle \|E\|^2 \rangle < \infty$. Then we get for the square of the effective degree of coherence of such fields from Eq. (6.2) the expression

$$\overline{\mu}_{\mathbf{E}}^2 = \frac{\mathrm{Tr}(\overline{\mathbf{W}}_{\mathbf{E}}^{\dagger}\overline{\mathbf{W}}_{\mathbf{E}})}{[\mathrm{Tr}(\overline{\mathbf{W}}_{\mathbf{E}})]^2} = \frac{\int_\Omega \int_\Omega \left\| \overline{\mathbf{W}}_{\mathbf{E}}(\mathbf{r}_1, \mathbf{r}_2, \omega) \right\|_F^2 \mathrm{d}\mathbf{r}_1 \mathrm{d}\mathbf{r}_2}{\left\{ \int_\Omega \mathrm{Tr}[\overline{\mathbf{W}}_{\mathbf{E}}(\mathbf{r}, \mathbf{r}, \omega)] \mathrm{d}\mathbf{r} \right\}^2}. \tag{6.5}$$

We observe that the result (6.5) coincides with the definition for the effective degree of coherence of an electric field as given in Ref. 62, where it is based on the electromagnetic degree-of-coherence function μ_{STF} as given by Eq. (3.28). It is noteworthy that two conceptually completely different approaches lead to exactly the same definition. This connection is further strengthened by the considerations in Sec. 6.7.

6.2 Invariance to scaled unitary mappings

As the effective degree of coherence gives an average measure of the coherence properties of a function (such as the scalar or electromagnetic field) in a Hilbert space, it is of interest to consider what mappings between Hilbert spaces leave the effective degree of coherence unchanged. For the electromagnetic scalar and vector fields these Hilbert spaces can for example correspond to a representation of the field within a volume, on a plane, on a surface, in the far field or simply be equal to the sequence space ℓ_2 of a sequence representation, such as the partial wave expansion, of the field. The mappings are not restricted to the change of representation basis, but may also correspond to transformations of the field when it passes through optical systems, is otherwise scattered, or simply changes with propagation, although the last case could also be interpreted just as a change of basis. In the electromagnetic case these mappings can also have a practical function, since it is for example seldom possible to evaluate the electromagnetic field inside a volume, whereas its far-field pattern is more readily available for measurement.

Let H_1 and H_2 be two Hilbert spaces and let $T : H_1 \to H_2$ be a bounded mapping between these spaces. For all fields $u \in H_1$ there is hence a corresponding field $v \in H_2$, given by

$$v = Tu. \tag{6.6}$$

Suppose then that the mapping T is what we call a *scaled unitary mapping*, viz.,

$$\|v\| = \|Tu\| = \sqrt{\alpha}\,\|u\|, \quad \forall u \in H_1, \tag{6.7}$$

for some constant $\alpha \in (0,\infty)$. By squaring the relation (6.7), we get

$$u^\dagger T^\dagger T u = \alpha u^\dagger u, \quad \forall u \in H_1 \quad \Leftrightarrow \quad T^\dagger T = \alpha I \quad \Leftrightarrow \quad T^\dagger S T = I, \tag{6.8}$$

where $S = \alpha^{-1} I'$ and I (I') is the identity mapping in H_1 (H_2).

When $\{u\}$ is an arbitrary ensemble of functions $u \in H_1$ that satisfies $\langle \|u\|^2 \rangle < \infty$ it follows readily that $\langle \|v\|^2 \rangle < \infty$, and we get from the definition (6.2) for the square of the effective degree of coherence of the ensemble $\{v\}$ of functions $v \in H_2$ the expression

$$\begin{aligned}\overline{\mu}_v^2 &= \frac{\mathrm{Tr}\left(\langle vv^\dagger\rangle\langle vv^\dagger\rangle\right)}{\left[\mathrm{Tr}\left(\langle vv^\dagger\rangle\right)\right]^2} = \frac{\mathrm{Tr}\left(T\langle uu^\dagger\rangle T^\dagger T\langle uu^\dagger\rangle T^\dagger\right)}{\left[\mathrm{Tr}\left(T\langle uu^\dagger\rangle T^\dagger\right)\right]^2} \\ &= \frac{\mathrm{Tr}\left(\langle uu^\dagger\rangle T^\dagger T\langle uu^\dagger\rangle T^\dagger T\right)}{\left[\mathrm{Tr}\left(\langle uu^\dagger\rangle T^\dagger T\right)\right]^2} = \frac{\alpha^2\,\mathrm{Tr}\left(\langle uu^\dagger\rangle\langle uu^\dagger\rangle\right)}{\alpha^2\left[\mathrm{Tr}\left(\langle uu^\dagger\rangle\right)\right]^2} = \overline{\mu}_u^2,\end{aligned} \tag{6.9}$$

where the last step again follows from the definition (6.2). The second step, in turn, follows from the identity $\mathrm{Tr}(ABC) = \mathrm{Tr}(CAB)$, which holds for bounded linear Hilbert-space operators A, B, and C, when at least one of them is a trace-class operator with a representation of the form (3.10). The result (6.9) implies:

Theorem 1 *The effective degree of coherence is invariant with respect to scaled unitary mappings between Hilbert spaces.*

As we have shown in Publication VI, this relationship is complemented by:

Theorem 2 *Of all bounded linear Hilbert space mappings, only the scaled unitary mappings leave the effective degree of coherence unchanged.*

Theorems 1 and 2 show that the effective degree of coherence is a consistent measure of the coherence properties of a function across different Hilbert-space representations as long as these representations are related by scaled unitary mappings. Thereby, as the definition (6.7) suggests, the norms of the respective representations should be proportional to each other with proportionality factors α that are constants independent of the functions. As we will show in the next section, this requirement for the invariance of the effective degree of coherence is relaxed enough so that the effective degree of coherence will have the same value in most of the representation Hilbert spaces of interest for electromagnetic fields.

6.3 Scaled unitary mappings in electromagnetics and the invariance of the effective degree of coherence

From Theorems 1 and 2 we know that the effective degree of coherence remains invariant when the field is transformed by scaled unitary mappings. Since most of the usually applied representation spaces for electromagnetic fields are, in fact, Hilbert spaces with norms, whose squares are proportional to the energy (intensity) of the field, it turns out that the effective degree of coherence is an invariant or intrinsic property of electromagnetic fields.

Indeed, as the results presented in Publication VI show, the coefficients of the partial wave expansions of the electric (E) and magnetic (H) components of an electromagnetic field $[E,H]$ have the same sequence space (ℓ_2) norm. Furthermore, this norm is proportional to the (potential) energy of the electric field, the energy of the magnetic field, and to their sum, the energy of the electromagnetic field in asymptotically large balls. Thus it follows from Theorem 1 that the effective degree of coherence has the same value in all of these representation spaces. This value can then be called the *effective degree of coherence of the electromagnetic field*, $\overline{\mu}_{[E,H]}$.

Furthermore, for fields carrying energy to infinity, or so-called outgoing fields, it can be shown that the total intensity of their far-field patterns is proportional to the square of their sequence space norm. On the other hand, the total intensity of a paraxial field, which is equal to the integral of the intensity across a plane transversal to the propagation direction, is also equal to the far-field intensity of that field. Consequently Theorem 1 implies that the effective degree of coherence evaluated from the far-field pattern of a field or from the transversal plane of a paraxial field will be equal to $\overline{\mu}_{[E,H]}$. In particular, it means that the effective degree of coherence as evaluated from the so-called Wigner distribution [128, 129] of the field will also produce this value.

It is furthermore useful to notice that for free fields, which can be used to model fields incident onto a scatterer, the incoming and outgoing parts of their partial wave expansions have the same coefficients and thus the same ℓ_2 norm, which is equal to exactly half the total ℓ_2 norm. Hence the effective degrees of coherence of the incoming and outgoing parts of a free field (incident field) are by Theorem 1 equal to each other and to the effective degree of coherence of the entire field, $\overline{\mu}_{[E,H]}$. When a free field is scattered, only the outgoing field coefficients are changed by the mapping related to the scatterer. For lossless scatterers this mapping is unitary and hence it follows that the squared norm of the outgoing part of the scattered field is equal to half the squared norm of the incident field. Theorem 1 then implies that the effective degree of coherence is invariant to scattering by a lossless scatterer.

However, we observe that the mapping (2.46), which maps the source distribution to the corresponding outgoing field, is not a scaled unitary mapping if the source current is taken to lie in the Hilbert space $L^2(\Omega)$, as is evident for example from the existence of so-called non-radiating sources [130]. Of course, we could choose a different representation Hilbert space for the sources, where the norm of a source would be proportional to the energy that the corresponding field carries out of Ω, but apart from such elaborately constructed Hilbert spaces, we can conclude on the basis of Theorem 2 that the effective degree of a source and the effective degree of the field generated by this source are generally not equal.

Finally, we mention that the invariance of the effective degree of coherence to (scaled) unitary mappings has earlier been established in special cases. For the propagation of paraxial fields through lossless optical systems this invariance was proven in Ref. 113, and for the Fourier transform mapping between the temporal and time–frequency representations of electromagnetic (scalar) fields, invariance was shown in Ref. 123. The equality of the electric and magnetic effective degrees of coherence was, in turn, determined in Ref. 124.

We note that all of these invariance results follow since the corresponding Hilbert spaces are related by scaled unitary mappings. In the following we explore what other measures, and in particular alternative definitions of an effective degree of coherence, are invariant to scaled unitary mappings.

6.4 Functionals invariant to scaled unitary mappings

First we note that the square of the effective degree of coherence as given in Eq. (6.2) is a functional of the covariance operator $W_u = \langle uu^\dagger \rangle$, that is,

$$\overline{\mu}_u^2 = \overline{\mu}_u^2(W_u) = \overline{\mu}_u^2(\langle uu^\dagger \rangle), \tag{6.10}$$

and we now consider what form such a functional must have to remain invariant with respect to scaled unitary mapping.

Since the operator W_u has a Mercer series of the form (3.10), we get from the definition (6.6) the result

$$\begin{aligned} W_v &= \langle vv^\dagger \rangle = \langle (Tu)(Tu)^\dagger \rangle = T\langle uu^\dagger \rangle T^\dagger = TW_uT^\dagger \\ &= T\big(\sum_n \lambda_n \phi_n {\phi_n}^\dagger\big)T^\dagger = \sum_n (\alpha\lambda_n)\left(\alpha^{-1/2}T\phi_n\right)\left(\alpha^{-1/2}T\phi_n\right)^\dagger, \end{aligned} \tag{6.11}$$

where the introduction of α at the last step is suggested by the orthonormality relation

$$\left(\alpha^{-1/2}T\phi_n\right)^\dagger\left(\alpha^{-1/2}T\phi_n\right) = \alpha^{-1}\phi_n{}^\dagger T^\dagger T\phi_m = \phi_n{}^\dagger\phi_m = \delta_{n,m}, \tag{6.12}$$

which is based on Eq. (6.8) and the orthonormality of the set $\{\phi_n\}_n$. The orthonormality relation (6.12) and the positivity of the coefficients $(\alpha\lambda_n)$ in the representation (6.11) imply that that expansion is, in fact, the Mercer series of W_v, where $\alpha\lambda_n$ are the eigenvalues and $\alpha^{-1/2}T\phi_n$ are the corresponding eigenfunctions. Because the mapping T is an arbitrary unitary mapping, these eigenfunctions can represent any complete basis of H_2 and since the coefficient α may equal any positive real number, it follows that the only invariants of the mapping are the normalized eigenvalues, given by

$$\nu_n = \frac{\lambda_n}{\sum_{n'}\lambda_{n'}}. \tag{6.13}$$

Accordingly, the functionals of the coherence operator that are unchanged by scaled unitary mappings must be of the form

$$\tau(W_u) = \tau_{\text{eig}}(\nu_1, \nu_2, \ldots), \tag{6.14}$$

where we assume the order $\nu_1 \geq \nu_2 \geq \cdots \geq 0$, and the subscript eig refers to the fact that the arguments of the functional are the (normalized) eigenvalues. Hence we have:

Theorem 3 *A functional of a covariance operator is invariant with respect to scaled unitary mappings if and only if it is a function of the normalized eigenvalues (6.13) of the covariance operator, that is, if it is of the form (6.14).*

Theorem 3 is important since it allows us to extend all the invariance results of the previous section to any measure (of coherence) of the form (6.14). Let us now consider a family $\{\overline{\mu}_{u,q}\}_q$ of such measures [114, 116, 117]. These overall degree-of-coherence functionals are given in terms of the normalized eigenvalues (6.13) by

$$\overline{\mu}_{u,q} = \left(\sum_n \nu_n^q\right)^{1/q}, \quad q > 1, \tag{6.15}$$

including the limits

$$\overline{\mu}_{u,\infty} = \lim_{q\to\infty}\overline{\mu}_{u,q} = \nu_1 \tag{6.16}$$

and

$$\overline{\mu}_{u,\text{ent}} = \lim_{q\to 1^+}\overline{\mu}_{u,q} = \exp\left(\sum_n \nu_n \log \nu_n\right), \tag{6.17}$$

where the subscript ent refers to entropy, since the argument of the exponential is the negative of the Shannon (information) entropy [131]. The overall degree-of-coherence functional $\overline{\mu}_{u,\text{ent}}$ can as a consequence be seen as a measure of the lack of information in the field, whence the larger this value is, the less information is needed to describe the field. This limiting form of the overall degree-of-coherence functionals has another interesting property. As is proven in Ref. 116, the family of effective degree-of-coherence functionals is strictly ordered in magnitude, so that

$$\overline{\mu}_{u,q} \leq \overline{\mu}_{u,p}, \quad \forall 1 \leq q \leq p \leq \infty, \tag{6.18}$$

and thus $\overline{\mu}_{u,\mathrm{ent}}$ always has the smallest value. As a result, the entropy degree of coherence, in particular, decreases from 1 more rapidly than any other of the family members, when the field goes from completely coherent to increasingly incoherent. The entropy overall degree-of-coherence functional is hence the most sensitive to deviations from complete coherence, and could in this sense be termed the 'best' overall degree-of-coherence functional [116, 117]. Since it follows from the definition (6.13) and a comparison of Eqs. (6.2) and (6.15) that

$$\overline{\mu}_u^2 = \overline{\mu}_{u,2}^2, \tag{6.19}$$

we could argue that the effective degree-of-coherence functional $\overline{\mu}_u$ is, for integer values of q, the 'second best' overall degree-of-coherence functional. Nevertheless, for practical purposes it is still the more appealing choice, because it can directly be evaluated from the pointwise values of the covariance operator [e.g., Eqs. (6.1) and (6.5)] and is thereby readily measurable, as is discussed in Sec. 6.8. Furthermore, the direct connection to the pointwise values of the covariance operator also leads to an important consistency property for $\overline{\mu}_u$ that will be explored in Sec. 6.7.

Finally, we note that the family $\{\overline{\mu}_{u,q}\}_q$ of overall degree-of-coherence functionals does not exhaust the possible choices of functionals of the form (6.14), even if all functions of the values $\overline{\mu}_{u,q}$ are included in the considerations.

6.5 Mappings and intrinsic properties (of electromagnetic fields)

It is of interest to note that scaled unitary mappings are, in fact, the most general class of mappings for which it is sensible to require an *intrinsic* property of a field, such as a measure of coherence, to stay unchanged. To see why that is so, let us begin by observing that a non-invertible mapping will irreversibly remove components of the field and such a mapping may even nullify the field completely. Thus it is not reasonable to assume that intrinsic properties of the field would be preserved by a non-invertible mapping. Accordingly, we restrict our considerations to invertible mappings. Suppose then that we have a field u, whose cross-spectral density operator W_u has a Mercer series of the form (3.10), where $\lambda_1 = \sum_{n=2}^{\infty} \lambda_n$, with many of the eigenvalues in the sum being of almost equal magnitude. Then W_u can be interpreted as the sum of a coherent part, as represented by the first term in the Mercer series, and an incoherent part, as represented by the rest of the Mercer series, of equal total intensity[§].

Let us now consider the linear mapping $M(\gamma)$ as given by

$$M(\gamma) = \gamma \phi_1 \phi_1{}^{\dagger} + (1-\gamma) \sum_{n=2}^{\infty} \phi_n \phi_n{}^{\dagger}, \tag{6.20}$$

where the functions ϕ_n are the orthonormal eigenfunctions of W_u. This operator attenuates the coherent part of the field by the factor γ and the incoherent part of the field by the factor $1-\gamma$, so that it is a invertible mapping when $0 < \gamma < 1$.

We put $u' = M(\gamma)u$, whereby the cross-spectral density operator $W_{u'}$ effectively represents the coherent part of W_u, when $\gamma \approx 1$, and the incoherent part of W_u, when $\gamma \approx 0$. Now, if an intrinsic measure of coherence is required to be unchanged by invertible mappings, it must assign the same degree of coherence to both of these extremes, which is clearly not reasonable if coherence is assumed to be related to correlations and/or fringe visibility in any useful way. Indeed, since all actual fields are 'contaminated' with noise, a general invertible mapping could suppress the actual field in favor of the noise, so that the intrinsic measure of coherence would essentially be the measure of coherence in the noise and not in the field of interest.

From the above (counter)example it is clear that the problem with general invertible mappings is that they can attenuate different components of the field in different ways, which may, as the example shows, lead to a situation where the same field can be made almost completely coherent and almost completely incoherent by such mappings. Hence it follows that, when invariance of the intrinsic concept of coherence of a field is considered, it only makes sense to include mappings that attenuate all components equally. The most general class of mappings with this property is precisely the set of scaled unitary mappings (which also includes those unitary mappings, which amplify all the components of the field by the same amount). On the other hand, as the set of scaled unitary mappings already leads to many reasonable possibilities for an invariant measure of coherence, as shown in the previous section, it is not useful to restrict this set of mappings further. These considerations give us:

Theorem 4 *The nontrivial intrinsic properties of a field are precisely those properties that are invariant with respect to scaled unitary mappings.*

[§]Such a division into a coherent and an incoherent part, although usually with different total intensities, is typical for supercontinuum fields [132, 133].

6.6 Local measure of coherence based on the effective degree of coherence

In view of the invariance property of the effective degree of coherence and the seemingly difficult task of defining a two-point degree-of-coherence function for vector-valued fields (see discussion in Sec. 3.2), it seems natural to try to define such a function in terms of the effective degree of coherence. The most direct approach then is to go back to Young's double pinhole experiment and use the effective degree of coherence of the total field behind the two apertures to *define* the two-point degree of coherence between the two points. It turns out that the function thus obtained is directly related to the function μ_{L} [72]. The only difference is that the quantity μ_{L} is shifted and normalized so that it attains the value 0 for complete incoherence in the sense defined in Ref. 72. As was discussed in Sec. 3.2, such differences are trivial and we can thus take the two measures of the degree of coherence to be equal in all important respects.

There is, however, a slight problem with μ_{L} and its relation to the effective degree of coherence. To see this, let us assume that $\overline{\mu}_u^2$ is computed for an ensemble $\{u\}$ of Hilbert space functions $u \in L^2(\Omega)$ (or more generally $u \in H$) and let us partition the region Ω into subregions Ω_j (divide the Hilbert space H into mutually orthogonal subspaces H_j). We correspondingly put

$$u = \sum_j u_j, \tag{6.21}$$

where each term vanishes outside its region Ω_j (it belongs to the subspace H_j), so that the terms are mutually orthogonal, or $u_{j'}{}^{\dagger} u_j = 0$ for $j' \neq j$. When we introduce the representation (6.21) into the definition (6.2), we get after applying the orthogonality property the expression

$$\overline{\mu}_u^2 = \frac{\sum_{j,j'} \mathrm{Tr}\left(\langle u_j u_{j'}{}^{\dagger}\rangle \langle u_{j'} u_j{}^{\dagger}\rangle\right)}{\left[\sum_j \mathrm{Tr}\left(\langle u_j u_j{}^{\dagger}\rangle\right)\right]^2} = \frac{\sum_{j,j'} \mathrm{Tr}\left(\langle u_j u_j{}^{\dagger}\rangle\right) \mathrm{Tr}\left(\langle u_{j'} u_{j'}{}^{\dagger}\rangle\right) \overline{\mu}_{u,j,j'}^2}{\left[\sum_j \mathrm{Tr}\left(\langle u_j u_j{}^{\dagger}\rangle\right)\right]^2}, \tag{6.22}$$

where we have defined the effective degree of coherence between two regions (Hilbert spaces) j and j' by its square as

$$\overline{\mu}_{u,j,j'}^2 = \frac{\mathrm{Tr}\left(\langle u_j u_{j'}{}^{\dagger}\rangle \langle u_{j'} u_j{}^{\dagger}\rangle\right)}{\mathrm{Tr}\left(\langle u_j u_j{}^{\dagger}\rangle\right) \mathrm{Tr}\left(\langle u_{j'} u_{j'}{}^{\dagger}\rangle\right)}. \tag{6.23}$$

We note that this is a direct generalization of the first part of the original scalar definition (6.1) of the effective degree of coherence. Now, for notational simplicity we consider, instead of μ_{L}^2, the square of the effective degree of coherence as obtained for two (separate) locations as would occur in Young's experiment.

From Eq. (6.22) we have for the two-region case the result

$$\overline{\mu}^2_{\text{Young}} = \frac{\mathrm{Tr}\left(\langle u_1 u_1{}^\dagger\rangle\langle u_1 u_1{}^\dagger\rangle\right) + \mathrm{Tr}\left(\langle u_2 u_2{}^\dagger\rangle\langle u_2 u_2\rangle\right) + 2\,\mathrm{Tr}\left(\langle u_1 u_2{}^\dagger\rangle\langle u_2 u_1{}^\dagger\rangle\right)}{\left[\mathrm{Tr}\left(\langle u_1 u_1{}^\dagger\rangle\right)\mathrm{Tr}\left(\langle u_2 u_2{}^\dagger\rangle\right)\right]^2}. \tag{6.24}$$

When we compare this representation to the two-region (two-point) component of the sum in Eq. (6.22) as given in Eq. (6.23), we note that the first two terms correspond to the coherence properties of each component separately. Since these terms are non-negative, it follows that if contributions of the form (6.24) are added pairwise, however weighted, the result will not equal the representation (6.22), because the 'additional' terms do not cancel.

Thereby a two point degree-of-coherence function, which is based on the effective degree of coherence, such as μ_{L}, will not be self-consistent in the sense that the effective degree of coherence cannot be obtained from it by weighted averaging. Instead it seems sensible to base a two point degree-of-coherence function on the definition (6.23) for the effective degree of coherence *between* two regions. Indeed, when the field is scalar and the regions are shrunk to singular points, this definition reproduces the squared modulus of the scalar two-point degree-of-coherence function (3.24). For an electromagnetic field, where the regions are shrunk to consist of the vector components at two points, the definition (6.23) reproduces the squared degree-of-coherence function μ_{STF}. This is of interest, since it sets that degree-of-coherence function of vector-valued fields apart from all the other degree-of-coherence functions for such fields as defined in Sec. 3.2.

6.7 Unique additivity property of the effective degree of coherence

We observe that the expressions (6.22) and (6.23) reveal a property of the effective degree-of-coherence functional $\overline{\mu}$ not shared by any other functional which is invariant to scalar unitary transformations. This property is that $\overline{\mu}$ can be computed for a sum of orthogonal fields if the value of the *pairwise* effective degree-of-coherence functional of every pair of components, including the effective degree of coherence of each component with itself, is known. No other functional of the form (6.14), considered in the previous sections shares this property. To prove this, it is sufficient to show that only $\overline{\mu}_u$ as given by Eq. (6.2) is additive for a specifically chosen set of random fields u. For that purpose, it is useful to note that the two largest normalized eigenvalues ν_1 and ν_2 (or any other two normalized eigenvalues) of a cross-spectral density operator W_u can be uniquely determined when $\overline{\mu}_u^2$ and all the other normalized eigenvalues are known. Thereby, we can rewrite Eq. (6.14) as

$$\tau(W_u) = \tau_{\overline{\mu}}\left(\overline{\mu}_u^2, \nu_3, \nu_4, \ldots\right), \tag{6.25}$$

where $\overline{\mu}_u^2 = \mathrm{Tr}(W_u^2)/[\mathrm{Tr}(W_u)]^2$.

Let us now consider the sum-field $u = u_1 + u_2$ of two orthogonal rank-1 fields $u_1 = \tilde{u}_1\phi_1$ and $u_2 = \tilde{u}_2\phi_2$, where $\{\phi_1, \phi_2\}$ is an orthonormal set. The cross-spectral density operator W_u can then be represented with respect to that set as the matrix

$$W_u = \begin{bmatrix} \langle|\tilde{u}_1|^2\rangle & \langle\tilde{u}_1\tilde{u}_2^*\rangle \\ \langle\tilde{u}_1\tilde{u}_2^*\rangle^* & \langle|\tilde{u}_2|^2\rangle \end{bmatrix}. \tag{6.26}$$

The two nonzero normalized eigenvalues of the operator W_u are given by the expressions

$$\nu_{1,2} = \frac{1}{2}\left\{1 \pm \frac{\sqrt{\left(\langle|\tilde{u}_1|^2\rangle - \langle|\tilde{u}_2|^2\rangle\right)^2 + 4\left|\langle\tilde{u}_1\tilde{u}_2^*\rangle\right|^2}}{\langle|\tilde{u}_1|^2\rangle + \langle|\tilde{u}_2|^2\rangle}\right\}, \tag{6.27}$$

which show that the (normalized) eigenvalues and any functional τ of these eigenvalues are independent of the phase of the correlation coefficient $\langle\tilde{u}_1\tilde{u}_2^*\rangle$. Therefore, for a functional τ to be additive, it must be independent of these phases for any number of added orthogonal fields, which means that $\tau(W_u)$ must be independent of the phases of the off-diagonal elements of the corresponding (matrix-form) representation of W_u.

We then take a normalized (unit trace) cross-spectral density operator W_u, whose nonzero eigenvalues ν_n are distinct, and define a family of cross-spectral density operators $W_j(\psi)$ based on the real parameter ψ and the normalized eigenvalues ν_j for $j \geq 3$ (so that $\nu_j \neq \nu_1, \nu_2$), by setting (in block-matrix notation)

$$W_j(\psi) = \begin{bmatrix} A_j(\psi) & & & & & \\ & \nu_3 & & & & \\ & & \ddots & & & \\ & & & \nu_{j-1} & & \\ & & & & \nu_{j+1} & \\ & & & & & \ddots \end{bmatrix}. \tag{6.28}$$

Here the matrices $A_j(\psi)$ are given by

$$A_j(\psi) = \begin{bmatrix} A_{j,11} & A_{j,12}e^{i\psi} & A_{j,13} \\ A_{j,21}e^{-i\psi} & A_{j,22} & A_{j,23} \\ A_{j,31} & A_{j,32} & A_{j,33} \end{bmatrix}, \quad A_j(0) = X_j \begin{bmatrix} \nu_1 & 0 & 0 \\ 0 & \nu_2 & 0 \\ 0 & 0 & \nu_j \end{bmatrix} X_j^\dagger \tag{6.29}$$

and X_j are the orthonormal (unitary) transformation matrices

$$X_j = \begin{bmatrix} \cos\theta_j & \sin\theta_j\cos\varphi_j & \sin\theta_j\sin\varphi_j \\ -\sin\theta_j & \cos\theta_j\cos\varphi_j & \cos\theta_j\sin\varphi_j \\ 0 & -\sin\varphi_j & \cos\varphi_j \end{bmatrix}. \tag{6.30}$$

The angles θ_j and φ_j will be fixed later. Since each operator $W_j(\psi)$ is obtained from W_u by a unitary transformation followed by a modification of the phase of an off-diagonal element, the assumptions on τ imply that $\tau[W_j(\psi)] = \tau(W_u)$. In addition, since the eigenvalues of W_u are distinct, a sufficiently small ψ will not affect their ordering and hence the eigenvalues $\nu_{j,k}(\psi)$ of $W_j(\psi)$ in particular satisfy $\nu_{j,k}(\psi) = \nu_k$ for $k \neq 1, 2, j$. Furthermore, the presented construction ensures that the normalization $\mathrm{Tr}[W_j(\psi)] = \mathrm{Tr}(W_u) = 1$ is preserved and that $\mathrm{Tr}[W_j^2(\psi)] = \mathrm{Tr}(W_u^2) = \overline{\mu}_u^2$. Therefore we have

$$\tau(W_u) = \tau\left[W_j(\psi)\right] = \tau_{\overline{\mu}}[\overline{\mu}_u^2, \nu_3, \nu_4, \ldots \nu_{j,j}(\psi), \ldots], \tag{6.31}$$

where, for each (sufficiently small) ψ and each j, the function $\nu_{j,j}(\psi)$ will be in the $(j-1)$st argument of the function $\tau_{\overline{\mu}}$.

From the definitions (6.28) and (6.29) we note that $\nu_{j,1}(\psi)$, $\nu_{j,2}(\psi)$ and $\nu_{j,j}(\psi)$ are the eigenvalues of the 3×3 Hermitian matrix $A_j(\psi)$. Because these eigenvalues are distinct, it follows that they are differentiable functions of the parameter ψ, with the first two derivatives at $\psi = 0$ given by (we have here dropped the explicit references to $\psi = 0$)

$$\partial_\psi \nu_{j,k} = x_{j,k}{}^\dagger (\partial_\psi A_j) x_{j,k} \tag{6.32}$$

and

$$\partial_\psi^2 \nu_{j,k} = x_{j,k}{}^\dagger (\partial_\psi^2 A_j) x_{j,k} - 2 x_{j,k}{}^\dagger (\partial_\psi A_j) \left(A_j - \nu_k I\right)^\# (\partial_\psi A_j) x_{j,k}, \tag{6.33}$$

where $k \in \{1, 2, j\}$, $x_{j,k}$ denotes the column-vector of X_j, which is the normalized eigenvector of $A_j(0)$ corresponding to ν_k, and the superscript # denotes the pseudoinverse. These equations can be obtained for example by differentiating the eigenequations and using the unitarity of the matrices X_j. From the expressions (6.29), (6.30), (6.32), and (6.33) we get explicitly at $\psi = 0$ the values

$$\partial_\psi \nu_{j,j} = 0 \tag{6.34}$$

and

$$\partial_\psi^2 \nu_{j,j} = 2 \frac{\nu_2 - \nu_j}{\nu_1 - \nu_j} (\sin\theta_j \cos\theta_j \sin\varphi_j \cos\varphi_j)^2 \left(\nu_1 - \nu_2 \cos^2\varphi_j - \nu_j \sin^2\varphi_j\right), \tag{6.35}$$

where the latter expression is nonzero if θ_j and φ_j are chosen appropriately, which we assume is done in what follows. Thereby $\nu_{j,j}(\psi)$ changes with ψ, when ψ is varied in the neighborhood of 0. Accordingly, if $\tau_{\overline{\mu}}$ in Eq. (6.31) is a function of its $(j-1)$st argument or ν_j $[\nu_{j,j}(\psi)]$ as indicated, it too will change with ψ, which, however, violates the assumption that τ is independent of the phases of off-diagonal elements. Hence we conclude that $\tau_{\overline{\mu}}$ as given by Eq. (6.25) cannot be a function of its $(j-1)$st argument. Because $j \geq 3$ was here chosen arbitrarily, it then follows that $\tau_{\overline{\mu}}$ cannot be a function of any of its $(j-1)$st arguments for $j \geq 3$ and for the W_u considered, the representation (6.25) reduces to

$$\tau(W_u) = \tau_{\overline{\mu}}\left(\overline{\mu}_u^2\right). \tag{6.36}$$

We have thus shown that for normalized cross-spectral density operators W_u with distinct eigenvalues, an additive τ must have the form (6.36). Since τ is by its definition (6.14) insensitive to the normalization of W_u and since the cross-spectral density operators with distinct eigenvalues are dense in the set of all cross-spectral density operators, we conclude that τ must have the same definition for all cross-spectral density operators and we arrive at:

Theorem 5 *The effective degree-of-coherence functional $\overline{\mu}$ is the only (modulo trivial modifications) functional of a cross-spectral density function, that is both invariant to scaled unitary mappings and additive in terms of constituent fields.*

6.8 Measuring the effective degree of coherence

As was noted in Sec. 6.3, the effective degree of coherence of a field as determined from its Wigner distribution agrees with the effective degree of coherence of the field related to most of the commonly used representations. This connection is useful, since the effective degree of coherence has been determined experimentally in conjunction with laser beam characterization from the Wigner distribution of the beam. In one such experiment the Wigner distribution itself was reconstructed from intensity measurements around the beam waist [118]. The theoretical aspects of this measurement procedure were also considered, and a numerical simulation of its performance was done. The simulation showed that the measurement procedure and, in particular, the effective degree of coherence computed from it, were robust against measurement and computational errors. Indeed, this is expected in view of the fact that the errors can be attributed to random noise, whose intensity is typically just a fraction of the intensity of the full field. Noise then only affects the smaller eigenvalues of the cross-spectral density operator and, in view of Eq. (6.1), has a minor effect on the effective degree of coherence. We mention that the effective degree of coherence has also been determined from the ambiguity function [134] of the field [119], as well as from a realization of Young's double pinhole experiment [120].

It is also interesting to observe that for beam characterization purposes it turns out that the product of the effective degree of coherence of the beam and the beam propagation factor equals the average width of the normalized power spectrum of the beam [119]. This is a useful result, since the latter quantity can be taken as a measure of the ‘roughness’ of the beam profile. Accordingly, one could in principle use the observed profile roughness together with the easily computable beam propagation factor to quickly obtain a coarse estimate of the effective degree of coherence. In practice this would of course require some way to assign roughness numbers to different kinds of beam profiles, which may not be practical.

To conclude, we observe that the example of beam propagation measurements suggests that the effective degree of coherence is not just a theoretical construct of only academic interest, but that it can be measured quite readily at least for laser beams, and that it is directly related to the roughness of the beam profile, that is, to a quantity which has actual significance. This connection to experiments is important, since it gives credence to the explicit and implicit choices that have been made on purely mathematical grounds when the effective degree of coherence has been defined in the way it has.

7. Conclusions and future work

7.1 Summary of main results

In this thesis we have presented work relating to three topics in second-order coherence theory: electromagnetic treatment of blackbody radiation, universality results for fields sourced by homogeneous and isotropic sources, and the effective degree of coherence. Below we summarize the main results of our work in this order, rather than chronologically.

We presented the first derivation of a coherent-mode representation in a three-dimensional volume for the cross-spectral density operator. Such expansions were determined both for a scalar field and for a vector-valued (three-component) electromagnetic field in Publication I. The derivations were done for the electromagnetic blackbody field and for the equivalent scalar field, which consists of uncorrelated plane-waves, and the obtained forms of the representations are valid in any spherical region, with the radius of the region only affecting the eigenvalues and the normalization of the eigenfunctions in the expansions. We also analyzed the distribution of the eigenvalues in these expansions and showed that the number of expansion coefficients needed for a specific relative accuracy in the vector-valued case is twice the number required for the same accuracy in the scalar case. This result nicely confirms the inkling one might have from studying the representation of vector-valued electromagnetic field in terms of scalar fields.

The old radiometric model of a surface blackbody emitter, which has been analyzed mainly in terms of scalar fields before, was studied by us in a full electromagnetic setting, where the blackbody radiation is produced in a cavity, from which it escapes through an aperture in the cavity wall. Although this setup had been considered before, the results we present in Publication II show that the earlier investigations contained an erroneous assumption about the cross-spectral density operator in the aperture, which made the operator presented incompatible with Maxwell's equations. Hence our work is the first in which the cross-spectral density of the blackbody field at a cavity opening has the correct form. We have since considered the problem further and noted that although we reach the correct form, this is done in a somewhat inconsistent way. Specifically, we have taken great care of computing the cross-spectral density correctly at the aperture, but we have done this starting from an expression for the cross-spectral density of the field valid only inside the cavity, far from the cavity walls. Near the cavity walls, where the aperture resides, their influence on the field must be taken into account in a rigorous electromagnetic analysis based on Maxwell's equations. However, as we show in Sec. 4.2, the appropriate modification of the cavity cross-spectral density for the boundary region when no aperture is present, is such that it vanishes for fields that are emitted through the aperture. We emphasize that although this result is exactly what one would expect intuitively, its derivation, which makes the analysis logically consistent, is nevertheless nontrivial.

We have investigated the so-called universality properties of fields sourced by homogeneous and isotropic sources in Publications III, IV, and V. In the first of these papers we prove that the universality result that holds for scalar fields has a counterpart, when a full vector-valued representation is used for electromagnetic fields. In fact, our results show that the universal cross-spectral density operator is proportional to the cross-spectral density operator of blackbody radiation inside a large cavity, or which is the same, the operator is proportional to the imaginary part of the free-space dyadic Green's tensor.

Since the universality results are derived either by assuming that the source region is asymptotically large, or by assuming that it is infinite but that space is asymptotically lossless, it is important to know how large, but finite regions, or small, but finite losses affect these results. We considered the latter of these questions in Publication IV, where we showed that the convergence of the field spatial density toward its universal form is sensitive to the exact functional form of the source region cross-spectral density operator. In particular, our results imply that it is not sufficient to assume that the losses are 'negligible' for the universal behavior to emerge, and we presented an example where at a fixed small relative loss, the cross-spectral density operator of the field can be made to change between the universal form and the cross-spectral density of the source by varying a parameter value. Thereby, since all actual systems have some amount of loss, one should not blindly assume that the loss is 'negligible' for the purposes of universality, even when it seems so from the point of view of other applications.

Furthermore, we have shown in Publication V that the convergence towards universality with an increasing source-region size is also not universal across different functional forms of the source cross-spectral density. Indeed, we prove that for finite regions of arbitrary size and composition, the universality results can be counteracted by a judicious construction of the source cross-spectral density at the boundary, where it does not influence the (local) homogeneity and isotropicity assumption. In fact, we show that the spatial density of the field inside the source region can be made to match the cross-spectral density operator of any twice continuously differentiable function.

Because our results imply that although the universality results appear when certain limits are taken, the emergence of the universal behavior can only be guaranteed when the limit is exactly reached. Therefore, it is likely that the universality results cannot be applied as such to actual systems, which are both finite and lossy. In particular, the inferral, which is often made based on the universality results (of the δ-correlated source), that the blackbody $\lambda/2$ coherence length is the shortest possible coherence length for fields sourced by homogeneous and isotropic sources is not true. Our results furthermore show that for systems of any size, with any amount (or lack of) losses, the coherence length can be made arbitrarily short.

We have proven that for any intrinsic property of an electromagnetic field to be nontrivial it should be invariant to scaled unitary mappings. In Publication VI we prove that the so-called effective (overall, global) degree of coherence has this invariance property. Therefore, as we also have shown, it follows that the effective degree of coherence can consistently be evaluated from all typical representations of the electromagnetic field, and that it remains unchanged when the field is scattered by lossless scatterers. Although the effective degree of coherence is not unique in being invariant to scaled unitary mappings, we have furthermore proven that it is the only intrinsic character, apart from the total energy of the field, that is also additive in the sense that it can be evaluated for a sum of orthogonal fields if the pairwise effective degrees of coherence of the component fields are known. Hence it is the only intrinsic property of the electromagnetic field that strictly obeys the 'pairwise' spirit of a second-order theory. Interestingly, the pairwise effective degree of coherence between orthogonal Hilbert spaces, as introduced by us, is actually the natural generalization of the degree-of-coherence function μ_{STF} from a pair-of-points setting to a pair-of-Hilbert spaces setting. This fact, together with the special role as an intrinsic property of the electromagnetic field enjoyed by the effective degree of coherence, sets the vector-valued degree-of-coherence function μ_{STF} apart from the other proposed vector-valued degree-of-coherence functions presented in Sec. 3.2.

7.2 Suggestions for future work

As was discussed in Sec. 3.3, the coherent-mode expansions are intimately related to the Karhunen–Lòeve expansions of random electromagnetic fields and specifically these expansions can be used to obtain the best possible approximations to such fields. These approximations find use for example when techniques are developed for the efficient transmission of 3D image fields. It is therefore of interest to understand the distribution of eigenvalues of cross-spectral density operators that correspond to typical image fields, because this provides information about the complexity of such fields and thereby gives lower bounds for the achievable approximation errors, which helps in the design of compression algorithms. For best results, such understanding should probably be based on both experimental work and theoretical models.

It is also of interest to study how the finite geometries of actual blackbody cavities influence the exact form of the cross-spectral density function in the cavity aperture. That there will be an influence is already known from prior work, but this body of knowledge has not yet been applied to model the field in the cavity aperture. Such a model should prove useful for electromagnetically rigorous analyses of experiments involving blackbody sources, since actual cavity dimensions are typically such that the infinite-cavity assumption is not completely warranted.

In general, the blackbody results can be expected to find uses, e.g., in the modeling of thermal sources and the propagation of natural light. A typical application is for example lighting design, where both old (incandescent bulbs) and modern (super-luminal leds) light sources produce light that is reminiscent of blackbody radiation. A good understanding of the behavior of blackbody radiation is also beneficial in the development of applications, where heat radiation needs to be controlled, such as solar energy systems and microchip cooling devices.

The construction we have presented for obtaining electromagnetic fields with arbitrarily short coherence lengths, both in a scalar setting and a rigorous electromagnetic setting, could be applied in the development of microscale and nanoscale devices. In particular, it could be used to design sources, which radiate fields with desired coherence and polarization properties. This is important since these properties influence for example the interaction between the fields and atoms. In this context it is of course important also to be able to construct the sources thus designed. One approach that could be used to yield realizations of such sources is to employ nonlinear optics, and in particular the process of difference-frequency generation [27] to convert two short-wavelength pump fields into a field, whose wavelength is much longer. Thereby, if the pump fields are structured so that their coherence lengths are at or near their respective diffraction limits or universal coherence lengths, they provide for the generated field a source, whose coherence length is much shorter than the wavelength of the sourced field.

So far the effective degree of coherence has mainly been studied from a theoretical point of view, with almost no emphasis on the practical applications this property of the electromagnetic field may have. Therefore, the behavior of the effective degree of coherence in optical systems should be investigated in more detail. In particular, it might be of interest to consider the possibility of measuring the effective degree of coherence between the fields at two finite apertures. On one hand, from the theory presented in Sec. 6.7, it follows that this information, together with the field energies at the apertures, is sufficient for determining the effective degree of coherence of entire fields. On the other hand, aperture sizes must be finite in any actual measurement, and hence the degree-of-coherence function μ_{STF} between a pair of points, will necessarily always be approximated by a corresponding effective degree-of-coherence functional between the Hilbert spaces at the two apertures.

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